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Bottleneck models and departure time problems

André de Palma, Ecole Normale Supérieure (ENS) Paris-Saclay ¹

Claude Lefèvre, Université Libre de Bruxelles (ULB) ²

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Abstract

This paper is concerned with the analysis of departure times in dynamic bottleneck models. First, the case of a set of individual drivers is discussed through both deterministic and stochastic approaches. Then, the analysis is extended to a new model that combines small and large agents. In the stochastic setting, our focus is mainly on the model building; simulations will be carried out in a near future.

Keywords: dynamic bottleneck models; congestion; equilibrium; deterministic version; stochastic approach; small and large agents; discrete choice models.

1 Introduction

Dynamic bottleneck models have become very popular in *Transportation*. This has not always been the case. In the early eighties, most transportation models were static, in the sense that congestion was assumed to be independent of the time of the day (see e.g. Sheffi (1981)). The static models were developed in different directions (heterogeneity, multiple types of vehicles, elastic demand, stochastic equilibrium ...). The treatment of congestion remained rather inappropriate because somewhat ad hoc peak and off-peak periods were introduced as a surrogate to take into account the time dependent nature of congestion. In reality, the choice between peak and off-peak must be endogenous as it depends on the level of congestion. This is also true with several policies, road pricing and change of the peak period lengths for instance, which have to be determined endogenously. One may assume cross-elasticities between the periods, but these elasticities should not be constant.

Intermediary models have been proposed by Nesterov and de Palma (2003) for a stable dynamic approach between the static and dynamic cases. These models are popular in operations research since they can be formulated through some convex problem. In the present work, we will focus on bottleneck models whose construction is fully dynamic.

By contrast, in that dynamic modelling, congestion depends endogenously on the time of the day. Such models extend the dynamic assignment models considered for problems of route

¹Corresponding author. CREST, Bâtiment Laplace, 61 avenue du Président Wilson, F-94230 Cachan, France. E-mail address: andre.depalma@ens-cachan.fr

²Département de Mathématique, Campus de la Plaine C.P. 210, Boulevard du Triomphe, B-1050 Bruxelles, Belgique. Email address: clefevre@ulb.ac.be

choice. The initial dynamic congestion model is due to the economist W.S. Vickrey (1969). In this deterministic model, there is an origin, a destination and a single bottleneck of finite capacity. The cost function for the driver is the sum of the travel time and the schedule delay cost (i.e. the cost due to an early or late arrival); it is piece-wise linear.

Variants of this dynamic model were discovered, independently, by Moshe Ben-Akiva, Nicolas Litinas, Pavlos Kanaroglou and the two authors of this paper. The reader is referred to de Palma, Ben-Akiva, Lefèvre and Litinas (1983), Ben-Akiva, de Palma and Kanaroglou (1985, 1986a, 1986b) and de Palma, Lefèvre and Ben-Akiva (1987). These authors generalized the deterministic model à la Vickrey in two ways. First, the demand side is represented by a discrete probabilistic model which is obtained from a continuous logit choice model. Second, the supply side is an extended bottleneck model inspired by Robert Herman, the pioneer of Transportation Science, who developed it with André during a visit at Austin, Texas.

In static bottleneck models, a major decision variable that was missing is the departure time choice. This variable is encapsulated in the fully dynamic models. The first computational model was developed with Pavlos who was then with André at McMaster University. A computer program was developed to compute a stochastic equilibrium in a small network. It was based on a day-to-day adjustment process (an iteration corresponds to a day). At the beginning of a day, the driver combines the experienced travel time with the expected travel time. At the end of the day, the driver has experienced another travel time. This allows the driver to revise its expectation of travel time for the next day according to a moving average process. That iteration procedure converges in general after about thirty steps.

This line of research work was continued and amplified in many papers. The early work in the economic literature is a seminal paper by Arnott, de Palma and Lindsey (1993). For a review, see e.g. de Palma and Fosgerau (2011). The literature on dynamic bottleneck models is now rich and diversified. In particular, a large scale dynamic model, METROPOLIS, was used in many cities all around the world; see de Palma and Marchal (1996) and de Palma, Marchal and Nesterov (1997). Recently, the driver simulation models DynaMit and MATSim were developed in a similar purpose by other researchers.

