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Kirillov’s orbit method: the case of discrete series representations

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Abstract

Let \( \pi \) be a discrete series representation of a real semi-simple Lie group \( G \) and let \( G' \) be a semi-simple subgroup of \( G' \). In this paper, we give a geometric expression of the \( G \)-multiplicities in \( \pi|_G \) when the representation \( \pi \) is \( G \)-admissible.

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1 Introduction

This paper is concerned by a central problem of non-commutative harmonic analysis: given a unitary irreducible representation $\pi$ of a Lie group $G'$, how does $\pi$ decompose when restricted to a closed subgroup $G \subset G'$? We analyse this problem for Harish-Chandra discrete series representations of a connected real semi-simple Lie group $G'$ with finite center, relatively to a connected real semi-simple subgroup $G$ (also with finite center).

We start with Harish-Chandra parametrization of the discrete series representations. We can attach an unitary irreducible representation $\pi_{G' G}$ of the group $G'$ to any regular admissible elliptic coadjoint orbit $O' \subset (g')^*$, and Schmid proved that the representation $\pi_{G' G}$ could be realize as the quantization of the orbit $O'$ [34, 35]. This is a vast generalization of Borel-Weil-Bott’s construction of finite dimensional representations of compact Lie groups. In the following, we denote $\tilde{G}_d$ and $\tilde{G}'_d$ the sets of regular admissible elliptic coadjoint orbits of our connected real semi-simple Lie groups $G$ and $G'$.

One of the rule of Kirillov’s orbit method [13] is concerned with the functoriality relatively to inclusion $G \hookrightarrow G'$ of closed subgroups. It means that, starting with discrete series representations representations $\pi_{G}$ and $\pi_{G'}$, attached to regular admissible elliptic orbits $O \subset g^*$ and $O' \subset (g')^*$, one expects that the multiplicity of $\pi_{G}$ in the restriction $\pi_{G'}|_G$ can be computed geometrically in terms of the space

$$O'/O := O' \cap p_{g,g'}^{-1}(O)/G,$$

where $p_{g,g'} : (g')^* \rightarrow g^*$ denotes the canonical projection. One recognises that (1.1) is a symplectic reduced space in the sense of Marsden-Weinstein, since $p_{g,g'} : O' \rightarrow g^*$ is the moment map relative to the Hamiltonian action of $G$ on $O'$.

In other words, Kirillov’s orbit method tells us that the branching laws $[\pi_{G} : \pi_{G'}]$ should be compute geometrically. So far, the following special cases have been achieved:

1. $G \subset G'$ are compact. In the 1980s, Guillemin and Sternberg [8] studied the geometric quantization of general $G$-equivariant compact Kähler
They proved the ground-breaking result that the multiplicities of this $G$-representation are calculated in terms of geometric quantizations of the symplectic reduced spaces. This phenomenon, which has been the center of many research and generalisations [22, 23, 37, 24, 21, 26, 33, 31, 10], is called nowadays “quantization commutes with reduction” (in short, “[Q,R]=0”).

2. $G$ is a compact subgroup of $G'$. In [25], we used the Blattner formula to see that the [Q,R]=0 phenomenon holds in this context when $G$ is a maximal compact subgroup. Duflo-Vergne have generalized this result for any compact subgroup [7]. Recently, Hochs-Song-Wu have shown that the [Q,R]=0 phenomenon holds for any tempered representation of $G'$ relatively to a maximal compact subgroup [11].

3. $\pi_{G'}^{G'}$ is an holomorphic discrete series. We prove that the [Q,R]=0 phenomenon holds with some assumption on $G$ [29].

However, one can observe that the restriction of $\pi_{G'}^{G'}$ with respect to $G$ may have a wild behavior in general, even if $G$ is a maximal reductive subgroup in $G'$ (see [15]).

In [15, 16, 17] T. Kobayashi singles out a nice class of branching problems where each $G$-irreducible summand of $\pi|_{G'}$ occurs discretely with finite multiplicity: the restriction $\pi|_{G}$ is called $G$-admissible.

