Ping-pong configurations and circular orders on free groups

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Dominique Malicet [*]   Kathryn Mann [†]   Cristóbal Rivas [‡]   Michele Triestino [§]

Abstract

We discuss actions of free groups on the circle with "ping-pong" dynamics, i.e. dynamics determined by a finite amount of combinatorial data. As a natural extension of the work started in [21], we show that the free group F_n admits an isolated circular order if and only if n is even, in stark contrast with the case for linear orders. Following [2], we also exhibit examples of "exotic" isolated points in the space of all circular orders on F_2. As an application, we obtain analogous results for the linear orders on the groups $F_n \times \mathbb{Z}$. [1]

1 Introduction

Let G be a group. A *linear order*, often called a *left order* on G is a total order invariant under left multiplication. Left-invariance immediately implies that the order is determined by the set of elements greater than the identity, called the *positive cone*. However, it is often far from obvious whether a given order is in fact determined by only finitely many inequalities, or whether a given group admits such a finitely determined order. This latter question turns out to be quite natural from an algebraic perspective, and can be traced back to [3] for the special case of free groups. McCleary answered this shortly afterwards, showing that F_n admits no finitely determined orders [25].

The question of finite determination gained a topological interpretation following Sikora's definition of the *space of linear orders* on G, denoted $\mathsf{LO}(G)$ [29]. This is the set of all linear orders on G endowed with the topology generated by open sets

$$U_{(\preceq, X)} := \{ \preceq' \mid x \preceq' y \text{ iff } x \preceq y \text{ for all } x, y \in X \}$$

as X ranges over all finite sets of G. Finitely determined linear orders on G are precisely the *isolated points* of $\mathsf{LO}(G)$. This is the simplest instance of the general theme that topological properties of $\mathsf{LO}(G)$ should correspond to algebraic properties of G.

Using both algebraic and dynamical methods, we now know many families of groups which do and do not admit isolated (i.e. finitely determined) orders. Examples that do *not* include free abelian groups [29], free groups [25, 26], free products of arbitrary linearly orderable groups [28], as well as some amalgamated free products such as fundamental groups of orientable closed surfaces [1]: they all fail to admit isolated left orders. Large families of groups which do have isolated orders include braid groups [11, 15], groups of the form $\langle x, y \mid x^n = y^m \rangle$ $(n, m \in \mathbb{Z})$ [19, 27], and groups

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with triangular presentations [10]. (In fact, all of these examples have orders for which the positive cone is finitely generated as a semi-group, a strictly stronger condition.)

As a consequence of our work here, we give a family of groups where, interestingly, *both* behaviors occur.

Theorem 1.1. *Let F_n denote the free group of n generators. The group $F_n \times \mathbb{Z}$ has isolated linear orders if and only if n is even.*

This result appears to give the first examples of any group G with a finite index subgroup H (in this case $F_n \times \mathbb{Z} \subset F_m \times \mathbb{Z}$, for $n \neq m$ mod 2) such that $\mathsf{LO}(G)$ and $\mathsf{LO}(H)$ are both infinite, but only one contains isolated points.

Theorem 1.1 also has an interesting consequence regarding the *space of marked groups* (see [8] for an introduction to the subject). It is known that, in the space of marked groups, the set of groups enjoying a linear order is closed. Surprisingly, it follows form Theorem 1.1 that enjoying an *isolated* linear order is neither a closed, nor an open property. Indeed, $F_2 \times \mathbb{Z}$ approaches $F_3 \times \mathbb{Z}$ (when changing the generating set) and also $F_3 \times \mathbb{Z}$ approaches $F_4 \times \mathbb{Z}$ (see [8, §2.3]). We state this observation as:

Corollary 1.2. *In the space of finitely generated marked groups, having an isolated linear order is neither a closed, nor an open property.*

Remark 1.3. In different terms, the previous corollary says that while the property of enjoying a linear order can be expressed in the first-order logic theory of a group, this is not the case for *isolated* linear orders. More precisely, the property

$$\mathcal{P}: \text{ ``the group } G \text{ admits an isolated linear order''}$$

is expressed by no set of first-order sentences (i.e. involving finitely many universal quantifiers). Indeed, as explained in [8], groups satisfying an equation given by a first-order sentence describe a closed set in the space of marked groups.

The main thrust of this work, giving the tools for Theorem 1.1, is the study of *circular orders* on F_n and actions of F_n on the circle. It is well known that, for countable G, admitting a linear order is equivalent to acting faithfully by orientation-preserving homeomorphisms on the line. By analogy, a *circular order* is an algebraic condition (specifically, a type of 2-cocycle) equivalent to acting faithfully by orientation-preserving homeomorphisms on $\mathbb{S}^1$. An action of G on $\mathbb{S}^1$ will *lift* to an action of a central extension of G by $\mathbb{Z}$ on the line, giving a correspondence between circular and linear orders and putting many dynamical tools at our disposal.

Analogous to $\mathsf{LO}(G)$, one can define a *space of circular orders* $\mathsf{CO}(G)$. In [21], the second and third authors showed that a circular order on F_n is isolated if and only if a corresponding action on the circle, has what they called *ping-pong dynamics*. They gave examples of isolated circular orders on free groups of even rank, but the odd rank case was left as an open problem, which we answer here in the negative:

Theorem 1.4. *F_n has isolated circular orders if and only if n is even.*

We prove this by developing a combinatorial tool for the study of ping-pong actions (similar to Markov partitions), inspired by the work in [13] and [2], which should have applications beyond the study of linear and circular orders (see Theorem 3.11). These are defined and motivated in the next section. Sections 3 and 4 give the application to the study of circular and linear orders, respectively, and the proofs of Theorems 1.1 and 1.4.

Remark 1.5. Similarly to Corollary 1.2, one can also prove that the set of groups admitting isolated circular orders is neither closed nor open in the space of marked groups.
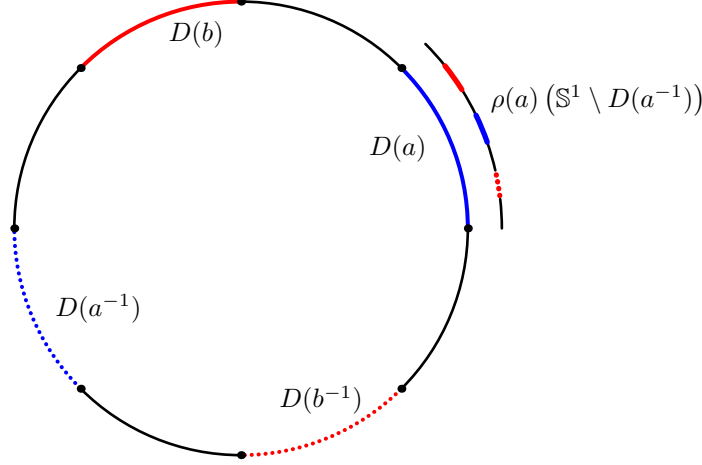
Figure 2.1: Classical ping-pong on two generators

2 Ping-pong actions and configurations

Definition 2.1. Let $\Gamma = F_n$ be the free group of rank n, freely generated by $S = \{a_1, \ldots, a_n\}$. A *ping-pong action* of (Γ, S) on $\mathbb{S}^1$ is a representation $\rho : \Gamma \to \mathsf{Homeo}_+(\mathbb{S}^1)$ such that there exist pairwise disjoint open sets $D(a) \subset \mathbb{S}^1$, $a \in S \cup S^{-1}$, each of which has finitely many connected components, and such that $\rho(a) \left(\mathbb{S}^1 \setminus D(a^{-1})\right) \subset \overline{D(a)}$. We further assume that if I and J are any connected components of $D(a)$, then $\bar{I} \cap \bar{J} = \emptyset$.