A first goal of the present paper is to provide both deterministic and *stochastic* versions of this dynamic setting. There is an enormous literature relying on a stochastic approach but it often neglects to describe endogenously the arrival rates. Besides that reason, we want to consider a stochastic bottleneck model because so far, there exists no simple day-to-day adjustment process which always leads to stable equilibria. Probabilistic birth-and-death master equations provide us with another angle of view to stability since they usually yield a stationarity distribution. As we know, a deterministic macroscopic stable solution is not necessarily the most likely outcome to the corresponding stochastic process. The reader is referred e.g. to the discussion made in de Palma and Lefèvre (1983). Let us also mention that stochasticity may play other roles in the bottleneck models. This is the case when considering stochastic capacities, as in Xiao, Huang and Liu (2015), or stochastic travel times, as in Xiao, Coulombel and de Palma (2017).

A second goal of this paper is to introduce in the model a combination of *small and large* transport agents. A driver is considered as small when its departure time has no influence

on the aggregate level of congestion. We consider a mix of small drivers and of a large fleet of vehicles which have similar sizes (the case of variable sizes could be discussed similarly). The departure time choices in the fleet are assumed to be taken by a single manager whose objective is to minimize his various costs by taking into account the consequences of his choices. This decision process is assumed here to occur continuously in time, so that a small number of vehicles of the fleet can be reallocated at each instant. The fleet is large to guarantee that the cost per vehicle is of the same order of magnitude as the externality. The case where the manager decides periodically, at discrete times, is left for a future research.

The structure of the paper is as follows. In Section 2, we construct the basic deterministic model: the supply side is the bottleneck model such as introduced by Vickrey (1969), while the demand side is a logit discrete choice model. In Section 3, we examine the stochastic version of this model by resorting to birth-and-death master equations (see e.g. Ross (1996)). In Section 4, we introduce a deterministic model which combines small and large agents. We compute the equilibrium state by showing that the individual and fleet vehicles behave rather differently. In Section 5, we formulate this model in a stochastic framework, with a special attention to the choice problem between two departure times.

2 Starting deterministic model

We first present a deterministic dynamic model for congestion. In this model, we adopt the traditional assumption that all the drivers have the same importance for the traffic. We also assume that each driver is arbitrarily small so that the equilibrium concept is of Nash with a continuum of users.

2.1 Definitions and notation

Let us introduce the key parameters of the model:

- t , the departure time ($t \in [0, T]$, continuous or not),
- ω , the iteration (day, $\omega = 0, \dots, \Omega$),
- $r_\omega(t)$, the number of drivers departing from the origin at time t at iteration ω ,
- $R_\omega(t)$, the cumulated inflow at time t at iteration ω , i.e. $R_\omega(t) = \int_0^t r_\omega(u) du$,
- N , the total number of drivers ($\int_0^T r_\omega(t) dt = N$),
- s , the capacity of the bottleneck (number of vehicles per hour),
- $Q_\omega(t)$, the number of users in the queue at time t at iteration ω ,
- $tt_\omega(t) = Q_\omega(t)/s$, the waiting time of a driver at the bottleneck at time t at iteration ω .

Without loss of generality, the free-flow travel time is assumed to be 0. Let $\tau_\omega(t)$ denote the last time before t when congestion started (provided it occurs) at iteration ω . So,

$$\tau_\omega(t) = \sup\{u \leq t : Q_\omega(u) > 0\}.$$

When there is congestion, the queue length is then given by

$$Q_\omega(t) = \int_{\tau_\omega(t)}^t r_\omega(u) du - s[t - \tau_\omega(t)]. \quad (2.1)$$

The cost for a driver departing at time t at iteration ω is defined by

$$C_\omega(t) = \alpha t t_\omega(t) + \beta[t^* - t_\omega^a(t)]^+ + \gamma[t_\omega^a(t) - t^*]^+, \quad (2.2)$$

in which

t^* is the desired time of destination,

$t_\omega^a(t)$ is the arrival time of a driver departing at time t at iteration ω , i.e. $t_\omega^a(t) = t + t t_\omega(t)$.