So we focus our attention to a discrete series $\pi_{G'}^{G'}$ that admit an admissible restriction relatively to $G$. It is well-known that we have then an Hilbertian direct sum decomposition

$$\pi_{G'}^{G'}|_{G} = \sum_{O\in \mathcal{G}_{d}} m_{O}^{G'} \pi_{O}^{G'}$$

where the multiplicities $m_{O}^{G'}$ are finite.

We will use the following geometrical characterization of the $G$-admissibility obtained by Duflo and Vargas [5, 6].

**Proposition 1.1** The representation $\pi_{G'}^{G'}$ is $G$-admissible if and only if the restriction of the map $p_{G'}^{G}$ to the coadjoint orbit $O'$ is a proper map.

Let $(O', O) \in \hat{\mathcal{G}}_{d} \times \hat{\mathcal{G}}_{d}$. Let us explain how we can quantize the compact symplectic reduced space $O'/O$ when the map $p_{G'}^{G}: O' \to \mathfrak{g}^*$ is proper.

If $O$ belongs to the set of regular values of $p_{G'}^{G}: O' \to \mathfrak{g}^*$, then $O'/O$ is a compact symplectic orbifold equipped with a spin$^c$ structure. We denote $Q^{spin}(O'/O) \in \mathbb{Z}$ the index of the corresponding spin$^c$ Dirac operator.
In general, we consider an elliptic coadjoint orbit \( O_\epsilon \) closed enough to \( O \), so that \( O'/O_\epsilon \) is a compact symplectic orbifold equipped with a spin\(^c\) structure. Let \( Q^{\text{spin}}(O'/O_\epsilon) \in \mathbb{Z} \) be the index of the corresponding spin\(^c\) Dirac operator. The crucial fact is that the quantity \( Q^{\text{spin}}(O'/O_\epsilon) \) does not depend on the choice of generic and small enough \( \epsilon \). Then we take
\[
Q^{\text{spin}}(O'/O) := Q^{\text{spin}}(O'/O_\epsilon)
\]
for generic and small enough \( \epsilon \).

The main result of this article is the following

**Theorem 1.2** Let \( \pi_{G'}^{O'} \) be a discrete series representation of \( G' \) attached to a regular admissible elliptic coadjoint orbits \( O' \). If \( \pi_{G'}^{O'} \) is \( G \)-admissible we have the Hilbertian direct sum
\[
(1.2) \quad \pi_{G'}^{O'}|_G = \sum_{O \in G_d} Q^{\text{spin}}(O'/O) \pi_{G}^{O}.
\]
In other words the multiplicity \( [\pi_{G}^{O} : \pi_{G'}^{O'}] \) is equal to \( Q^{\text{spin}}(O'/O) \).

In a forthcoming paper we will study Equality (1.2) in further details when \( G \) is a symmetric subgroup of \( G' \).

Theorem 1.2 give a positive answer to a conjecture of Duflo-Vargas.

**Theorem 1.3** Let \( \pi_{G'}^{O'} \) be a discrete series representation of \( G' \) that is \( G \)-admissible. Then all the representations \( \pi_{G}^{O} \) which occurs in \( \pi_{G'}^{O'} \) belongs to a unique family of discrete series representations of \( G \).

## 2 Restriction of discrete series representations

Let \( G \) be a connected real semi-simple Lie group with finite center. A discrete series representation of \( G \) is an irreducible unitary representation that is isomorphic to a sub-representation of the left regular representation in \( L^2(G) \). We denote \( \hat{G}_d \) the set of isomorphism class of discrete series representation of \( G \).

We know after Harish-Chandra that \( \hat{G}_d \) is non-empty only if \( G \) has a compact Cartan subgroup. We denote \( K \subset G \) a maximal compact subgroup and we suppose that \( G \) admits a compact Cartan subgroup \( T \subset K \). The Lie algebras of the groups \( T, K, G \) are denoted respectively \( \mathfrak{t}, \mathfrak{k} \) and \( \mathfrak{g} \).

In this section we recall well-know facts concerning restriction of discrete series representations.

\(^1\)The precise meaning will be explain in Section 5.2.
2.1 Admissible coadjoint orbits

Here we recall the parametrization of $\hat{G}_d$ in terms of regular admissible elliptic coadjoint orbits. Let us fix some notations. We denote $\Lambda \subset t^*$ the weight lattice: any $\mu \in \Lambda$ defines a 1-dimensional representation $C_{\mu}$ of the torus $T$.

