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Nonlocal \( p \)-Laplacian evolution problems on graphs

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Abstract

The non-local \( p \)-Laplacian evolution equation, governed by given kernel, has various applications to model diﬀusion phenomena, in particular in signal and image processing. In practice, such an evolution equation is implemented in discrete form (in space and time) as a numerical approximation to a continuous problem, where the kernel is replaced by an adjacency matrix of graph. The natural question that arises is to understand the structure of solutions to the discrete problem, and study their continuous limit. This is the goal pursued in this work. Combining tools from graph theory and non-linear evolution equations, we give a rigorous interpretation to the continuous limit of the discrete \( p \)-Laplacian on graphs. More specifically, we consider a sequence of deterministic weighted graphs converging to a so-called graphon. The continuous \( p \)-Laplacian evolution equation is then discretized on this graph sequence both in space and time. We therefore prove that the solutions of the sequence of discrete problems converge to the solution of the continuous evolution problem governed by the graphon, when the number of graph vertices grows to infinity. We exhibit the corresponding convergence rate.

Keywords

Nonlocal diffusion; \( p \)-Laplacian; graphs; graph limits; numerical approximation.

1 Introduction

In its continuous form, the nonlocal \( p \)-Laplacian problem with homogeneous Neumann boundary conditions governed by a given kernel \( K \) corresponds to the following nonlinear evolution equation

\[
\left\{ \begin{array}{ll}
\frac{\partial}{\partial t} u(x,t) = -\Delta^K_p u(x,t), & (x,t) \in \Omega \times [0,T], \\
u(x,0) = g(x), & x \in \Omega,
\end{array} \right. \quad (P)
\]

where

\[\Delta^K_p = -\int_n K(x,y) |u(y,t) - u(x,t)|^{p-2} (u(y,t) - u(x,t))dy.\]

\( \Omega = [0, 1] \) (without loss of generality), \( K(\cdot, \cdot) \) is a symmetric, nonnegative and bounded mapping and \( p \in [1, +\infty] \). The problem of existence and uniqueness of a solution to (P) is non-trivial. Despite the fact that we will not include the details here, we can show, relying on the theory of nonlinear semi-groups\([2]\), that for \( p \in [1, +\infty] \), (P) admits a unique strong solution in \( L^p(\Omega) \). There are many applications that integrate equation (P) to model nonlocal diﬀusion processes. For \( p \neq 2 \), the discrete \( p \)-Laplacian on graphs was studied for the semi-supervised classiﬁcation, as well as for various image processing applications such as simpliﬁcation and unsupervised segmentation (see Figure 1 and 2 for some illustrations). These practical considerations naturally lead to a discrete time and space approximation of (P). To do this, we ﬁx \( n \in \mathbb{N} \) and consider a partition \( \mathcal{Q}_n \) on \( \Omega \)

\[\{i-1\}/n, i/n], \quad i \in [n], \quad \mathcal{Q}_n = \{\Omega^{(n)}_i, i \in [n]\},\]
We consider the totally discrete counterpart of (\(P\)) on \(K/\mathcal{Q}_n\):

\[
\begin{align*}
\left\{ \right. \\
\left. \begin{array}{l}
\hat{u}^{h}_{i,j} - \hat{u}^{h-1}_{i,j} = \frac{1}{h} \sum_{i=1}^{n} (K_n)_{i,j} \left[ u^{h-1}_{i,j} - u^{h-1}_{i,j} \right], \\
u(0) = g^0, \quad i, j \in [n].
\end{array} \right.
\end{align*}
\]

\(K_n\) represents the adjacency matrix of a given convergent graph sequence \(\{G_n\}\) converging to a limit object called graphon \(K(\cdot, \cdot)\) (see \[1\] for more details about graph limits). Our goal is to study the continuum limit of the discrete \(p\)-Laplacian on graphs and quantify the rate convergence and the error estimates. All the proofs of the results can be found in the long version \[3\].

**Theorem 1.1.** Suppose that \(p \in [1, +\infty]\), \(K : \Omega^2 \to [0, 1]\) is a symmetric measurable function, and \(g \in L^\infty(\Omega)\). Let \(u\) and \(\tilde{u}\) be the solutions of \((P)\) and \((P^w)\) respectively. Then

\[
\|u - \tilde{u}\|_{C(0,T;L^p(\Omega))} \to 0, \quad \Delta \to 0.
\]

To quantify the rate of convergence in \((3)\), we need to add some supplementary assumptions on the kernel \(K\) and the initial data \(g\).

**Definition 1.1.** The total variation of a function \(K\) is defined by duality: For \(K \in L^{1}_{\infty}(\Omega^2)\) it is given by

\[
J(K) = \sup_{\phi \in S} \left\{ -\int_{\Omega^2} K \text{div}(\phi) \, dx \, dy \right\},
\]

where \(\phi \in S := \{\phi \in C^\infty(\Omega^2; \mathbb{R}^N), |\phi(x,y)| \leq 1 \forall (x,y) \in \Omega^2\}\).

A function is said to have bounded variation whenever \(J(K) < +\infty\). We call BV(\(\Omega^2\)) the set of functions with bounded variation \(K \in L^1(\Omega^2)\) such that \(J(K) < +\infty\).

**Theorem 1.2.** Suppose that \(p \in [1, +\infty]\), \(K : \Omega^2 \to [0, 1]\) is a symmetric and measurable function in BV(\(\Omega^2\)), and \(g \in L^\infty(\Omega) \cap \text{BV}(\Omega)\). Let \(u\) and \(\tilde{u}\) be the solutions of \((P)\) and \((P^w)\) respectively. Then

\[
\|u - \tilde{u}\|_{C(0,T;L^p(\Omega))} \leq O(n^{-\frac{1}{2}}) + O(\tau).
\]

**References**