2.2 Departure pattern

Denote by $P_\omega(t|t')$ the transition probability from departure choice t' at iteration $\omega - 1$ to departure time t at iteration ω . Note that t may be larger or smaller than t' . That probability is assumed to be of the form

$$P_\omega(t|t') = \frac{h_\omega(t|t')}{H_\omega(t')}, \quad (2.3)$$

where $H_\omega(t') = \int_0^T h_\omega(t|t') dt$ is the aggregator at iteration ω and yields, of course,

$$\int_0^T P_\omega(t|t') dt = 1.$$

Then, the number of drivers departing at time t at iteration ω is given by

$$r_\omega(t) = \int_0^T P_\omega(t|t') r_{\omega-1}(t') dt'. \quad (2.4)$$

The drivers are assumed to be small enough so that they may consider that their departure time decisions do not affect the value of the aggregator. That is, during each period, the value of the aggregator is constant from the driver point of view. This is an assumption similar to the monopolistic competition (see Dixit and Stiglitz (1977)).

2.3 Logit probabilities

In (2.3), the function $h_\omega(t|t')$ depends on the cost incurred at iteration $\omega - 1$ at time t' and on the cost incurred at iteration ω at time t .

We consider a logit specification for the transition probabilities. The logit model is a classical tool in Transportation (McFadden (2001)) and Industrial Organization (Anderson, de Palma and Thisse (1992)). Recently, it was also used in the study of Big Data, especially for sentiment analysis (Yessenalina and Cardie (2011)). Specifically, we choose

$$h_\omega(t|t') = e^{-[C_\omega(t) - C_{\omega-1}(t')]}.$$

In this case, (2.3) is given by the continuous logit formula

$$P_\omega(t|t') = \frac{e^{-C_\omega(t)}}{\int_0^T e^{-C_\omega(u)} du} \equiv P_\omega(t). \quad (2.5)$$

It is worth recalling that when the error terms in the logit are of zero variance, this model reduces to the deterministic case. From (2.5), we observe that the transition probability becomes here independent of t' , which makes easier the model analysis. Using (2.4), we then obtain that

$$r_\omega(t) = \int_0^T P_\omega(t) r_{\omega-1}(t') dt' \equiv N P_\omega(t). \quad (2.6)$$

Let us check that this system has $r_\omega(t)$ as a fixed point. Indeed, from (2.5) and (2.6),

$$r_\omega(t) = N \frac{e^{-C_\omega(t)}}{\int_0^T e^{-C_\omega(u)} du}.$$

We postulate that the equilibrium solution has the following intuitive pattern over time: first, departures occur without congestion early (arrival before t^*); then, departures occur with congestion early; next, departures occur with congestion late; finally, departures occur without congestion late. Let t_ω^i be the first early departure with congestion and let t_ω^f be the first late departure without congestion. For early arrivals, we have

$$C_\omega^E(t) = (\alpha - \beta) tt_\omega(t) + \beta(t^* - t), \quad t < \hat{t}_\omega,$$

where $t_\omega^a(\hat{t}_\omega) = t^*$. Likewise, for late arrivals,

$$C_\omega^L(t) = (\alpha + \gamma) tt_\omega(t) + \gamma(t - t^*), \quad t > \hat{t}_\omega.$$

When there is no congestion, the travel time is constant (normalized to 0 with wlog). When there is congestion, the travel time is given by

$$tt_\omega(t) = \left[\int_{t_\omega^i}^t r_\omega(u) du - s(t - t_\omega^i) \right] \frac{1}{s}, \quad t_i \leq t \leq t_\omega^f.$$

So, the choice probabilities $P_\omega(t)$ depend on the cost function which depends on the travel time, which itself depends on the departure time schedule $\{r_\omega(u)\}$. This shows that the departure rate is a solution of the fixed point problem.

We have assumed above that each driver at iteration ω selects its departure time at the beginning of the iteration with full knowledge of travel time for each departure time and solves the fixed point accordingly. Alternatively, we assume that a driver has not a perfect foresight and builds at the beginning of iteration ω an estimate of the travel times for the whole period. The estimation made for period ω depends on the estimation made for period $\omega - 1$ and on the observed travel times during period $\omega - 1$, i.e.

$$tt_\omega^{ap}(t) = g[tt_{\omega-1}^{ap}(t), tt_{\omega-1}^{ob}(t)].$$

In the general case, the transition probabilities might depend explicitly on the previous choice t' at iteration $\omega - 1$.