Let $R_c \subset R \subset \Lambda$ be respectively the set of (real) roots for the action of $T$ on $\mathfrak{k} \otimes \mathbb{C}$ and $\mathfrak{g} \otimes \mathbb{C}$. The non-compact roots are those belonging to the set $R_n := R \setminus R_c$. We choose a system of positive roots $R_c^+$ for $R_c$, we denote by $t^*_c$ the corresponding Weyl chamber. Recall that $\Lambda \times t^*_c$ is the set of dominant weights.

We denote by $B$ the Killing form on $\mathfrak{g}$. It induces a scalar product (denoted by $(\cdot, \cdot)$) on $t$, and then on $t^*$. An element $\lambda \in t^*$ is called $G$-regular if $(\lambda, \alpha) \neq 0$ for every $\alpha \in R$, or equivalently, if the stabilizer subgroup of $\lambda$ in $G$ is $T$. For any $\lambda \in t^*$ we denote

$$\rho(\lambda) := \frac{1}{2} \sum_{\alpha \in R, (\alpha, \lambda) > 0} \alpha.$$ 

We denote also $\rho_c := \frac{1}{2} \sum_{\alpha \in R_c^+} \alpha$.

**Definition 2.1**

1. A coadjoint orbit $O \subset \mathfrak{g}^*$ is elliptic if $O \cap t^* \neq \emptyset$.

2. An elliptic coadjoint orbit $O$ is admissible\(^2\) when $\lambda - \rho(\lambda) \in \Lambda$ for any $\lambda \in O \cap t^*$.

Harish-Chandra has parametrized $\hat{G}_d$ by the set of regular admissible elliptic coadjoint orbits of $G$. In order to simplify our notation, we denote $\hat{G}_d$ the set of regular admissible elliptic coadjoint orbits of $G$. For an orbit $O \in \hat{G}_d$ we denote $\pi_O^G$ the corresponding discrete series representation of $G$.

Consider the subset $(t^*_c)_se := \{ \xi \in t^*_c, (\xi, \alpha) \neq 0, \forall \alpha \in R_n \}$ of the Weyl chamber. The subscript means strongly elliptic, see Section 5.1. By definition any $O \in \hat{G}_d$ intersects $(t^*_c)_se$ in a unique point.

**Definition 2.2** The connected component $(t^*_c)_se$ are called chambers. If $C$ is a chamber, we denote $\hat{G}_d(C) \subset \hat{G}_d$ the subset of regular admissible elliptic orbits intersecting $C$.

\(^2\)Duflo has defined a notion of admissible coadjoint orbits in a much broader context [4].
Notice that the Harish-Chandra parametrization has still a meaning when $G = K$ is a compact connected Lie group. In this case $\tilde{K}$ corresponds to the set of regular admissible coadjoint orbits $\mathcal{P} \subset \mathfrak{k}^*$, i.e. those of the form $\mathcal{P} = K\mu$ where $\mu - \rho_c \in \Lambda \cap \mathfrak{t}_+^*$; the corresponding representation $\pi^K_{\mathcal{P}}$ is the irreducible representation of $\tilde{K}$ with highest weight $\mu - \rho_c$.

### 2.2 Spinor representation

Let $\mathfrak{p}$ be the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$: the Killing form of $\mathfrak{g}$ defines a $K$-invariant Euclidean structure on it. Note that $\mathfrak{p}$ is even dimensional since the groups $G$ and $K$ have the same rank.

We consider the two-fold cover $\text{Spin}(\mathfrak{p}) \to \text{SO}(\mathfrak{p})$ and the morphism $K \to \text{SO}(\mathfrak{p})$. We recall the following basic fact.

**Lemma 2.3** There exists a unique covering $\tilde{K} \to K$ such that

1. $\tilde{K}$ is a compact connected Lie group,
2. the morphism $K \to \text{SO}(\mathfrak{p})$ lifts to a morphism $\tilde{K} \to \text{Spin}(\mathfrak{p})$.