We call the sets $D(a)$ the *ping-pong domains* for ρ.

Remark 2.2. In [21] the ping-pong domains are assumed to be closed. The definition above is slightly more general. For this reason later we will introduce Convention 3.6, to recover the definition in [21].

The reader may notice that, for a given ping-pong action ρ of (Γ, S), there can be many choices of sets $D(a)$ satisfying the property in Definition 2.1. For instance, if ρ is a ping-pong action such that $\bigcup_{a \in S \cup S^{-1}} \overline{D(a)} \neq \mathbb{S}^1$, then one may choose an arbitrary open set I disjoint from $\bigcup_{a \in S \cup S^{-1}} \overline{D(a)}$ and replace $D(a_1)$ with $D(a_1) \cup I$, leaving the other domains unchanged. These new domains still satisfy $\rho(a) \left(\mathbb{S}^1 \setminus D(a^{-1})\right) \subset \overline{D(a)}$. Later we will adopt a convention to avoid this kind of ambiguity.

Why ping-pong actions? The classical ping-pong lemma implies that ping-pong actions are always faithful, and a little more work shows that the action is determined up to semi-conjugacy by a finite amount of combinatorial data coming from the cyclic ordering and the images of the connected components of the sets $D(a)$ (see Definition 2.6, [23, Thm. 4.7] or Lemma 3.4 below). In particular, one can think of ping-pong actions as the family of "simplest possible" faithful actions of F_n on $\mathbb{S}^1$, and it is very easy to produce a diverse array of examples. Perhaps the best-known examples are the actions of discrete, free subgroups of $\mathsf{PSL}(2, \mathbb{R})$ on $\mathbb{RP}^1$. For these actions, one can choose domains $D(a)$ with a single connected component. Figure 2.1 shows an example of the dynamics of such an action of $F_2 = \langle a, b \rangle$.

Despite their simplicity, ping-pong actions are quite useful. For instance, in [2] ping-pong actions were used in the discovery of the first examples of discrete groups of real-analytic circle diffeomorphisms that act minimally and are not conjugate to a subgroup in a cover of $\mathsf{PSL}(2, \mathbb{R})$ (Example 3.9 originates from [2]). This was a by-product of a series of works on longstanding open conjectures by Hector, Ghys and Sullivan, concerning the relationship between minimality and

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ergodicity of a codimension-one foliation (see for instance [12, 13, 16]). In general, it is very easy to study the dynamic and ergodic properties of a ping-pong action, and this program has been carried out by many authors [6, 7, 18, 20, 22].

2.1 Basic properties

Lemma 2.3. *Given a ping-pong action of (Γ, S), there exists a choice of ping-pong domains $D(a)$ such that $\rho(a)\left(\mathbb{S}^1 \setminus \overline{D(a^{-1})}\right) = D(a)$ holds for all $a \in S \cup S^{-1}$.*

Proof. Let ρ be a ping-pong action with sets $D(a)$ given. We will modify these domains to satisfy the requirements of the lemma. For each generator $a \in S$ (*not* the symmetric generating set), we shrink the domain $D(a)$, setting

$$D'(a) := \rho(a)\left(\mathbb{S}^1 \setminus \overline{D(a^{-1})}\right).$$

Applying a^{-1} to both sides of the above expression, we clearly get $\rho(a^{-1})(D'(a)) = \mathbb{S}^1 \setminus \overline{D(a^{-1})}$. Moreover, since the connected components of $D(a)$ have disjoint closures, and the same holds for $D(a^{-1})$, and hence also for $D'(a)$, we also have $\rho(a^{-1})\left(\overline{D'(a)}\right) = \mathbb{S}^1 \setminus D(a^{-1})$, or equivalently

$$\rho(a^{-1})\left(\mathbb{S}^1 \setminus \overline{D'(a)}\right) = D(a^{-1}).$$

This is what we needed to show. □

Convention 2.4. From now on, we assume all choices of domains $D(a)$ for every ping-pong action are as in Lemma 2.3.

Remark 2.5. In particular, Convention 2.4 implies that for each $a \in S$, $\pi_0(D(a))$, the set of connected components of $D(a)$, has the same cardinality than $\pi_0(D(a^{-1}))$.

Definition 2.6. Let ρ be a ping-pong action of (Γ, S). The *ping-pong configuration* of ρ is the data consisting of

1. The cyclic order of the connected components of $\bigcup_{a \in S \cup S^{-1}} D(a)$ in $\mathbb{S}^1$, and

2. For each $a \in S \cup S^{-1}$, the assignment of connected components

$$\lambda_a : \pi_0\left(\bigcup_{b \in S \cup S^{-1} \setminus \{a^{-1}\}} D(b)\right) \to \pi_0\left(D(a)\right)$$

induced by the action.

Note that not every abstract assignment λ_a as in the definition above can be realized by an action $F_2 \to \mathsf{Homeo}_+(\mathbb{S}^1)$. However, one way to produce some large families of examples is as follows.

Example 2.7 (An easy construction of ping-pong actions). For $a \in S$, let X_a and $Y_a \subset \mathbb{S}^1$ be disjoint sets each of cardinality $k(a)$ for some integer $k(a) \geq 1$ such that every two points of X_a are separated by a point of Y_a. Choose these so that all the sets $X_a \cup Y_a$ are pairwise disjoint as a ranges over S. Let $D(a)$ and $D(a^{-1})$ be neighborhoods of X_a and Y_a, respectively, chosen small enough so that all these sets remain pairwise disjoint. Now one can easily construct a piecewise linear homeomorphism (or even a smooth diffeomorphism) $\rho(a)$ with X_a as its set of attracting periodic points, and Y_a as the set of repelling periodic points such that $\rho(a)\left(\mathbb{S}^1 \setminus D(a^{-1})\right) = D(a)$. The assignments λ_a are now dictated by the period of $\rho(a)$ and the cyclic order of the sets X_a and Y_a.

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While the reader should keep the method above in mind as a source of examples, we will show in Example 3.12 that not every ping-pong configuration can be obtained in this manner. However, the regularity (PL or smooth) in the construction is attainable in general:

Lemma 2.8. *Given a ping-pong action ρ_0 of (Γ, S) with domains $\{D_0(a)\}_{a \in S \cup S^{-1}}$ following Convention 2.4, one can find another ping-pong action ρ of (Γ, S) such that:*

1. *The action ρ is piecewise linear.*

2. *Denoting by $D(a)$ the domains of ρ, there exists $\mu > 1$ such that for any $a \in S \cup S^{-1}$, one has*

$$\rho(a)'|_{D(a^{-1})} \geq \mu.$$

3. *The actions ρ_0 and ρ have the same ping-pong configuration.*

Proof. Given a ping-pong action of (Γ, S) with domains $\{D_0(a)\}_{a \in S \cup S^{-1}}$ following Convention 2.4, for each $a \in S \cup S^{-1}$, replace the original domains by smaller domains $D(a) \subset D_0(a)$, chosen sufficiently small so that the largest connected component of $D(a)$ is at most half the length of the smallest connected component of $\mathbb{S}^1 \setminus D(a^{-1})$. We require also that $D(a)$ has exactly one connected component in each connected component of $D_0(a)$.

