Inequality decomposition values : the trade-off between marginality and consistency

Frederic Chantreuil, Alain Trannoy

Inequality decomposition values : the trade-off between marginality and consistency

Frédéric Chantreuil*
Alain Trannoy**

N° 99-24

July, 1999

* INRA-ESR, Rennes
** THEMA (UMR CNRS 7536), Université de Cergy-Pontoise

Abstract

A general procedure inspired by the Shapley value is proposed for decomposing any inequality index by factor components or by populations subgroups. To do so we define two types of inequality games. Although these games cannot be expressed in terms of unanimity games, an axiomatization of the Shapley decomposition is provided in this context by using the Potential function pioneered by Hart and Mas-Colell (89). This result proves to be useful to illustrate a trade-off between the desirable properties of consistency and marginality. A comparison with the method promoted by Shorrocks (82) is performed in case of the factor components decomposition. Refinement of the Shapley decomposition is investigated when the set of sources is nested. An application to population subgroups decomposition problem is also discussed.

Résumé

Une procédure générale inspirée par la valeur de Shapley est proposée pour décomposer n'importe quel indice d'inégalité par sources de revenu ou sous-groupes de population. Deux types de jeux d'inégalité sont ainsi définis. Bien que ces jeux ne puissent pas être exprimés en termes de jeux d'unanimité, une axiomatisation de la décomposition de Shapley est fournie dans ce contexte en utilisant la fonction "potentiel" mise en avant par Hart et Mas-Colell (1989). Ce résultat est utilisé pour illustrer le conflit entre les propriétés de cohérence et de marginalité. Une comparaison avec la méthode proposée par Shorrocks (1982) est offerte pour le cas de la décomposition par sources de revenu. Un raffinement de la décomposition de Shapley est introduit lorsque l'ensemble des sources a une structure emboîtée. Une application à la décomposition par sous-groupes de population est également discutée.

Classification JEL: D 63, C 71

Mots-clés : Shapley value, inequality, decomposition

1 Introduction[1]

This paper considers the conceptual issue of decomposition of inequality indices, namely, the decomposition of the aggregate inequality value into some relevant component contributions. The issue to which this kind of analysis has been applied falls into two broad cases. The first one deals with the influence of population subgroups such as these defined by age, sex or race (see, for example, the analytical explorations in Bourguignon (1979), Cowell (1980) and Shorrocks (1980, 1984 and 1988)). It is legitimate to ask for the contribution of each subgroup to the overall inequality. In this framework the additive decomposable property seems sensible. It requires that the overall inequality in a cross section is just the sum of two terms: a weighted sum of *within*-group inequalities, and a *between*-group term computed as if each person within a given group received the group's mean income. This property has proved to be very powerful since it leads to the use of a specific class of inequality measures namely the so-called generalized entropy class. Hence the requirement of additive decomposability prevents the use of some of the most popular inequality index like the Gini index. This raises the question of the existence of broader methods of decomposition valid for the full class of inequality measures.

The second category covers situations in which different components of total income are examined. If we disaggregate the total income into several factor components, such as earnings, property incomes and transfers, we wish to evaluate the contribution of each income source to the aggregate inequality. We can give Shorrocks (1982) credit for setting up the axiomatic foundations of the decomposition by income components[2]. Shorrocks shows that the natural decomposition of the variance is the only decomposition rule satisfying six properties for any inequality measure. By the way for each income component, the assessment of its relative contribution to the total income inequality must be independent of the inequality measure chosen. The author adds "This is a particularly attractive feature for those involved in applied research on income distribution (Shorrocks, 1982 p. 209)". Maybe, but for a theoretical point of view, the result is debatable: one can argue that the relative contribution of a component must depend on the inequality measure chosen. More specifically it must depend on how this measure weights a progressive transfer with a regressive one, and whether the index is relative or absolute. To illustrate the first point, let us take an income distribution coming from two identical distributed sources except for a Pigou Dalton transfer which has been performed in the second distribution between two rich individuals. There is no question that the relative contribution of the first source must be greater than one half for all inequality measures, but it will be sensible to add a further requirement: it has to be all the more since the index is sensitive to progressive transfers performed at the top of the income distribution. For instance the relative contribution of the first source must be larger with the coefficient of variation than with the Gini coefficient, the former being known to be specifically sensitive to transfers among two rich individuals. The difference concerning the relative or absolute character of the indexes provides another reason to dispute the validity of the decomposition rule proposed by Shorrocks. The following example illustrates that the contribution

[1]We thank A. Shorrocks for a discussion which has revived our interest in this topic, Nicolas Gravel, Christophe Muller, Nathalie Picard and Vincent Feltkamp, Shmuel Zamir and participants in Crest seminar for helpfull comments. Earlier versions of the paper has been presented at the ESEM congress in Berlin and at the forth conference *Social choice and Welfare* inVancouver. The financial support of *The Distribution and Redistribution of Income Network*, contrat ERBCHRXCT940647 is gratefully aknowledged. The usual *caveat* applies.

