The Newtonian Potential and the Demagnetizing Factors of the General Ellipsoid

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Abstract. The objective of this paper is to present a modern and concise new derivation for the explicit expression of the interior and exterior Newtonian potential generated by homogeneous ellipsoidal domains in $\mathbb{R}^N$ (with $N \geqslant 3$). The very short argument is essentially based on the application of Reynolds transport theorem in connection with Green-Stokes integral representation formula for smooth functions on bounded domains of $\mathbb{R}^N$, which permits to reduce the N-dimensional problem to a 1-dimensional one. Due to its high physical relevance, a separate section is devoted to the derivation of the *demagnetizing factors* of the general ellipsoid which are one of the most fundamental quantities in ferromagnetism.

1 Historical Introduction, Motivations

The computation of the gravitational potential induced by an homogeneous ellipsoid was one of the most important problems in mathematics for more than two centuries after Newton enunciated his universal law of gravitation [6, 8, 9, 24, 30, 31].

Once fixed an ellipsoidal domain Ω of $\mathbb{R}^3$, the problem consists in finding an *explicit* expression for the Newtonian potential induced by a constant mass/charge density on Ω [14]:

$$\mathcal{N}_\Omega[1_\Omega](x) := \frac{1}{4\pi}\int_\Omega \frac{1}{|x-y|}\mathrm{d}y, \tag{1}$$

in the internal points of Ω (*interior problem*) and in the exterior points of Ω (*exterior problem*).

In the case of a homogeneous *spherically symmetric* region, Newton (in 1687) proved what nowadays is known as Newton's shell theorem [3, 24]: *if Ω is an homogeneous*[1] *spherical region centered at the origin, then for all $t > 1$, $\mathcal{N}_{t\Omega\backslash\Omega}[1_{t\Omega\backslash\Omega}]$ is constant in Ω, i.e.* $t\Omega\backslash\Omega$ (the so called *hollow ball*) *induces no gravitational force inside Ω. Moreover, a spherically symmetric body affects external objects gravitationally as though all of its mass were concentrated at a point at its center.*

For what concerns an ellipsoidal domain Ω of $\mathbb{R}^3$, the *ellipsoidal interior* problem was for the first time solved by Gauss (in 1813) by the means of what it is in present days known as the Gauss divergence theorem [16]. Later, in 1839, Dirichlet proposed a solution of the interior problem based on the theory of Fourier integrals [21].

The results of Gauss and Dirichlet can be summarized by saying that if Ω is an ellipsoidal domain centered at the origin, then the gravitational potential induced by an homogeneous ellipsoid in its internal points is a second order polynomial. In other terms:

$$\mathcal{N}_\Omega[1_\Omega](x) = c - Px\cdot x \quad \forall x\in\Omega, \tag{2}$$

[1] By an *homogeneous* region of $\mathbb{R}^N$ we mean an open and connected subset of $\mathbb{R}^N$ endowed by a constant density of masses/charges.

for some constant $c \in \mathbb{R}$ and some matrix $P \in \mathbb{R}^{3\times 3}$ whose values can be expressed in terms of elliptic integrals [20, 35].

Remark 1.1. For the sake of completeness we recall that the converse statement (the *inverse homogeneous ellipsoid problem*) is also true [11, 14, 18], namely: *if Ω is a bounded domain of $\mathbb{R}^N$ such that $\mathbb{R}^N \backslash \Omega$ is connected and* (2) *holds, then Ω is an ellipsoid.* Historically speaking the inverse homogeneous ellipsoid problem was for the first time solved by Dive [12] in 1931 for $N = 3$ and in 1932 by Hölder [17] for $N = 2$. A modern proof of this result can be found in DiBenedetto and Friedman [11] who, in 1985, extended it to all $N \geqslant 2$. In 1994, Karp [18], by the means of certain topological methods, obtained an alternative proof of the inverse homogeneous ellipsoid problem.

Despite the well-knownness of (2) in the mathematical and physical community, and its importance in theoretical and applied studies [1, 5, 7, 32, 33, 10, 25], rigorous proofs of that result are not readily available in the literature: to the best knowledge of the author, relative modern treatments of the interior problem can be found in [20] and [35], and more recently in [22] and [26] where also the the exterior problem is investigated. However, in all the cited references, the solution of the problem is always based on the use of ellipsoidal coordinates which tends to focus the attention on the computational details of the question rather than on its geometric counterpart. Eventually, modern proofs of Newton's theorem and relation (2), as well as far reaching beautiful generalizations, can be found in [19] where neverthless the problem of finding an anlytic expression for the coeficients c and P is not touched.