Now define $\rho(a)$ as a piecewise linear homeomorphism that maps connected components of $\mathbb{S}^1 \setminus D(a^{-1})$ onto connected components of $D(a)$ linearly following the assignment λ_a. □

Definition 2.9. For each $a \in S$, we define an oriented bipartite graph Γ_a as follows:

- The set of vertices is formed by the connected components of $D(a)$ and $D(a^{-1})$.

- There is an oriented edge from an interval $I^+ \in \pi_0(D(a))$ to an interval $I^- \in \pi_0(D(a^{-1}))$ if and only if, denoting by J^+ the connected component of $\mathbb{S}^1 \setminus D(a)$ that is right adjacent to I^+, one has $\rho(a^{-1})(J^+) = \overline{I^-}$.

- Similarly, there is an oriented edge from $I^- \in \pi_0(D(a^{-1}))$ to $I^+ \in \pi_0(D(a))$ if and only if, denoting by J^- the connected component of $\mathbb{S}^1 \setminus D(a^{-1})$ that is right adjacent to I^-, one has $\rho(a)(J^-) = \overline{I^+}$.

Proposition 2.10. *Consider a ping-pong action $\rho : \Gamma \to \mathsf{Homeo}_+(\mathbb{S}^1)$ of a free group (Γ, S). Then for each generator $a \in S$ there exists an integer $k(a)$ such that the graph Γ_a is an oriented $2k(a)$-cycle.*

We give the proof of this proposition dividing the arguments into different lemmas.

Lemma 2.11. *Each vertex in Γ_a has exactly one outgoing edge.*

Proof. As a direct consequence of Convention 2.4, we have that for any $s \in S \cup S^{-1}$ and connected component J of $\mathbb{S}^1 \setminus D(s^{-1})$ there exists a connected component $I \in \pi_0(D(s))$ such that $\rho(s)(J) = \overline{I}$. This implies that each vertex of Γ_a has at least one edge going out from it. On the other hand, it is clear from Definition 2.9 of the graph that there is at most one edge going out from any vertex. □

Lemma 2.12. *The graph Γ_a is a disjoint union of nontrivial cycles.*

Proof. After Lemma 2.11, we need to prove that each vertex $I \in \pi_0(D(s))$, $s = a^{\pm 1}$ has a unique incoming edge. This is clear, since there cannot be two distinct connected components of $\mathbb{S}^1 \setminus D(s^{-1})$ whose image by $\rho(s)$ is exactly $\overline{I}$. □

Lemma 2.13. *The graph Γ_a is connected.*

Proof. Let I^- be a connected component of $D(a^{-1})$ and consider the connected component J^+ of $\mathbb{S}^1 \setminus D(a)$ such that $\rho(a^{-1})(J^+) = \overline{I^-}$. Let I_1^+ and I_2^+ be the connected components of $D(a)$ (possibly the same) which are adjacent to J^+.

By Definition 2.9 of the graph Γ_a, the intervals I_1^+, I^-, I_2^+ are consecutive vertices in a same cycle of the graph. And vice versa: if three intervals I_1^+, I^-, I_2^+ are consecutive vertices, then $J^+ := \rho(a)(\overline{I^-})$ is the connected component of $\mathbb{S}^1 \setminus D(a)$ adjacent to both I_1^+ and I_2^+.

This proves that if I_1^+ and I_2^+ are consecutive connected components of $D(a)$ in $\mathbb{S}^1$, then they belong to the same cycle in Γ_a. Hence we easily deduce that *all* connected components of $D(a)$ are in the same cycle in Γ_a. Hence the same holds for the components of $D(a^{-1})$. $\qquad\square$

Proof of Proposition 2.10. Fix a generator $a \in S$. The previous Lemmas 2.12 and 2.13 imply that the graph Γ_a is a connected cycle. As the graph is bipartite, it must have an even number of edges, whence the statement. $\qquad\square$

3 Left-invariant circular orders

We begin this section by recalling definitions and basic properties. A reader familiar with circular orders may skip to Section 3.1.

Definition 3.1. Let G be a group. A *left-invariant circular order* is a function $c : G \times G \times G \to \{0, \pm 1\}$ such that

1. c is *homogeneous*: $c(\gamma g_0, \gamma g_1, \gamma g_2) = c(g_0, g_1, g_2)$ for any $\gamma, g_0, g_1, g_2 \in G$;

2. c is a *2-cocycle* on G:

$$c(g_1, g_2, g_3) - c(g_0, g_2, g_3) + c(g_0, g_1, g_3) - c(g_0, g_1, g_2) = 0 \quad \text{for any } g_0, g_1, g_2, g_3 \in G;$$

3. c is *non-degenerate*: $c(g_0, g_1, g_2) = 0$ if and only if $g_i = g_j$ for some $i \neq j$.

We denote by $\mathsf{CO}(G)$ the *space* of all left-invariant circular orders on G, endowed with the topology as a subset of $\{0, \pm 1\}^{G \times G \times G}$ (with the product topology).

Although spaces of left-invariant linear orders have been well-studied, there are very few cases in which understand completely the topology of $\mathsf{CO}(G)$. A classification of groups such that $\mathsf{CO}(G)$ is finite is given in [9]. Other sporadic examples are known, for instance it is fairly easy to see that $\mathsf{CO}(\mathbb{Z})$ is homeomorphic to a Cantor set. Given that left-orders on free groups are well understood, a natural next case of circular orders to study is $\mathsf{CO}(F_n)$.

Our main tool here is the following classical relationship between circular orders and actions on $\mathbb{S}^1$ (see [5, 21]).

Proposition 3.2. *Given a left-invariant circular order c on a countable group G, there is an action $\rho_c : G \to \mathsf{Homeo}_+(\mathbb{S}^1)$ such that $c(g_0, g_1, g_2) = \mathsf{ord}\,(\rho_c(g_0)(x), \rho_c(g_1)(x), \rho_c(g_2)(x))$ for some $x \in \mathbb{S}^1$, where ord denotes cyclic orientation.*

Moreover, there is a canonical procedure for producing ρ_c which gives a well-defined conjugacy class of action. This conjugacy class is called the dynamical realization *of c with basepoint x.*

See [21] for a description of this procedure, or [17] for the analogous linear order case. Note that modifying a dynamical realization by blowing up an orbit of some point $y \notin \rho(G)(x)$ results in a non-conjugate action that still satisfies the property $c(g_0, g_1, g_2) = \mathrm{ord}\,(\rho_c(g_0)(x), \rho_c(g_1)(x), \rho_c(g_2)(x))$. However, this blown-up action cannot be obtained through the canonical procedure.

