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Abstract: Minimum time control of slow-fast systems is considered. In the case of only one fast angle, averaging techniques are available for such systems. The approach introduced in Dargent (2014) and Bombrun et al. (2013) is recalled, then extended to time dependent systems by means of a suitable filtering operator. The process relies upon approximating the dynamics by means of sliding windows. The size of these windows is an additional parameter that provides intermediate approximations between averaging over the whole fast angle period and the original dynamics. The method is illustrated on problems coming from space mechanics.

Keywords: minimum time control, slow-fast dynamical systems, averaging, filtering, space mechanics

INTRODUCTION

When dealing with slow-fast dynamical systems, averaging is a well know approach to devise an approximation of the original system. In the case of systems with only one fast angle, standard averaging is well understood and used in many applications (see, *e.g.*, Lochak (1988); Sanders et al. (2007)). The application of averaging to optimal control is a more delicate issue; one cannot average naively the control depending dynamical system, as the control must keep track of the fast angle (see Chaplais (1987)). We adopt here the point of view of Bombrun et al. (2013) which provides a suitable framework for the approach developed in Geffroy (1997); Tarzi (2012); Dargent (2014, 2015) for space mechanical applications in the minimum time case. (See also Edelbaum (1974); Bonnard et al. (2007, 2009, 2006) for energy minimization.) Then we extend the method to filtering: Instead of eliminating the fast angle by averaging over an entire period of the system, we use a partial average on a sliding window whose size is only a fraction of the period. For another approach based on filtering over fixed rectangular windows, see Bernard (2015).

The paper is organized as follows. In the first section we recall the averaging procedure of Bombrun et al. (2013) for minimum time control systems with one fast angle. Then we present a new filtering method that extends the previous computation to time dependent systems, time being another fast variable. Numerical simulations based on an implementation of filtering into the industrial code *T3D* are finally presented. We restrict to time minimization and do not touch the more complicated problem of fuel minimization. (See Chen et al. (2016) for a recent mathematical analysis of the problem, and Dargent (2014, 2015) for numerical results using averaging techniques.)

1. AVERAGING FOR MINIMUM TIME

We consider the following slow-fast control system:

$$\dot I(t) = \varepsilon F_0(I(t),\varphi(t)) + \varepsilon \sum_{i=1}^m u_i(t) F_i(I(t),\varphi(t)), \quad (1)$$

$$\begin{aligned}\dot\varphi(t) &= \omega(I(t),\varphi(t)) + \varepsilon G_0(I(t),\varphi(t))\\ &\quad + \varepsilon\sum_{i=1}^m u_i(t)G_i(I(t),\varphi(t)), \quad |u(t)|\le 1,\end{aligned} \quad (2)$$

where $x=(I,\varphi)$ belongs to $M\times\mathbf{S}^1$; the data are so supposed to be periodic wrt. the angle φ, and smooth on a smooth n-dimensional manifold (that we may treat as an open subset of $\mathbf{R}^n$, up to some choice of local coordinates). As $\varepsilon>0$ is to be interpreted as small parameter, $I\in M$ represents the slow variables while φ is the fast angle. The Euclidean norm of the control is bounded by one, and we consider time minimization, typically for fixed endpoints of the slow variables and unprescribed initial and final angles. The pulsation $\omega(I,\varphi)$ is assumed to be uniformly bounded from below by some positive constant. We denote by $f(x,u)$ the right hand side of (1-2) so that the dynamical system on $M\times\mathbf{S}^1$ writes $\dot x(t)=f(x(t),u(t))$, $|u(t)|\le 1$.

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For a smooth function g on $M \times \mathbf{S}^1 \times \mathbf{R}^m$, and a given control function $u \in \mathrm{L}^\infty(\mathbf{S}^1, B)$ where B is the closed unit Euclidean ball of $\mathbf{R}^m$, one defines

$$\mu(g)(x, u(\cdot)) := \frac{1}{T(x)} \int_0^{2\pi} g(x, u(\varphi)) \frac{\mathrm{d}\varphi}{\omega(x)} \tag{3}$$

where

$$T(x) := \int_0^{2\pi} \frac{\mathrm{d}\varphi}{\omega(x)},$$

that is $2\pi/T(x) = \overline{\omega}(x) = \overline{\omega}(I)$, the harmonic average of $\omega(x)$ (which, as $\mu(g)$ and T, only depends on I—see Lemma 5.) Note that, as $x = (I, \varphi)$, the integration in (3) is

$$\mu(g)(I, \varphi, u(\cdot)) = \frac{1}{T(I,\varphi)} \int_0^{2\pi} g(I, \psi, u(\psi)) \frac{\mathrm{d}\psi}{\omega(I, \psi)}.$$

