Well-Composedness in Alexandrov Spaces Implies Digital Well-Composedness in Z^n

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Well-Composedness in Alexandrov Spaces implies Digital Well-Composedness in $\mathbb{Z}^n$

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Abstract. In digital topology, it is well-known that, in 2D and in 3D, a digital set $X \subseteq \mathbb{Z}^n$ is *digitally well-composed (DWC)*, *i.e.*, does not contain any critical configuration, if its immersion in the Khalimsky grids $\mathbb{H}^n$ is *well-composed in the sense of Alexandrov (AWC)*, *i.e.*, its boundary is a disjoint union of discrete $(n-1)$-surfaces. We show that this is still true in n-D, $n \geq 2$, which is of prime importance since today 4D signals are more and more frequent.
Keywords: well-composed · discrete surfaces · Alexandrov spaces · critical configurations · digital topology.

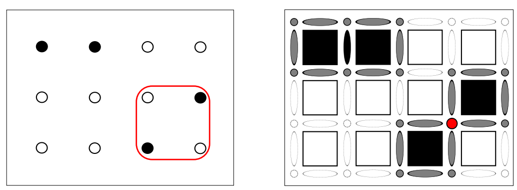


Fig. 1: From a digital set $X \subseteq \mathbb{Z}^2$ to its immersion $\mathcal{I}(X)$ in $\mathbb{H}^2$; X and $\mathcal{I}(X)$ are depicted in black, and the boundary of $\mathcal{I}(X)$ in $\mathbb{H}^2$ is depicted in gray and red.

1 Introduction

We recall that a subset of $\mathbb{Z}^n$, $n \geq 1$, is said to be *digital* if it is finite or if its complement in $\mathbb{Z}^n$ is finite. When $n \in \{2, 3\}$, the *immersion* $\mathcal{I}(X)$ into the *Khalimsky grid* $\mathbb{H}^n$ of a digital subset X of $\mathbb{Z}^n$ based on the *miss-strategy* is known to be *well-composed in the Alexandrov sense (AWC)* [13], *i.e.*, the connected components of the *boundary* of $\mathcal{I}(X)$ are *discrete* $(n-1)$*-surfaces*, iff X is *digitally well-composed (DWC)* [5], *i.e.*, X does not contain any *critical configuration* (see Figure 1). The aim of this paper is to show that AWCness (of the immersion of a set) implies DWCness (of the initial set) in n-D, $n \geq 2$.

In order to do so, we show that we can reformulate AWCness in a local way: a subset Y of $\mathbb{H}^n$ is AWC iff, for any z in the boundary $\mathfrak{N}$ of Y in $\mathbb{H}^n$, the subspace $|\beta^{\Box}_{\mathfrak{N}}(z)|$ is a discrete $(n-2-\dim(z))$-surface. This property is of prime importance since we compare AWCness with DWCness, which is a local property. It is then sufficient to proceed by counterposition and to show that, if a digital subset X of $\mathbb{Z}^n$ contains a critical configuration in a block S such that $X \cap S = \{p, p'\}$ (primary case) or $S \setminus X = \{p, p'\}$ (secondary case) with p and p' two antagonists in S, then there exists some z^* in the boundary $\mathfrak{N}$ of $\mathcal{I}(X)$ satisfying that $|\beta^{\Box}_{\mathfrak{N}}(z^*)|$ is not a $(n-2-\dim(z^*))$-surface. In fact, by choosing a particular z^* related to $\frac{p+p'}{2}$, we obtain that $|\beta^{\Box}_{\mathfrak{N}}(z^*)|$ is the union of two disjoint $(n-2-\dim(z^*))$-surfaces, the first is a function of p and z^* and the second is a function of p' and z^*. This way, $|\beta^{\Box}_{\mathfrak{N}}(z^*)|$ is not a $(n-2-\dim(z^*))$-surface, which concludes the proof.

In Section 2, we recall some needed basics principles to formalize DWCness and AWCness. In Section 3, we present some new mathematical tools supporting our proof. In Section 4, after having detailed some properties of $\mathbb{Z}^n$ and $\mathbb{H}^n$, we show how they are related. In Section 5, we outine the proof of the paper's main result, while we conclude our work in Section 6.

2 Basic concepts of digital topology

Let us reintroduce the notions of DWCness and AWCness.

2.1 Digital topology and DWCness

Let $\mathbb{B} = \{e^1, \dots, e^n\}$ be the canonical basis of $\mathbb{Z}^n$. We use the notation x_i, where i belongs to $[\![1, n]\!]$, to determine the i^{th} coordinate of $x \in \mathbb{Z}^n$. We recall that the L^1-*norm* of a point $x \in \mathbb{Z}^n$ is denoted by $\|.\|_1$ and is equal to $\sum_{i \in [\![1,n]\!]} |x_i|$ where $|.|$ is the *absolute value*. Also, the L^∞-*norm* is denoted by $\|.\|_\infty$ and is equal to $\max_{i \in [\![1,n]\!]} |x_i|$. For a given point $x \in \mathbb{Z}^n$, an element of the set $\mathcal{N}^*_{2n}(x) = \{y \in \mathbb{Z}^n \ ; \ \|x - y\|_1 = 1\}$ (resp. of the set $\mathcal{N}^*(x) = \{y \in \mathbb{Z}^n \ ; \ \|x - y\|_\infty = 1\}$) is a $2n$-*neighbor* (resp. a $(3^n - 1)$-*neighbor*) of x. For any $z \in \mathbb{Z}^n$ and any $\mathcal{F} = (f^1, \dots, f^k) \subseteq \mathbb{B}$, we denote by $S(z, \mathcal{F})$ the set $\left\{ z + \sum_{i \in [\![1,k]\!]} \lambda_i f^i \ \middle| \ \lambda_i \in \{0, 1\}, \forall i \in [\![1, k]\!] \right\}$. We call this set the *block* associated with the pair $(z, \mathcal{F})$; its *center* is $z + \sum_{f \in \mathcal{F}} \frac{f}{2}$, and its *dimension*, denoted by $\dim(S)$, is equal to k. More generally, a set $S \subset \mathbb{Z}^n$ is said to be a *block* iff there exists a pair $(z, \mathcal{F}) \in \mathbb{Z}^n \times \mathcal{P}(\mathbb{B})$ such that $S = S(z, \mathcal{F})$. Then, we say that two points $p, q \in \mathbb{Z}^n$ belonging to a block S are *antagonists* in S iff the distance between them equals the maximal distance using the L^1 norm between two points in S; in this case we write $p = \mathrm{antag}_S(q)$. Note that the antagonist of a point p in a block S containing p exists and is unique. Two points that are antagonists in a block of dimension $k \geq 0$ are said to be k-*antagonists*; k is then called the *order of antagonism* between these two points. We say that a digital subset X of $\mathbb{Z}^n$ contains a *critical configuration* in a block S of dimension $k \in [\![2, n]\!]$ iff there exists two points $\{p, p'\} \in \mathbb{Z}^n$ that are antagonists in S s.t. $X \cap S = \{p, p'\}$ (*primary case*) or s.t. $S \setminus X = \{p, p'\}$ (*secondary case*). Then, a digital set $X \subset \mathbb{Z}^n$ is said to be *digitally well-composed (DWC)* [5] iff it does not contain any critical configuration.

2.2 Axiomatic digital topology and AWCness

Let X be any set, and let $\mathcal{U}$ be a set of subsets of X satisfying that $X, \emptyset$ belong to $\mathcal{U}$, any union of any family of elements of $\mathcal{U}$ belongs to $\mathcal{U}$, and any finite intersection of any family of elements of $\mathcal{U}$ belongs to $\mathcal{U}$. Then $\mathcal{U}$ is a *topology* [10,1], and the pair $(X,\mathcal{U})$ is called a *topological space*. We abusively say that X is a topological space, assuming it is supplied with a topology $\mathcal{U}$. The elements of $\mathcal{U}$ are called the *open sets* of $(X,\mathcal{U})$, and the complement of an open set is said to be a *closed set* [1]. A set N containing an element p of a topological space X s.t. there exists $U \in \mathcal{U}$ satisfying $p \in U \subseteq N$ is said to be a *neighborhood* of p in X. We say that a topological space $(X,\mathcal{U})$ satisfies the T_0 *axiom of separation* [3,10,1] iff for any two different elements in X, for at least one of them there is an open neighborhood not containing the other element. A topological space which satisfies the T_0 axiom of separation is said to be a T_0*-space*, a topological space X is called *discrete* [2] iff the intersection of any family of open sets of X is open in X, and a discrete T_0-space is said to be an *Alexandrov space* [8].