Remark 3.3. The above proposition actually characterizes countable circularly orderable groups as groups admitting a faithful action on $\mathbb{S}^1$. Indeed, given a faithful action ρ of G on $\mathbb{S}^1$, we can *induce* a circular order on G. If for the action ρ there is a point x with trivial stabilizer, then the induced order is simply defined by looking at the cyclic order of points in the orbit of x:

$$c(g_1, g_2, g_3) := \mathrm{ord}(\rho(g_1)(x), \rho(g_2)(x), \rho(g_3)(x)).$$

However it is not always possible to find such a point (for instance, this is impossible for the natural action of Thompson's group T on $\mathbb{S}^1$) and we refer the reader to [5, Thm. 2.2.14] for the general case.[2]

In the case of ping-pong actions, however, there is always a point with trivial stabilizer. Precisely, we have:

Lemma 3.4. *Suppose that ρ is a ping-pong action of (Γ, S) with domains $D(a)$. If $x_0 \in \mathbb{S}^1 \setminus \bigcup_{a \in S \cup S^{-1}} D(a)$, then the orbit of x_0 is free and its cyclic order is completely determined by the cyclic order of the elements of $\left\{ \pi_0 \left(\bigcup_{a \in S \cup S^{-1}} D(a) \right), \{x_0\} \right\}$ and the assignments λ_a.*

The proof is obtained by a careful reading of the standard proof of the classical ping-pong lemma. See [21, Lemma 4.2] for details.

3.1 Isolated circular orders on free groups

In this section we will use ping-pong actions to answer a question of [21] that asked which free groups admit isolated circular orders, proving Theorem 1.4 from the introduction. First, we introduce some tools.

Let G be any group, and $\rho: G \to \mathsf{Homeo}_+(\mathbb{S}^1)$. Recall that, if G does not have a finite orbit, then there is a unique closed, G-invariant set contained in the closure of every orbit, called the *minimal set* of $\rho(G)$ and denoted $\Lambda(\rho)$. If $\Lambda(\rho) = \mathbb{S}^1$, the action is called *minimal*. Otherwise, $\Lambda(\rho)$ is homeomorphic to a Cantor set and ρ permutes the connected components of $\mathbb{S}^1 \setminus \Lambda(\rho)$. In general the permutation action can have many disjoint cycles, however, this is not the case for dynamical realizations:

Lemma 3.5 ([21] Lemma 3.21 and Cor. 3.24). *Let $\rho: G \to \mathsf{Homeo}_+(\mathbb{S}^1)$ be a dynamical realization of a circular order c. Suppose that ρ has a minimal invariant Cantor set Λ. Then ρ acts transitively on the set of connected components of $\mathbb{S}^1 \setminus \Lambda$.*

In the case of ping-pong actions of free groups (of rank at least 2), there are no finite orbits, and invariance of the minimal set immediately implies that $\Lambda(\rho) \subset \bigcup_{a \in S \cup S^{-1}} \overline{D(a)}$. When the union $\bigcup_{a \in S \cup S^{-1}} \overline{D(a)}$ does not cover the whole circle, $\Lambda(\rho)$ is a Cantor set. Up to blowing up one orbit (which does not change the class of semi-conjugacy of the action), we may assume that this condition holds, and more strongly, from now on we assume the following.

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[2]In [4, Prop. 2.4] the authors propose an alternative way of inducing an ordering of G. However their method is incorrect, as the following example shows: suppose to have three distinct homeomorphisms f, g, h, with f coinciding with g on one half circle and with h on the other half. Then for any point $x \in \mathbb{S}^1$, there are always two equal points in the triple $(f(x), g(x), h(x))$.

Convention 3.6. In a ping-pong action of (Γ, S), we will assume that $\overline{D(s)} \cap \overline{D(t)} = \emptyset$ holds for every two distinct $s, t \in S \cup S^{-1}$.

Remark 3.7. Under Convention 3.6 we have the stronger inclusion $\Lambda(\rho) \subseteq \bigcup_{a \in S \cup S^{-1}} D(a)$.

The following theorem relates circular orders and ping-pong actions.

Theorem 3.8 ([21] Thm. 1.5). *Let $\Gamma = F_n$ be a free group. A circular order $c \in \mathsf{CO}(\Gamma)$ is isolated if and only if its dynamical realization $\rho_c : \Gamma \to \mathsf{Homeo}_+(\mathbb{S}^1)$ is a ping-pong action satisfying Convention 3.6.*

As a concrete example we have

Example 3.9. Let $\rho : F_2 \to \mathsf{PSL}(2, \mathbb{R})$ be a discrete, faithful representation corresponding to a hyperbolic structure on a genus 1 surface with one boundary component. Then ρ is the dynamical realization of an isolated circular order.

More generally, take ρ as above, and let $\hat{\rho}$ be any lift of the action above to $\mathsf{PSL}^{(k)}(2, \mathbb{R})$. Then $\hat{\rho}$ is the dynamical realization of an isolated circular order. (See [21, Lemma 4.14]). For ρ, and hence for $\hat{\rho}$, we can choose ping-pong domains such that the connected components of the sets $D(a) \cup D(a^{-1})$ and $D(b) \cup D(b^{-1})$ alternate.

With the tools above, we may now provide the

Proof of Theorem 1.4. If n is even, then the representation of Γ into $\mathsf{PSL}(2, \mathbb{R})$ coming from a hyperbolic structure on a genus $n/2$ surface with one boundary component gives an isolated circular order, as explained in [21]. (By taking lifts to cyclic covers, one can in fact obtain infinitely many isolated circular orders in distinct equivalence classes under the action of $\mathsf{Aut}(F_n)$ on $\mathsf{CO}(G)$.)

For the case where n is odd, we need more work. Suppose that $\rho : F_n \to \mathsf{Homeo}_+(\mathbb{S}^1)$ is a dynamical realization of an isolated circular order on $\Gamma = F_n$ (we do not assume yet n is odd), and fix a generating set $S = \{a_1, \ldots, a_n\}$. In particular we have that the ping-pong action satisfies Convention 3.6 and that the connected components of $\mathbb{S}^1 \setminus \Lambda(\rho)$ form a unique orbit. Let $c_0, \ldots, c_r$ be the connected components of $\mathbb{S}^1 \setminus \Lambda(\rho)$ that are not contained in any domain $D(s)$.

Suppose that c_i has endpoints in $D(s)$ and $D(t)$, for some $s \neq t$. Then, for any generator $u \notin \{s^{-1}, t^{-1}\}$, we have that $\rho(u)(c_i) \in D(u)$. Also, from Remark 3.7 we have that $\rho(s^{-1})(c_i)$ and $\rho(t^{-1})(c_i)$ belong to $\{c_0, \ldots, c_r\}$. This implies that c_i and c_j are in the same orbit (which they do) if and only if they are equivalent under the relation $\sim$ generated by

$$c_i \sim \rho(t^{-1})c_i \quad \text{if } c_i \cap D(t) \neq \emptyset, \text{ for } t \in S \cup S^{-1}.$$

We now use a clever argument to show that if n is odd, then the number of equivalence classes under this relation cannot be 1.