[2]Foster and Sen (1997 p.149) even wrote that Shorrocks (1982) has provided a definitive study of the alternative methodologies !

requirement, marginal contribution has to be computed according to some function different from the inequality index. This idea leads to the following desirable property. There must be some function related to the inputs of the model, the set of components and the inequality index, such that the contribution of any component is just equal to the marginal contribution according to this *new* function. This property would be meaningless if such a function did not exist for some inequality index or if we could build an infinity of such functions. Fortunately Hart and Mas-Colell (1989) proved that such a function exists and is unique. Furthermore they proved that the only decomposition rule satisfying this "marginality" property and the requirement of consistency is the Shapley inequality decomposition. In other words, we only have one way to extend the problem of decomposition such that the marginalist interpretation remains valid under the efficiency constraint. Our one merit here lies in a slight rewriting of the axioms and of the proof in order that it would be clear that no linearity assumption of the games space is involved in the characterization. While this point may seem obvious to game theorists, it is worth emphasizing this point when one addresses economists primarily interested in inequality measurement. Indeed a crucial feature of inequality games is that they cannot be decomposed in a linear combination of unanimity games. Maybe first attempts like Auvray and Trannoy (1992) or Rongve (1993) in applying Shapley value to inequality issues failed to tackle this issue properly.

As far as decomposition by sources is concerned, the Shapley value inequality decomposition offers some advantages. It is sensitive to the choice of inequality index. For example, the contribution to inequality of an equally distributed factor component is zero if the index is absolute and negative is the index is relative. But this property does not preclude the Shapley inequality decomposition by sources to correspond to the natural decomposition of the variance if the chosen inequality index is the variance. Moreover the Shapley contribution of a specific source can be regarded as an inequality index when it is studied as a function of an income distribution. But these properties have been obtained at some cost. The Shapley decomposition does not respect a somewhat natural axiom of independence introduced by Shorrocks (1982). The contribution of a source must be independent from the level of disaggregation, i.e, the number of other sources considered. A partial answer can be found in an extension of the Shapley value promoted by Chantreuil (1998). The Nested-Shapley value leads to a decomposition of inequality indices which satisfies a milder request of independence, once a more general framework consisting in some partition of the set of sources has been introduced. For instance some income sources can be labelled as market incomes while others can be considered as transfers. With the Nested-Shapley inequality decomposition value the contribution of a labor income would be independent from the number of sources gathered under the label of transfers.

The application of Shapley value to population subgroups decomposition problem presents some attractive features even though it is more questionable. Indeed the between group term disappears and is absorbed by the within group terms. But on the other hand the Shapley decomposition satisfies a number of nice properties at least in case of two subgroups.

The outline of the rest of the paper is as follows. The first three sections deal with what one can see as the most natural application of Shapley decomposition, namely the decomposition by factor components. We begin in section 2 with some general results about the inconsistency of marginal decomposition. Section 3 is devoted to the definitions of what we call the sources inequality games and we state the main result about the Shapley inequality decomposition and its properties. Section 4 extends the Shapley value decomposition to inequality games in which the set of components is a priori decomposed into a partition of subgroups of components. Section 5 applies the previous results to the population sub-

Since I is relative

$$I(\frac{k-1}{k}X) = I(X)$$

We deduce

$$\sum_{j=1}^{k} I(X_{-j}) \geq kI(X) > (k-1)I(X)$$

or

$$\sum_{j=1}^{k}[I(X) - I(X_{-j})] < I(X)$$

■

When we enlarge our attention to quasi-convex inequality indices, the landscape is more intricate.

Proposition 2.2 *Let K be a set of two sources. If the inequality index is strictly quasi-convex and relative, the marginalist decomposition rule is strictly underconsistent.*

Proof. From the definition of strict quasi-convexity, we deduce that

$$I[\frac{x^1 + x^2}{2}] < max[I(x^1), I(x^2)]$$

Then

$$I[\frac{x^1 + x^2}{2}] < I(x^1) + I(x^2)$$

On the other hand, since I is relative

$$I[\frac{x^1 + x^2}{2}] = I(x^1 + x^2)$$

Therefore

$$I(x^1 + x^2) < I(x^1) + I(x^2)$$

or

$$I(X) - I(x^1) + I(X) - I(x^2) < I(X)$$

■

The following counterexample shows that the above proposition cannot be extended to more than two factor components.