Aim of this paper is to give a modern and concise derivation for the expression of the *interior* and *exterior* Newtonian potential (induced by homogeneous ellipsoids). The very short argument is essentially based on the application of Reynolds transport theorem in connection with Green-Stokes integral representation formula for smooth functions on bounded domains of $\mathbb{R}^N$. This approach permits to reduce the N-dimensional problem to a 1-dimensional one, providing (in particular) at once a proof of (2) together with an explicit expression of the coefficients P and c in terms of 1-dimensional integrals. More precisely, the paper is organized as follows:

Section 2 is devoted to the main result of the paper. We give a concise proof of the homogeneous ellipsoid problem. For completensee, we then derive Newton's shell theorem as an immediate corollary.

In Section3 we focus the attention to the three dimensional case. An expression in terms of the elliptic integrals is given for the coefficients of P. Due to its physical relevance, particular attention is paid to the eigenvalues of P. Indeed, when $N = 3$, the matrix P and its eigenvalues, known in the theory of ferromagnetism respectively as *the demagnetization tensor* and *the demagnetizing factors*, are one of the most important and well-studied quantities of ferromagnetism [1, 2, 5, 7, 10, 25]. In fact, the following magnetostatic counterpart of the homogeneous ellipsoid problem holds: *given a uniformly magnetized ellipsoid, the induced magnetic field is also uniform inside the ellipsoid*. This result was for the first time showed by Poisson [28], while an explicit expression for the demagnetizing factors was for the first time obtained by Maxwell [23]. Their importance is in that they encapsulate the self-interaction of magnetized bodies: their knowledge being equivalent to the one of the corresponding demagnetizing (stray) fields [5].

2 The Interior and Exterior Potential of an Homogeneous Ellipsoid

In what follows we denote by Ω the ellipsoidal domain of $\mathbb{R}^N$ ($N \geqslant 3$) having $a_1, a_2, \ldots, a_N$ as semi-axes lengths. We then denote by $(\Omega_t)_{t\in[0,+\infty)}$ the family of ellipsoidal domains of

$\mathbb{R}^N$, given by the inverse image $\phi_t^{-1}(B_N)$ of the unit ball of $\mathbb{R}^N$ under the one parameter family of diffeomorphisms $\phi_t : x \in \mathbb{R}^N \mapsto \sqrt{A_t}x \in \mathbb{R}^N$, where

$$\sqrt{A_t} := \operatorname{diag}\left[\frac{1}{\sqrt{a_1^2+t}}, \frac{1}{\sqrt{a_2^2+t}}, \dots, \frac{1}{\sqrt{a_N^2+t}}\right]. \tag{3}$$

Note that each diffeomorphism ϕ_t^{-1} maps the unit ball of $\mathbb{R}^N$ onto the ellipsoidal domain of $\mathbb{R}^N$ defined by the position $\Omega_t = \{x \in \mathbb{R}^N \ : \ |\phi_t(x)|^2 \leqslant 1\}$. In particular $\partial\Omega_t = \{x \in \mathbb{R}^N \ : \ |\phi_t(x)|^2 = 1\}$ and $\Omega \equiv \Omega_0$. Finally, we denote by $\mathcal{N}_{\Omega_t}[1_{\Omega_t}]$ the Newtonian potential generated by the uniform space density of masses or charges on Ω_t:

$$\mathcal{N}_{\Omega_t}[1_{\Omega_t}](x) = c_N \int_{\Omega_t} \frac{1}{|x-y|^{N-2}} \mathrm{d}\tau(y), \tag{4}$$

with $c_N := [(N-2)\omega_N]^{-1}$ and ω_N the surface measure of the unit sphere in $\mathbb{R}^N$ (cfr. [13, 14]).