We aim to build a surface Σ by starting with the disc $\mathbb{D}$ and gluing another disc to $\partial \mathbb{D}$ for each generator $a \in S$ as we next explain. For each generator $a \in S$, let $k(a)$ be the integer given by Proposition 2.10. Let P_a be a $4k(a)$-gon (topologically a disc) with cyclically ordered vertices $v_1, v_2, \ldots, v_{4k(a)}$. Choose a connected component $I = [x_1, y_1]$ of $\overline{D(a)}$ and glue the oriented edge $v_1 v_2$ to I so as to agree with the orientation of $I \subset \mathbb{S}^1$. Then glue the edge $v_3 v_4$ to the connected component of $\overline{D(a^{-1})}$ containing $\rho(a^{-1})(x_1)$, according to the orientation in $\mathbb{S}^1$. Let y_2 denote the other endpoint of this connected component, and glue $v_5 v_6$ to the connected component of $\overline{D(a)}$ containing $\rho(a)(y_2)$. Iterate this process until all edges $v_{2j-1} v_{2j}$ have been glued to $\mathbb{S}^1 = \partial \mathbb{D}$. Note that the remaining (unglued) edges of P_a correspond exactly to the edges of the graph Γ_a from Definition 2.9. Repeat this procedure for each generator in S. (A cartoon of the result of this procedure for the ping-pong action of Example 3.12 is shown in Figure 3.1.)

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Because of Proposition 2.10, the gluing of P_a described adds one face and $2k(a)$ edges to the existing surface. Therefore after all the polygons P_a (as a ranges over the generators) have been glued, we will have a surface Σ of Euler characteristic $\chi(\Sigma) \equiv n + 1$ mod 2. Since $\chi(\Sigma)$ is also congruent to the number of its boundary components mod 2, to finish the proof of Theorem 1.4 it is enough to show the following

Claim. *The number of boundary components of the surface Σ is exactly the number of $\sim$-equivalence classes of the c_i's.*

Indeed, since the number of $\sim$-equivalence classes of the c_i's is 1, the Claim and the above discussion implies that $1 \equiv n + 1$ mod 2, which implies that $n = \mathsf{rank}(F_n)$ is even.

Proof of Claim. Indeed, $\partial\Sigma \cap \{c_0, \ldots, c_r\}$ has one connected component in each interval c_i. By construction, if c_i has endpoints in $D(s)$ and $D(t)$, then $\partial\Sigma \cap c_i$ is joined to $\rho(t^{-1})c_i \cap \partial\Sigma$ and $\rho(s^{-1})c_i \cap \partial\Sigma$ by edges of P_s and P_t respectively. Moreover, each edge of $P_s \cap \partial\Sigma$ plays the role of one of these connectors. It follows that components of $\partial\Sigma$ captures the equivalence classes described above. $\qquad\square$

Remark 3.10. Note that n even is not a sufficient condition to ensure that a ping pong action of F_n comes from a dynamical realization (equivalently, that Σ has one boundary component). For instance one may consider a group generated by two sufficiently large powers of hyperbolic elements of $\mathsf{PSL}(2, \mathbb{R})$ with non-crossing axes when seeing them acting on $\mathbb{H}^2$.

The proof above can be improved to give a statement about general ping-pong actions:

Theorem 3.11. *Let $\Gamma = F_n$ be the free group of rank n with generating system S. Consider a ping-pong action ρ of (Γ, S) satisfying Conventions 2.4 and 3.6. Let $\Lambda(\rho)$ be the minimal invariant Cantor set for the action. Then the number of orbits of connected components of the complement $\mathbb{S}^1 \setminus \Lambda(\rho)$ is congruent to $n + 1$ mod 2.*

Proof. Compared to the proof of Theorem 1.4, we only need to observe the following:

Claim. *Let $c_0, c_1, \ldots, c_r$ be the connected components of $\mathbb{S}^1 \setminus \Lambda(\rho)$ that are not contained in any domain $D(s)$. Then each orbit of the connected components of $\mathbb{S}^1 \setminus \Lambda(\rho)$ under the action of Γ contains some c_i.*

Proof of Claim. Suppose I is a connected component contained in some $D(s)$. By Lemma 2.8, we can take $\rho(s)$ to be piecewise linear, and such that each $\rho(s^{-1})$ expands $D(s)$ uniformly, increasing the length of each connected component by a factor of some $\mu > 1$, independent of s. Iteratively, assuming that $\rho(s_k s_{k-1} \cdots s_1)(I) \subset D(s_{k+1}^{-1})$, then the length of $\rho(s_{k+1} s_k \cdots s_1)(I)$ is at least $\mu^{k+1} \operatorname{length}(I)$. This process cannot continue indefinitely, so some image of I is not contained in a ping-pong domain. $\qquad\square$

As a consequence, to count the number of orbits of the action of Γ on connected components of $\mathbb{S}^1 \setminus \Lambda(\rho)$, we need only to count the number of equivalence classes of the c_i's (with respect to the orbit equivalence relation). This is done as in the proof of Theorem 1.4, with the help of the surface Σ defined by gluing polygons. $\qquad\square$
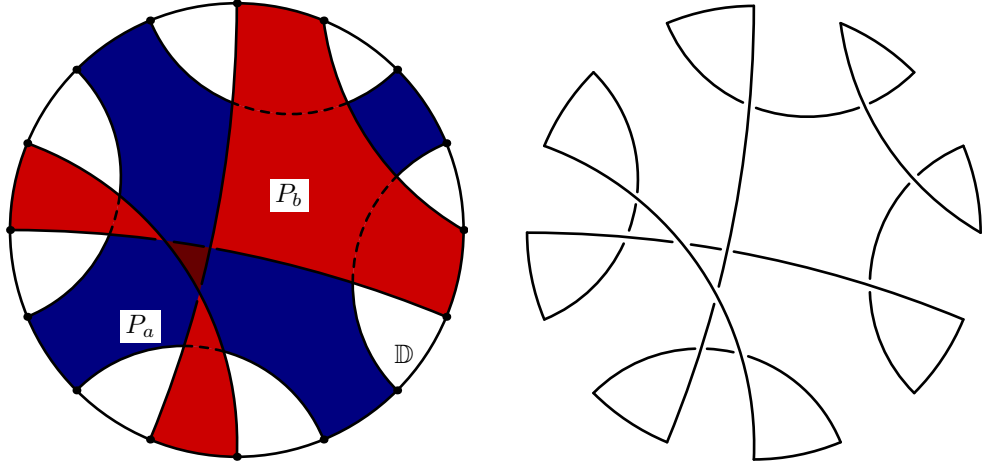
Figure 3.1: The surface associated with the exotic example (left), and its boundary component (right).
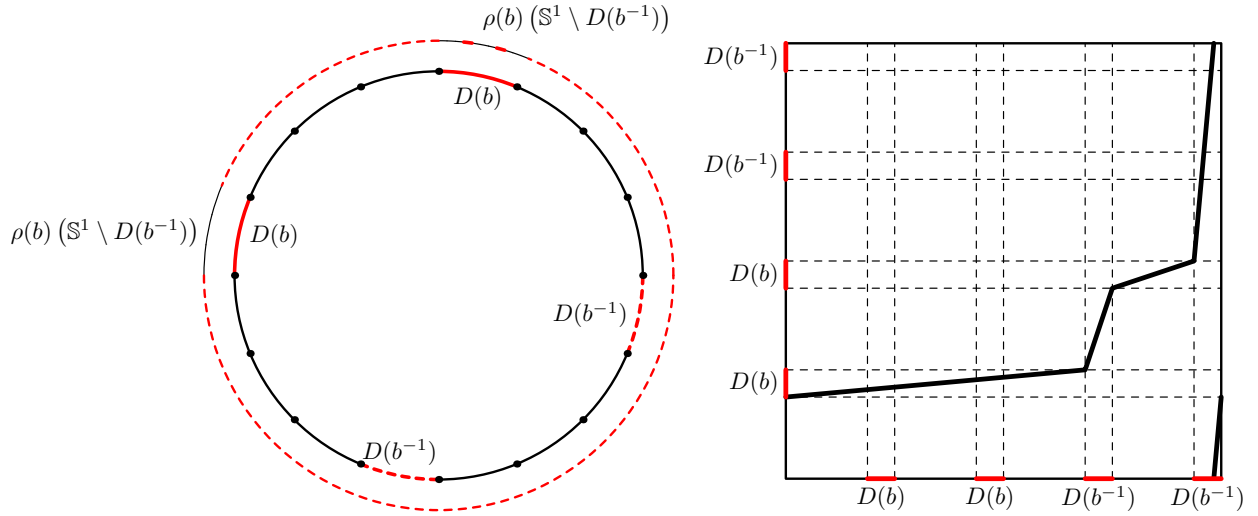