Let Λ be an arbitrary set. A *binary relation* [4] R on Λ is a subset of $\Lambda \times \Lambda$, and for any $x, y \in \Lambda$, we denote by $x \; R \; y$ the fact that $(x, y) \in R$, or equivalently $x \in R(y)$. A binary relation R is called *reflexive* iff, $\forall x \in \Lambda$, $x \; R \; x$, is called *antisymmetric* iff, $\forall x, y \in \Lambda$, $x \; R \; y$ and $y \; R \; x$ imply $x = y$, and is called *transitive* iff, $\forall x, y, z \in \Lambda$, $x \; R \; y$ and $y \; R \; z$ imply $x \; R \; z$. Also, we denote by $R^{\square}$ the binary relation defined such that, $\forall x, y \in \Lambda$, $\left\{x \; R^{\square} \; y\right\} \Leftrightarrow \{x \; R \; y \text{ and } x \neq y\}$. An *order relation* [4] on Λ is a binary relation which is reflexive, antisymmetric, and transitive; a set Λ of arbitrary elements supplied with an order relation R on Λ is denoted (Λ, R) or $|\Lambda|$ and is called a *poset* [4]; Λ is called the *domain* of $|\Lambda|$. According to Alexandrov (Th. 6.52, p. 28 of [1]), we can identify any poset $|X| = (X, R)$ with the Alexandrov space *induced* by the order relation R. Let (X, α_X) be a poset and p an element of X, the *combinatorial closure* $\alpha_X(p)$ of p in $|X|$ is the set $\{q \in X \; ; \; (q, p) \in \alpha_X\}$, the *combinatorial opening* $\beta_X(p)$ of p in $|X|$ is the set $\{q \in X \; ; \; (p, q) \in \alpha_X\}$, and $\theta_X(p) := \alpha_X(p) \cup \beta_X(p)$; $\alpha_X(p)$ (resp. $\beta_X(p)$) is then the smallest closed (resp. open) set containing $\{p\}$ in X. Also, $\forall S \subseteq X$, $\alpha_X(S) := \cup_{p \in S} \alpha_X(p)$, $\beta_X(S) := \cup_{p \in S} \beta_X(p)$, and $\theta_X(S) := \cup_{p \in S} \theta_X(p)$. Assuming that $|X|$ is a poset and S is a subset of X, the *suborder* [4] of $|X|$ relative to S is the poset $|S| = (S, \alpha_S)$ with $\alpha_S := \alpha_X \cap S \times S$; we have then, for any $x \in S$, $\alpha_S(x) = \alpha_X(x) \cap S$, $\beta_S(x) = \beta_X(x) \cap S$, and $\theta_S(x) = \theta_X(x) \cap S$. For any suborder $|S|$ of $|X|$, we denote by $\mathrm{Int}_X(S)$ the open set $\{h \in X \; ; \; \beta_X(h) \subseteq S\}$. A set $S \subseteq X$ is said to be a *regular open set* (resp. a *regular closed set*) iff $S = \mathrm{Int}_X(\alpha_X(S))$ (resp. $S = \alpha_X(\mathrm{Int}_X(S))$). We call *relative topology* [8] induced in S by $\mathcal{U}$ the set of all the sets of the form $U \cap S$ where $U \in \mathcal{U}$. A set which is open in the relative topology of S is said to be a *relatively open set* [8]. A set $S \subseteq X$ is then said to be *connected* iff it is not the disjoint union of two non-empty relatively open subsets w.r.t. S. The largest connected set in $(X,\mathcal{U})$ containing $p \in X$ is called the *component* [1] of the point p in $(X,\mathcal{U})$ and we denote it $\mathcal{CC}(X, p)$. When $(X,\mathcal{U})$ is non-empty, the set of maximal components of X in the inclusion sense is denoted by $\mathcal{CC}(X)$ and is called the set of *connected components* of X. We call *path* [4] into $S \subseteq X$ a finite sequence $(p^0, \ldots, p^k)$ such that for all $i \in [\![1, k]\!]$, $p^i \in \theta_X^{\square}(p^{i-1})$, and we say that a set $S \subseteq X$ is *path-connected* [4] iff for any points p, q in S, there exists a path into S joining them. When $|X|$ is an Alexandrov space, any subset S of X is connected iff it is path-connected [8,4].

The *Khalimsky grid* [11] of dimension n is denoted $|\mathbb{H}^n| = (\mathbb{H}^n, \subseteq)$ and is the poset defined such that $\mathbb{H}^1_0 = \{\{a\} \,;\, a \in \mathbb{Z}\}$, $\mathbb{H}^1_1 = \{\{a, a+1\} \,;\, a \in \mathbb{Z}\}$, $\mathbb{H}^1 = \mathbb{H}^1_0 \cup \mathbb{H}^1_1$, and $\mathbb{H}^n = \left\{h_1 \times \cdots \times h_n \,;\, \forall i \in [\![1, n]\!], h_i \in \mathbb{H}^1\right\}$. For any $h \in \mathbb{H}^n$, we have the following equalities: $\alpha(h) := \alpha_{\mathbb{H}^n}(h) = \{h' \in \mathbb{H}^n \,;\, h' \subseteq h\}$, $\beta(h) := \beta_{\mathbb{H}^n}(h) = \{h' \in \mathbb{H}^n \,;\, h \subseteq h'\}$, and $\theta(h) := \theta_{\mathbb{H}^n}(h) = \{h' \in \mathbb{H}^n \,;\, h' \subseteq h \text{ or } h \subseteq h'\}$. For any suborder $|X|$ of $|\mathbb{H}^n|$, we obtain that $\alpha_X(h) = \{h' \in X \,;\, h' \subseteq h\}$, $\beta_X(h) = \{h' \in X \,;\, h \subseteq h'\}$, and $\theta_X(h) = \{h' \in X \,;\, h' \subseteq h \text{ or } h \subseteq h'\}$. Any element h of $\mathbb{H}^n$ which is the Cartesian product of k elements, with $k \in [\![0, n]\!]$, of $\mathbb{H}^1_1$ and of $(n-k)$ elements of $\mathbb{H}^1_0$ is said to be of *dimension* k [12], which is denoted by $\dim(h) = k$, and the set of all the elements of $\mathbb{H}^n$ which are of dimension k is denoted by $\mathbb{H}^n_k$. Furthermore, for any $n \geq 1$, $|\mathbb{H}^n|$ is an Alexandrov space [4]. Finally, let A, B be two subsets of $\mathbb{H}^n$; we say that A and B are *separated* iff $(A \cap (\beta(B)) \cup (\beta(A) \cap B) = \emptyset$, or equivalently iff $A \cap \theta(B) = \emptyset$. The *rank* $\rho(x, |X|)$ of an element x in $|X|$ is 0 if $\alpha^\Box_X(x) = \emptyset$ and is equal to $\max_{y \in \alpha^\Box_X(x)}(\rho(y, |X|)) + 1$ otherwise. The *rank* of a poset $|X|$ is denoted by $\rho(|X|)$ and is equal to the maximal rank of its elements. An element x of X such that $\rho(x, |X|) = k$ is called *k-face* [4] of X. In Khalimsky grids, the dimension is equal to the rank.