The main result of the paper is stated in the following:

Theorem 2.1. *Let $\Omega = \{x \in \mathbb{R}^N \ : \ |\phi_0(x)|^2 \leqslant 1\}$ be the ellipsoidal domain of $\mathbb{R}^N$ having $(a_1, a_2, \dots, a_N) \in \mathbb{R}_+^N$ as semi-axes lengths. For every $x \in \mathbb{R}^N$*

$$\mathcal{N}_\Omega[1_\Omega](x) = \frac{1}{4}\int_{\tau(x)}^{+\infty} \gamma_t(1 - A_t x\cdot x)\mathrm{d}t, \qquad \gamma_t := \prod_{i=1}^N \frac{a_i}{\sqrt{a_i^2+t}}, \tag{5}$$

where we have denoted by τ the non-negative real valued function

$$\tau : x \in \mathbb{R}^N \mapsto \begin{cases} 0 & \text{if } x \in \bar{\Omega} \\ \tau(x) & \text{if } x \in \Omega^c \cap \partial\Omega_{\tau(x)} \end{cases}. \tag{6}$$

Remark 2.1. For general ellipsoidal domains the integral in (5) can be evaluated by the means of the theory of elliptic integrals (cfr. [4, 29]). The function τ defined by (6) can be computed by solving the equation $\partial\Omega_{\tau(x)} = \{x \in \mathbb{R}^N \ : \ |\phi_{\tau(x)}(x)|^2 = 1\}$. In particular, when Ω is a spherical region of radius a (centered around the origin), the function τ reduces to the function equal to $|x|^2 - a^2$ if $x \in \mathbb{R}^N \backslash \Omega$, zero otherwise, and the integral in (5) can be readily computed.

Proof. For every $t \in \mathbb{R}^+$, the function $|\phi_t|^2 - 1$ is the unique solution of the homogeneous Dirichlet problem for the Poisson equation $u = 2\operatorname{tr}(A_t)$ in Ω_t, $u \equiv 0$ on $\partial\Omega_t$. Thus, according to the Green-Stokes representation formula (see [14]), for every $t \in \mathbb{R}^+$ we have

$$(|\phi_t|^2 - 1)1_{\Omega_t} = -2\operatorname{tr}(A_t)\mathcal{N}_{\Omega_t}[1_{\Omega_t}] + \mathcal{S}_{\partial\Omega_t}[\partial_{\boldsymbol{n}}|\phi_t|^2] \quad \text{in } \mathbb{R}^N \backslash \partial\Omega_t, \tag{7}$$

where we have denoted by

$$\mathcal{S}_{\partial\Omega_t}[\partial_{\boldsymbol{n}}|\phi_t|^2](x) := c_N \int_{\partial\Omega_t} \frac{\partial_{\boldsymbol{n}(y)}|\phi_t(y)|^2}{|x-y|^{N-2}} \mathrm{d}\sigma(y), \tag{8}$$

the simple layer potential generated by the space density of masses or charges $\partial_{\boldsymbol{n}}|\phi_t|^2$ concentrated on $\partial\Omega_t$ (cfr. [13, 14]). Next, we observe that the N-dimensional Newtonian kernel $|x|^{2-N}$ is in $W^{1,1}_{\mathrm{loc}}(\mathbb{R}^N)$, therefore due to Reynolds transport theorem,

$$\partial_t\mathcal{N}_{\Omega_t}[1_{\Omega_t}] = c_N \int_{\partial\Omega_t} \frac{\boldsymbol{v}_t(y)\cdot\boldsymbol{n}(y)}{|x-y|^{N-2}} \mathrm{d}\sigma(y) = \frac{1}{4}\mathcal{S}_{\partial\Omega_t}[\partial_{\boldsymbol{n}}|\phi_t|^2], \tag{9}$$

where we have denoted by $\boldsymbol{v}_t := \partial_t[\phi_t^{-1}] \circ \phi_t$ the eulerian velocity field associated to the motion ϕ_t^{-1}, for which one has

$$\boldsymbol{v}_t(y) := \partial_t[\phi_t^{-1}] \circ \phi_t(y) = \frac{1}{2} A_t y = \frac{1}{4} \nabla |\phi_t(y)|^2 \quad \forall (t, y) \in \mathbb{R}^+ \times \partial \Omega_t. \tag{10}$$

Hence, substituting (9) into (7) we get

$$\frac{1}{4}(|\phi_t(x)|^2 - 1) 1_{\Omega_t}(x) = -\frac{1}{2} \operatorname{tr}(A_t) \mathcal{N}_{\Omega_t}[1_{\Omega_t}](x) + \partial_t \mathcal{N}_{\Omega_t}[1_{\Omega_t}](x) \quad \forall x \in \mathbb{R}^N \backslash \partial \Omega_t. \tag{11}$$