Estimation of departure time choices is standard using an ordered probit model (see e.g. Small (1982)). A logit model for the departure time choices is possible too, on the basis of stated or revealed data. In the next section, we propose a stochastic approach which is more natural to describe the departure time process. It has not been considered so far.

3 A stochastic framework

This time, we assume that the departure times are discrete ($t = 0, \dots, T$) and the iteration process is Markovian in continuous time ($\omega \in \mathbb{R}_+$). This provides us with a framework which is standard and convenient in probability theory (see e.g. Ross (1996)). Note that considering a continuous time ω yields an approximation to the daily decision process which is reasonable in the long term.

Let $X_\omega(t)$ be the number of drivers departing at time t at iteration ω . We want to derive the joint distribution of the vector $\mathbf{X}_\omega = \{X_\omega(t), t = 0, \dots, T\}$. Let $p_\omega(\mathbf{x}) = P(\mathbf{X}_\omega = \mathbf{x})$, with $p_0(\mathbf{x})$ supposed to be fixed.

Using the Markovian property, we get the master equation

$$\begin{aligned} \frac{dp_\omega(\mathbf{x})}{d\omega} = & \sum_{j|x_j \geq 1} \sum_{i \neq j | x_i \leq N-1} (x_i + 1) p_\omega(\mathbf{x} + \mathbf{e}_i - \mathbf{e}_j) g_\omega^{j|i}(\mathbf{x} + \mathbf{e}_i - \mathbf{e}_j) \\ & - p_\omega(\mathbf{x}) \sum_{j|x_j \geq 1} \sum_{i \neq j | x_i \leq N-1} x_i g_\omega^{j|i}(\mathbf{x}), \end{aligned} \quad (3.1)$$

where $\mathbf{e}_i$ is the vector with 1 at the i -th position and 0 elsewhere, and where $g_\omega^{j|i}(\mathbf{x})$ denotes the transition rate from departure time i to departure time j at iteration ω when the present state is $\mathbf{x}$. This rate is a function of the form

$$g_\omega^{j|i}(\mathbf{x}) = f[C_\omega(i), C_\omega(j)], \quad (3.2)$$

where the cost $C_\omega(t)$ is defined as before by (2.2) and depends on t and $\{X_\omega(u), u = 0, \dots, t\}$. The latter property comes from the fact that the cost at time t depends on travel times at time t , which depend themselves on departures before time t . Thus, $g_\omega^{j|i}(\mathbf{x})$ depends in fact on $\mathbf{x}_{i \vee j} = (x_1, \dots, x_{i \vee j})$ only. In (3.2), the function $f(x, y)$ is expected to be increasing in x and decreasing in y .

The process $\{\mathbf{X}_\omega, \omega \in \mathbb{R}_+\}$ corresponds to a multi-dimensional birth-and-death process on $[0, N]^{T+1}$. As the state space is finite, the equations (3.1) admit, for all ω , a unique solution $p_\omega(\mathbf{x})$ under certain conditions on the transition rates. Moreover, this solution can then converge towards a stationary distribution $p(\mathbf{x})$. Further details can be found e.g. in Zeifman (1995).

Note that a single driver might be considered as small with respect to the total mass of drivers. In such a case, it can neglect the impact of its decision on the travel pattern. Mathematically, this means that in (3.1), the transition rate $g_\omega^{j|i}(\mathbf{x} + \mathbf{e}_i - \mathbf{e}_j)$ can be approximated by $g_\omega^{j|i}(\mathbf{x})$.

4 Small and large agents: a deterministic setting

Let us pass to a more general model where the drivers can have a variable importance. In this section, we deal with the deterministic approach, directly at the stationary state (so that the index ω becomes superfluous).

There are now a number N of individual drivers and a large fleet of M vehicles that is managed by a single decision maker. The manager of the fleet takes into account the congestion induced by its decision. The choice process is made in two stages. At the first step, all the individual drivers may choose their departure time according to the method described in Section 3. At the next step, the manager decides to reallocate the vehicles of its fleet by estimating the generated congestion. The fleet is large so that the cost per vehicle is approximately of the same level as the externality (see below).