Example 2.1 *Let $n = 2$, $k = 3$, $x^1 = (3,1)$, $x^2 = (1,3)$, $x^3 = (5,3)$ and therefore $X = (9,7)$. $I(x^1+x^2) = I(x^2+x^3) = 0$ for any inequality index. The issue of consistency depends of the relative magnitude of $I(x^1 + x^3)$ and $2I(X)$. Let us take $I(x) = 1 - \Pi_{i=1}^{3}(\frac{x_i}{\mu_X})^{\frac{1}{3}}$. It is well known that $I(X)$ is quasi-convex but not convex for X belonging to a given simplex. It turns out that $2I(X) = 2 - \frac{3\sqrt{7}}{8} > 1 - \frac{\sqrt{2}}{3} = I(x^2 + x^3)$ which proves that the marginal decomposition is overconsistent for these values of parameters.*

Proposition 2.4 *Consider two sources with equal mean. If the inequality index is relative and if there exists a source which is regarded more unequally distributed than the other for every relative inequality index, then the marginal decomposition is strictly underconsistent.*

Proof. Let $x_{(1)} \leq \ldots \leq x_{(i)} \ldots \leq x_{(n)}$ denote the components of x in increasing order. Without lost of generality, assume that $I(x^2) \geq I(x^1)$, $\forall I \in I_r$. It implies that

$$\frac{1}{n\mu}\sum_{i=1}^{l} x_{(i)}^1 \geq \frac{1}{n\mu}\sum_{i=1}^{l} x_{(i)}^2, \quad l = 1, \ldots, n-1 \quad (3)$$

with a strict inequality for at least some l.
Obviously, we have

$$\frac{1}{n\mu}\sum_{i=1}^{l} x_{(i)}^2 = \frac{1}{n\mu}\sum_{i=1}^{l} x_{(i)}^2, \quad l = 1, \ldots, n-1 \quad (4)$$

Adding (3) and (4) we obtain

$$\sum_{i=1}^{l}[\frac{x_{(i)}^1}{n\mu} + \frac{x_{(i)}^2}{n\mu}] \geq 2\sum_{i=1}^{l}[\frac{x_{(i)}^2}{n\mu}] \quad l = 1, \ldots, n-1 \quad (5)$$

with a strict inequality for at least some l.
From a result proved by Day (1972) (see Marshall and Olkin (1979)) we deduce

$$\sum_{i=1}^{l}[\frac{x_{(i)}^1}{n\mu} + \frac{x_{(i)}^2}{n\mu}]_{(i)} \geq \sum_{i=1}^{l}[\frac{x_{(i)}^2}{n\mu} + \frac{x_{(i)}^2}{n\mu}] \quad l = 1, \ldots, n-1 \quad (6)$$

with a strict inequality for at least some l.
Combining (5) and (6) we deduce that

$$\frac{1}{2}\sum_{i=1}^{l}[\frac{x^1}{n\mu} + \frac{x^2}{n\mu}]_{(i)} \geq \sum_{i=1}^{l}[\frac{x_{(i)}^2}{n\mu}] \quad l = 1, \ldots, n-1 \quad (7)$$

with a strict inequality for at least some l, which means that for every relative inequality index $I \in I_r$ we must find $I(x^2) > I(x^1 + x^2)$ and therefore $I(x^1) + I(x^2) > I(x^1 + x^2)$ ∎

As a consequence of the above proposition, we obtain that underconsistency holds when there are only two individuals in the society and two sources since all relative inequality agree on the ranking of sources in that case. A generalization in case of three sources can be obtained.

Proposition 2.5 *Consider three sources with equal mean . If x^1, x^2 and x^3 are similarly ordered and the ranking of the three sources is identical whatever the relative inequality index $I \in I_r$ is, then the marginal decomposition is strictly underconsistent.*

Proof. Without lost of generality, let assume that $I(x^1) > I(x^2) > I(x^3)$. From the proof of proposition 4, we are able to establish that

$$I(x^2) > I(x^1 + x^2) \;\; \forall\, I \in I_r \quad (8)$$

$V_I(\emptyset) = 0$. Therefore $V_I(K) = I(X)$ and $V_I(S) = I(y(S))$, for all S different of $\emptyset$. Let us define

$$\mathbf{V}_I = \{V_I : \exists y : 2^K \to R^n | V_I = I \circ y\}$$

In the second computation the distribution of income among subsets of sources is obtained by equalizing complementary sources, i.e. we define: $y^e : 2^K \to R^n$ such that, $y^e(\emptyset) = 0$, and for all $S \in 2^K$, $S \neq \emptyset$,

$$y^e(S) = (\sum_{j\in S} x_1^j + \mu_X - \mu_{Y(S)}, ..., \sum_{j\in S} x_n^j + \mu_X - \mu_{Y(S)})$$

The characteristic function is given by $V_I(S) = I(y^e(S))$ for all S different of $\emptyset$ and $V_I(\emptyset) = 0$. Let define

$$\mathbf{V}_I^e = \{V_I : \exists y^e : 2^K \to R^n | V_I = I \circ y^e\}$$

The following definition summarizes the above discussion.