Let $|X| = (X, \alpha_X)$ be a poset. $|X|$ is said to be *countable* iff its domain X is countable. Also, $|X|$ is called *locally finite* iff for any element $x \in X$, the set $\theta_X(x)$ is finite. A poset which is countable and locally finite is said to be a *CF-order* [4]. Now let us recall the definition of *n-surfaces* [9]. Let $|X| = (X, \alpha_X)$ be a CF-order; the poset $|X|$ is said to be a (-1)-surface iff $X = \emptyset$, or a 0-surface iff X is made of two different elements $x, y \in X$ such that $x \notin \theta^\Box_X(y)$, or an n-surface, $n \geq 1$, iff $|X|$ is connected and for any $x \in X$, $|\theta^\Box_X(x)|$ is a $(n-1)$-surface. According to Evako *et al.* [9], $|\mathbb{H}^n|$ is an n-surface. Also, any n-surface $|X|$ is *homogeneous* [6], *i.e.*, $\forall x \in X$, $\beta_X(x)$ contains an n-face. The *boundary* [13] of a digital subset S in an Alexandrov space X is defined as $\alpha_X(S) \cap \alpha_X(X \setminus S)$, and S is said to be *well-composed in the sense of Alexandrov (AWC)* iff the connected components of its boundary are discrete $(n-1)$-surfaces where $n \geq 0$ is the rank of X. Also, let us recall some properties about n-surfaces that will be useful in the sequel. Let $|X|, |Y|$ be two posets; it is said that $|X|$ and $|Y|$ can be *joined* [4] if $X \cap Y = \emptyset$. If $|X|$ and $|Y|$ can be joined, the *join* of $|X|$ and $|Y|$ is denoted $|X| * |Y|$ and is equal to $(X \cup Y, \alpha_X \cup \alpha_Y \cup X \times Y)$.

Property 1 ([7]) *Let $|X|$ and $|Y|$ be two orders that can be joined, and let n be an integer. The poset $|X| * |Y|$ is a $(n+1)$-surface iff there exists some $p \in [\![-1, n+1]\!]$ such that $|X|$ is a p-surface and $|Y|$ is a $(n-p)$-surface.*

Property 2 (Property 10 in [6]) *Let $|X| = (X, \alpha_X)$ be a poset. Then $|X|$ is an n-surface iff for any $x \in X$, $|\alpha^\Box_X(x)|$ is a $(k-1)$-surface and $|\beta^\Box_X(x)|$ is a $(n-k-1)$-surface, with $k = \rho(x, |X|)$.*

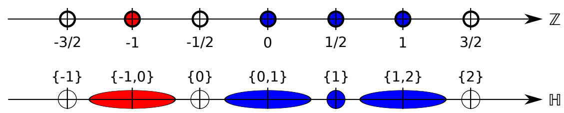


Fig. 2: Bijection between $(\mathbb{Z}/2)$ and $\mathbb{H}^1$.

2.3 A bijection between $(\mathbb{Z}/2)^n$ and $\mathbb{H}^n$

For A, B two arbitrary families of sets, we set $A \otimes B := \{a \times b\ ;\ a \in A, b \in B\}$, where $\times$ is the *Cartesian product*. For any $a \in \mathbb{H}^n$ and any $i \in [\![1,n]\!]$, we denote by a_i the i^{th} coordinate of a into $\mathbb{H}^n$. Then, as a consequence of the Cartesian product, we obtain that $\forall a \in \mathbb{H}^n$, $\alpha(a) = \otimes_{m \in [\![1,n]\!]} \alpha(a_m)$ (resp. $\beta(a) = \otimes_{m \in [\![1,n]\!]} \beta(a_m)$). Also, we define the bijection $\mathcal{H} : (\mathbb{Z}/2) \to \mathbb{H}^1$ s.t. $\forall z \in (\mathbb{Z}/2)$, $\mathcal{H}(z) = \{z, z+1\}$ if $z \in \mathbb{Z}$ and $\mathcal{H}(z) = \{z + 1/2\}$ otherwise (see Figure 2). Its inverse is denoted by $\mathcal{Z}$. Finally, we define the bijection $\mathcal{H}_n : \left(\frac{\mathbb{Z}}{2}\right)^n \to \mathbb{H}^n$ as the n-ary Cartesian product of $\mathcal{H}$ and $\mathcal{Z}_n : \mathbb{H}^n \to \left(\frac{\mathbb{Z}}{2}\right)^n$ its inverse.

3 Introducing a new mathematical background

Let us introduce new mathematical properties which show how $\mathbb{Z}^n$ and $\mathbb{H}^n$ are related to each other.

3.1 Complements about antagonism in $\mathbb{Z}^n$

Lemma 1 *Let x, y be two elements of $\mathbb{Z}^n$. Then, x and y are antagonists in a block of $\mathbb{Z}^n$ of dimension $k \in [\![0,n]\!]$ iff:*

$$\begin{cases} \mathrm{Card}\,\{m \in [\![1,n]\!]\ ;\ x_m = y_m\} = n - k, & (1) \\ \mathrm{Card}\,\{m \in [\![1,n]\!]\ ;\ |x_m - y_m| = 1\} = k. & (2) \end{cases}$$

Proof: Let x, y be two elements of $\mathbb{Z}^n$ satisfying (1) and (2) with $k \in [\![0,n]\!]$. Now, let us take $c \in \mathbb{Z}^n$ such that $\forall i \in [\![1,n]\!]$, $c_i := \min(x_i, y_i)$, $\mathcal{I}_x := \{i \in [\![1,n]\!]\ ;\ c_i \neq x_i\}$ and $\mathcal{I}_y := \{i \in [\![1,n]\!]\ ;\ c_i \neq y_i\}$. Obviously, $\mathcal{I}_x \cap \mathcal{I}_y = \emptyset$, and then by (1), $\mathrm{Card}(\mathcal{I}_x \cup \mathcal{I}_y) = k$. Since by (2) we have $x = c + \sum_{i \in \mathcal{I}_x} e^i$ and $y = c + \sum_{i \in \mathcal{I}_y} e^i$, then x and y belong to $S(c, \mathcal{F})$ where $\mathcal{F} := \left\{e^i \in \mathbb{B}\ ;\ i \in \mathcal{I}_x \cup \mathcal{I}_y\right\}$ is of cardinality k. Furthermore, the L^1 norm of $x - y$ is equal to k, and thus x and y maximize the L^1-distance between two points into $S(c, \mathcal{F})$. So, x and y are antagonists in $S(c, \mathcal{F})$. Conversely, let us assume that $x, y \in \mathbb{Z}^n$ are antagonists in a block $S(c, \mathcal{F})$ of dimension $k \in [\![0,n]\!]$. For any $i \in [\![1,n]\!]$, e^i belongs to $\mathcal{F}$ and hence $|x_i - y_i| = 1$, or it does not belong to $\mathcal{F}$ and hence $x_i = y_i$. Since $\mathrm{Card}(\mathcal{F}) = k$ by hypothesis, this concludes the proof. $\square$

3.2 General facts between $\mathbb{Z}^n$ and $\mathbb{H}^n$

Let us present some properties relating $\left(\frac{\mathbb{Z}}{2}\right)^n$ and $\mathbb{H}^n$ that are induced by our bijection $\mathcal{H}_n$.

Lemma 2 *Let c be a value in $(\mathbb{Z}/2) \setminus \mathbb{Z}$, and let y be a value in $\mathbb{Z}$. Then, $y \in \left\{c - \frac{1}{2}, c + \frac{1}{2}\right\}$ iff $\beta(\mathcal{H}(y)) \subseteq \beta(\mathcal{H}(c))$.*

Proof: When c belongs to $(\mathbb{Z}/2) \setminus \mathbb{Z}$, $\mathcal{H}(c) = \{c + \frac{1}{2}\} \in \mathbb{H}_0^1$, and $\beta(\mathcal{H}(c)) = \{\{c - 1/2, c + 1/2\}, \{c + 1/2\}, \{c + 1/2, c + 3/2\}\}$. Also, when $y \in \mathbb{Z}$, $\mathcal{H}(y) = \{y, y + 1\} \in \mathbb{H}_1^1$, and $\beta(\mathcal{H}(y)) = \{\{y, y + 1\}\}$. If y belongs to $\{c - \frac{1}{2}, c + \frac{1}{2}\}$, we obtain that $\beta(\mathcal{H}(y)) \subseteq \beta(\mathcal{H}(c))$. Conversely, if $\{\{y, y + 1\}\} \subseteq \{\{c - 1/2, c + 1/2\}, \{c + 1/2\}, \{c + 1/2, c + 3/2\}\}$, it means that $y \in \{c - 1/2, c + 1/2\}$. □