Why do keep we φ in $\mu(g)(I, \varphi, u(\cdot))$ and $T(I, \varphi)$? (see the above remark between parenthesis)

The same remark holds for the integration to compute $T(x)$, and for the integrals involved in the definition of filtering in the next section. The linear operator μ on smooth functions times essentially bounded controls defined by (3) readily extends to functions defined on $T^*(M \times \mathbf{S}^1) \times \mathbf{R}^m$. Instead of the original problem, one then considers the following differential inclusion:

$$\dot{x}(t) \in \{\mu(f)(x(t), u(\cdot)),\ u(\cdot) \in \mathrm{L}^\infty(\mathbf{S}^1, B)\}. \tag{4}$$

Convergence of the slow coordinates of time minimum trajectories of (1) towards those of (4) when the small parameter ε tends to zero is proven in Bombrun et al. (2013) under mild assumptions. Note that, in accordance with Dargent (2014), we keep track of the fast angle as there is a still a dynamics on φ (to be interpreted as a mean angle, see Remark 2). It is proven in Bombrun et al. (2013) that the right-hand side multiapplication is locally Lipschitz so that, for time minimization, the maximum principle for differential inclusions of Clarke et al. (1998) holds. In order to avoid possible singularities, we restrict the discussion to the open subset Ω defined as the complement in the cotangent bundle of

$$\Sigma := \{(x,p) \in T^*(M \times \mathbf{S}^1) \mid H_i(x,p) = 0,\ i = 1, \ldots, m\}$$

where $p = (p_I, p_\varphi)$ and $H_i(x,p) := \langle p_I, F_i(x) \rangle + p_\varphi G_i(x,p)$, $i = 0, 1, \ldots, m$. Then, if x is an absolutely continuous solution of (4), time minimizing for prescribed boundary conditions, there exists an absolutely continuous covector function $p = (p_I, p_\varphi)$ such that

$$\dot{x}(t) = \frac{\partial \overline{H}}{\partial p}(x(t), p(t)), \quad \dot{p}(t) = -\frac{\partial \overline{H}}{\partial x}(x(t), p(t)),$$

where the Hamiltonian maximized over the field of velocities parameterized by controls depending on the fast angle

$$\overline{H}(x,p) := \max_{u(\cdot) \in \mathrm{L}^\infty(\mathbf{S}^1, B)} \mu(H)(x, p, u(\cdot))$$

$$(\text{with } H(x,p,u) := \langle p, f(x,u) \rangle)$$

is well defined and smooth. Actually,

Proposition 1. For $(x,p) \in \Omega$,

$$\overline{H}(x,p) = p_\varphi \overline{\omega}(I) + \varepsilon \overline{K}(x,p)$$

with

$$\overline{K}(x,p) := \frac{1}{T(x)} \int_0^{2\pi} K(x,p) \frac{\mathrm{d}\varphi}{\omega(x)}$$

and

$$K(x,p) := H_0(x,p) + \sqrt{\sum_{i=1}^m H_i^2(x,p)}.$$

Proof. On Ω, $\overline{H}(x,p) = \mu(H)(x,p,u(\cdot))$ evaluated at $u(\varphi) = u(x,p)$ ($= u(I, \varphi, p_I, p_\varphi)$) with

$$u(x,p) := \frac{(H_1(x,p), \ldots, H_m(x,p))}{|(H_1(x,p), \ldots, H_m(x,p))|} \cdot \qquad \square \tag{5}$$

Remark 2. As φ becomes a cyclic variable of $\overline{H}$ (see Lemma 5), p_φ is constant. For $p_\varphi = 0$, one has $\dot{\varphi} = \overline{\omega}(I)$ which allows to interpret the angle of the averaged system as a mean angle. Keeping track of this angle is important in practice as this allows to treat more complex boundary conditions in applications (see Dargent (2014, 2015)).

Remark 3. For $p_\varphi = 0$, the same system can be obtained by suitably averaging the extremal system associated with the original optimal control problem after identifying carefully the slow and fast variables (CNES (2015)).

In a more compact form with $z = (x,p)$, the extremal system writes

$$\dot{z}(t) = \overrightarrow{\nabla} \overline{H}(z(t)), \tag{6}$$

where $\overrightarrow{\nabla}$ denotes the symplectic gradient. It turns out that

Lemma 4.

$$\overrightarrow{\nabla} \overline{H}(z) = \left[\overrightarrow{\nabla} \mu(H)(z, u(\cdot)) \right]_{|u(\cdot) = u(z)}$$

with $u(z)$ defined by (5).