Definition 3.1 *An income sources inequality game is a pair (K, V_I), where K is the set of players and V_I is a function defined on all subsets $S \in 2^K$ such that $V_I \in \mathbf{V}_I \cup \mathbf{V}_I^e$. A zero-income (respectively an equalized) source inequality game coins for an element of $\mathbf{V}_I$, (resp for an element of $\mathbf{V}_I^e$). Let us denote by G_I the set of all income sources inequality games generated by an inequality index I.*

Let us remark that in each case, the characteristic function is not assumed to be superadditive[7]. Furthermore some *unanimity* games do not belong to G_I. Let us remind that a S-unanimity game is a game (K, V) such that there exists a nonempty set $S \subset K$ interpreted as a single minimum winning coalition such that $V(T) = V(S)$ if $S \subset T$, and $V(T) = 0$ otherwise. For example K can never be a single minimum winning coalition for some inequality game. If the inequality for the overall set of sources is strictly positive, it means that there must exist at least one unequally distributed source. Therefore the inequality for all subsets of sources including this source is strictly positive which invalids that K can be the minimum winning coalition. By the way a zero-income or an equalized inequality game one cannot be expressed as a linear combination of *unanimity* games. The absence of linearity is a specific feature of the inequality games and it has some severe consequences. Most axiomatizations of the Shapley value use this property of the space of TU-games, and then cannot be supposed to be true when the restriction of domain to inequality games is considered[8].

We now turn to the axiomatization of the Shapley inequality decomposition which allows to conciliate a marginalist interpretation with the consistency requirement.

Definition 3.2 *A consistent decomposition rule is a function φ that assigns to every income sources inequality game $(K, V_I) \in G_I$ a vector of k components which are not restricted to be positive, such that:*

$$\sum_{j=1}^{k} \varphi_j(K, V_I) = I(X) \quad (11)$$

[7]We say that a characteristic function Z is superadditive if and only if, for every pair of coalitions S and T, if $S \bigcap T = \emptyset$, then $Z(S \bigcup T) \geq Z(S) + Z(T)$.

[8]In particular the axiomatizations of Shapley (1953) and Young (1985) cannot be supposed to be true.

The proof is straightforward. Let us assume that some source j is equally distributed. In case of an absolute index, $V_I(S) - V_I(S - \{j\}) = 0$ for all $S \in 2^K$. In case of a relative index, $V_I(S) - V_I(S - \{j\}) < 0$ for all $S \in 2^K$, $S \neq \{j\}$ and for $S = \{j\}$, we have $V_I(\{j\}) - V_I(\emptyset) = 0$. Now for an equalized income sources inequality game the Shapley contribution of source j is computed as:

$$Sh_j(K, V_I) = \sum_{\substack{S \subset K \\ j \in S}} \frac{(s-1)!k-s)!}{k!} [I\ (\sum_{h \in S} x^h + \mu_X - \mu_{Y(S)}) - I(\sum_{\substack{h \in S \\ h \neq j}} x^h + \mu_X - \mu_{Y(S-\{j\})})] (14)$$

The result comes from the fact that if x^j is equally distributed we have for all S including j

$$\sum_{h \in S} x^h + \mu_X - \mu_{Y(S)} = \sum_{\substack{h \in S \\ h \neq j}} x^h + \mu_X - \mu_{Y(S-\{j\})}$$

We deepen this study of the Shapley inequality decomposition by wondering whether Sh_j (as a function of x_j) is an inequality index ? The answer depends of the relative or absolute type of the inequality index. More precisely we can state:

Proposition 3.2 *(i) For a zero income sources inequality game, if the inequality index is absolute, $Sh_j(x^j)$ is an absolute inequality index, while in case of a relative inequality index $Sh_j(x^j)$ satisfies all the properties of an inequality index except the equality to zero when the source is equally distributed. (ii) For an equalized income sources inequality game, $Sh_j(x^j)$ is an absolute (resp. relative) inequality index if the inequality index is absolute (resp. relative).*

Proof. For a zero income -sources inequality game the Shapley contribution attributed to source j is computed as:

$$Sh_j(K, V_I) = \sum_{\substack{S \subset K \\ j \in S}} \frac{(s-1)!k-s)!}{k!} [I\ (\sum_{h \in S} x^h) - I(\sum_{\substack{h \in S \\ h \neq j}} x^h)] (15)$$

Symmetry: I is invariant to a permutation of the individuals and so Sh_j.

Schur-convexity: Let us assume that x'^j has been obtained from x^j through a finite sequence of progressive transfers while $x'^h = x^h \, \forall h \neq j$. By the way $\sum_{h \in S} x'^h$ is obtained from $\sum_{h \in S} x^h$ through a finite sequence of progressive transfers while $\sum_{\substack{h \in S \\ h \neq j}} x'^h = \sum_{\substack{h \in S \\ h \neq j}} x^h$. Therefore $I\ (\sum_{h \in S} x'^h) \leq I\ (\sum_{h \in S} x^h)$ while $I(\sum_{\substack{h \in S \\ h \neq j}} x'^h) = I(\sum_{\substack{h \in S \\ h \neq j}} x^h)$. Then $Sh_j(x'^j) \leq Sh_j(x^j)$.

Principle of population: I is invariant to a replication of individuals and so Sh_j.