Proposition 1 *Let S be a block in $\mathbb{Z}^n$, and let c be its center in $\left(\frac{\mathbb{Z}}{2}\right)^n$. Then $S = \mathcal{Z}_n(\beta(\mathcal{H}_n(c)) \cap \mathbb{H}_n^n)$.*

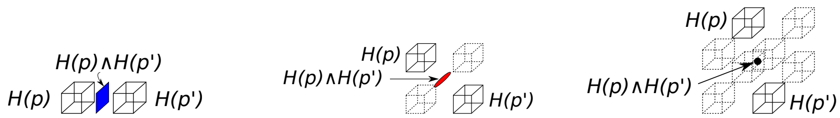
Proof: Let us remark that $S = \left\{c + \sum_{i \in \frac{1}{2}(c)} \lambda_i e^i \; ; \; \forall i \in \frac{1}{2}(c), \lambda_i \in \left\{-\frac{1}{2}, \frac{1}{2}\right\}\right\}$ where $\frac{1}{2}(c)$ denotes the set of indices of the coordinates $i \in [\![1, n]\!]$ satisfying $c_i \in (\mathbb{Z}/2) \setminus \mathbb{Z}$. Then, for any $y \in S$, if $i \in [\![1, n]\!] \setminus \frac{1}{2}(c)$, then $y_i = c_i$, if $i \in \frac{1}{2}(c)$ such that $\lambda_i = 1/2$, then $y_i = c_i + 1/2$ with $c_i \in (\mathbb{Z}/2) \setminus \mathbb{Z}$, and if $i \in \frac{1}{2}(c)$ such that $\lambda_i = -1/2$, hence $y_i = c_i - 1/2$ with $c_i \in (\mathbb{Z}/2) \setminus \mathbb{Z}$. Then, for any $i \in [\![1, n]\!]$, by Lemma 2, $\mathcal{H}(y_i) \in \beta(\mathcal{H}(c_i))$, and then $\mathcal{H}_n(y) \in \beta(\mathcal{H}_n(c))$. Because $y \in \mathbb{Z}^n$, $\mathcal{H}_n(y) \in \mathbb{H}_n^n$, and then $\mathcal{H}_n(y) \in \beta(\mathcal{H}_n(c)) \cap \mathbb{H}_n^n$, which leads to $y \in \mathcal{Z}_n(\beta(\mathcal{H}_n(c)) \cap \mathbb{H}_n^n)$. Conversely, let us assume that $y \in \mathcal{Z}_n(\beta(\mathcal{H}_n(c)) \cap \mathbb{H}_n^n)$. Then, $\mathcal{H}_n(y) \in \beta(\mathcal{H}_n(c)) \cap \mathbb{H}_n^n$, which means that $y \in \mathbb{Z}^n$, and $\mathcal{H}_n(y) \in \beta(\mathcal{H}_n(c))$. In other words, for any $i \in [\![1, n]\!]$, $\mathcal{H}(y_i) \in \beta(\mathcal{H}(c_i))$. Two cases are then possible: $c_i \in \mathbb{Z}$, hence $y_i = c_i$, or $c_i \in (\mathbb{Z}/2) \setminus \mathbb{Z}$ and thus by Lemma 2, $y_i \in \{c_i - \frac{1}{2}, c_i + \frac{1}{2}\}$. This way, $y \in S$. □

3.3 Infimum of two faces in $\mathbb{H}^n$

Let X be a subset of $\mathbb{H}^n$. If there exists one element $x \in X$ such that for any $y \in X$, $y \subseteq x$, we say that x is the *supremum* of X, and we denote it $\sup(X)$. Now, let a, b be two elements of $\mathbb{H}^n$. When $\sup(\alpha(a) \cap \alpha(b))$ is well-defined, we denote it $a \wedge b$ and we call it the *infimum* between a and b.

Lemma 3 *Let a, b be two elements of $\mathbb{H}^n$. Then, $\alpha(a) \cap \alpha(b) \neq \emptyset$ iff $a \wedge b$ is well-defined. Furthermore, when $a \wedge b$ is well-defined, $a \wedge b = \times_{i \in [\![1,n]\!]}(a_i \wedge b_i)$, and $\alpha(a \wedge b) = \alpha(a) \cap \alpha(b)$.*

Proof: Let a_1, b_1 be two elements of $\mathbb{H}^1$, then it is easy to show by a case-by-case study that $\alpha(a_1) \cap \alpha(b_1) \neq \emptyset$ is equivalent to saying that $a_1 \wedge b_1$ is well-defined, and that $\alpha(a_1) \cap \alpha(b_1) = \alpha(a_1 \wedge b_1)$ when $a_1 \wedge b_1$ is well-defined. Then, when a, b belong to $\mathbb{H}^n$, $n \geq 1$ with $\alpha(a) \cap \alpha(b) \neq \emptyset$, we obtain that $\alpha(a) \cap \alpha(b)$ is equal to $\otimes_{i \in [\![1,n]\!]} (\alpha(a_i) \cap \alpha(b_i))$ which is non-empty, which means that for any $i \in [\![1, n]\!]$, $\alpha(a_i) \cap \alpha(b_i)$ is not empty, and then $a_i \wedge b_i$ is well-defined and $\alpha(a_i) \cap \alpha(b_i) = \alpha(a_i \wedge b_i)$. This way, $\alpha(a) \cap \alpha(b)$ is equal to $\otimes_{i \in [\![1,n]\!]} \alpha(a_i \wedge b_i)$, and then is equal to $\alpha(\times_{i \in [\![1,n]\!]}(a_i \wedge b_i))$, and then the supremum of $\alpha(a) \cap \alpha(b)$ is $\times_{i \in [\![1,n]\!]}(a_i \wedge b_i)$, *i.e.*, exists and is unique, and so can be denoted by $a \wedge b$. Furthermore, it satisfies $\alpha(a \wedge b) = \alpha(a) \cap \alpha(b)$. Conversely, when $a \wedge b$ is well-defined, the supremum of $\alpha(a) \cap \alpha(b)$ exists and thus $\alpha(a) \cap \alpha(b) \neq \emptyset$. □



(a) $k = 1$ (b) $k = 2$ (c) $k = 3$

Fig. 3: Infima between images by $\mathcal{H}_n$ of the k-antagonists p and p' in 3D.

Lemma 4 *$\forall p, p' \in \mathbb{Z}^n$, p and p' are k-antagonists, $k \in [\![0, n]\!]$, iff $\mathcal{H}_n(p) \wedge \mathcal{H}_n(p')$ is well-defined and belongs to $\mathbb{H}^n_{n-k}$.*