Proof. As $\overrightarrow{\nabla}$ commutes with the integration over φ, it suffices to check that the symplectic gradient also commutes with evaluating at $u(z)$, the unique maximizer of $H(x,p,\cdot)$ on B, for z in Ω. But this is obvious since, in coordinates, if $A(z) \in \mathbf{R}^m$ is a non-vanishing smooth function of $z \in \mathbf{R}^{2n}$, one has the following: $h(z,v) := (A(z)|v)$ is maximized wrt. v on the unit Euclidean ball B of $\mathbf{R}^m$ by $v(z) := A(z)/|A(z)|$, and

$$\frac{\partial h}{\partial z}(z, v(z)) = \frac{\mathrm{d}}{\mathrm{d}z}(h(z, v(z))). \qquad \square$$

As a result, the averaged extremal system (6) can be also written

$$\dot{x}(t) = \left[\frac{\partial}{\partial p} \mu(H)(x, p, u(\cdot)) \right]_{|u(\cdot) = u(x,p)} \tag{7}$$

$$\dot{p}(t) = \left[-\frac{\partial}{\partial x} \mu(H)(x, p, u(\cdot)) \right]_{|u(\cdot) = u(x,p)} \tag{8}$$

and it is this from this form that we depart to define an approximation by filtering in the next section. We conclude by indicating the effect of the choice of the fast angle on the computation of the adjoint equation.

Lemma 5. For any smooth function g on $M \times \mathbf{S}^1 \times \mathbf{R}^m$,

$$\frac{\partial}{\partial I} \cdot \mu(g) = \mu\left(\frac{\partial g}{\partial I}\right) + \mu(g) \cdot \mu\left(\frac{\partial \omega/\partial I}{\omega}\right) - \mu\left(g \cdot \frac{\partial \omega/\partial I}{\omega}\right),$$

$$\frac{\partial}{\partial \varphi} \cdot \mu(g) = 0.$$

The proof is obvious, and the result emphasizes that the non-commutativity of the averaging operator μ with taking derivatives is due to the non-canonical choice for the fast angle. Whenever φ is such that its pulsation, ω, only depends on the slow variable I, the remainder in $(\partial/\partial I)\cdot\mu$ vanishes and commutativity is retrieved. Such a choice is always possible but may not be convenient in practice, as the change of angle may involve to solve some transcendent equation (*e.g.*, Kepler equation in two-body problems).

2. FILTERING BY SLIDING WINDOWS

We now consider the following time dependent slow-fast control system:

$$\dot{I}(t) = \varepsilon F_0(t, I(t), \varphi(t)) + \varepsilon \sum_{i=1}^{m} u_i(t) F_i(t, I(t), \varphi(t)), \quad (9)$$

$$\begin{aligned}\dot{\varphi}(t) &= \omega(I(t), \varphi(t)) + \varepsilon G_0(t, I(t), \varphi(t)) \\ &\quad + \varepsilon \sum_{i=1}^{m} u_i(t) G_i(t, I(t), \varphi(t)), \quad |u(t)| \le 1, \quad (10)\end{aligned}$$

and denote

$$\omega(t, x, u) := \omega(x) + \varepsilon G_0(t, x) + \varepsilon \sum_{i=1}^{m} u_i G_i(t, x)$$

(the dependence on ε is kept implicit). Time is another fast variable, but no periodicity wrt. t is assumed. Therefore it is less relevant to use an approximation such as (3) that would freeze time over a whole angle period. Given a window size $\Delta \in (0, 2\pi]$ and $u \in \mathrm{L}^\infty(\mathbf{S}^1, B)$, we define

$$\mu_\Delta(g)(t, x, u(\cdot)) := \frac{1}{T_\Delta(t, x, u(\cdot))} \int_{\varphi-\Delta/2}^{\varphi+\Delta/2} g(t, x, u(\varphi)) \frac{\mathrm{d}\varphi}{\omega(t, x, u(\varphi))} \quad (11)$$

with

$$T_\Delta(t, x, u(\cdot)) := \int_{\varphi-\Delta/2}^{\varphi+\Delta/2} \frac{\mathrm{d}\varphi}{\omega(t, x, u(\varphi))}$$

and where g is any smooth function on $\mathbf{R} \times M \times \mathbf{S}^1 \times \mathbf{R}^m$.

Remark 6. In addition to time dependence, another difference with averaging as described in the previous section is the integration wrt. $\mathrm{d}\varphi/\omega(t, x, u(\varphi))$ instead of $\mathrm{d}\varphi/\omega(x)$; this also results in a functional dependence on $u(\cdot)$ for T_Δ. So when $\Delta = 2\pi$, one retrieves $\mu_\Delta = \mu$ provided the system is autonomous and such that $G_0 = G_1 = \cdots = G_m = 0$ to kill the dependence on u in $\omega(t, x, u)$. Note that the difference between $\omega(x)$ and $\omega(t, x, u)$ is of order one in ε, though.