By using remark 3.1 and the facts that the absolute or relative property of I are kept by Sh, (i)'s proof is complete. Now for an equalized income sources inequality game the proof is analogue except the fact that an equal income distribution has a null contribution whatever the inequality index is absolute or relative. ■

Considering now the equalized inequality games, an important remark has to be done:

Proof. Let us prove in case of a zero income sources inequality game. By remark 3.2 the result extends to the case of an equalized inequality game. If inequality is measured with the variance, then the computation of the characteristic function gives for all $S \subseteq K$ where $\rho_{hh'}$ stands as the coefficient of correlation between source h and h':

$$V_{\sigma^2}(S) = \sum_{h \in S} \sigma^2(x^h) + \sum_{h \in S} \sum_{\substack{h' \in S \\ h' \neq h}} \rho_{hh'} \sigma(x^h) \sigma(x^{h'})$$

and

$$V_{\sigma^2}(S - \{j\}) = \sum_{\substack{h \in S \\ h \neq j}} \sigma^2(x^h) + \sum_{\substack{h \in S \\ h \neq j}} \sum_{\substack{h' \in S \\ h' \neq h,,j}} \rho_{hh'} \sigma(x^h) \sigma(x^{h'})$$

thus

$$V_{\sigma^2}(S) - V_{\sigma^2}(S - \{j\}) = \sigma^2(x^j) + 2 \sum_{\substack{h,j \in S \\ h \neq j}} \rho_{jh} \sigma(j) \sigma(h)$$

Then, the contribution of source j to the overall inequality reads:

$$Sh_j(K, V_{\sigma^2}) = \sum_{\substack{S \subseteq K \\ j \in S}} \frac{(s-1)!(h-s)!}{h!} [\sigma^2(x^j) + 2 \sum_{\substack{h \in S \\ h' \neq j}} \rho_{jh} \sigma(x^j) \sigma(x^h)]$$

since

$$\sum_{\substack{S \subseteq K \\ j \in S}} \frac{(s-1)!(k-s)!}{k!} = 1$$

and

$$\sum_{\substack{S \subseteq K \\ j,h \in S, h \neq j}} \frac{(s-1)!(k-s)!}{k!} = \frac{1}{2}$$

we conclude that

$$Sh_j(K, V_{\sigma^2}) = \sigma^2(x^j) + \sum_{h \neq j} \rho_{jh} \sigma(x^j) \sigma(x^h) = Cov(x^j, X)$$

■

The variance is seldom used as a measure of inequality primarily because it is not mean independent. The relative index linked to the variance is the coefficient of variation or the square of the coefficient of variation which belongs to the entropy family and is a monotone transformation of the former. The *natural decomposition of the square of the coefficient of variation* attributes to each source the same proportion of total inequality that attributed by the *natural decomposition of the variance*. In that sense they are identical[10]. This is no longer true for Shapley decomposition in case of a zero income sources inequality game.

[10] See Shorrocks (1982) p 195.

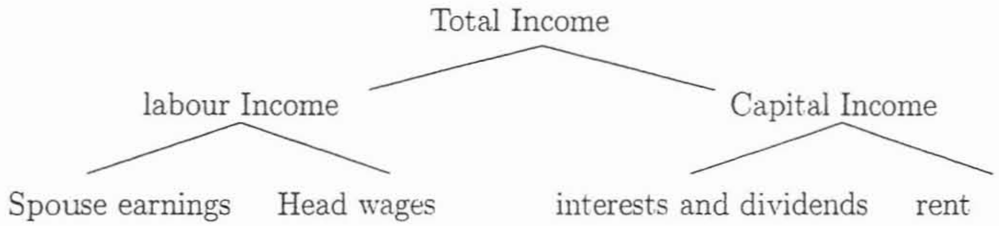

Figure 1

Consider for example the contribution of spouse's earnings. With the Shapley inequality decomposition, the contribution of this source will depend of the number of sources considered in the disaggregation of both labor income and capital income. A milder requirement would be that the contribution of spouse's earnings would at least be independent on the disaggregation of capital incomes and by way of consequence, the contribution of labor income would also be independent on the disaggregation of capital income. It turns out that this milder requirement of independence on the level of disaggregation leads to applications of another value of cooperative games which is derived from the Shapley value: the Nested-Shapley value.

Some further notations are needed to be introduced. A partition of the set of income sources K is the set $\mathcal{P}_K = \{S_1, ..., S_l, ..., S_m\}$ such that for all S_h, $S_l \in \mathcal{P}_K$

$$S_h \bigcap S_l = \emptyset \text{ and } \bigcup_{h=1}^{m} S_h = K \text{ with } 1 < m \leq k$$

A partitioned inequality game is a triple $(K, \mathcal{P}_K, V_I)$ and let G_I^P be the set of these games when (K, V_I) belongs to G_I. Let observe that (K, K, V_I) is a partitioned game where the set of components is the partition itself. Therefore (K, K, V_I) is identical to (K, V_I). At the opposite $(\mathcal{P}_K, \mathcal{P}_K, V_I)$ will be labelled as the "the top" game, the game where the players are the subgroups themselves. Of course $(\mathcal{P}_K, \mathcal{P}_K, V_I)$ is identical to $(\mathcal{P}_K, V_I)$. Let G_I^T be the set of top games when (K, V_I) belongs to G_I. We face again a dilemma between the requirement of consistency and marginality. In the context of partitioned inequality games it is tempting to demand consistency not only over the whole set of components but also within each subset of components.