Let $\xi \in \mathfrak{t}^*$ be a regular element and consider

$$\rho_n(\xi) := \frac{1}{2} \sum_{\alpha \in \mathfrak{n}_+, (\alpha, \xi) > 0} \alpha.$$  

Note that

$$\tilde{\Lambda} = \Lambda \cup \{\rho_n(\xi) + \Lambda\}$$

is a lattice that does not depends on the choice of $\xi$.

Let $\mathcal{T} \subset K$ be a maximal torus and $\tilde{\mathcal{T}} \subset \tilde{K}$ be the pull-back of $\mathcal{T}$ relatively to the covering $\tilde{K} \to K$. We can now precise Lemma 2.3.

**Lemma 2.4** Two situations occur:

1. if $\rho_n(\xi) \in \Lambda$ then $\tilde{K} \to K$ and $\tilde{\mathcal{T}} \to \mathcal{T}$ are isomorphisms, and $\tilde{\Lambda} = \Lambda$.
2. if $\rho_n(\xi) \notin \Lambda$ then $\tilde{K} \to K$ and $\tilde{\mathcal{T}} \to \mathcal{T}$ are two-fold covers, and $\tilde{\Lambda}$ is the lattice of weights for $\tilde{\mathcal{T}}$.

Let $\mathcal{S}_p$ the spinor representation of the group $\text{Spin}(\mathfrak{p})$. Let $c : \text{Cl}(\mathfrak{p}) \to \text{End}_\mathbb{C}(\mathcal{S}_p)$ be the Clifford action. Let $o$ be an orientation on $\mathfrak{p}$. If $e_1, e_2, \cdots, e_{\dim \mathfrak{p}}$ is an oriented orthonormal base of $\mathfrak{p}$ we define the element

$$e_o := (i)^{\dim \mathfrak{p}/2}e_1e_2\cdots e_{\dim \mathfrak{p}} \in \text{Cl}(\mathfrak{p}) \otimes \mathbb{C}.$$
that depends only of the orientation. We have \( \epsilon_o^2 = -1 \) and \( \epsilon_o v = -v \epsilon \) for any \( v \in \mathfrak{p} \). The element \( c(\epsilon_o) \) determines a decomposition \( S_p = S_p^{+,o} \oplus S_p^{-,o} \) into irreducible representations \( S_p^{\pm,o} = \ker(c(\epsilon_o) \mp Id) \) of \( \text{Spin}(\mathfrak{p}) \). We denote \( S_p^o := S_p^{+,o} \oplus S_p^{-,o} \)

the corresponding virtual representation of \( \tilde{K} \).

**Remark 2.5** If \( o \) and \( o' \) are two orientations on \( \mathfrak{p} \), we have \( S_p^o = \pm S_p^{o'} \), where the sign \( \pm \) is the ratio between \( o \) and \( o' \).

**Example 2.6** Let \( \lambda \in \mathfrak{k} \) such that the map \( \text{ad}(\lambda) : \mathfrak{p} \to \mathfrak{p} \) is one to one. We get a symplectic form \( \Omega_\lambda \) on \( \mathfrak{p} \) defined by the relations \( \Omega_\lambda(X,Y) = \langle \lambda, [X,Y] \rangle \) for \( X,Y \in \mathfrak{p} \). We denote \( o(\lambda) \) be the orientation of \( \mathfrak{p} \) defined by the top form \( \Omega_\lambda^{\dim \mathfrak{p}/2} \).

### 2.3 Restriction to the maximal compact subgroup

We start with a definition.

**Definition 2.7** • We denote \( \hat{R}(G,d) \) the group formed by the formal (possibly infinite) sums

\[
\sum_{\mathcal{O} \in \mathcal{G}_d} a_{\mathcal{O}} \pi_{\mathcal{O}}^G
\]

where \( a_{\mathcal{O}} \in \mathbb{Z} \).

• Similarly we denote \( \hat{R}(K) \) the group formed by the formal (possibly infinite) sums \( \sum_{\mathcal{P} \in \mathcal{K}} a_{\mathcal{P}} \pi_{\mathcal{P}}^K \) where \( a_{\mathcal{P}} \in \mathbb{Z} \).