Figure 3.2: The ping-pong domains for $\rho(b)$ (left) and its graph (right). The circle is oriented counterclockwise.

3.2 Exotic examples

We conclude this section with discussion of new examples of ping-pong configurations and isolated orders, i.e. those that, even after an automorphism of the group, cannot arise from the construction in Example 2.7.

Example 3.12. The example described in Figure 3.1 (left) is not realized by a ping-pong action in $\mathsf{PSL}(2,\mathbb{R})$, neither in any cover of it. Indeed, $\rho(b)$ has 2 hyperbolic fixed points, while $\rho(a)$ has four.

Observe that the ping-pong configuration for $\rho(b)$ alone is atypical, in the sense that it is not the classical ping-pong configuration for a hyperbolic element in $\mathsf{PSL}(2,\mathbb{R})$ – $\rho(b)$ has a "slow" contraction on the left half of the circle, as two iterations of $\rho(b)$ are needed in order to bring the external gaps of $D(b) \cup D(b^{-1})$ into the component of $D(b)$ with the attracting fixed point. See Figure 3.2.

This ping-pong configuration gives a surface with one boundary component, so it corresponds to an isolated circular order in $\mathsf{CO}(F_2)$. See Figure 3.1 (right).

Observe that one can create several examples of this kind, by choosing $\rho(b)$ to have two hyperbolic fixed points, but with an arbitrarily slow contraction (i.e. with N connected components for $D(b)$, $N \in \mathbb{N}$ arbitrary) and then choosing $\rho(a)$ to be a N-fold lift of a hyperbolic element in $\mathsf{PSL}(2,\mathbb{R})$.

The classification of all isolated circular orders for F_2 does not seem an easy task, at least at first glance.

4 Left-invariant linear orders on $F_n \times \mathbb{Z}$

The main purpose of this section is to prove Theorem 1.1, stating that $F_n \times \mathbb{Z}$ admits an isolated linear order if and only if n is even.

4.1 Preliminaries

Before proceeding to the proof of Theorem 1.1, we recall some (more or less) classical tools.

As for circular orders, linear orders on countable groups have a *dynamical realization* (see for instance [14, Prop. 1.1.8]). One quick way of seeing this is thinking of a linear order as a special caso of circular order. Indeed, given a linear order $\preceq$ on a group G, one defines the cocycle $c_{\preceq}$ by setting, for distinct $g_1, g_2, g_3 \in G$

$$c_{\preceq}(g_1, g_2, g_3) = \mathsf{sign}(\sigma),$$

where σ is the permutation of the indices such that $g_{\sigma(1)} \prec g_{\sigma(2)} \prec g_{\sigma(3)}$. In more sophisticated terms, one can show that linear orders correspond exactly to 2-cocycles c which are *coboundaries* (see for instance [21, §2]). Thus, the construction of the dynamical realization sketched in the proof of Proposition 3.2 may be performed also for a linear order. The result is an action on the circle with only one global fixed point, which one can view as an action on the line with no global fixed point. Conversely a faithful action on the real line $\rho : G \to \mathsf{Homeo}_+(\mathbb{R})$ can be viewed as a faithful action on the circle with a single fixed point, and from it the induced orders will be linear orders on G (*cf.* Remark 3.3).

Next, we recall the notion of convex subgroups and their relationship to isolated orders.

Definition 4.1. A subgroup C in a linearly-ordered group $(G, \preceq)$ is *convex* if for any two elements $h, k \in C$, and for any $g \in G$, the condition $h \preceq g \preceq k$ implies $g \in C$.

Remark 4.2. Let G be a countable group and consider a dynamical realization ρ of $(G, \preceq)$ with basepoint x. Then, if C is a convex subgroup, the interval

$$I := \left(\inf_{h \in C} \rho(h)(x), \ \sup_{h \in C} \rho(h)(x) \right)$$

has the following property:

$$\text{for any } g \in G, \ \rho(g) \text{ either fixes } I \text{ or } \rho(g)(I) \cap I = \emptyset. \tag{4.1}$$

Moreover, the stabilizer of I is precisely C.

Conversely, given a faithful action on the real line $\rho : G \to \mathsf{Homeo}_+(\mathbb{R})$, if an interval I has the property (4.1), then the stabilizer $C = \mathsf{Stab}_G(I)$ is convex in any induced order with basepoint $x \in I$.

It is easy to see that the family of convex subgroups of a linearly ordered group $(G, \preceq)$ forms a *chain*: if C_1, C_2 are two convex subgroups of $(G, \preceq)$, then either $C_1 \subset C_2$ or $C_2 \subset C_1$. Moreover, for any convex subgroup $C \subset G$, the group G acts on the quotient $(G/C, \preceq_C)$ by order-preserving transformation, where, by definition, $fC \prec_C gC$ if and only if $fc \prec gc'$ for every $c, c' \in C$ (this definition makes sense because C is convex). In particular, this implies that if C is convex in $(G, \preceq)$, then *any* linear order $\preceq'$ on C may be extended to a (new) order $\preceq''$ on G by declaring

$$id \preceq'' g \Leftrightarrow \begin{cases} C \prec_C gC & \text{if } g \notin C, \\ id \preceq' g & \text{if } g \in C. \end{cases}$$

Elaborating on this, one can show the following lemma (see [14, Prop. 3.2.53] or [24, Thm. 2] for details).