Definition 4.1 *A consistent decomposition rule is a function φ that assigns to every partitioned inequality game $(K, \mathcal{P}_K, V_I) \in G_I^P$ a vector of k components, such that:*

$$\sum_{j=1}^{k} \varphi_j(K, \mathcal{P}_K, V_I) = V_I(K) = I(X) \quad (17)$$

Furthermore we strengthen the consistency property by requiring it should be satisfied among the elements of the same subset of components. Precisely we require:

Definition 4.2 *A decomposition rule φ satisfies the within-consistency property if for any S_l, $l = 1, ..., m$*

$$\sum_{j \in S_l} \varphi_j(K, \mathcal{P}_K, V_I) = \varphi_l(\mathcal{P}_K, \mathcal{P}_K, V_I) \quad (18)$$

Equation (18) requires that the sum of the contributions of components belonging to some element of the partition adds up to the contribution of this specific subset of contribution in the top game.

Proof. From proposition 3.1 we can deduce that the function P is unique. Furthermore, we know that the contribution of any subset of sources $S_l \in \mathcal{P}_K$ equals the Shapley value of this subset of sources in the top game $(\mathcal{P}_K, V_I)$. By the same token, we can deduce that for all l, $l = 1, ..., m$ the function P_l is unique and the contribution of any component $j \in S_l$ equals the Shapley value of this component in the subgame (S_l, V_I^s). Hence, the searched decomposition corresponds to the product of the Shapley values of the games (S_l, V_I^s) and $(\mathcal{P}_K, V_I)$, that is the Nested Shapley decomposition[12].
■

A generalization of this result can be obtained at the price of the introduction of additional notations. In this section we have only consider situations in which the set of sources is decomposed into a partition of subgroups of sources. A richer setting would be to dispose of a "level structure" in the lingo of cooperative game literature, that is a sequence of partitions of subgroups of sources. The contribution of each source would be computed through another value of cooperative games, usually called a level structure value (see for example Calvo *et al.* 1996). It can be argued that this kind of decomposition will be less and less dependent on the level of disaggregation all the more since the partition of sources is nested. Finally it must be noticed that propositions 3.2, 3.3 and remarks 3.1 and 3.2 remain valid for the Nested Shapley decomposition.

5 Decomposition by population subgroups

Let us consider an income distribution x among a set of individuals $N = \{1, ..., i, ..., n\}$, namely, an application $x : N \rightarrow R_+$. N is supposed to be partitioned between l population subgroups indexed by $j = 1, ..., l$, L denoting the set of all population subgroups x_j stands as the income vector of subpopulation j, μ_j stands as the mean of subpopulation j. For all subsets of population subgroups $S \in 2^L$, we will denote $x(S) = x|_{N_S}$ the income distribution restricted to the population N_S. We adopt the same presentation as served with income components. The marginal decomposition can be underconsistent (for example in case of the union of two subgroups one of which being the replication of the other) or overconsistent (for instance in case of equalized subgroups) as well.

The same idea prevails in defining the population subgroups games. They differ in the treatment of subgroups not included in the considered subset. In the first one, the value of the characteristic function for some subset of population subgroups S is simply the value of the inequality index when subgroups not included in S are removed, while in the second computation it is given by the value of the inequality index when *inequality* is removed from all subgroups not in S.

More precisely in the first computation the distribution x help us to build a distribution of income among subsets of population subgroups, namely an application $y : 2^l \rightarrow \bigcup_{p=1}^{n} R^p$ such that $y(S) = x(S)$, for all $S \in 2^L$ and $y(\emptyset) = 0$ by convention.

We assume that inequality is measured by an inequality index: $I : \bigcup_{p=1}^{n} R^p \rightarrow [0, +\infty)$ with $I(R) = 0$. The first computation of the characteristic function with respect to the inequality index, would be the composite function $W_I : 2^L \rightarrow [0, +\infty)$ such that $W_I = I \circ y$, with $W_I(\emptyset) = 0$. Thus $W_I(S) = I(y(S))$ for all S different of $\emptyset$ and $W_I(N) = I(x)$.