The following technical fact will be used in the proof of Theorem 1.2.

**Proposition 2.8** Let \( o \) be an orientation on \( \mathfrak{p} \).

• The restriction morphism \( V \to V|_K \) defines a map \( \hat{R}(G,d) \to \hat{R}(K) \).

• The map \( r^o : \hat{R}(G,d) \to \hat{R}(K) \) defined by \( r^o(V) := V|_K \otimes S_p^o \) is one to one.

**Proof.** When \( \mathcal{O} = G\lambda \in \mathcal{G}_d \), with \( \lambda \in \mathfrak{t}^* \), we denote \( c_{\mathcal{O}}^G = \|\lambda + \rho(\lambda)\| \).

Similarly when \( \mathcal{P} = K\mu \in \mathcal{K} \), with \( \mu - \rho_c \in \Lambda \cap \mathfrak{t}_+^* \), we denote \( c_{\mathcal{P}}^K = \|\mu + \rho_c\| \).

Note that for each \( r > 0 \) the set \( \{\mathcal{O} \in \mathcal{G}_d, c_{\mathcal{O}}^G \leq r\} \) is finite.
Consider now the restriction of a discrete series representation \( \pi_G^O \) relatively to \( K \). The Blattner’s formula [9] tells us that the restriction \( \pi_G^O |_K \) admits a decomposition

\[
\pi_G^O |_K = \sum_{\mathcal{P} \in \mathcal{K}} m_\mathcal{O}(\mathcal{P}) \pi^K_\mathcal{P}
\]

where the (finite) multiplicities \( m_\mathcal{O}(\mathcal{P}) \) are non-zero only if \( c^K_{\mathcal{P}} \geq c^O_{\mathcal{O}} \).

Consider now an element \( V = \sum_{\mathcal{O} \in \mathcal{G}_d} a_\mathcal{O} \pi^O_\mathcal{O} \in \hat{\mathcal{R}}(G, d) \). The multiplicity of \( \pi^K_\mathcal{P} \) in \( V|_K \) is equal to

\[
\sum_{\mathcal{O} \in \mathcal{G}_d} a_\mathcal{O} m_\mathcal{O}(\mathcal{P})
\]

Here the sum admits a finite number of non zero terms since \( m_\mathcal{O}(\mathcal{P}) = 0 \) if \( c^O_{\mathcal{O}} > c^K_{\mathcal{P}} \). So we have proved that the \( K \)-multiplicities of \( V|_K : = \sum_{\mathcal{O} \in \mathcal{G}_d} a_\mathcal{O} \pi^O_\mathcal{O} |_K \) are finite. The first point is proved.

The irreducible representation of \( \hat{K} \) are parametrized by the set \( \hat{\mathcal{K}} \) of regular \( \hat{K} \)-admissible coadjoint orbits \( \mathcal{P} \subset t^* \), i.e. those of the form \( \mathcal{P} = K\mu \) where \( \mu - \rho_c \in \Lambda \cap t^*_+ \). It contains the set \( \hat{K} \) of regular \( K \)-admissible coadjoint orbits. We define

\[
\hat{K}_{\text{out}} \subset \hat{K}
\]

as the set of coadjoint orbits \( \mathcal{P} = K\mu \) where\(^3\) \( \mu - \rho_c \in \{\rho_n(\xi) + \Lambda\} \cap t^*_+ \). Here \( \xi \) is any regular element of \( t^* \) and \( \rho_n(\xi) \) is defined by (2.3).

We notice that \( \hat{K}_{\text{out}} = \hat{K} \) when \( \hat{K} \cong K \) and that \( \hat{K} = \hat{K} \cup \hat{K}_{\text{out}} \) when \( \hat{K} \rightarrow K \) is a two-fold cover.

We will use the following basic facts.

**Lemma 2.9**

1. \( \mathcal{O} \mapsto \mathcal{O}_K : = \mathcal{O} \cap t^* \) defines an injective map between \( \mathcal{G}_d \) and \( \hat{K}_{\text{out}} \).

2. We have \( \pi^O_G |_K \otimes S^o_p = \pm \pi^\hat{K}_{\mathcal{O}_K} \) for all \( \mathcal{O} \in \mathcal{G}_d \).