Proof: The intuition of the proof is depicted on Figure 3. Let p, p' be defined in $\mathbb{Z}^n$ and $k \in [\![0, n]\!]$ such that p and p' are antagonists in a block of dimension $k \in [\![0, n]\!]$. By Lemma 1, there exists $\mathfrak{I} \subseteq [\![1, n]\!]$ or cardinality k, and s.t. $\forall i \in \mathfrak{I}$, $|p_i - p'_i| = 1$, and $\forall i \in [\![1, n]\!] \setminus \mathfrak{I}$, $p_i = p'_i$. Since for each $i \in [\![1, n]\!]$, we have $p_i, p'_i \in \mathbb{Z}$, then $\mathcal{H}(p_i) = \{p_i, p_i + 1\}$, and $\mathcal{H}(p'_i) = \{p'_i, p'_i + 1\}$. Let us denote $z_i = \mathcal{H}(p_i)$, and $z'_i = \mathcal{H}(p'_i)$, then $z_i, z'_i \in \mathbb{H}^1_1$. When i is in $\mathfrak{I}$, $p'_i = p_i - 1$, and $\alpha(z_i) \cap \alpha(z'_i) = \{\{p_i\}\}$, and then $z_i \wedge z'_i = \{p_i\} \in \mathbb{H}^1_0$, or $p'_i = p_i + 1$, and $\alpha(z_i) \cap \alpha(z'_i) = \{\{p'_i\}\}$ and then $z_i \wedge z'_i = \{p'_i\} \in \mathbb{H}^1_0$. When i belongs to $[\![1, n]\!] \setminus \mathfrak{I}$, $z_i = z'_i$ and $\alpha(z_i) \cap \alpha(z'_i) = \alpha(z_i)$ and then $z_i \wedge z'_i = z_i \in \mathbb{H}^1_1$. It follows then that $\times_{i \in [\![1,n]\!]}(z_i \wedge z'_i)$ belongs to $\mathbb{H}^n_{n-k}$. Also, since $\alpha(z_i) \cap \alpha(z'_i) \neq \emptyset$ for any $i \in [\![1, n]\!]$, $\alpha(\mathcal{H}_n(p)) \cap \alpha(\mathcal{H}_n(p'))$ is equal to $\otimes_{i \in [\![1,n]\!]}(\alpha(z_i) \cap \alpha(z'_i))$ which is non-empty, and then, by Lemma 3, $\mathcal{H}_n(p) \wedge \mathcal{H}_n(p')$ exists and is equal to $\times_{i \in [\![1,n]\!]}(z_i \wedge z'_i)$, which belongs to $\mathbb{H}^n_{n-k}$. Let us now proceed to the converse implication. Let p, p' be two points of $\mathbb{Z}^n$, and $z = \mathcal{H}_n(p), z' = \mathcal{H}_n(p')$ such that $z \wedge z'$ is well-defined and belongs to $\mathbb{H}^n_{n-k}$. Then, we define $\mathfrak{I} = \{i \in [\![1, n]\!] \ ; \ z_i \wedge z'_i \in \mathbb{H}^1_0\}$, whose cardinality is equal to k thanks to Lemma 3. Now, let us observe that, for any $i \in [\![1, n]\!]$, $p_i \in \{p'_i - 1, p'_i + 1\}$ iff $z_i \wedge z'_i \in \mathbb{H}^1_0$, then p and p' have exactly k different coordinates, and they differ from one. Then, p and p' are antagonists in a block of dimension k by Lemma 1. □

Lemma 5 *Let a, b be two elements of $\mathbb{Z}^n$ such that a and b are $(3^n - 1)$-neighbors in $\mathbb{Z}^n$ or equal. Then, $\mathcal{H}_n((a + b)/2) = \mathcal{H}_n(a) \wedge \mathcal{H}_n(b)$.*

Proof: Since a and b are $(3^n - 1)$-neighbors in $\mathbb{Z}^n$, they are antagonists in a block of dimension $k \in [\![0, n]\!]$, and then by Lemma 4, $\mathcal{H}_n(a) \wedge \mathcal{H}_n(b)$ is well-defined. Now, let us prove that $(a + b)/2 = \mathcal{Z}_n(\mathcal{H}_n(a) \wedge \mathcal{H}_n(b))$. This is equivalent to say that for any $i \in [\![1, n]\!]$, we have $(a_i + b_i)/2 = \mathcal{Z}(\mathcal{H}(a_i) \wedge \mathcal{H}(b_i))$ by Lemma 3. Starting from the equality $\mathcal{H}(a_i) \wedge \mathcal{H}(b_i) = \{a_i, a_i + 1\} \wedge \{b_i, b_i + 1\}$ and observing that, since a and b are $(3^n - 1)$-neighbors in $\mathbb{Z}^n$ or equal, they satisfy for any $i \in [\![1, n]\!]$ that $a_i \in \{b_i - 1, b_i, b_i + 1\}$, we have 3 possible cases: $a_i = b_i - 1$, and then $\mathcal{H}(a_i) \wedge \mathcal{H}(b_i) = \{b_i - 1, b_i\} \wedge \{b_i, b_i + 1\} = \{b_i\}$, whose image by $\mathcal{Z}$ is equal to $b_i - \frac{1}{2} = (a_i + b_i)/2$, or we have $b_i = a_i - 1$, and then a symmetrical reasoning leads to the same result, or $b_i = a_i$, and then the result is immediate. □

Proposition 2 *Let S be a block and let $p, p' \in S$ be any two antagonists in S. Then the center of the block S is equal to $\frac{p+p'}{2}$. Furthermore, its image by $\mathcal{H}_n$ into $\mathbb{H}^n$ is equal to $\mathcal{H}_n(p) \wedge \mathcal{H}_n(p')$.*

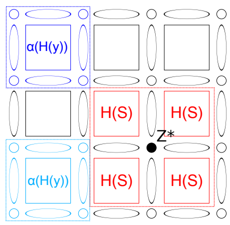


Fig. 4: When $y \notin S$ centered at $\mathcal{Z}_n(z^*)$, $\alpha(\mathcal{H}_n(y)) \cap \beta(z^*) = \emptyset$.

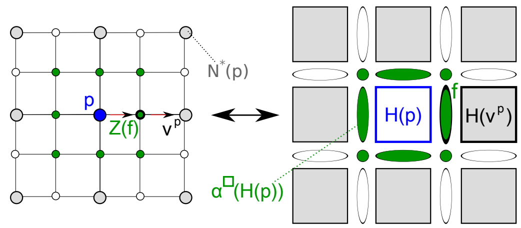


Fig. 5: $\alpha^{\Box}(\mathcal{H}_n(p))$ is composed of the faces $\mathcal{H}_n(p) \wedge \mathcal{H}_n(v^p)$ such that $v^p \in \mathcal{N}^*(p)$.

Proof: Starting from the two antagonists p, p' in S, we can compute $z \in \mathbb{Z}^n$ and $\mathcal{F} \subseteq \mathbb{B}$ such that $S = S(z, \mathcal{F})$. In fact, for all $i \in [\![1, n]\!]$, $z_i = \min(p_i, p'_i)$, and $\mathcal{F} = \{e^i \,;\, i \in [\![1, n]\!],\ p_i \neq p'_i\}$. Then, it is clear that $p = (p - z) + z = z + \sum_{p_i \neq z_i} e^i$, and that $p' = (p' - z) + z = z + \sum_{p'_i \neq z_i} e^i$. Then, $p + p' = 2z + \sum_{f \in \mathcal{F}} f$, which shows that $\frac{p+p'}{2}$ is the center of S in $(\mathbb{Z}/2)^n$. The second part of the proposition follows from Lemma 5. $\Box$

Lemma 6 *Let S be a block of $\mathbb{Z}^n$, and let $z^* \in \mathbb{H}^n$ be the image by $\mathcal{H}_n$ of the center of S. For all $y \in \mathbb{Z}^n$, $y \notin S$ implies that $\alpha(\mathcal{H}_n(y)) \cap \beta(z^*)$ is empty.*

Proof: This proof can be followed on Figure 4. Let y be an element of $\mathbb{Z}^n$ s.t. $\alpha(\mathcal{H}_n(y)) \cap \beta(z^*)$ is not empty. Then, for all $i \in [\![1, n]\!]$, $\alpha(\mathcal{H}(y_i)) \cap \beta(z^*_i)$ is not empty. Now, let us show that y belongs to S. Since there exists $p_i \in \alpha(\mathcal{H}(y_i)) \cap \beta(z^*_i)$, then $\mathcal{H}(y_i) \in \beta(p_i)$ and $p_i \in \beta(z^*_i)$, which leads to $\mathcal{H}(y_i) \in \beta(z^*_i)$, and then $\mathcal{H}_n(y) \in \beta(z^*)$. Since $y \in \mathbb{Z}^n$, $\mathcal{H}_n(y) \in \mathbb{H}^n_n$, and then $\mathcal{H}(y) \in \beta(z^*) \cap \mathbb{H}^n_n$, which is equivalent to $y \in \mathcal{Z}_n(\beta(z^*) \cap \mathbb{H}^n_n)$, which is the reformulation of a block centered at z^* by Lemma 1. $\Box$

Lemma 7 $\forall p \in \mathbb{Z}^n$, $\alpha^{\Box}(\mathcal{H}_n(p)) = \bigcup_{v \in \mathcal{N}^*(p)} \alpha(\mathcal{H}_n(p) \wedge \mathcal{H}_n(v))$.