[12] See Chantreuil (1998) for a more detailed proof.

done. Let us recall that an entropy inequality measures are the only inequality measures satisfying the additive decomposability property, precisely,

$$I(x) = \sum_{j=1}^{l} w_j(n_j, \mu_j) I(x_l) + I(\mu(N_{1)}, ..., \mu(N_j), ..., \mu(N_l))$$

where the $w_j(n_l, \mu_l)$ stand as the positive weights which depend only of subgroups sizes and means and add up to 1. It can be imagined that the split of the *between group term* would be organized according to the weights or on an equally basis. It turns out that in case of an equalized inequalized inequality game the second option is chosen by the Shapley value in case of two population subgroups,

$$Sh_1(2, V_{I_e}) = w_1(n_1, \mu_1)) I(x_1) + \frac{1}{2} I(\mu(N_{1)}, \mu(N_2))$$

but in the general case the *between group term* enters in a more intricate way. For example with three population subgroups we obtain

$$\begin{aligned} Sh_1(3, V_{I_e}) &= w_1(n_1, \mu_1)) I(x_1) + \frac{1}{3} I(\mu(N_{1)}, \mu(N_2), \mu(N_3)) + \frac{1}{3} I(\mu(N_{1)}, \mu(N_2 + N_3), \mu(N_2 + N_3)) \\ &\quad - \frac{1}{6} I(\mu(N_{2)}, \mu(N_1 + N_3), \mu(N_1 + N_3)) - \frac{1}{6} I(\mu(N_{3)}, \mu((N_1 + N_2)), \mu((N_1 + N_2))) \end{aligned}$$

It is hard to imagine a situation where the sum of the three last terms would vanish without equalizing the mean of the three sources. If this last condition is satisfied it would mean in turn that the *between group term* will also be equal to zero. Then the simple decomposition formula which emerges for two population subgroups is definitely lost in almost situations in the general case.

In case of a zero-income inequality game the Shapley decomposition seems a bit odd since the contribution depends also of the *within term* of other subgroups ; for example in case of two population subgroups we get:

$$Sh_1(2, V_{I_e}) = \frac{1}{2}(1 + w_1(n_1, \mu_1)) I(x_1) - \frac{1}{2}(1 - w_1(n_1, \mu_1)) I(x_2) + \frac{1}{2} I(\mu(N_{1)}, \mu(N_2))$$

This property casts doubt on the interest of the zero-income subpopulation Shapley decomposition. This negative impression is reinforced by the study of the Shapley decomposition in case of an equalized subpopulation whose the mean is different from the mean of the population.

In case of two population subgroups we get:

$$Sh_1 = \frac{1}{2}[I(\mu_{1,} x_2) - I(x_2)]$$

which can be clearly negative or positive. In particular we know that for $\mu_1 = \mu_2$, $I(\mu_{1,} x_2) < I(x_2)$. Therefore for a continuous inequality index there exists a continuum of values of μ_1 close from μ_2 such that the above inequality prevails. A different response is brought when the equalized modelisation is chosen.

For an equalized income population subgroups inequality game, the Shapley contribution attributed to subpopulation j is computed as follows:

$$Sh_j(L, W_I^e) = \sum_{\substack{S \subset 2^L \\ j \in S}} \frac{(s-1)!(l-s)!}{l!} [I(x(S), \mu(N-S))] - I(x(S-\{j\}), \mu(N-(S-\{j\}))] \quad (24)$$

Then, we have

$$Sh_1 = \frac{1}{2}[I(x_1, \mu(N-\{1\})) - I(\mu(N))] + \frac{1}{2}[I(x) - I(x_2, \mu(N-\{1\}))].$$

Since $\mu(N-\{1\}) = \mu_2$, we can conclude that $Sh_1 = 0$. ■

It is instructive to know whether Sh_j (as a function of x_j) is an inequality index.

Proposition 5.2 *The properties of a inequality index are kept by the Shapley decomposition satisfies except the value equal to zero when the subpopulation is equally distributed and the principle of population.*

Proof. Let's us start by a zero-income population subgroups inequality game.

From equation (23) it is clear that if inequality is measured by an absolute index, then the Shapley inequality decomposition is invariant to a translation of the income distribution. Similarly, it is invariant to an homothety of the income distribution if inequality is measured by a relative index.

Symmetry: Indeed let $\prod(L)$ denotes the set of all permutation of N which are consistent with the set of population subgroups L considered. That is, for all $\pi \in \prod(L)$ and for all $i \in N$, $i \in \{j\}$ implies $\pi(i) \in \pi(\{j\})$. We define the population subgroups inequality game $\pi(L, W_I)$ by $\pi(L, W_I(S)) = (L, W_I(\pi(S)))$. Since I is invariant to a permutation of individuals,then $Sh_j(L, V_I) = Sh_{\pi j}(\pi(L, V_I))$.

Schur-convexity: let us assume that x'_j has been obtained from x_j through a finite sequence of progressive transfers while $x'_h = x_h$ $\forall h \neq j \in L$. By the way for any S including j, $x'(S)$ is obtained from $x(S)$ through a finite sequence of progressive transfers while for any S not including j.$x'(S) = x(S)$. Therefore in the former case $I\ (x'(S)) \leq I\ (x(S))$ while in the latter $I(x'(S)) = I(x(S))$. Therefore $Sh_j(x'_j) \leq Sh_j(x_j)$.

