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From the Adler–Moser polynomials to the polynomial tau functions of KdV

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In 1978, M. Adler and J. Moser proved that there exists a unique change of variables that transforms the Adler–Moser polynomials into the polynomial tau functions of the KdV hierarchy. In this paper we exhibit this change of variables.

Introduction

The Korteweg–de Vries hierarchy (or KdV) is a sequence of pairwise commuting partial differential equations first discovered by A. Lenard in 1967. Each equation admits a Lax pair as follows [Lax68]. Let a function u which depends on infinitely many time variables indexed by only odd integers $\mathrm{t} = (x = t_1, t_3, t_5, \ldots)$. We denote $\mathrm{D}\,u = \partial_x u = u'$. We associate to u the Schrödinger operator $L = \mathrm{D}^2 + u$. Then the KdV hierarchy is defined by the flows

$$\frac{\partial L}{\partial t_{2i-1}} = \left[\left(P^{2i-1}\right)_+, L\right],$$

where $P = L^{1/2}$ is the square root of L and for any pseudo-differential operator X, X_+ is its purely differential part. To any solution u we associate a tau function defined by $u = -2(\log\tau)''$.

Among the formal solutions of KdV there are no polynomial solutions so that only rational solutions depend on finitely many times. It was proven by Airault, McKean and Moser [AMM77] that there are denumerably many of them, each one being the orbit of a single function $u_n = \frac{n(n+1)}{x^2}$ under the flows of the hierarchy. In [AM78], the authors constructed the Adler–Moser polynomials $\theta_n(x = q_1, q_3 \ldots, q_{2n-1})$ for $n \geq 0$, defined by the recursion

$$\theta'_{n+1}\theta_{n-1} - \theta_{n+1}\theta'_{n-1} = (2k-1)\theta_n^2.$$

An important result of [AM78] (cf. Theorem 1.1) is that there exists a unique change of variables that transforms the Adler–Moser polynomials into polynomial tau functions of KdV and that we recover all rational solutions of KdV. But we did not know what this change of variables was.

In this paper, we show that the following change of variables transforms the Adler–Moser polynomials into the polynomial tau functions of KdV: $q_1 = t_1 = x$, and

$$\sum_{i\geq 2} \frac{q_{2i-1}}{\alpha_{2i-1}} z^{2i-1} = \tanh\left(\sum_{i\geq 2} t_{2i-1} z^{2i-1}\right).$$

where $\alpha_{2i-1} = (-1)^{i-1}3^2 5^2 \dots (2i-3)^2(2i-1)$. To do so, we apply this change of variables to the Adler–Moser polynomials to get some polynomials $\tau_n(t_1, t_3, t_5, \dots)$. Then by seeing them as functions $\tau_n(t_1, t_2, t_3, \dots)$ of even and odd times we show that they are tau functions of the Kadomstev–Petviasvhili hierarchy (KP). Then since the τ_n's actually depend only on odd times, they are indeed tau functions of KdV.

It is well known how to compute the polynomial tau functions of KdV without using the Adler–Moser polynomials. For instance, in [Hir04] R. Hirota constructs a family of tau functions of KP in terms of Wronskian of the elementary Schur polynomials, which can be reduced to recover the polynomial tau functions of KdV. But the Adler–Moser polynomials reveal a recursive structure in the space of rational solutions of KdV. It would be interesting to investigate how to generalize this to the Drinfeld–Sokolov hierarchies.