4.1 Binary choice

Let us first discuss the case where there are only two possible departure times 1 and 2. The cost per vehicle in the fleet are given at time 1 by $C_1^A = h_1(X_1 + Y_1)$, and at time 2 by $C_2^A = h_2(\theta X_1 + \theta Y_1 + X_2 + Y_2)$ where $0 < \theta < 1$. The functions h_1 and h_2 are increasing since the costs increase with the number of vehicles. The parameter θ translates the fact that the vehicles departing at time 1 are linearly less costly for those departing later at time 2. The cost of the whole fleet of vehicles is thus equal to $C^A = Y_1 C_1^A + Y_2 C_2^A$.

Suppose now that δ_Y vehicles from the fleet change their previous choice and decide to depart at time 2 instead of time 1. The direct cost generated by these δ_Y vehicles is

$$\begin{aligned}\Delta^D(C^A) &= \delta_Y [-h_1(X_1 + Y_1 - \delta_Y) + h_2(\theta X_1 + \theta Y_1 - \theta \delta_Y + X_2 + Y_2 + \delta_Y)] \\ &= \delta_Y [-h_1(X_1 + Y_1) + h_2(\theta X_1 + \theta Y_1 + X_2 + Y_2)] + O(\delta_Y^2).\end{aligned}$$

Therefore,

$$\Delta^D(C^A) = \delta_Y [-C_1^A + C_2^A + O(\delta_Y)],$$

and as δ_Y is small with respect to the costs C_i^A , $i = 1, 2$,

$$\Delta^D(C^A) \approx \delta_Y (-C_1^A + C_2^A). \quad (4.1)$$

Let us examine the impact of the change made by the δ_Y vehicles on the other vehicles of the fleet. Using as before first-order approximations, we obtain for the generated indirect cost

$$\begin{aligned}\Delta^I(C^A) &= (Y_1 - \delta_Y) [h_1(X_1 + Y_1 - \delta_Y) - h_1(X_1 + Y_1)] + \\ &\quad + (Y_2 + \delta_Y) [h_2(\theta X_1 + \theta Y_1 - \theta \delta_Y + X_2 + Y_2 + \delta_Y) - h_2(\theta X_1 + \theta Y_1 + X_2 + Y_2)] \\ &= \delta_Y [-Y_1 h_1'(X_1 + Y_1) + Y_2 (1 - \theta) h_2'(\theta X_1 + \theta Y_1 + X_2 + Y_2)] + O(\delta_Y^2),\end{aligned}$$

where f' denotes the derivative of a function f . Therefore,

$$\Delta^I(C^A) \approx \delta_Y [Y_1 C_1'^A + Y_2 (1 - \theta) C_2'^A]. \quad (4.2)$$

The total impact due to the δ_Y changes is $\Delta(C^A) = \Delta^D(C^A) + \Delta^I(C^A)$. From (4.1) and (4.2), we then get

$$\Delta(C^A) \approx \delta_Y \{C_2^A - C_1^A + [Y_2 (1 - \theta) C_2'^A - Y_1 C_1'^A]\}. \quad (4.3)$$

We are in a position to make precise the assumption that the fleet is large enough. By this, we mean that

Assumption: C_i^A is of the same order of magnitude as $Y_i C_i'^A$, $i = 1, 2$.

Let us recall that the externality, denoted by e_i^A , is the difference between the marginal social cost, $(Y_i C_i^A)' = Y_i C_i'^A + C_i^A$, and the individual cost C_i^A ; so, $e_i^A = Y_i C_i'^A$. Therefore, the assumption above states that the cost C_i^A per vehicle in the fleet is of the same order of magnitude as the externality e_i^A .

4.2 Equilibrium

The choice process begins with the individual drivers. They select between the departure times 1 and 2 and at the equilibrium, the associated costs are equal. We consider a Nash equilibrium, so that each driver considers that the decision of the others is fixed. The costs per individual are given at time 1 by $C_1^I = g_1(X_1 + Y_1)$, and at time 2 by $C_2^I = g_2(\theta X_1 + \theta Y_1 + X_2 + Y_2)$ where the parameter θ is defined as for the fleet vehicles. The functions g_1 and g_2 are again increasing.