The Hamiltonian associated to time minimization of this control system is

$$H(t, x, p, u) := p_\varphi \omega(x) + \varepsilon H_0(t, x, p) + \varepsilon \sum_{i=1}^{m} u_i H_i(t, x, p),$$

$$H_i(t, x, p) := \langle p_I, F_i(t, x)\rangle + p_\varphi G_i(t, x), \quad i = 0, m.$$

Similarly to what was done in the previous section, we restrict to the open complement Ω in $\mathbf{R} \times T^*(M \times \mathbf{S}^1)$ of

$$\Sigma := \{(t, x, p) \in \mathbf{R} \times T^*(M \times \mathbf{S}^1) \mid H_i(t, x, p) = 0,\ i = 1, \dots, m\}.$$

For any $(t, z) = (t, x, p) \in \Omega$, the unique maximizer of $H(t, x, p, \cdot)$ on the unit Euclidean ball B of $\mathbf{R}^m$ is

$$u(t, z) := \frac{(H_1(t, z), \dots, H_m(t, z))}{|(H_1(t, z), \dots, H_m(t, z))|}.$$

By analogy with the averaged system (7-8), we consider the following filtered extremal system

$$\dot{z}(t) = \left[\overrightarrow{\nabla} \mu_\Delta(H)(t, z(t), u(\cdot))\right]_{|u(\cdot)=u(t,z)}. \quad (12)$$

Remark 7. As opposed to Lemma 4, taking gradient and evaluating at the maximizing control need not commute anymore. Indeed, for a given Δ, evaluating at $u(t, z)$ does not necessarily maximize $\mu_\Delta(H)(t, z, u(\cdot))$ over $u(\cdot)$ in $\mathrm{L}^\infty(\mathbf{S}^1, B)$ because of the dependence in $u(\cdot)$ in the filtered expression (due to the presence of the control in $\omega(t, x, u)$, and hence in $T_\Delta(t, x, u(\cdot))$). As a consequence, the dynamical system (12) is not Hamiltonian, in general.

We now review the basic properties of the linear operator μ_Δ with in mind convergence properties when Δ tends to zero. The motivation is that, for the time minimization of (9-10), filtering over a $\Delta = 2\pi$ window (which is close to averaging, notwithstanding time dependency—see Remark 6) might not provide a good enough approximation to initialize a convergent numerical resolution of the original system; having a continuous set of intermediate approximations as Δ range from 2π to 0 may allow to ensure this convergence. In this respect, see the final section for numerical experiments on problems stemming from space mechanics. The computation below is completely similar to Lemma 5 for the part concerning the slow variables; notice that filtering does not kill anymore dependency on the fast angle, though. (The corresponding expression in brackets below means taking the value at $\varphi + \Delta/2$ minus the one at $\varphi - \Delta/2$.) The function ω involved is the whole right-hand side $\omega(t, x, u)$ of $\dot{\varphi}$, in accordance with definition (11).

Lemma 8. Let Δ in $(0, 2\pi]$. For any smooth function g on $\mathbf{R} \times M \times \mathbf{S}^1 \times \mathbf{R}^m$,

$$\begin{aligned}\frac{\partial}{\partial I} \cdot \mu_\Delta(g) &= \mu_\Delta\left(\frac{\partial g}{\partial I}\right) + \mu_\Delta(g) \cdot \mu_\Delta\left(\frac{\partial \omega/\partial I}{\omega}\right) \\ &\quad - \mu_\Delta\left(g \cdot \frac{\partial \omega/\partial I}{\omega}\right), \\ \frac{\partial}{\partial \varphi} \cdot \mu_\Delta(g) &= \frac{1}{T_\Delta}\left[\frac{f}{\omega}\right]_{\varphi-\Delta/2}^{\varphi+\Delta/2} + \mu_\Delta(g) \cdot \mu_\Delta\left(\frac{\partial \omega/\partial \varphi}{\omega}\right).\end{aligned}$$

Before giving the next lemma (or maybe in the lemma's statement), I would define δ_φ.