Lemma 4.3. *If $(G, \preceq)$ has an infinite chain of convex subgroups, then $\preceq$ is non-isolated in $\mathsf{LO}(G)$.*

Let us also introduce a dynamical property that implies that an order is non-isolated:

Definition 4.4. Let G be a discrete, countable group and consider the space of representations $\mathsf{Rep}(G, \mathsf{Homeo}_+(\mathbb{R}))$ endowed with the topology of pointwise convergence. We denote by $\mathsf{Rep}_\#(G, \mathsf{Homeo}_+(\mathbb{R}))$ the set of representations with no global fixed points.

A representation $\rho_0 \in \mathsf{Rep}_\#(G, \mathsf{Homeo}_+(\mathbb{R}))$ is *flexible* if there are arbitrarily close representations $\rho \in \mathsf{Rep}_\#(G, \mathsf{Homeo}_+(\mathbb{R}))$ which are not semi-conjugate to ρ_0.

It was implicit in the work by Navas [26] as well as in [28], that if the dynamical representation coming from an order is flexible, then the order is non-isolated. An explicit proof can be found in [1, Prop. 2.8]. Also, a characterization of isolated (circular or linear) orders in terms of a strong form of rigidity (i.e. strong non-flexibility) can be found in [21].

Lemma 4.5. *Let G be a discrete, countable group and let ρ_0 the dynamical realization of an order $\preceq$ with basepoint $x \in \mathbb{R}$.[3] If ρ_0 is flexible, then $\preceq$ is non-isolated in $\mathsf{LO}(G)$.*

As mentioned in the introduction, in order to prove Theorem 1.1 we consider the interplay with circular and linear orders on groups. For this purpose we now recall the notion of cofinal elements.

Definition 4.6. An element h in a linearly-ordered group $(G, \preceq)$ is called *cofinal* if:

$$\text{for all } g \in G, \text{ there exists } m, n \in \mathbb{Z} \text{ such that } h^m \preceq g \preceq h^n. \tag{4.2}$$

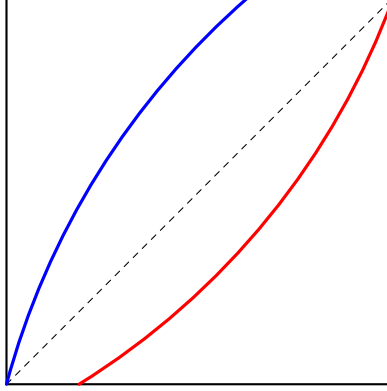[3]See Proposition 3.2 for the definition.

Figure 4.1: A crossing in the dynamical realization.

Remark 4.7. When considering a dynamical realization ρ with basepoint x, a cofinal element is an element with no fixed point: if (4.2) is not verified for $h \in G$, take

$$x_* := \inf\{\rho(g)(x) \mid h^n \preceq g \text{ for every } n \in \mathbb{Z}\}.$$

This is a fixed point for h.

Given a group G with a circular order c, there is a natural procedure to *lift c to* a linear order $\preceq_c$ on a central $\mathbb{Z}$-extension on G [21,30]. With respect to this order, the central $\mathbb{Z}$ – or more precisely a generator of $\mathbb{Z}$ – is cofinal. The following statement appears as Proposition 5.4 in [21]:

Proposition 4.8. *Assume that G is finitely generated and c is an isolated circular order on G. If $\preceq_c$ is the lift of c to $\widehat{G}$, a central $\mathbb{Z}$-extension of G, then the induced linear order $\preceq_c$ is isolated in $\mathsf{LO}(\widehat{G})$.*

One last crucial notion is that of crossings:

Definition 4.9. Let G be a group acting on a totally ordered space $(\Omega, \preceq)$. The action has *crossings* if there exist $f, g \in G$ and $u, v, w \in \Omega$ such that:

1. $u \prec w \prec v$.

2. $g^n u \prec v$ and $f^n v \succ u$ for every $n \in \mathbb{N}$, and

3. there exist M, N in $\mathbb{N}$ such that $f^N v \prec w \prec g^M u$.

Remark 4.10. When considering the dynamical realization, a crossing is simply given by two elements whose graphs are crossing, as pictured in Figure 4.1.

Lemma 4.11 ([14] Cor. 3.2.28)**.** *Let C be a convex subgroup of $(G, \preceq)$ and suppose that the (natural) action of G on $(G/C, \preceq_C)$ has no crossings. Then there exists a homomorphism $\tau : G \to \mathbb{R}$ with C in its kernel. Moreover, if C is the maximal convex subgroup of $(G, \preceq)$, then C agrees with the kernel of τ.*

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4.2 Isolated linear orders on $F_n \times \mathbb{Z}$

As a consequence of Proposition 4.8, we know that $F_{2n} \times \mathbb{Z}$ admits isolated linear orders, namely, any that comes from lifting an isolated circular order on F_{2n}. Further, if

$$0 \to \mathbb{Z} \to \hat{G} \xrightarrow{\pi} G \to 1$$

is a central $\mathbb{Z}$-extension of G, then any linear order $\preceq$ on $\hat{G}$ in which $\mathbb{Z}$ is cofinal gives a canonical circular order on G, which we denote by $\pi^*(\preceq)$, as follows. Let z be the generator of $\mathbb{Z}$ such that $z \succ id$. Since z is cofinal, for each $g \in G$, there exists a unique representative $\hat{g} \in \pi^{-1}(g)$ such that $id \preceq \hat{g} \prec z$. Given distinct elements $g_1, g_2, g_3 \in G$, let σ be the permutation such that

$$id \preceq \hat{g}_{\sigma(1)} \prec \hat{g}_{\sigma(2)} \prec \hat{g}_{\sigma(3)} \prec z.$$

Define $\pi^*(\preceq)(g_1, g_2, g_3) := \mathsf{sign}(\sigma)$. One checks that this is a well defined circular order on G. In the proof of Proposition 5.4 of [21], it is shown that π^* is locally injective when G is finitely generated. This implies that an isolated linear order of $F_{2n+1} \times \mathbb{Z}$ with cofinal center would produce an isolated circular order of F_{2n+1}. This we know is not possible after Theorem 1.4. Thus, to finish the proof of Theorem 1.1 it is enough to show the following:

Proposition 4.12. *Let F be a free group, and $\preceq$ a linear order on $G = F \times \mathbb{Z}$ in which the central factor is not cofinal. Then $\preceq$ is non-isolated.*

As a warm-up and tool in the proof, we give a short proof of a special case.

Lemma 4.13. *Let F be the free group of infinite rank and $G = F \times \mathbb{Z}$. Then no order in $\mathsf{LO}(G)$ is isolated.*

Proof. Let $f_1, f_2, \ldots$ be a set of free generators of the free factor F and g the generator of the central factor $\mathbb{Z}$. Let $\preceq$ be any order on G and ρ_0 its dynamical realization. For any fixed $n \in \mathbb{N}$, we can define a representation $\rho_n : G \to \mathsf{Homeo}_+(\mathbb{R})$ by setting

$$\rho_n(g) = \rho_0(g), \quad \rho_n(f_k) = \begin{cases} \rho_0(f_k) & \text{if } k \neq n, \\ \rho_0(f_n)^{-1} & \text{if } k = n. \end{cases}$$

It is easy to see that the actions ρ_n are all free and pairwise non-semi-conjugate one to another. So they determine distinct orders $\preceq_n$, which converge to $\preceq$ in $\mathsf{LO}(G)$ as $n \to \infty$. $\qquad\square$

Proof of Proposition 4.12. If F has rank one or infinite rank, it is easy to see that they do not admit isolated orders. Indeed, in the former case the group G is free abelian hence it has no isolated order [29], in the latter case we have seen this in Lemma 4.13.