At equilibrium, we have

$$C_1^I = g_1(X_1 + Y_1) = C_2^I = g_2(\theta X_1 + \theta Y_1 + X_2 + Y_2). \quad (4.4)$$

Since $X_1 + X_2 = N$, (4.4) becomes

$$g_1(X_1 + Y_1) = g_2(\theta X_1 + \theta Y_1 + N - X_1 + M - Y_1). \quad (4.5)$$

Let us assume that there is an interior solution. It is unique since the left-hand side is increasing in X_1 , while the right-hand side is decreasing in X_1 . That solution is denoted by $X_1^{br} = \gamma(Y_1; N, M)$, with $dX_1^{br}/dY_1 < 0$.

Then, it is the turn of the drivers in the fleet. The fleet manager minimizes the total cost of the fleet taking constant the decisions of the individual drivers. At the optimum, it is necessary and sufficient that any change has no impact on the cost. This means that $\Delta(C^A) = 0$, i.e. from (4.3),

$$\delta_Y(-C_1^A + C_2^A) + \delta_Y[-Y_1 C_1'^A + Y_2(1 - \theta) C_2'^A] = 0.$$

After simplification by δ_Y , we get

$$C_1^A + Y_1 C_1'^A = C_2^A + Y_2(1 - \theta) C_2'^A. \quad (4.6)$$

Equation (4.6) is equivalent to

$$\begin{aligned} h_1(X_1 + Y_1) + Y_1 h_1'(X_1 + Y_1) &= h_2(\theta X_1 + \theta Y_1 + N - X_1 + M - Y_1) \\ &\quad + Y_2(1 - \theta) h_2'(\theta X_1 + \theta Y_1 + N - X_1 + M - Y_1). \end{aligned} \quad (4.7)$$

As before, assume that there is an interior solution. It is unique here too and denoted by $Y_1^{br} = \mu(X_1; N, M)$, with $dY_1^{br}/dX_1 < 0$.

From (4.5) and (4.7), a Nash equilibrium state (X_1^*, Y_1^*) will satisfy the following system

$$\begin{cases} X_1^* = \gamma(Y_1^*; N, M), \\ Y_1^* = \mu(X_1^*; N, M). \end{cases}$$

Of course, $X_2^* = N - X_1^*$ and $Y_2^* = M - Y_1^*$. Let us pursue with a more analytical formulation.

Linear costs. We now assume that the costs are linear with, for instance, $g_1 = g_2 = h_1 = h_2$ the identity function. Then, the individual condition (4.5) becomes

$$(X_1 + Y_1) = (\theta X_1 + \theta Y_1 + X_2 + Y_2),$$

which can be rewritten as

$$X_1(2 - \theta) + Y_1(2 - \theta) = M + N. \quad (4.8)$$

Moreover, the fleet condition (4.7) becomes

$$(X_1 + Y_1) + Y_1 = \theta X_1 + \theta Y_1 + N - X_1 + M - Y_1 + (M - Y_1)(1 - \theta),$$

which reduces to

$$X_1(2 - \theta) + Y_1(4 - 2\theta) = N + M(2 - \theta). \quad (4.9)$$

Combining (4.8) and (4.9), we obtain a system of two linear equations with two unknowns. Its solution is found to be

$$\begin{cases} X_1^* = (N + \theta)/(2 - \theta), \text{ and thus } X_2^* = [N(1 - \theta) - \theta]/(2 - \theta), \\ Y_1^* = (M - \theta)/(2 - \theta), \text{ and thus } Y_2^* = [M(1 - \theta) + \theta]/(2 - \theta). \end{cases}$$

Note that $0 < Y_1^*, Y_2^* < M$, while $0 < X_1^*, X_2^* < N$ if $\theta < N/(N + 1)$, which will be often satisfied in practice.