**Proof.** Let \( \mathcal{O} : = G\lambda \in \mathcal{G}_d \) where \( \lambda \) is a regular element of the Weyl chamber \( t^*_+ \). Then \( \mathcal{O}_K = K\lambda \) and the term \( \lambda - \rho_c \) is equal to the sum \( \lambda - \rho(\lambda) + \rho_n(\lambda) \) where \( \lambda - \rho(\lambda) \in \Lambda \) and \( \rho_n(\lambda) \in \Lambda \) (see (2.4)), so \( \lambda - \rho_c \in \{\rho_n(\xi) + \Lambda\} \). The element \( \lambda \in t^*_+ \) is regular and admissible for \( \hat{K} \); this implies that \( \lambda - \rho_c \in t^*_+ \). We have proved that \( \mathcal{O}_K \in \hat{K}_{\text{out}} \).

\(^3\)The set \( \{\rho_n(\xi) + \Lambda\} \cap t^*_+ \) does not depend on the choice of \( \xi \).
The second point is a classical result (a generalisation is given in Theorem 5.7). Let us explain the sign \( \pm \) in the relation. Let \( O \in \hat{G}_d \) and \( \lambda \in O \cap \mathfrak{t}^* \). Then the sign \( \pm \) is the ratio between the orientations \( o \) and \( o(-\lambda) \) of the vector space \( \mathfrak{p} \) (see Example 2.6).

We can now finish the proof of the second point of Proposition 2.8. If \( V = \sum_{O \in \hat{G}_d} a_O \pi^G_O \in \hat{R}(G, d) \), then \( r^o(V) = \sum_{O \in \hat{G}_d} \pm a_O \pi^K_{\hat{O} \cap K} \). Hence \( r^o(V) = 0 \) only if \( V = 0 \). \( \square \)

2.4 Admissibility

Let \( \pi^G_{O'} \) be a discrete series representation of \( G \) attached to a regular admissible elliptic orbit \( O' \subset (\mathfrak{g}')^* \).

We denote \( \text{As}(O') \subset (\mathfrak{g}')^* \) the asymptotic support of the coadjoint orbit \( O' \): by definition \( \xi \in \text{As}(O') \) if \( \xi = \lim_{n \to \infty} t_n \xi_n \) with \( \xi_n \in O' \) and \( (t_n) \) is a sequence of positive number tending to 0.

We consider here a closed connected semi-simple Lie subgroup \( G \subset G' \). We choose maximal compact subgroups \( K \subset G \) and \( K' \subset G' \) such that \( K \subset K' \). We denote \( \mathfrak{k} \subset (\mathfrak{k}')^* \) the orthogonal (for the duality) of \( \mathfrak{k} \subset \mathfrak{k}' \).

The moment map relative to the \( G \)-action on \( O' \) is by definition the map \( \Phi_G : O' \to \mathfrak{g}^* \) which is the composition of the inclusion \( O' \to (\mathfrak{g}')^* \) with the projection \( (\mathfrak{g}')^* \to \mathfrak{g}^* \). We use also the moment map \( \Phi_K : O' \to \mathfrak{k}^* \) which the composition of \( \Phi_G \) with the projection \( \mathfrak{g}^* \to \mathfrak{k}^* \).

Let \( p\mathfrak{r}_{\mathfrak{g}', \mathfrak{g}} : (\mathfrak{g}')^* \to (\mathfrak{k}')^* \) be the canonical projection. The main objective of this section is the proof of the following result that refines Proposition 1.1.

**Theorem 2.10** The following facts are equivalent:

1. The representation \( \pi^G_{O'} \) is \( G \)-admissible.
2. The moment map \( \Phi_G : O' \to \mathfrak{g}^* \) is proper.
3. \( p\mathfrak{r}_{\mathfrak{g}', \mathfrak{g}}(\text{As}(O')) \cap \mathfrak{k}^\perp = \{0\} \).

Theorem 2.10 is a consequence of different equivalences. We start with the following result that is proved in [5, 29].

**Lemma 2.11** The map \( \Phi_G : O' \to \mathfrak{g}^* \) is proper if and only if the map \( \Phi_K : O' \to \mathfrak{k}^* \) is proper.