The principle of population is violated since a replication of individuals belonging to $\{j\}$ changes the value of the inequality index for any S including $\{j\}$.

For an equalized population subgroups inequality games the proof is similar.

■

A consequence of the respect of Schur-Convexity is that if inequality is reduced in some subgroup, all things being equal, then it must be true that the Shapley contribution of this subgroup falls down.

6 Conclusion

The Shapley value and others values of cooperative game theory have been proved to be useful in many applications. One of the most famous applications concerns cost allocation which has been pioneered by Shubik(1962). (For a recent survey and references see Young (1994)). Here we have tried to propose an application of the Shapley value to decomposition

Now, let

$$F(K, V_I) = \sum_{S \subseteq K} \frac{(s-1)!(k-s)!}{k!} V_I(S)$$

Then,

$$F(K - \{j\}, V_I) = \sum_{S \subseteq K - \{j\}} \frac{(s-1)!(k-s-1)!}{(k-1)!} V_I(S)$$

and

$$\sum_{j=1}^{k} F(K - \{j\}, V_I) = \sum_{j=1}^{k} \left[\sum_{S \subseteq K} \frac{(s-1)!(k-s-1)!}{(k-1)!} V_I(S) - \sum_{\substack{S \subseteq K \\ j \in S}} \frac{(s-1)!(k-s-1)!}{(k-1)!} V_I(S) \right]$$

$$\sum_{j=1}^{k} F(K - \{j\}, V_I) = k \sum_{S \subseteq k} \frac{(s-1)!(k-s-1)!}{(k-1)!} V_I(S) - \sum_{\substack{S \subseteq K \\ j \in S}} \sum_{j=1 j \in S}^{k} \frac{(s-1)!(k-s-1)!}{(k-1)!} V_I(S)$$

Since for all $S \subseteq K$, $\sum_{j=1 j \in S}^{k} = s$ and for $S = K$ the right term of the previous equality is null, we deduce that:

$$\sum_{j=1}^{k} F_i(k - \{j\}, V_I) = (k-s) \sum_{S \subset K} \frac{(s-1)!(k-s-1)!}{(k-1)!} V_I(S)$$

Thus we have $F = P$.
Finally, for every inequality game (k, V_I) and each component $j \in K$

$$\varphi_j(K, V_I) = \sum_{S \subseteq K} \frac{(s-1)!(k-s)!}{k!} V_I(S) - \sum_{S \subseteq K - \{j\}} \frac{(s-1)!(k-1-s)!}{(k-1)!} V_I(S)$$

$$= Sh_j(K, V_I)$$

References

[1] Aumann R. and Dreze J. (1974) "Cooperative Games with Coalition Structures" International Journal of Game Theory, 3, pp 217-238.

[2] Auvray C. and Trannoy A. (1992) "Décomposition par source de l'inégalité des revenus à l'aide de la valeur de Shapley", Journées de Microéconomies Appliquées, Sfax.

[3] Bourguignon F. (1979) "Decomposable Income Inequality Measures" Econometrica, 47, pp 901-920.

[21] Shapley L.S. (1953) "A Value for n-person Games", in Contributions to the Theory of Games, Vol. 2, Annals of Mathematics Studies 28) ed. by H.W. Kuhn and A.W. Tucker. Princeton University Press, pp 307-317.

[22] Shorrocks A.F. (1980) "The Class of Additively Decomposable Inequality Measures", Econometrica, 48, pp 613-625.

[23] Shorrocks A.F. (1982) "Inequality Decomposition by Factor Components", Econometrica, 50, pp 193-211.

[24] Shorrocks A.F. (1984) "Inequality Decomposition by Population Subgroups", Econometrica, 51, pp 1369-1385.

[25] Shorrocks A.F. (1988) " Aggregation Issues in Inequality Measurement" in Eichhorn W. (ed.) Measurement in Economics. Physica-Verlag, New York.

[26] Shubik M. .(1962) "Incentives, Decentralized Control , the Assignment of Joint Costs and Internal Pricing" Management Science, 8, pp 325-343.

[27] Young H.P. (1985) "Monotonic Solutions of Cooperative Game Theory" International Journal of Game Theory. Vol.14, 2, pp 65-72.

[28] Young H.P. (1994) "Cost Allocation" in Handbook of Game Theory. Vol 2, Edited by Aumann R.J. and Hart S. Elsevier Science BV.

UNIVERSITÉ DE CERGY-PONTOISE

UNIVERSITÉ DE PARIS X - NANTERRE

Inequality decomposition values : the trade-off between marginality and consistency

Frédéric Chantreuil*
Alain Trannoy**

N° 99-24

Thema

Théorie Economique, Modélisation et Applications
UMR 7536 - Centre National de la Recherche Scientifique

CERGY - PONTOISE université

UNIVERSITÉ Paris X NANTERRE