4.3 Extension

Let us now examine a more general model with K possible departure times. The vehicles departed before any time j , $1 \leq j \leq K$, influence the costs for the fleet vehicles and the individual drivers through the polynomial function

$$\sum_{r=0}^{j-1} \theta^{j-r} x_r.$$

A similar reasoning will allow us to obtain the equilibrium state by recursion. For instance, take $K = 3$. We start by considering the first two departure times. By the previous method, we obtain a solution $\{X_1^*, Y_1^*, X_2^*, Y_2^*\}$. For the individual drivers, the common cost, denoted by C^I , satisfies (4.4), i.e.

$$g_1(X_1^* + Y_1^*) = C^I = g_2(\theta X_1^* + \theta Y_1^* + X_2^* + Y_2^*).$$

At equilibrium, the costs for the individuals are the same for the K times, which yields

$$g_3(\theta^2 X_1^* + \theta^2 Y_1^* + \theta X_2^* + \theta Y_2^* + X_3 + Y_3) = C^I. \quad (4.10)$$

For the vehicles in the fleet, the common cost, denoted by MSC^A (marginal social cost), is given by

$$MSC^A = h_1(X_1^* + Y_1^*) + Y_1^* h_1'(X_1^* + Y_1^*),$$

and from (4.7), we also have

$$MSC^A = h_2(\theta X_1^* + \theta Y_1^* + X_2^* + Y_2^*) + Y_2^* (1 - \theta) h_2'(\theta X_1^* + \theta Y_1^* + X_2^* + Y_2^*).$$

Here again, at equilibrium, the marginal social costs are identical for the K times, so that

$$\begin{aligned} h_3(\theta^2 X_1^* + \theta^2 Y_1^* + \theta X_2^* + \theta Y_2^* + X_3 + Y_3) + Y_3 h_3'(\theta^2 X_1^* + \theta^2 Y_1^* + \theta X_2^* + \theta Y_2^* + X_3 + Y_3) \\ = MSC^A. \end{aligned} \quad (4.11)$$

The system (4.10) and (4.11) will have a unique solution X_3^* and Y_3^* , under some regulatory conditions. The procedure can then continue recursively.

5 Small and large agents: a stochastic setting

This time, we will use a Markovian model in continuous-time (see e.g. Ross (1996)) to describe the decision process of the individual drivers and the fleet manager.

5.1 Binary choice

We start again by considering two possible departure times 1 and 2. Let (X_ω, Y_ω) be the numbers of individual drivers and vehicles in the fleet that choose to depart at time 2 at iteration ω . Note that the number of departures at time 1 is $(N - X_\omega, M - Y_\omega)$. We are interested in the joint distribution of the vector (X_ω, Y_ω) . Let $p_\omega(x, y) = P(X = x, Y = y)$, with $p_0(x, y)$ supposed to be fixed.

From the Markovian assumption, we obtain the master equations

$$\begin{aligned} \frac{dp_\omega(x, y)}{d\omega} = & p_\omega(x, y + 1) f_\omega^{2|1}(x, y + 1) + p_\omega(x, y - 1) f_\omega^{1|2}(x, y - 1) \\ & + p_\omega(x + 1, y) (x + 1) g_\omega^{2|1}(x + 1, y) + p_\omega(x - 1, y) (N - x + 1) g_\omega^{1|2}(x - 1, y) \\ & - p_\omega(x, y) \left[f_\omega^{2|1}(x, y) + f_\omega^{1|2}(x, y) + x g_\omega^{2|1}(x, y) + (N - x) g_\omega^{1|2}(x, y) \right], \end{aligned} \quad (5.1)$$

where, for instance, $f_\omega^{2|1}(x, y)$ (resp. $g_\omega^{2|1}(x, y)$) is the fleet transition rate (resp. individual transition rate) from departure time 1 to departure time 2 at iteration ω when the state is (x, y) at iteration $\omega - 1$. The transition rates at state (x, y) equal 0 when x is outside $[0, N]$ or y is outside $[0, M]$.

The transition rates can be constructed by means of a logit model. Let us omit the argument ω to simplify the notation. For the fleet, arguing as for (4.2), this leads to

$$f^{2|1}(x, y) = \lambda \frac{e^{-[C_2^A(x, y) + y(1-\theta)C_2^{A'}(x, y)]}}{e^{-[C_1^A(x, y) + (M-y)C_1^{A'}(x, y)]} + e^{-[C_2^A(x, y) + y(1-\theta)C_2^{A'}(x, y)]}}, \quad (5.2)$$

where λ is the decision-making rate of the manager and $C_i^A(x, y)$, $i = 1, 2$, is the cost of a vehicle in the fleet departing at time i . For the individual drivers, we have

$$g^{2|1}(x, y) = \mu \frac{e^{-C_2^I(x, y)}}{e^{-C_1^I(x, y)} + e^{-C_2^I(x, y)}}, \quad (5.3)$$

where μ is the individual decision-making rate and $C_i^I(x, y)$, $i = 1, 2$, is the cost of a driver departing at time i .