Lemma 9. As Δ tends to zero,

$$\begin{aligned}\mu_\Delta &\to \delta_\varphi, \\ \frac{\partial}{\partial I} \cdot \mu_\Delta &\to \delta_\varphi \cdot \frac{\partial}{\partial I}, \\ \frac{\partial}{\partial \varphi} \cdot \mu_\Delta &\to \delta_\varphi \cdot \frac{\partial}{\partial \varphi}.\end{aligned}$$

Proof. Given u in $\mathrm{L}^\infty(\mathbf{S}^1, B)$ and a smooth function g on $\mathbf{R} \times M \times \mathbf{S}^1 \times \mathbf{R}^m$, one readily gets

$$\mu_\Delta(g)(t, x, u(\cdot)) \to g(t, x, u(\varphi)) \quad (\text{with } x = (I, \varphi))$$

when Δ tends to zero, which we denote $\mu_\Delta \to \delta_\varphi$. Using the previous lemma, the limit of $\frac{\partial}{\partial I} \cdot \mu_\Delta$ follows. In the case of $\frac{\partial}{\partial\varphi} \cdot \mu_\Delta$, one has

$$\frac{1}{T_\Delta}\left[\frac{f}{\omega}\right]_{\varphi-\Delta/2}^{\varphi+\Delta/2} \to \omega \cdot \frac{\partial}{\partial\varphi}\left(\frac{f}{\omega}\right) = \frac{\partial f}{\partial\varphi} - f \cdot \frac{\partial\omega/\partial\varphi}{\omega}$$

when Δ tends to zero, whence the result. $\square$

We set $\mu_0 :=$ id, which amounts to extend (12) by

$$\dot{z}(t) = \left[\overrightarrow{\nabla} H(t, z(t), u(\cdot))\right]_{|u(\cdot)=u(t,z)} \tag{13}$$

for $\Delta = 0$. (This system being Hamiltonian as taking gradient and evaluating at the maximizing control actually commute.)

Proposition 10. Let (t_0, z_0) be an arbitrary initial condition in Ω. There exists $\eta > 0$ such that, for all Δ in $[0, 2\pi]$, the differential equation (12) admits a unique solution of class $\mathscr{C}^1$ on $[t_0 - \eta, t_0 + \eta]$; this solution depends continuously on Δ. In particular, one has convergence towards a solution of the original extremal system (13) when Δ tends to zero.

Proof. For (t, z) in Ω and Δ in $[0, 2\pi]$, denote by $g(t, z, \Delta)$ the right-hand side of (12) (extended by the right-hand side of (13) when $\Delta = 0$). Partial functions $\Delta \mapsto g(t, z, \Delta)$ are continuous on $[0, 2\pi]$, the previous lemma ensuring continuity at $\Delta = 0$. Given the expression of (12) (see Lemma 8), the result below suffices to show that all functions g are locally Lipschitz wrt. z with local Lipschitz constants that are uniform in Δ.

Lemma 11. Let $f : \mathbf{R}^n \to \mathbf{R}^m$ be a smooth function, and fix d in $\mathbf{R}^n$. For δ in $(0, 1]$, set

$$g_\delta(x) := \frac{1}{\delta}\left[f\left(x + \frac{\delta}{2}d\right) - f\left(x - \frac{\delta}{2}d\right)\right].$$

The family $\{g_\delta\}_\delta$ is locally equilipschitzian.

Proof. Fix x_0 in $\mathbf{R}^n$; $g'_\delta(x) \to f''(x) \cdot d$ when δ tends to zero, and the convergence is uniform on a compact neighbourhood V of x_0 (Taylor-Lagrange inequality). So one can find $\delta_0 > 0$ such that, for all δ in $(0, \delta_0]$ and x in V, $|g'_\delta(x)| \leq |f''(x) \cdot d| + 1$. The family $\{g_\delta\}_\delta$ is so equilipschitzian on V with Lipschitz constant the maximum of $|f''(x) \cdot d| + 1$ and $\max_{(x,\delta)\in V\times[\delta_0,1]} |g'_\delta(x)|$. $\square$

A standard application of the parameterized fixed point theorem allows to prove that, given (t_0, z_0) in Ω, there exists $\eta > 0$ such that

$$\dot{z}(t) = g(t, z(t), \Delta), \quad z(t_0) = z_0,$$

admits a unique $\mathscr{C}^1$ solution defined on $[t_0 - \eta, t_0 + \eta]$, continuously depending on the parameter Δ in $[0, 2\pi]$. $\square$

3. APPLICATION IN SPACE MECHANICS

Unperturbed dynamics. We want to compute minimum time trajectories for a spacecraft orbiting around the Earth. The control is provided by a new generation electro-ionic engine delivering small to very small thrust levels, as opposed to older chemical propulsion. The literature on this topic is now well established, and we refer to it for further details on the problem (see, *e.g.*, Caillau et al. (2012a,b)). Having in mind to take into account perturbations of the Keplerian motion, we first recall the unperturbed dynamics describing the controlled two-body problem. The state is made of slow variables $I = (P, e_x, e_y, h_x, h_y)$ that characterize the geometry of the osculating ellipse (we restrict to periodic free motion), and of one fast angle, the longitude ℓ, that defines the position of the spacecraft on the current orbit. Note that $(e_x, e_y) = e\exp(\imath(\Omega + \omega))$, where e is the eccentricity, Ω the longitude of the ascending node, ω the argument of the pericenter, while $(h_x, h_y) = \tan(i/2)\exp(\imath\Omega)$, where i is the inclination of the orbit plane wrt. the equatorial plane; P is the semi-latus rectum of the ellipse. In these coordinates, the dynamics is