So from now on we assume that F has finite rank ≥ 2. Looking for a contradiction, we let $\preceq$ be a linear order on G which is isolated and in which the center is not cofinal. This means that in the dynamical realization ρ of $(G, \preceq)$, the $\mathbb{Z} = \langle z \rangle$ factor acts with fixed points (Remark 4.7). In fact, the fixed points of z accumulate to both ∞ and $-\infty$, because F commutes with z and if this was not the case, there would be a global fixed point for G.

Call I the connected component of $\mathbb{R} \setminus \mathsf{Fix}(z)$ that contains the basepoint x_0 of ρ (which is a bounded interval), and let $C = \mathsf{Stab}_G(I)$.

Claim 1. *C is a convex subgroup of $(G, \preceq)$. Moreover, z is cofinal in $(C, \preceq)$.*

Proof of Claim. If $h, k \in C$ and $g \in G$ verify

$$h(x_0) < g(x_0) < k(x_0),$$

then for any n we also have

$$z^n h(x_0) < z^n g(x_0) < z^n k(x_0);$$

as z is central, this implies

$$h z^n(x_0) < g z^n(x_0) < k z^n(x_0). \tag{4.3}$$

Assume, without loss of generality, that $z(x_0) > x_0$. Then, as $n \to \infty$, the sequence of points $z^n(x_0)$ converges to the rightmost point of I, which is fixed by both h and k. We deduce from (4.3) that also g fixes this point. Repeating this argument considering the limit in (4.3) as $n \to -\infty$, we get that also the leftmost point of I is fixed by g. Hence $g \in C$, as wanted.

Finally by Remark 4.7, the fact that z has no fixed points in I implies that z is cofinal in $(C, \preceq)$. □

Claim 2. *C has infinite index in G.*

Proof of Claim. If C had finite index, then the G-orbit of the interval I would be bounded. This would imply that the dynamical realization has a global fixed point, which is absurd. □

Since C is a convex subgroup, if the restriction of $\preceq$ to C is non-isolated, then we can approach $\preceq$ by approaching its restriction to C. So going forward we assume that the restriction of $\preceq$ to C is isolated.

If the chain of convex subgroups of $\preceq$ is infinite, then Lemma 4.3 implies that $\preceq$ is non-isolated and we are done. So, we also assume from now on that the chain of convex subgroups of G is finite. Hence we can let $\overline{C}$ be the smallest convex subgroup properly containing C. In this way, to show that $\preceq$ is non-isolated, it is enough to show that $\preceq$ is non-isolated in $\overline{C}$, or, what is the same, we can restrict to the case where C is the *maximal* convex subgroup of G.

For our first claim observe that C admits a decomposition of the form $F^* \times \mathbb{Z}$, where F^* is a subgroup of F.

Claim 3. *F^* is a non trivial free group of even rank.*

Proof of Claim. Since the restriction of $\preceq$ to $C = F^* \times \mathbb{Z}$ is isolated, as before, Lemma 4.13 implies that F^* cannot have infinite rank. Moreover, F^* cannot be trivial as then the action of G would be semi-conjugated to an action of F, thus making very easy to perturb the action of $F \times \mathbb{Z}$ and thus the order $\preceq$ (recall that free groups have no isolated orders [25]). Finally, as z is cofinal in the ordering in C, the argument above Proposition 4.12 shows that F^* must have even rank. □

Since every (non trivial) normal, infinite index subgroup of F has infinite rank, we conclude from Claim 3 that F^* (and thus C) is not a normal subgroup of G. Lemma 4.11 implies that the action of G on $(G/C, \preceq_C)$ has crossings, as otherwise C would be normal. In particular, if we collapse I and its G-orbit, we obtain a semi-conjugate action $\bar{\rho} : G \to \mathsf{Homeo}_+(\mathbb{R})$ which is minimal and has crossings. Note that the center $\mathbb{Z} = \langle z \rangle \subset C$ acts trivially in this action.

Claim 4. *For any compact $K \subset \mathbb{R}$, there exists ρ' agreeing with ρ on K, but not semi-conjugate to ρ.*

Proof of Claim. We show this by showing how to perturb ρ in a neighborhood of $+\infty$. For this we fix a compact set K. We want to perturb the generators of G outside K in such a way that the perturbed action is not semi-conjugate to ρ.

Suppose as an initial case that there is a generator, say a, that has a fixed point p outside – and on the right of – K. Then we can change $\rho(a)$ on each component of $\mathbb{R} \setminus \mathsf{Fix}(\rho(a))$ to the right of p, in such a way that the perturbed action, say a', satisfies that $a'(x) \geq x$ for all $x \geq p$, or $a'(x) \leq x$ for all $x \geq p$, while leaving the rest of the generators untouched. Since these two possible perturbations give actions that are not semi-conjugate, this ensures that at least one of them is not semi-conjugate to ρ. Moreover, since p is also a fixed point of $\rho(z)$, one can perform this modification in such a way that $a' = c\rho(a)$ where c is an element commuting with $\rho(z)$. In particular, the new action extends to a representation of $F \times \mathbb{Z}$. This concludes the proof in the case one generator has a fixed point outside K.

For the remaining case, suppose that no generator has a fixed point outside K. We claim that then we can perturb the action ρ to obtain one generator with a fixed point outside K; showing this will be the content of the remainder of the proof. First, since $\bar{\rho}$ has crossings, for any compact $K \subset \mathbb{R}$ there is $g \in G$ such that $\mathbb{R} \setminus \mathsf{Fix}(\rho(g))$ has a component outside (and on the right of) K. Let $J = (j^-, j^+)$ denote one of those components.

Claim ($*$)**.** *There is a free generator a (or an inverse) of F such that $\rho(a)(j^-) \in J$, but $\rho(a)(j^+) \notin J$.*

Proof of Claim ($*$)*.* If this was not the case, J would satisfy property (4.1) and as observed in Remark 4.2 this would give a convex subgroup properly containing a conjugate of C, which is not possible by the maximality of C. $\qquad\square$

Let $\bar{g}$ be the homeomorphism defined as the identity outside J and agreeing with $\rho(g)$ on J. Let a be the generator given by Claim ($*$) and define $\rho^{\bar{g}}$ by

$$\rho^{\bar{g}}(a) = \bar{g}\rho(a), \text{ and } \rho^{\bar{g}}(b) = \rho(b) \text{ for any other generator of } F \text{ and for } b = z.$$

Since $\bar{g}$ commutes with z, the new action $\rho^{\bar{g}}$ is a representation of G. Moreover, by changing $\bar{g}$ by some power if necessary, we have that $\rho^{\bar{g}}(a)$ has a fixed point in J. This ends the proof of Claim 4. $\qquad\square$

To finish the proof of Proposition 4.12 (and thus that of Theorem 1.1), note that the *flexibility* of ρ from Claim 4 implies that the order is non-isolated (Lemma 4.5). $\qquad\square$

Acknowledgments

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