Proof: This proof is depicted on Figure 5. Since $p \in \mathbb{Z}^n$, it can be easily proved that $\alpha^{\Box}(\mathcal{H}_n(p))$ is equal to the set of elements f of $\mathbb{H}^n$ satisfying $\|\mathcal{Z}_n(f) - p\|_\infty = \frac{1}{2}$, *i.e.*, $\|v^p - p\|_\infty = 1$ with $v^p := 2\mathcal{Z}_n(f) - p$. Then, $\alpha^{\Box}(\mathcal{H}_n(p))$ is equal to the set of elements $f \in \mathbb{H}^n$ satisfying $v^p \in \mathcal{N}^*(p)$ and $f = \mathcal{H}_n((v^p + p)/2)$. By Lemma 5, we obtain that $\alpha^{\Box}(\mathcal{H}_n(p))$ is equal to $\{\mathcal{H}_n(v^p) \wedge \mathcal{H}_n(p) \in \mathbb{H}^n \,;\, v^p \in \mathcal{N}^*(p)\}$, which leads to the required formula by applying the α operator. $\Box$

Lemma 8 *Let S be a block in $\mathbb{Z}^n$ of dimension $k \geq 2$. Now, let p, p' be two antagonists in S, and v be a $2n$-neighbor of p in S. Then, we have the following relation: $\mathcal{H}_n(p) \wedge \mathcal{H}_n(p') \in \alpha(\mathcal{H}_n(p) \wedge \mathcal{H}_n(v))$.*

Proof: By Lemma 4, $\mathcal{H}_n(p) \wedge \mathcal{H}_n(p')$ and $\mathcal{H}_n(p) \wedge \mathcal{H}_n(v)$ are well-defined (p and v are antagonists in a block of dimension 1). By Lemma 3, the first term of the relation is equal to $\times_{i \in [\![1,n]\!]}(\mathcal{H}(p_i) \wedge \mathcal{H}(p'_i))$. Likewise, the second term is equal to $\otimes_{i \in [\![1,n]\!]} (\alpha(\mathcal{H}(p_i)) \cap \alpha(\mathcal{H}(v_i)))$. Then we want to show that for all

$i \in [\![1,n]\!]$, $\mathcal{H}(p_i) \wedge \mathcal{H}(p'_i)$ belongs to $\alpha(\mathcal{H}(p_i)) \cap \alpha(\mathcal{H}(v_i))$. Let $\mathfrak{I}$ be the family of indices $\{i \in [\![1,n]\!] \,;\, p_i \neq p'_i\}$. Since v is a $2n$-neighbor of p into S, there exists an index i^* in $\mathfrak{I}$ such that $v_{i^*} \neq p_{i^*}$, *i.e.*, $v_{i^*} = p'_{i^*}$, and $\forall i \in [\![1,n]\!] \setminus \{i^*\}, v_i = p_i$. When $i \in [\![1,n]\!] \setminus \mathfrak{I}$ or when $i = i^*$, the property is obviously true. When $i \in \mathfrak{I} \setminus \{i^*\}$, then $v_i = p_i$, which implies $\alpha(\mathcal{H}(p_i)) \cap \alpha(\mathcal{H}(v_i)) = \alpha(\mathcal{H}(p_i)) = \{\{p_i\}, \{p_i+1\}, \{p_i, p_i+1\}\}$. However, either $\mathcal{H}(p_i) \wedge \mathcal{H}(p'_i) = \{p_i\}$ (if $p'_i = p_i - 1$) or $\mathcal{H}(p_i) \wedge \mathcal{H}(p'_i) = \{p_i + 1\}$ (if $p'_i = p_i + 1$), then $\mathcal{H}(p_i) \wedge \mathcal{H}(p'_i) \in \alpha(\mathcal{H}(p_i)) \cap \alpha(\mathcal{H}(v_i))$. $\square$

3.4 Some additional background concerning n-surfaces

The following proposition results from the proof of Property 11 (p. 55) in [6].

Proposition 3 *Let $|X| = (X, \alpha_X)$ and $|Y| = (Y, \alpha_Y)$ be two n-surfaces, $n \geq 0$. Then, if $|X|$ is a suborder of $|Y|$, then $|X| = |Y|$.*

Proof: Let us proceed by induction. Initialization ($n = 0$): when $|X|$ and $|Y|$ are two 0-surfaces, the inclusion $X \subseteq Y$ implies directly that $X = Y$ since they have the same cardinality, and then $|X| = |Y|$. Heredity ($n \geq 1$): we assume that when two $(n-1)$-surfaces satisfy an inclusion relationship, they are equal. Now, let $|X|$ and $|Y|$ be two n-surfaces, $n \geq 1$, such that $|X|$ is a suborder of $|Y|$. Then, for all $x \in X$, $x \in Y$ and so we can write $\theta^{\square}_X(x) \subseteq \theta^{\square}_Y(x)$ since $X \subseteq Y$. However, $|\theta^{\square}_X(x)|$ and $|\theta^{\square}_Y(x)|$ are $(n-1)$-surfaces and $|\theta^{\square}_X(x)|$ is a suborder of $|\theta^{\square}_Y(x)|$, then $|\theta^{\square}_X(x)| = |\theta^{\square}_Y(x)|$. Now, let us assume that we have $X \subsetneq Y$. Then let x be a point of X and y a point of $Y \setminus X$. Since $|Y|$ is connected as an n-surface with $n \geq 1$, it is connected by path, and so $x, y \in Y$ implies that there exists a path π joining them into Y. This way, there exist $x' \in X$ and $y' \in Y \setminus X$ s.t. $y' \in \theta^{\square}(x')$. In other words, $y' \in \theta^{\square}_Y(x') = \theta^{\square}_X(x')$ since $x' \in X$. This leads to $y' \in X$. We obtain a contradiction. Thus we have $X = Y$, and consequently $|X| = |Y|$. $\square$

Corollary 1 *Let $|X_1|$ and $|X_2|$ be two k-surfaces, $k \geq 0$, with $X_1 \cap X_2 = \emptyset$. Then $|X_1 \cup X_2|$ is not a k-surface.*

Proposition 4 *Let a, b be two elements of $\mathbb{H}^n$ with $a \in \beta^{\square}(b)$. Then $|\alpha^{\square}(a) \cap \beta^{\square}(b)|$ is a $(\dim(a) - \dim(b) - 2)$-surface.*

Proof: Since $|\mathbb{H}^n|$ is an n-surface, then $|\alpha^{\square}(a)|$ is a $(\rho(a, |\mathbb{H}^n|) - 1)$-surface by Property 2, and then is a $(\dim(a) - 1)$-surface. Now, we can remark that because b belongs to $\alpha^{\square}(a)$, we can write that $\alpha^{\square}(a) \cap \beta^{\square}(b) = \beta^{\square}_{\alpha^{\square}(a)}(b)$, and then, again by Property 2, $|\alpha^{\square}(a) \cap \beta^{\square}(b)|$ is a $((\dim(a) - 1) - \rho(b, |\alpha^{\square}(a)|) - 1)$-surface. Since $\rho(b, |\alpha^{\square}(a)|) = \rho(b, |\mathbb{H}^n|) = \dim(b)$, the proof is done. $\square$

4 Properties specific to the proof

From now on, we suppose n is an integer greater than or equal to 2, that X is a digital subset of $\mathbb{Z}^n$, that Y is the complement of X into $\mathbb{Z}^n$; also, we define the sets $\mathcal{X} := \mathcal{H}_n(X)$ and $\mathcal{Y} := \mathcal{H}_n(Y)$, and the *immersion* of X into $\mathbb{H}^n$ using the *miss strategy*: $\mathcal{I}(X) := \mathrm{Int}(\alpha(\mathcal{X}))$; its boundary is $\mathfrak{N} := \alpha(\mathcal{I}(X)) \cap \alpha(\mathbb{H}^n \setminus \mathcal{I}(X))$.

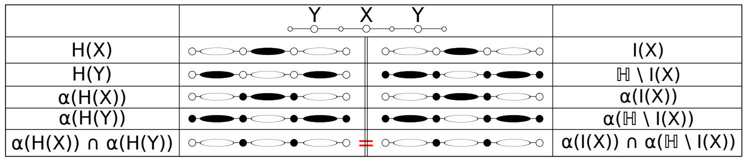


Fig. 6: $\alpha(\mathcal{H}_n(X)) \cap \alpha(\mathcal{H}_n(Y))$ vs. $\alpha(\mathcal{I}(X)) \cap \alpha(\mathbb{H}^n \setminus \mathcal{I}(X))$ in 1D.