We have the same kind of equivalence for the admissibility.
Lemma 2.12 The representation \( \pi_{O'} \) is \( G \)-admissible if and only if it is \( K \)-admissible.

Proof. The fact that \( K \)-admissibility implies \( G \)-admissibility is proved by T. Kobayashi in [15]. The opposite implication is a consequence of the first point of Proposition 2.8.

At this stage, the proof of Theorem 2.10 is complete if we show that the following facts are equivalent:

(a) The representation \( \pi_{O'} \) is \( K \)-admissible.

(b) The moment map \( \Phi_K : O' \rightarrow \mathfrak{k}^* \) is proper.

(c) \( p_{r,g'} (\text{As}(O')) \cap \mathfrak{t}^\perp = \{0\} \).

We start by proving the equivalence \( (b) \iff (c) \).

Proposition 2.13 ([29]) The map \( \Phi_K : O' \rightarrow \mathfrak{k}^* \) is proper if and only

\[ p_{r,g'} (\text{As}(O')) \cap \mathfrak{t}^\perp = \{0\} \]

Proof. The moment map \( \Phi_K' : O' \rightarrow (\mathfrak{k}')^* \) relative to the action of \( K' \) on \( O' \) is a proper map that corresponds to the restriction of the projection \( p_{r,g'} \) to \( O' \).

Let \( T' \) be a maximal torus in \( K' \) and let \((\mathfrak{k}')^*_+ \subset (\mathfrak{k}')^* \) be a Weyl chamber. The convexity theorem [14, 20] tells us that \( \Delta_{K'}(O') = p_{r,g'}(O') \cap (\mathfrak{k}')^*_+ \) is a closed convex polyedral subset. We have proved in [29][Proposition 2.10], that \( \Phi_K : O' \rightarrow \mathfrak{k}^* \) is proper if and only

\[ K' \cdot \text{As}(\Delta_{K'}(O')) \cap \mathfrak{t}^\perp = \{0\} \]

A small computation shows that \( K' \cdot \text{As}(\Delta_{K'}(O')) = p_{r,g'} (\text{As}(O')) \) since \( K' \cdot \Delta_{K'}(O') = p_{r,g'} (O') \). The proof of Proposition 2.13 is completed. \( \square \)

We denote \( \text{AS}_{K'}(\pi_{O'}) \subset (\mathfrak{k}')^* \) the asymptotic support of the following subset of \((\mathfrak{k}')^*\):

\[ \{ p' \in \widehat{K}, [\pi_{K'}^{O'} : \pi_{O'}] \neq 0 \} \]

The following important fact is proved by T. Kobayashi (see Section 6.3 in [18]).

Proposition 2.14 The representation \( \pi_{O'} \) is \( K \)-admissible if and only if

\[ \text{AS}_{K'}(\pi_{O'}) \cap \mathfrak{t}^\perp = \{0\} \]
We will use also the following result proved by Barbasch and Vogan (see Propositions 3.5 and 3.6 in [2]).

**Proposition 2.15** Let $\pi_{G'}$ be a representation of the discrete series of $G'$ attached to the regular admissible elliptic orbit $O'$. We have

$$\text{AS}_{K'}(\pi_{G'}) = \text{p}_{\nu_{K'}}(\text{As}(O')).$$

Propositions 2.14 and 2.15 give the equivalence $\text{(a)} \iff \text{(c)}$. The proof of Theorem 2.10 is completed. □

In fact Barbasch and Vogan proved also in [2] that the set $\text{As}(O')$ does not depend on $O'$ but only on the chamber $C'$ such that $O' \in \tilde{G}'_d(C')$. We obtain the following corollary.

**Corollary 2.16** The $G$-admissibility of a discrete series representation $\pi_{G'}$ does not depend on $O'$ but only on the chamber $C'$ such that $O' \in \tilde{G}'_d(C')$.

## 3 Spin$^c$ quantization of compact Hamiltonian manifolds

### 3.1 Spin$^c$ structures

Let $N$ be an even dimensional Riemannian manifold, and let $\text{Cl}(N)$ be its Clifford algebra bundle. A complex vector bundle $E \to N$ is a $\text{Cl}(N)$-module if there is a bundle algebra morphism $c_E : \text{Cl}(N) \to \text{End}(E)$.