Moreover, by the assignment

$$\gamma_t := \exp\left(-\frac{1}{2} \int_0^t \operatorname{tr}(A_s) \mathrm{d}s\right) = \prod_{i=1}^N \frac{a_i}{\sqrt{a_i^2 + t}} \quad , \quad \gamma_0 = 1, \tag{12}$$

the equality (11) reads as

$$\partial_t(\gamma_t \mathcal{N}_{\Omega_t}[1_{\Omega_t}]) = \frac{1}{4} \gamma_t (|\phi_t|^2 - 1) 1_{\Omega_t} \quad \forall x \in \mathbb{R}^N \backslash \partial \Omega_t. \tag{13}$$

Thus, once introduced the non-negative real function defined by (6) we have, for every $t \in \mathbb{R}^+$ and for every $x \in \mathbb{R}^N$

$$\partial_t(\gamma_t \mathcal{N}_{\Omega_t}[1_{\Omega_t}])(x) = \frac{1}{4} \gamma_t (|\phi_t(x)|^2 - 1) 1_{[\tau(x), +\infty)}(t). \tag{14}$$

Integrating both members of (14) on $[0, +\infty)$; taking into account that due to the well-known decay at infinity of the Newtonian potential [34] one has $\lim_{t \to +\infty} \gamma_t \mathcal{N}_{\Omega_t}[1_{\Omega_t}] = 0$; we finish with (5). □

Corollary 2.1. (Newton's shell theorem) *Let $\Omega \subseteq \mathbb{R}^3$ be an homogeneous spherical region (centered around the origin) of radius a and of total mass M. For every $x \in \mathbb{R}^3 \backslash \Omega$ the induced gravitational potential is the same as though all of its mass were concentrated at a point at its center. Moreover, for all $t > 1$, $\mathcal{N}_{t\Omega \backslash \Omega}[1_{t\Omega \backslash \Omega}]$ is constant in Ω, i.e. the hollow ball induces no gravitational force inside Ω.*

Proof. We denote by $\rho 1_\Omega(x)$, $(\rho := M/|\Omega|)$ the uniform density of mass in Ω. The gravitational potential induced by ρ in $\mathbb{R}^3$ is then given by $u_\rho = 4\pi G \mathcal{N}_\Omega[\rho] = 4\pi G \mathcal{N}_\Omega[1_\Omega]$, where we have denoted by G the *gravitational constant*. In this geometrical setting the function τ defined by (6) reduces to the function $\tau(x) := (|x|^2 - a^2) 1_{\mathbb{R}^N \backslash \Omega}(x)$ and the integral in (5) immediately gives

$$u_\rho(x) = \frac{2}{3} \pi G \rho \left(\frac{a^2}{2} - |x|^2\right) \quad \text{if } x \in \bar{\Omega} \quad , \quad u_\rho(x) = \frac{4}{3} \pi a^3 \frac{G\rho}{|x|} = \frac{GM}{|x|} \quad \text{if } x \in \mathbb{R}^3 \backslash \Omega. \tag{15}$$

Thus, for every $x \in \mathbb{R}^3 \backslash \Omega$ the induced gravitational potential is equal to the one induced by a Dirac mass concentrated in the center of Ω. The gravitational field is given by $\boldsymbol{g} := -\nabla u_\rho$ and the fact that the hollow ball $t\Omega \backslash \Omega$, $t > 1$, induces no gravitational force inside Ω can be immediately seen by splitting the uniform density of mass in $t\Omega \backslash \Omega$ in the form $\rho = (M/|t\Omega \backslash \Omega|) 1_{t\Omega} - (M/|t\Omega \backslash \Omega|) 1_\Omega$. Indeed by linearity we get that u_ρ is constant in Ω and therefore $\boldsymbol{g} = \mathbf{0}$. □

3 The demagnetizing factors of the general ellipsoid

We know focus on the three-dimensional framework ($N = 3$) and, in particular, on the so-called demagnetizing factors of the general ellipsoid [25]. To this end we recall that the demagnetizing (stray) field $\boldsymbol{h}[\boldsymbol{m}]$ associated to a magnetization $\boldsymbol{m} \in C^\infty(\bar{\Omega})$ can be expressed as the gradient field of a suitable magnetostatic potential $\varphi_{\boldsymbol{m}}$ (see [15, 27]). Precisely $\varphi_{\boldsymbol{m}} := -\operatorname{div} \mathcal{N}_\Omega[\boldsymbol{m}]$ and $\boldsymbol{h}[\boldsymbol{m}] := -\nabla\varphi_{\boldsymbol{m}}$ in $\mathbb{R}^3$. In particular, if $\boldsymbol{m}$ is constant in Ω, then $\varphi_{\boldsymbol{m}} = -\boldsymbol{m} \cdot \nabla\mathcal{N}_\Omega[1_\Omega]$. Thus from (5):