Proposition 5 *$\mathfrak{N}$ is equal to $\alpha(\mathcal{X}) \cap \alpha(\mathcal{Y})$.*

Proof: An intuition of the proof is given in Figure 6. Let us first remark that $\alpha(\mathcal{X})$ is a regular closed set. Effectively, $\mathrm{Int}(\alpha(\mathcal{X}))) \subseteq \alpha(\mathcal{X})$ implies that $\alpha(\mathrm{Int}(\alpha(\mathcal{X}))) \subseteq \alpha(\mathcal{X})$ by monotonicity of α. Conversely, any element $x \in \mathcal{X}$ satisfies $\beta(x) = \{x\} \subseteq \mathcal{X}$, and so $\mathrm{Int}(\alpha(\mathcal{X}))$, which is equal to $\{h \in \alpha(\mathcal{X})\ ;\ \beta(h) \subseteq \alpha(\mathcal{X})\}$, contains $\mathcal{X}$. This implies that $\alpha(\mathrm{Int}(\alpha(\mathcal{X}))) \supseteq \alpha(\mathcal{X})$. Thus $\alpha(\mathcal{X})$ is a regular closed set. We can then simplify the formula of $\mathfrak{N}$; by definition, $\mathfrak{N}$ is equal to $\alpha(\mathcal{I}(X)) \cap \alpha(\mathbb{H}^n \setminus \mathcal{I}(X))$, which is then equal to $\alpha(\mathcal{X}) \cap \alpha(\mathbb{H}^n \setminus \mathcal{I}(X))$. Since $\mathcal{I}(X)$ is open, $\alpha(\mathbb{H}^n \setminus \mathcal{I}(X)) = \mathbb{H}^n \setminus \mathcal{I}(X)$. Thus, $\mathfrak{N} = \alpha(\mathcal{X}) \setminus \mathcal{I}(X)$, which is equal to $\alpha(\mathcal{X}) \cap (\mathrm{Int}(\alpha(\mathcal{X})))^c$, and so $\mathfrak{N} = \alpha(\mathcal{X}) \cap \alpha(\mathrm{Int}(\mathcal{X}^c))$. Let us show that $\alpha(\mathcal{X}) \cap \alpha(\mathrm{Int}(\mathcal{X}^c))$ is equal to $\alpha(\mathcal{X}) \cap \alpha(\mathcal{Y})$. Since $\mathcal{Y} = \mathbb{H}^n_n \setminus \mathcal{X} \subseteq \mathbb{H}^n \setminus \mathcal{X}$, it is clear that $\mathrm{Int}(\mathcal{Y}) \subseteq \mathrm{Int}(\mathbb{H}^n \setminus \mathcal{X})$. Since $\mathcal{Y}$ is open as a set of n-faces, we obtain $\mathcal{Y} \subseteq \mathrm{Int}(\mathbb{H}^n \setminus \mathcal{X})$, and thus $\alpha(\mathcal{Y}) \subseteq \alpha(\mathrm{Int}(\mathbb{H}^n \setminus \mathcal{X})) = \alpha(\mathrm{Int}(\mathcal{X}^c))$. This way, $\alpha(\mathcal{X}) \cap \alpha(\mathcal{Y}) \subseteq \alpha(\mathcal{X}) \cap \alpha(\mathrm{Int}(\mathcal{X}^c))$. Now, let z be an element of $\alpha(\mathcal{X}) \cap \alpha(\mathrm{Int}(\mathcal{X}^c))$, then $\beta(z) \cap \mathbb{H}^n_n \subseteq \mathcal{X}$ (1), or $\beta(z) \cap \mathbb{H}^n_n \subseteq \mathcal{Y}$ (2), or $\beta(z) \cap \mathbb{H}^n_n \cap \mathcal{X} \neq \emptyset \neq \beta(z) \cap \mathbb{H}^n_n \cap \mathcal{Y}$ (3). Before treating the first case, let us prove that $\alpha(\beta(z)) = \alpha(\beta(z) \cap \mathbb{H}^n_n)$ (P). The converse inclusion is obvious. Concerning the direct inclusion, let a be an element of $\alpha(\beta(z))$. There exists $p \in \beta(z)$ such that $a \in \alpha(p)$. Also, $\mathbb{H}^n$ is an n-surface, and so is homogeneous. This implies that there exists $p^n \in \beta(p)$ s.t. $p^n \in \mathbb{H}^n_n$. Since $p^n \in \beta(p)$ and $p \in \beta(z)$, $p^n \in \beta(z) \cap \mathbb{H}^n_n$, and the fact that a belongs to $\alpha(p)$ implies that $a \in \alpha(\beta(z) \cap \mathbb{H}^n_n)$. This way, (P) is true. Now, we can treat the first case: $\beta(z) \cap \mathbb{H}^n_n \subseteq \mathcal{X}$ implies that $\mathrm{Int}(\alpha(\beta(z) \cap \mathbb{H}^n_n)) \subseteq \mathrm{Int}(\alpha(\mathcal{X}))$. Using (P), we obtain $\mathrm{Int}(\alpha(\beta(z))) \subseteq \mathrm{Int}(\alpha(\mathcal{X}))$. Since $\beta(z)$ is an open regular set, we obtain $\beta(z) \subseteq \mathrm{Int}(\alpha(\mathcal{X}))$. Yet, $\beta(z) \subseteq \alpha(\beta(z)) \subseteq \alpha(\mathcal{X})$, since $\alpha(\mathcal{X})$ is a regular closed set. However, this imples that $\beta(z) = \mathrm{Int}(\beta(z)) \subseteq \mathrm{Int}(\alpha(\mathcal{X}))$, and so $z \notin \alpha(\mathrm{Int}(\mathcal{X}^c))$, which is a contradiction. In the second case, $\beta(z) \cap \mathbb{H}^n_n \subseteq \mathcal{Y}$, which means that no $x \in \mathcal{X}$ exists such that $x \in \beta(z)$, which means that $z \notin \alpha(\mathcal{X})$, which leads once more to a contradiction. In the third case, $\beta(z) \cap \mathbb{H}^n_n \cap \mathcal{X} \neq \emptyset$ and $\beta(z) \cap \mathbb{H}^n_n \cap \mathcal{Y} \neq \emptyset$ implies that there exists some $x \in \mathcal{X}$ and $y \in \mathcal{Y}$ such that $z \in \alpha(x) \cap \alpha(y)$, and so $z \in \alpha(\mathcal{X}) \cap \alpha(\mathcal{Y})$. $\square$

Proposition 6 *For any $z \in \mathfrak{N}$, $|\alpha^{\square}_{\mathfrak{N}}(z)|$ is a $(\dim(z) - 1)$-surface.*

Proof: Since $\mathfrak{N}$ is closed, $\forall z \in \mathfrak{N}$, $|\alpha^{\square}_{\mathfrak{N}}(z)| = |\alpha^{\square}(z)|$, which is a $(\rho(z, |\mathbb{H}^n|) - 1)$-surface by Property 2 since $\mathbb{H}^n$ is an n-surface. Since $\rho(z, |\mathbb{H}^n|) = \dim(z)$, $|\alpha^{\square}_{\mathfrak{N}}(z)|$ is a $(\dim(z) - 1)$-surface. $\square$

Lemma 9 *$\mathcal{I}(X)$ is AWC iff $\forall z \in \mathfrak{N}$, $|\beta^{\square}_{\mathfrak{N}}(z)|$ is a $(n - 2 - \dim(z))$-surface.*

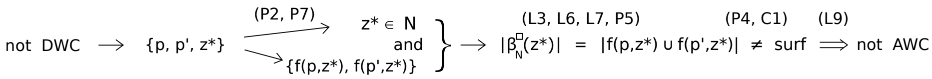


Fig. 7: Summary of the proof; $\mathsf{f}(q, z^*) := \alpha^{\Box}(\mathcal{H}_n(q)) \cap \beta^{\Box}(z^*)$, with $q \in \{p, p'\}$.