$$\begin{aligned}
\dot{P} &= 2\sqrt{P^3/\mu}\, u_2/W,\\
\dot{e}_x &= \sqrt{P/\mu}(1/W)(W \sin\ell\, u_1 + Au_2 - e_y C u_3),\\
\dot{e}_y &= \sqrt{P/\mu}(1/W)(-W\cos\ell\, u_1 + Bu_2 + e_x C u_3),\\
\dot{h}_x &= \sqrt{P/\mu}(D/2W)\cos\ell\, u_3,\\
\dot{h}_y &= \sqrt{P/\mu}(D/2W)\sin\ell\, u_3,\\
\dot{\ell} &= \sqrt{\mu/P^3}\, W^2 + \sqrt{P/\mu}(C/W)u_3,
\end{aligned}$$

with $|u| \leq T_{\max}$,

$$W = 1 + e_x\cos\ell + e_y\sin\ell,$$
$$A = e_x + (1 + W)\cos\ell, \quad B = e_y + (1 + W)\sin\ell,$$
$$C = h_x\sin\ell - h_y\cos\ell, \quad D = 1 + h_x^2 + h_y^2,$$

and where the control is expressed in a radial-orthoradial-out of plane local frame. Note that this unperturbed model is of the form (1-2) with the thrust modulus $T_{\max}$ playing the role of the small parameter ε, and no drift terms F_0, G_0. The constant μ is the Earth gravitational constant ($\mu \simeq 3.9860047 \times 1e{+}14$ in m^2/s^2). The complete dynamics also includes the mass of the spacecraft as an additional state variable (see Dargent (2014) for further details). For the numerical tests below, the initial orbit has semi-major axis 24505.9 kilometers, eccentricity 0.72, inclination 7.05 degrees, argument of pericenter 180 degrees, null longitude of the ascending node and anomaly. (The anomaly, ν, is such that $\ell = \Omega + \omega + \nu$.) The target orbit is the geostationary one. The thrust level is very low, $T_{\max} = 0.175$ Newtons, for a spacecraft of mass 2000 kilograms. We use the software *T3D* developed at Thales Alenia Space (see Dargent (2014, 2015)) for the computation. Filtering as described in Section 2 has been incorporated into the code. For these specific boundary conditions, single shooting initialized by the solution of the averaged problem (defined as in Section 1) does not converge. Using an intermediate step of filtering at $\Delta = \pi/2$ allows shooting on the true dynamics to converge (Figure 1 and 2).

Dynamics with J_2 perturbation. We now add a first perturbation to the dynamics to take into account the J_2 effect (higher order term in the Earth potential). This amounts to adding drift terms F_0, G_0 in the previous equations. For the numerical tests, the initial orbit has semi-major axis 26600 kilometers, eccentricity 0.75, inclination 30 degrees, argument of pericenter 10 degrees, longitude of the ascending node 10 degrees and null anomaly. The target

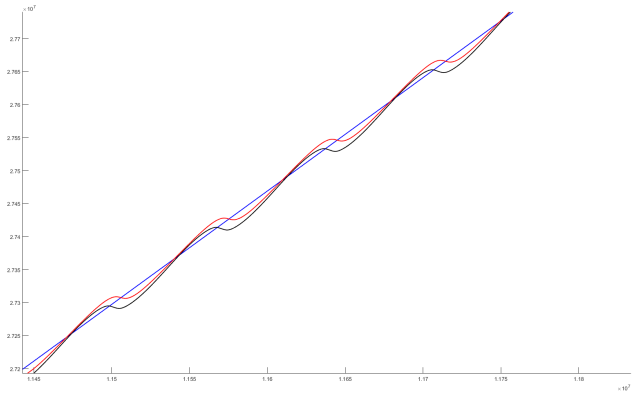

Fig. 1. Unperturbed dynamics, case of a rendez-vous in $(a, e, \omega, \Omega, i)$. Plot of the semi-latus rectum P for $\Delta = 2\pi$ (blue curve), $\Delta = \pi/2$ (red curve), and for the true dynamics (black curve). For $T_{\max} = 0.175$ Newtons, the final time is 273.54 days.