1 The Adler–Moser Polynomials

Let a set of evolution variables $\mathrm{q} = (x = q_1, q_3, q_5, \dots)$[1]. We save the variables $\mathrm{t} = (x = t_1, t_3, t_5, \dots)$ for the tau functions of KdV. The Adler–Moser polynomials form a sequence $\theta_n(q_1, q_3 \dots, q_{2n-1})$ for $n \geq 0$, defined by the following recursion [AM78]: $\theta_0 = 1$, $\theta_1 = x$ and for $n \geq 1$,

$$\theta'_{n+1}\theta_{n-1} - \theta_{n+1}\theta'_{n-1} = (2k-1)\theta_n^2, \tag{1}$$

where $f' = \mathrm{D}f = \partial_x f$. This recursion leaves an integration constant that is chosen to be q_{2n-1} when computing θ_n. We can check that θ_n has degree $d_n = \frac{1}{2}n(n+1)$ in x and q_{2n-1} is actually the coefficient of $x^{d_{n-2}}$ in this polynomial. The first few polynomials read

$$\theta_0 = 1, \qquad \theta_1 = x, \qquad \theta_2 = x^3 + q_3, \qquad \theta_3 = x^6 + 5q_3x^3 + q_5x - 5q_3^2.$$

In that same article ([AM78], pp. 17–18), the authors state the following theorem.

Theorem 1.1 (Adler–Moser, 1978)**.** *There exists a unique change of variables* $\mathrm{q} \mapsto \mathrm{t}$ *that transforms the Adler–Moser polynomials* $\theta_n(\mathrm{q})$ *into the polynomial tau functions* $\tau_n(\mathrm{t})$ *of KdV. That is, the rational functions* $u_n = -2(\log\tau)''$ *define operators* $L_n = \mathrm{D}^2 + u_n$ *that satisfy the Lax system of KdV:*

$$\frac{\partial L_n}{\partial t_{2i-1}} = \left[\left(L_n^{\frac{2i-1}{2}}\right)_+, L_n\right].$$

In this article we prove that the desired change of variables is given by: $q_1 = t_1 = x$, and

$$\sum_{i\geq 2} \frac{q_{2i-1}}{\alpha_{2i-1}} z^{2i-1} = \tanh\left(\sum_{i\geq 2} t_{2i-1} z^{2i-1}\right).$$

where $\alpha_{2i-1} = (-1)^{i-1}3^2 5^2 \dots (2i-3)^2(2i-1)$. The latter coefficients α_{2i-1} where already given in Adler and Moser's article. Notice that this change of variables amounts to simply change the choice of the integration constant in the differential recursion (1). To prove the statement, let us first recall and prove some results stated in [AM78]. These lemmas relate the Adler–Moser polynomials to a Wronskian representation through a multiplicative factor and a simple rescaling of the variables.

Let another set of variables $\mathrm{s} = (x = s_1, s_3, s_5, \dots)$ and functions $\psi_j(\mathrm{s})$, $j \geq 0$, defined by:

$$\sum_{j\geq 1} \psi_j z^{2j-1} = \sinh(xz) + \cosh(xz) \sum_{i\geq 2} s_{2i-1} z^{2i-1}, \tag{2}$$

and $\psi_0 = 0$. It readily implies the recursion $\psi_j'' = \psi_{j-1}$.

[1] In their original paper [AM78] the authors use the variables τ_i instead. But since then, the letter τ has been used rather for the tau functions. Moreover, we use odd indices so that we can later complete the set into the variables of KP.

Lemma 1.2. *The Wronskians of the functions ψ_j, defined by*

$$W_n := \mathrm{Wr}(\psi_1, \dots, \psi_n) = \det\left(\mathrm{D}^{i-1}\psi_j\right)_{i,j=1,\dots,n},$$

satisfy the recursion

$$W'_{n+1}W_{n-1} - W_{n+1}W'_{n-1} = W_n^2. \tag{3}$$

Proof. For any smooth function $\chi(\mathrm{s})$ with respect to x, denote $W_n(\chi) := \mathrm{Wr}(\psi_1, \dots, \psi_n, \chi)$. Then one has Jacobi's identity:

$$W'_n(\chi)W_{n+1} - W_n(\chi)W'_{n+1} = W_{n+1}(\chi)W_n. \tag{4}$$

This can be proven by noticing that the left-hand side is a linear differential operator of order $n+1$ which vanishes for $\chi = \psi_1, \dots, \psi_n$, as well as for ψ_{n+1}. Yet the functions ψ_j being linearly independent, the left-hand side must be proportional to $W_{n+1}(\chi)$. Comparing the highest coefficient, one obtain Equation (4). Now thanks to the relation $\psi_j'' = \psi_{j-1}$ and $\psi_1 = x$, we can compute that for $\chi = 1$,

$$W_n(1) = (-1)^k W_{n-1}. \tag{5}$$

Then setting $\chi = 1$ in Jacobi's identity (4), one finds Equation (3). □

Now since $W_0 = \theta_0 = 1$ and $W_1 = \theta_1 = x$, the two sequences of polynomials differ only by a multiplicative factor that can be computed:

$$\theta_n(\mathrm{q}) = \mu_n W_n(\mathrm{s}), \qquad \mu_n = \prod_{j=1}^{k}(2k-2j+1)^j. \tag{6}$$

Lemma 1.3. *The parameters of the Adler–Moser polynomials $\theta_n(\mathrm{q})$ and those of the Wronskians $W_n(\mathrm{s})$ are related via a rescaling:*

$$s_{2i-1} = \frac{q_{2i-1}}{\alpha_{2i-1}}, \qquad \alpha_{2i-1} = (-1)^{i-1}3^2 5^2 \dots (2i-3)^2(2i-1) \tag{7}$$

Proof. It all has to do with the choice of the normalization in the recursions (1) and (3). For θ_n, the choice is such that q_{2n-1} is the coefficient of $x^{d_{n-2}}$. Moreover if θ_n is a solution of (1), so is $\theta_n + c\theta_{n-2}$ so that the normalization can be expressed as

$$\theta_n = \mathring{\theta}_n + q_{2n-1}\theta_{n-2},$$

where $\mathring{\theta}_n := \theta_n|_{q_{2n-1}=0}$. Similarly, define $\mathring{W}_n := W_n|_{s_{2n-1}=0}$. Since $\psi_n = \mathring{\psi}_n + s_{2n-1}$, then by Equation (5),

$$W_n = \mathring{W}_n + s_{2n-1}\mathrm{Wr}(\psi_1, \dots, \psi_n, 1) = \mathring{W}_n + (-1)^{k-1}s_{2n-1}W_{n-2}.$$

Comparing the last two equations and using Equation (6) one obtains the result. □

2 A Change of Variables to the Polynomial Tau Functions KdV

The aim of this section is to prove the following theorem.

Theorem 2.1. *The following change of variables transforms the Adler–Moser polynomials into the polynomial tau functions of KdV: $q_1 = t_1 = x$, and*

$$\sum_{i\geq 2}\frac{q_{2i-1}}{\alpha_{2i-1}}z^{2i-1} = \tanh\left(\sum_{i\geq 2}t_{2i-1}z^{2i-1}\right). \tag{8}$$

where $\alpha_{2i-1} = (-1)^{i-1}3^2 5^2 \dots (2i-3)^2(2i-1)$.

To prove Theorem 2.1, we introduce another sequence of functions defined by the generating series.

$$\sum_{j\geq 1}\phi_j z^{2j-1} = \sinh(xz) + \cosh(xz)\tanh\left(\sum_{i\geq 2} t_{2i-1}z^{2i-1}\right). \tag{9}$$

It amounts to applying the change of variables of Equation (8) to the functions ψ_j of Equation (2). With these functions, define another sequence of Wronskians:

$$\tau_n := \mathrm{Wr}(\phi_1, \ldots, \phi_n), \tag{10}$$

In what follows, we prove that these Wronskians are tau functions of the KP hierarchy. Yet because they depend only on odd times, they are tau functions of KdV, which proves Theorem 2.1. We use the same approach as in [Caf08] and [IZ92]. First we need the following lemma.