The process $\{(X_\omega, Y_\omega), \omega \in \mathbb{R}_+\}$ corresponds to a bivariate birth-and-death process on $[0, N] \times [0, M]$. The associated equations (5.1) admit a unique solution $p_\omega(x, y)$ for all ω and a stationary solution $p(x, y)$, under some regularity conditions on the rates.

5.2 General case

Let $X_\omega(t)$ be the number of individual drivers departing at time t at iteration ω , and let $Y_\omega(t)$ be the number of vehicles of the fleet departing at time t at iteration ω . We want to derive the joint distribution of the vector $(\mathbf{X}_\omega, \mathbf{Y}_\omega) = \{[X_\omega(t), t = 0, \dots, T], [Y_\omega(t), t = 0, \dots, T]\}$. Let $p_\omega(\mathbf{x}, \mathbf{y}) = P(\mathbf{X}_\omega = \mathbf{x}, \mathbf{Y}_\omega = \mathbf{y})$, with $p_0(\mathbf{x}, \mathbf{y})$ supposed to be fixed.

In this case, the master equations become

$$\begin{aligned} \frac{dp_\omega(\mathbf{x}, \mathbf{y})}{d\omega} = & \sum_{\mathbf{z} \neq \mathbf{y}} p_\omega(\mathbf{x}, \mathbf{z}) f_\omega^{y|z}(\mathbf{x}, \mathbf{z}) \\ & + \sum_{j|x_j \geq 1} \sum_{i \neq j | x_i \leq N-1} (x_i + 1) p_\omega(\mathbf{x} + \mathbf{e}_i - \mathbf{e}_j, \mathbf{y}) g_\omega^{j|i}(\mathbf{x} + \mathbf{e}_i - \mathbf{e}_j, \mathbf{y}) \\ & - p_\omega(\mathbf{x}, \mathbf{y}) \left[\sum_{\mathbf{z} \neq \mathbf{y}} f_\omega^{y|z}(\mathbf{x}, \mathbf{y}) + \sum_{j|x_j \geq 1} \sum_{i \neq j | x_i \leq N-1} x_i g_\omega^{j|i}(\mathbf{x}, \mathbf{y}) \right], \end{aligned} \quad (5.4)$$

using the same notation as for (5.1). To the best of our knowledge, such a birth-and-death process is a new framework for the congestion problems. It would be interesting to simulate this stochastic process, assuming a smooth behavior for the transition rates.

6 Conclusions

We have developed two main ideas in this work. The first one is the use of a stochastic approach to tackle dynamic transportation bottleneck models. Equilibrium solutions can be determined in various situations but very little is known concerning the adjustment process and

the stability. We believe that the problem is more amenable to numerical simulation when it is formulated in a truly stochastic environment (i.e., not only with discrete choice models but with birth-and-death master equations).

Secondly, we have taken into account the possible presence of a fleet of vehicles in the model. The technology of tomorrow suggests that fleets of vehicles will become more frequent in the real world (e.g., vehicles on demand that are centrally managed ...). These issues were rarely addressed in the literature. We have considered here a tractable way to combine individual vehicles and a fleet of vehicles which are centrally managed.

Such a model can raise questions that are mathematically difficult. As shown by Silva, Lindsey, de Palma and van den Berg (2017), a combination of small and large vehicles can lead to some non-equilibrium in the bottleneck problem. This situation cannot be easily avoided and is very unfortunate. The stochastic approach is more realistic and flexible, and it is expected to provide us with an equilibrium solution even in such a case.

The relation between stochastic and deterministic models is an important but difficult topic (see e.g. the book of Ethier and Kurtz (2005)). In our context, two relevant questions are the links between the local extrema of (stochastic) stationary distributions and the (deterministic) equilibrium states, and the existence of a solution in a stochastic model while there is no solution in the deterministic case.

At this point, we were mainly concerned with the modelling problems. Simulation of these processes and stationarity properties are topics that will be the object of further research.

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