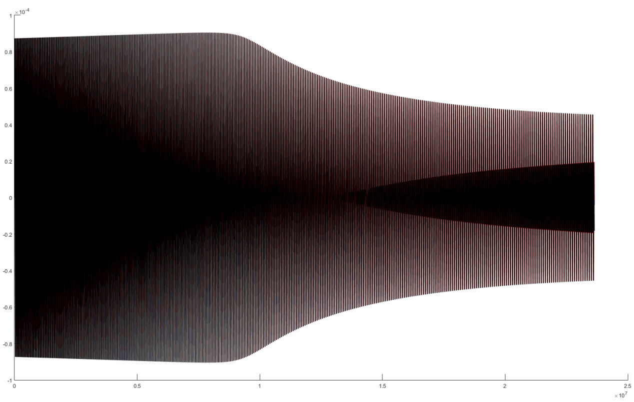

Fig. 2. Unperturbed dynamics, case of a rendez-vous in $(a, e, \omega, \Omega, i)$. Plot of the radial component of the control, illustrating the highly oscillatory behaviour of the minimum time control. At this scale, the differences between $\Delta = 2\pi$, $\Delta = \pi/2$ and the true dynamics are indiscernible.

orbit is the geostationary orbit. The thrust level varies from $T_{\max} = 0.8$ to 0.25 Newtons, for a spacecraft of mass 1000 kilograms. Convergence of shooting on the true dynamics is obtained thanks to intermediate filtering steps (see Table 1 and Figures 3,4). Also note that if a rendez-vous in (a, e, i) only is considered (letting free Ω and ω, which makes sense for a circular target), filtering does improve convergence as several steps with intermediate $\Delta < 2\pi$ allow shooting to find a solution whereas direct initialization by the $\Delta = 2\pi$ solution fails. (See Table 2).

Table 1. Convergence: Dynamics with J_2 only, case of a rendez-vous in $(a, e, \omega, \Omega, i)$.

Thrust (Newtons)	Converging sequence
0.80	$\Delta = 0 \leftarrow \Delta = \pi/2 \leftarrow \Delta = 2\pi$
0.70	$\Delta = 0 \leftarrow \Delta = \pi/2 \leftarrow \Delta = \pi \leftarrow \Delta = 2\pi$
0.60	$\Delta = 0 \leftarrow \Delta = \pi \leftarrow \Delta = 2\pi$
0.50	$\Delta = 0 \leftarrow \Delta = \pi \leftarrow \Delta = 2\pi$
0.25	— (no convergence)

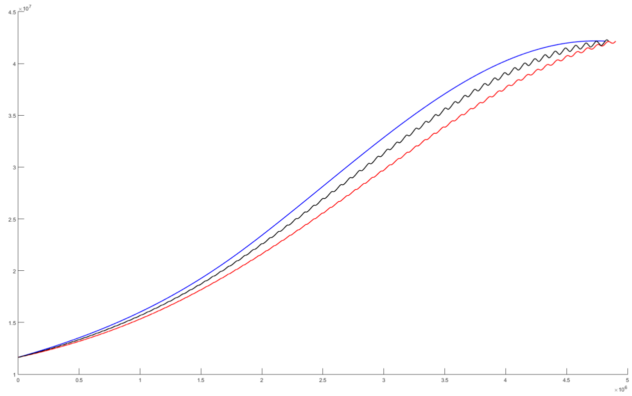

Fig. 3. Dynamics with J_2 only, case of a rendez-vous in $(a, e, \omega, \Omega, i)$. Plot of the semi-latus rectum P for $\Delta = 2\pi$ (blue curve), $\Delta = \pi$ (red curve), and for the true dynamics (black curve).

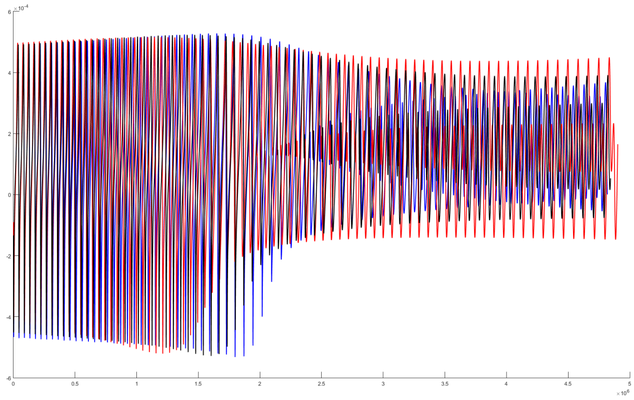

Fig. 4. Dynamics with J_2 only, case of a rendez-vous in $(a, e, \omega, \Omega, i)$. Plot of the radial component of the control for $\Delta = 2\pi$ (blue curve), $\Delta = \pi$ (red curve), and for the true dynamics (black curve).

Table 2. Convergence: Dynamics with J_2 only, case of a rendez-vous in (a, e, i) only.