Lemma 2.2. *The functions ϕ_j satisfy the following relation for any integers $i, j \geq 1$:*

$$\phi_j^{(2i-1)} - \frac{\partial \phi_j}{\partial t_{2i-1}} = \sum_{k=1}^{j-i-1} \phi_k a_{j-i-k+1}, \tag{11}$$

where the a_j's are functions that do not depend on x and are defined by

$$\sum_{j\geq 2} a_j z^{2j-1} = \tanh\left(\sum_{i\geq 2} t_{2i-1}z^{2i-1}\right).$$

Proof. It is a direct calculation: differentiating Equation (9) and using a Cauchy product, one obtains

$$\sum_{j\geq 1}\left(\phi_j^{(2i-1)} - \frac{\partial \phi_j}{\partial t_{2i-1}}\right) z^{2j-1} = \sum_{j\geq 1} z^{2j-1} \sum_{k=0}^{j-i} \phi_k a_{j-i-k+1}.$$

Then, noticing that $\phi_0 = 0$ and $a_0 = a_1 = 0$, one gets the correct boundaries in the last sum. □

Remark 2.3. This relation is to be compared with the one satisfied by the elementary Schur polynomials defined by

$$\exp\left(\sum_{i\geq 1} t_i z^i\right) = \sum_{j\geq 1} P_j z^j.$$

These polynomials satisfy the relation $P_j^{(i)} = \partial_{t_i} P_j$. Now define p_j to be P_{2j-1} where all even times are set to 0. They satisfy the relation

$$p_j^{(2i-1)} = \frac{\partial p_j}{\partial t_{2i-1}}.$$

The last relation allows to prove that their Wronskians $\mathrm{Wr}(p_1, \ldots, p_n)$ are the tau functions of KdV the same way we prove that the Wronskians of the ϕ_j's are (see for instance [IZ92]). In particular the Wronskians of the p_j's and those of the ϕ_j's coincide.

Now let us introduce all the times ($x = t_1, t_2, t_3, \ldots$) of the KP hierarchy (that is, odd and even). To prove that the τ_n's are tau functions of KP we use Sato's equation, which we state in a equivalent form in the following proposition.

Proposition 2.4. *Let the following differential operator*

$$\Delta_n(\chi) := \frac{\mathrm{Wr}\,(\chi, \phi_1, \ldots, \phi_n)}{\mathrm{Wr}\,(\phi_1, \ldots, \phi_n)} = \frac{1}{\tau_n}\mathrm{Wr}\,(\chi, \phi_1, \ldots, \phi_n),$$

for any differentiable function χ with respect to x. The following equation holds for any $i \geq 1$:

$$\frac{\partial \Delta_n}{\partial t_i} = \left(\Delta_n \mathrm{D}^i \Delta_n^{-1}\right)_+ \Delta_n - \Delta_n \mathrm{D}^i. \tag{12}$$

Proof. It is sufficient to prove the equality when acting on $\phi_1, \dots, \phi_n$ which are n linearly independent functions. Yet these functions are solutions of the equation $\Delta_n(\phi_j) = 0$, so it amounts to proving that

$$\frac{\partial \Delta_n}{\partial t_i}(\phi_j) + \Delta_n\left(\phi_j^{(i)}\right) = 0.$$

If $i = 2\ell$ is even, then we only have to prove that $\Delta_n\left(\phi_j^{(2\ell)}\right) = 0$. Yet $\phi_j'' = \phi_j$ so that $\phi_j^{(2\ell)} = \phi_{j-\ell}$, or 0 if $j \leq \ell$. Eventually, it amounts to $\Delta_n\left(\phi_{j-\ell}\right) = 0$ which holds true. If $i = 2\ell - 1$ is odd, by Lemma 2.2, it amounts to proving that

$$\frac{\partial \Delta_n}{\partial t_{2\ell-1}}(\phi_j) + \Delta_n \frac{\partial}{\partial t_{2\ell-1}}(\phi_j) + \sum_{k=1}^{j-\ell-1} \Delta_n\left(\phi_k a_{j-\ell-k+1}\right) = 0.$$