Proof: Let us recall that two disjoint components C_1 and C_2 of $\mathfrak{N}$ are separated: $C_1 \cap \theta(C_2) = \emptyset$. For this reason, for any $z \in \mathfrak{N}$, $|\theta^{\Box}_{\mathfrak{N}}(z)| = |\theta^{\Box}(z) \cap \bigcup_{C \in \mathcal{CC}(\mathfrak{N})} C| = |\bigcup_{C \in \mathcal{CC}(\mathfrak{N})} (\theta^{\Box}(z) \cap C)| = |\theta^{\Box}_{\mathcal{CC}(\mathfrak{N},z)}(z)|$. Since $n \geq 2$, $\mathcal{I}(X)$ is AWC iff $\forall C \in \mathcal{CC}(\mathfrak{N})$, C is a $(n-1)$-surface, *i.e.*, $\forall C \in \mathcal{CC}(\mathfrak{N})$, $\forall z \in C$, $|\theta^{\Box}_{C}(z)|$ is a $(n-2)$-surface, which means that $\forall C \in \mathcal{CC}(\mathfrak{N})$, $\forall z \in C$, $|\theta^{\Box}_{\mathfrak{N}}(z)|$ is a $(n-2)$-surface, or, in other words, by Property 1 and Proposition 6, $\forall z \in \mathfrak{N}$, $|\beta^{\Box}_{\mathfrak{N}}(z)|$ is a $(n-2-\dim(z))$-surface. □

Proposition 7 *Let S be a block of dimension $k \in [\![2, n]\!]$ s.t. $X \cap S = \{p, p'\}$ (resp. $Y \cap S = \{p, p'\}$) and $p' = \mathrm{antag}_S(p)$, then $\mathcal{H}_n\left(\frac{p+p'}{2}\right) \in \mathfrak{N}$.*

Proof: Let v be a $2n$-neighbor of p in S, then, by Lemma 4, $\mathcal{H}_n(p) \wedge \mathcal{H}_n(v)$ is well-defined, and by Lemma 3, $\alpha(\mathcal{H}_n(p) \wedge \mathcal{H}_n(v)) = \alpha(\mathcal{H}_n(p)) \cap \alpha(\mathcal{H}_n(v))$. Since $\dim(S) \geq 2$, $v \in Y$, and so $\alpha(\mathcal{H}_n(p)) \cap \alpha(\mathcal{H}_n(v)) \subseteq \mathfrak{N}$ by Proposition 5. Now, using Proposition 2, $\mathcal{H}_n\left(\frac{p+p'}{2}\right)$ is equal to $\mathcal{H}_n(p) \wedge \mathcal{H}_n(p')$, which belongs to $\alpha(\mathcal{H}_n(p) \wedge \mathcal{H}_n(v))$ by Lemma 8, and thus to $\mathfrak{N}$. □

5 Proof of the main result

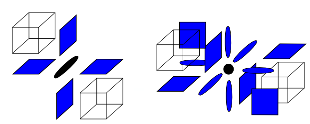


Fig. 8: $\beta^{\Box}_{\mathfrak{N}}(\mathcal{H}_n(c))$ (in blue) when X admits a 2D/3D critical configuration in the block of center c (whose image by $\mathcal{H}_n$ is depicted in black) when $n = 3$.

We want to prove that AWCness implies DWCness in n-D (see Figure 7 for the summary of the proof). To this aim, we will show that if X is not DWC, then $\mathcal{I}(X)$ is not AWC. Since by Lemma 9, $\mathcal{I}(X)$ is AWC iff $\forall z \in \mathfrak{N}$, $|\beta^{\Box}_{\mathfrak{N}}(z)|$ is a $(n-2-\dim(z))$-surface, it is sufficient to prove that when X is not DWC, then there exists an element z^* of $\mathfrak{N}$ such that $|\beta^{\Box}_{\mathfrak{N}}(z^*)|$ is not a $(n-2-\dim(z^*))$-surface. So, let us assume that X is not DWC. Now, X admits a critical configuration. Let us treat the primary case, since the reasoning for the secondary case is similar: let us assume that there exists a block S of dimension $k \in [\![2, n]\!]$ such that $X \cap S = \{p, p'\}$ with $p' = \mathrm{antag}_S(p)$. This way, we can compute the image z^* by $\mathcal{H}_n$ into $\mathbb{H}^n$ of the center of S. By Proposition 2, $z^* = \mathcal{H}_n(p) \wedge \mathcal{H}_n(p')$. Let us show that $|\beta^{\Box}_{\mathfrak{N}}(z^*)|$ is not a $(n-2-\dim(z))$-surface. By Proposition 7, $z^* \in \mathfrak{N}$, so the expression $\beta^{\Box}_{\mathfrak{N}}(z^*)$ is well-defined. Now, let us compute $|\beta^{\Box}_{\mathfrak{N}}(z^*)|$. Using Lemma 7 and Lemma 6, we obtain that $\mathsf{f}(p, z^*) := \alpha^{\Box}(\mathcal{H}_n(p)) \cap \beta^{\Box}(z^*)$ is equal to $\bigcup_{y \in S \setminus \{p\}} \alpha(\mathcal{H}_n(p) \wedge \mathcal{H}_n(y)) \cap \beta^{\Box}(z^*)$. Then, since we know that $\alpha(\mathcal{H}_n(p) \wedge \mathcal{H}_n(p')) \cap \beta^{\Box}(z^*) = \emptyset$, thus $\mathsf{f}(p, z^*)$ is equal to $\bigcup_{y \in S \setminus \{p,p'\}} \alpha(\mathcal{H}_n(p) \wedge \mathcal{H}_n(y)) \cap \beta^{\Box}(z^*)$. By Lemma 3, $\mathsf{f}(p, z^*) = \alpha(\mathcal{H}_n(p)) \cap \alpha(\mathcal{H}_n(Y \cap S)) \cap \beta^{\Box}(z^*)$. With a similar calculation based on p', we obtain that

$\mathsf{f}(p', z^*) := \alpha^{\square}(\mathcal{H}_n(p')) \cap \beta^{\square}(z^*)$ is equal to $\alpha(\mathcal{H}_n(p')) \cap \alpha(\mathcal{H}_n(Y \cap S)) \cap \beta^{\square}(z^*)$. Next, $\mathsf{f}(p, z^*) \cup \mathsf{f}(p', z^*)$ is equal to $\alpha(\mathcal{H}_n(X \cap S)) \cap \alpha(\mathcal{H}_n(Y \cap S)) \cap \beta^{\square}(z^*)$, which is equal by Lemma 6 to $\alpha(\mathcal{X}) \cap \alpha(\mathcal{Y}) \cap \beta^{\square}(z^*)$, and then to $\beta^{\square}_{\mathfrak{N}}(z^*)$ by Proposition 5. Finally, we have that $|\beta^{\square}_{\mathfrak{N}}(z^*)|$ is equal to $|\mathsf{f}(p, z^*) \cup \mathsf{f}(p', z^*)|$. Figure 8 depicts examples of $\beta^{\square}_{\mathfrak{N}}(z^*)$ in the case $n = 3$. Let us now remark that $|\beta^{\square}_{\mathfrak{N}}(z^*)|$ is the disjoint union of $|\mathsf{f}(p, z^*)|$ and of $|\mathsf{f}(p', z^*)|$: $\alpha^{\square}(\mathcal{H}_n(p)) \cap \alpha^{\square}(\mathcal{H}_n(p')) \cap \beta^{\square}(z^*) = \emptyset$. However, by Proposition 4, $|\mathsf{f}(p, z^*)|$ and $|\mathsf{f}(p', z^*)|$ are both $(n - \dim(z^*) - 2)$-surfaces. Finally, by Corollary 1, $|\beta^{\square}_{\mathfrak{N}}(z^*)|$ is not a $(n - \dim(z^*) - 2)$-surface, and then $\mathcal{I}(X)$ is not AWC. $\square$

6 Conclusion

Now that we have proved that AWCness implies DWCness for sets, we naturally conclude that, thanks to *cross-section topology*, this implication is also true for gray-level images: a gray-level image $u : \mathbb{Z}^n \to \mathbb{Z}$ will be DWC if the *span-based immersion* of u is AWC. As future work, we propose to study the converse implication, *i.e.*, if DWCness implies AWCness in n-D, $n \geq 2$.

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