$$\varphi_{\boldsymbol{m}}(x) = Px \cdot \boldsymbol{m} \quad , \quad \boldsymbol{h}[\boldsymbol{m}] = -P\boldsymbol{m} \quad \forall x \in \Omega, \tag{16}$$

where we have denote by $P := \nabla\mathcal{N}_\Omega[1_\Omega]$ the diagonal matrix, known in literature as the *demagnetizing tensor*, whose diagonal i-entry (the i-th *demagnetizing factor*), by virtue of (5), is given by

$$P_i := \frac{1}{2}\int_0^{+\infty} \frac{1}{(a_i^2+t)} \prod_{j=1}^{3} \frac{a_j}{\sqrt{a_j^2+t}} \mathrm{d}t \qquad \forall i \in \mathbb{N}_3. \tag{17}$$

Proposition 3.1. *We have $P_i \geqslant 0$ for every $i \in \mathbb{N}_3$ and if $a_1 \geqslant a_2 \geqslant a_3$ then $P_1 \leqslant P_2 \leqslant P_3$. The trace of P satisfy the relation* $\operatorname{tr}(P) = 1$.

Proof. The first statement is obvious. The relation $\operatorname{tr}(P) = 1$ can of course be verified by a direct evaluation of the integrals in (17), but it is also possible to observe that since the Newtonian potential $\mathcal{N}_\Omega[1_\Omega]$ satisfies the Poisson equation $\Delta\mathcal{N}_\Omega[1_\Omega] = -1_\Omega$, one has $\operatorname{tr}(P) = \operatorname{div}(Px) = -\Delta\mathcal{N}_\Omega[1_\Omega] = 1_\Omega$. □

Assuming $a_1 \geqslant a_2 \geqslant a_3$, from (17) and the theory of elliptic integrals we get

$$P_1 = 1 - P_2 - P_3 \tag{18}$$

$$P_2 = -\frac{a_3}{a_2^2-a_3^2}\left[a_3 - \frac{a_1a_2}{(a_1^2-a_2^2)^{1/2}} E\left(\arccos\left(\frac{a_2}{a_1}\right)\middle| \frac{a_1^2-a_3^2}{a_1^2-a_2^2}\right)\right] \tag{19}$$

$$P_3 = +\frac{a_2}{a_2^2-a_3^2}\left[a_2 - \frac{a_1a_3}{(a_2^2-a_3^2)^{1/2}} E\left(\arccos\left(\frac{a_3}{a_1}\right)\middle| \frac{a_1^2-a_2^2}{a_1^2-a_3^2}\right)\right], \tag{20}$$

where, for every $y \in \mathbb{R}$ and every $0 < p < 1$ we have denoted by

$$E(y|p) := \int_0^y (1 - p\sin^2\theta)^{1/2} \mathrm{d}\theta, \tag{21}$$

the incomplete elliptic integral of the second kind expressed in *parameter form* [4, 29]. In particular, in the case of a *prolate spheroid* ($a_1 \geqslant a_2 = a_3$) we get

$$P_1 = -\frac{a_3^2}{(a_1^2-a_3^2)^{3/2}}\left[(a_1^2-a_3^2)^{1/2} + a_1 \operatorname{arccoth}\left(\frac{a_1}{(a_1^2-a_3^2)^{1/2}}\right)\right], \tag{22}$$

$$P_2 = P_3 = \frac{1-P_1}{2}, \tag{23}$$

while in the case of an *oblate spheroid* ($a_1 = a_2 \geqslant a_3$)

$$P_1 = P_2 = \frac{1-P_3}{2}, \tag{24}$$

$$P_3 = \frac{a_1^2}{(a_1^2-a_3^2)^{3/2}}\left[(a_1^2-a_3^2)^{1/2} + a_3 \arctan\left(\frac{a_3}{(a_1^2-a_3^2)^{1/2}}\right) - a_3\frac{\pi}{2}\right]. \tag{25}$$

Finally, in the case of a *sphere* ($a_1 = a_2 = a_3$) one finish with $P_1 = P_2 = P_3 = \frac{1}{3}$.

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