Yet the functions a_j do not depend on x, so the last sum vanishes. And for the other terms it amounts to

$$\frac{\partial \Delta_n}{\partial t_{2\ell-1}}(\phi_j) + \Delta_n \frac{\partial}{\partial t_{2\ell-1}}(\phi_j) = \frac{\partial}{\partial t_{2\ell-1}} \Delta_n(\phi_j) = 0.$$

□

Now the fact that Equation (12) implies the KP hierarchy via Sato's Equation is a classical result which we sum up in the following proposition.

Proposition 2.5. *Let the pseudo-differential operator $S_n = \Delta_n \mathrm{D}^{-n}$. Then S_n satisfies Sato's equation*

$$\frac{\partial S_n}{\partial t_i} + \left(S_n \mathrm{D}^i S_n^{-1}\right)_- S_n = 0. \tag{13}$$

Moreover, the order 1 monic pseudo-differential operator $\mathcal{L}_n = S_n \mathrm{D} S_n^{-1}$ satisfies the Lax system of the KP hierarchy

$$\frac{\partial \mathcal{L}_n}{\partial t_i} = \left[\left(\mathcal{L}_n^i\right)_+, \mathcal{L}_n\right]. \tag{14}$$

Proof. Equation (12) on differential operators readily implies Sato's equation (13) on purely pseudo-differential operators. Then it is well known how to derive the Lax system of KP from Sato's equation. One only has to differentiate the relation $\mathcal{L}_n S_n = S_n \mathrm{D}$ by t_i, then use the relation $\left[\mathcal{L}_n, \mathcal{L}_n^i\right] = 0$. □

Finally, the following proposition states that the dressing operator S_n is indeed related to the Wronskians τ_n via the usual shift equation, ie, that τ_n is a tau function of KP.

Proposition 2.6. *The dressing operator S_n and the Wronskian τ_n satisfy the following relation*

$$S_n\left(\mathrm{e}^{\xi(\mathrm{t};\lambda)}\right) = \mathrm{e}^{\xi(\mathrm{t};\lambda)} \frac{\tau_n\left(\mathrm{t} - [\lambda^{-1}]\right)}{\tau_n(\mathrm{t})}, \tag{15}$$

where $\xi(\mathrm{t};\lambda) = \sum_{i\geq 1} t_i \lambda^i$ and

$$\tau_n\left(\mathrm{t} - [\lambda^{-1}]\right) := \tau_n\left(t_1 - \tfrac{1}{\lambda}, t_3 - \tfrac{1}{3\lambda^3}, t_5 - \tfrac{1}{5\lambda^5}, \dots\right). \tag{16}$$

To prove that, we need the following lemma.

Lemma 2.7. *The shift of the functions ϕ_j reads the following triangular relation for any $j \geq 1$,*

$$\phi_j(\mathrm{t} - [\lambda^{-1}]) = \phi_j(\mathrm{t}) - \lambda^{-1}\phi_j'(\mathrm{t}) + \sum_{i=1}^{j-1} \left(\phi_i(\mathrm{t}) - \lambda^{-1}\phi_i'(\mathrm{t})\right) b_{j-i}(\mathrm{t}), \tag{17}$$

where the b_j's are functions that do not depend on x and are defined by

$$\sum_{j\geq 0} b_j z^{2j} = \operatorname{sech}\left(z\lambda^{-1}\right)\left[1 - z\lambda^{-1}\tanh\left(z\lambda^{-1}\right) - z\lambda^{-1}\tanh(\eta) + \tanh\left(z\lambda^{-1}\right)\tanh(\eta)\right]^{-1}, \tag{18}$$

with $b_0 = 1$, and

$$\eta(\mathrm{t}; z) := \sum_{i\geq 2} t_{2i-1} z^{2i-1}.$$

Here sech $= 1/\cosh$ *and the exponent -1 stands for the multiplicative inverse of formal power series.*