Thrust (Newtons)	Converging sequence
6.00	$\Delta = 0 \leftarrow \Delta = \pi \leftarrow \Delta = 2\pi$
5.00	$\Delta = 0 \leftarrow \Delta = \pi \leftarrow \Delta = 2\pi$
1.00	$\Delta = 0 \leftarrow \Delta = \pi/2 \leftarrow \Delta = \pi \leftarrow \Delta = 2\pi$
0.50	$\Delta = 0 \leftarrow \Delta = \pi \leftarrow \Delta = 2\pi$
0.25	— (no convergence)

Dynamics with full perturbations. We now consider the full model with several perturbations to take into account not only the J_2 but also higher order terms of the Earth potential, the Luni-Solar potential, solar pressure and atmospheric drag. All these additional effects are built in *T3D*. This amounts to adding some time dependency in the previous equations, in accordance with (9-10). For the numerical tests, the initial orbit has semi-major axis 26600 kilometers, eccentricity 0.75, inclination 30 degrees, argument of pericenter 10 degrees, longitude of the ascending node 10 degrees and null anomaly. The target orbit is the geostationary orbit. The thrust level varies from $T_{\max} = 1.5$ to 0.5 Newtons, for a spacecraft of mass 1000 kilograms. Convergence of shooting on the true dynamics is again obtained thanks to intermediate filtering steps (see Table 3 and Figures 5,6). This confirms the interest of being able to perform a discrete continuation on the window size Δ, in order to connect the solution for $\Delta = 2\pi$ provided by averaging to the solution of the true dynamics. Further work on these approaches include refined approximations by averaging based on corrector terms for the boundary conditions of the averaged system. (See Dargent et al. (2017).)

Table 3. Convergence: Dynamics with full perturbations, case of a rendez-vous in $(a, e, \omega, \Omega, i)$.

Thrust (Newtons)	Converging sequence
1.50	$\Delta = 0 \leftarrow \Delta = \pi/2 \leftarrow \Delta = \pi \leftarrow \Delta = 2\pi$
1.00	$\Delta = 0 \leftarrow \Delta = \pi \leftarrow \Delta = 2\pi$
0.50	$\Delta : 0 \leftarrow \pi/4 \leftarrow \pi/2 \leftarrow \pi \leftarrow 2\pi$

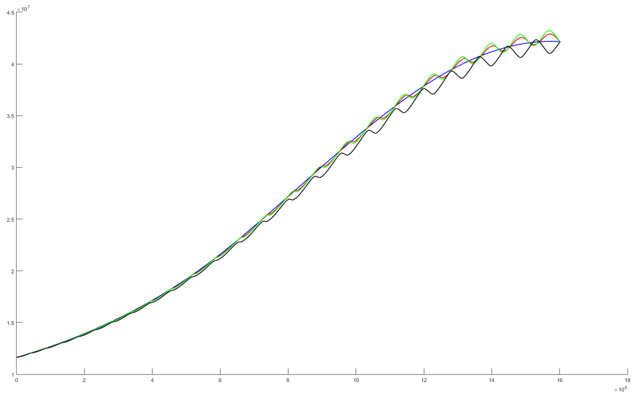

Fig. 5. Dynamics with full perturbations, case of a rendez-vous in $(a, e, \omega, \Omega, i)$. Plot of the semi-latus rectum P for $\Delta = 2\pi$ (blue curve), $\Delta = \pi$ (red curve), $\Delta = \pi/2$ (green), and for the true dynamics (black curve).

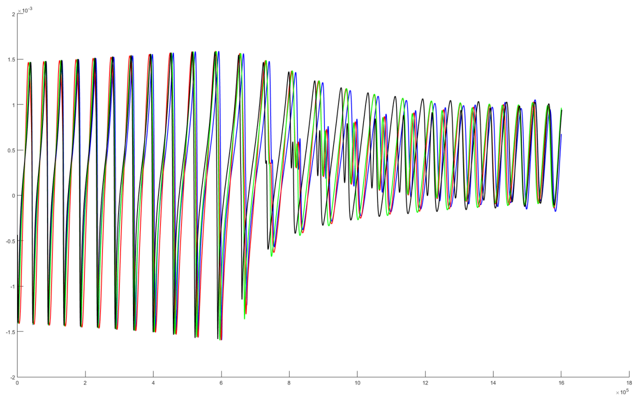

Fig. 6. Dynamics with full perturbations, case of a rendez-vous in $(a, e, \omega, \Omega, i)$. Plot of the radial component of the control for $\Delta = 2\pi$ (blue curve), $\Delta = \pi$ (red curve), $\Delta = \pi/2$ (green), and for the true dynamics (black curve).

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