Proof. Using the fact that

$$\tanh^{-1}(z\lambda^{-1}) = z\lambda^{-1} + \sum_{i\geq 2} \frac{z^{2i-1}}{(2i-1)\lambda^{2i-1}}$$

for $z\lambda^{-1} \in (-1, 1)$, and applying the sum formulae of hyperbolic functions, one obtains that

$$\sum_{j\geq 1} \phi_j\left(\mathrm{t} - [\lambda^{-1}]\right) z^{2j-1} = \frac{\operatorname{sech}\left(z\lambda^{-1}\right)\sum_{j\geq 1}\left(\phi_j - \lambda^{-1}\phi_j'\right) z^{2j-1}}{1 - z\lambda^{-1}\tanh\left(z\lambda^{-1}\right) - z\lambda^{-1}\tanh(\eta) + \tanh\left(z\lambda^{-1}\right)\tanh(\eta)}.$$

Moreover, the above denominator has its constant term equal to 1 and only even powers, so is its multiplicative inverse. Therefore, the series $\sum_{j\geq 0} b_j z^{2j}$ in Equation (18) is well defined and has indeed $b_0 = 1$, hence the triangular relation (17). □

Remark 2.8. As in Lemma (2.2), this relation is to be compared with the one satisfied by the elementary Schur polynomials, namely,

$$P_j(\mathrm{t} - [\lambda^{-1}]) = P_j(\mathrm{t}) - \lambda^{-1} P_{j-1}(\mathrm{t}).$$

We can now prove Proposition 2.6 which concludes the proof of Theorem 2.1.

Proof of Proposition 2.6. We prove an equivalent equation which only uses differential operator:

$$\Delta_n\left(\mathrm{e}^{\xi(\mathrm{t};\lambda)}\right) = \lambda^n \mathrm{e}^{\xi(\mathrm{t};\lambda)} \frac{\tau_n\left(\mathrm{t} - [\lambda^{-1}]\right)}{\tau_n(\mathrm{t})}.$$

Thanks to Lemma 2.7, we can rewrite the right-hand side as

$$\frac{\lambda^n \mathrm{e}^{\xi(\mathrm{t};\lambda)}}{\tau_n} \begin{vmatrix} \phi_1 - \lambda^{-1}\phi_1' & \cdots & \phi_1^{(n-1)} - \lambda^{-1}\phi_1^{(n)} \\ \vdots & & \vdots \\ \phi_n - \lambda^{-1}\phi_n' + \sum_{i=1}^{j-1}\left(\phi_i - \lambda^{-1}\phi_i'\right) b_{j-i} & \cdots & \phi_n^{(n-1)} - \lambda^{-1}\phi_n^{(n)} + \sum_{i=1}^{j-1}\left(\phi_i^{(n-1)} - \lambda^{-1}\phi_i^{(n)}\right) b_{j-i} \end{vmatrix}.$$

On the other hand, the left-hand side reads

$$\Delta_n\left(\mathrm{e}^{\xi(\mathrm{t};\lambda)}\right) = \frac{1}{\tau_n} \begin{vmatrix} \mathrm{e}^{\xi(\mathrm{t};\lambda)} & \lambda\mathrm{e}^{\xi(\mathrm{t};\lambda)} & \cdots & \lambda^n \mathrm{e}^{\xi(\mathrm{t};\lambda)} \\ \phi_1 & \phi_1' & \cdots & \phi_1^{(n)} \\ \vdots & & & \vdots \\ \phi_n & \phi_n' & \cdots & \phi_n^{(n)} \end{vmatrix}.$$

Using operations on rows and columns, we can easily check that these two expressions are equal. □

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