**Definition 3.1** Let $S \to M$ be a $\text{Cl}(N)$-module such that the map $c_S$ induces an isomorphism $\text{Cl}(N) \otimes_{\mathbb{R}} \mathbb{C} \to \text{End}(S)$. Then we say that $S$ is a spin$^c$-bundle for $N$.

**Definition 3.2** The determinant line bundle of a spin$^c$-bundle $S$ on $N$ is the line bundle $\det(S) \to M$ defined by the relation

$$\det(S) := \text{hom}_{\text{Cl}(N)}(\overline{S}, S)$$

where $\overline{S}$ is the $\text{Cl}(N)$-module with opposite complex structure.

Basic examples of spin$^c$-bundles are those coming from manifolds $N$ equipped with an almost complex structure $J$. We consider the tangent bundle $TN$ as a complex vector bundle and we define

$$S_J := \bigwedge^C TN.$$
It is not difficult to see that $S_J$ is a spin$^c$-bundle on $N$ with determinant line bundle $\det(S_J) = \bigwedge^\text{max}_C TN$. If $L$ is a complex line bundle on $N$, then $S_J \otimes L$ is another spin$^c$-bundle with determinant line bundle equal to $\bigwedge^\text{max}_C TN \otimes L^{\otimes 2}$.

3.2 Spin$^c$-prequantization

In this section $G$ is a semi-simple connected real Lie group.

Let $M$ be a Hamiltonian $G$-manifold with symplectic form $\Omega$ and moment map $\Phi_G : M \to \mathfrak{g}^*$ characterized by the relation

$$\iota(X_M)\omega = -d\langle \Phi_G, X \rangle, \quad X \in \mathfrak{g},$$

where $X_M(m) := \left. \frac{d}{dt} \right|_{t=0} e^{-tX} \cdot m$ is the vector field on $M$ generated by $X \in \mathfrak{g}$.

In the Kostant-Souriau framework [19, 36], a $G$-equivariant Hermitian line bundle $L_\Omega$ with an invariant Hermitian connection $\nabla$ is a prequantum line bundle over $(M, \Omega, \Phi_G)$ if

$$L(X) - \nabla_X M = i\langle \Phi_G, X \rangle \quad \text{and} \quad \nabla^2 = -i\Omega,$$

for every $X \in \mathfrak{g}$. Here $L(X)$ is the infinitesimal action of $X \in \mathfrak{k}$ on the sections of $L_\Omega \to M$. The data $(L_\Omega, \nabla)$ is also called a Kostant-Souriau line bundle.

**Definition 3.3 ([28])** A $G$-Hamiltonian manifold $(M, \Omega, \Phi_G)$ is spin$^c$ prequantized if there exists an equivariant spin$^c$ bundle $S$ such that its determinant line bundle $\det(S)$ is a prequantum line bundle over $(M, 2\Omega, 2\Phi_G)$.

Consider the case of a regular elliptic coadjoint orbit $O = G\lambda$: here $\lambda \in \mathfrak{t}^*$ has a stabilizer subgroup equal to $T$. The tangent space $T_\lambda O \simeq \mathfrak{g}/\mathfrak{t}$ is an even dimensional Euclidean space, equipped with a linear action of $T$ and an $T$-invariant antisymmetric endomorphism $^4 \text{ad}(\lambda)$. Let $J_\lambda := \text{ad}(\lambda)(-\text{ad}(\lambda)^2)^{-1/2}$ be the corresponding $T$-invariant complex structure on $\mathfrak{g}/\mathfrak{t}$: we denote $V$ the corresponding $T$-module. It defines an integrable $G$-invariant complex structure on $O \simeq G/T$.

As we have explained in the previous section, the complex structure on $O$ defines the spin$^c$-bundle $S_\circ := \bigwedge_C T\mathcal{O}$ with determinant line bundle

$$\det(S_\circ) = \bigwedge^\text{max}_C T\mathcal{O} \simeq G \times_T \bigwedge^\text{max}_C V.$$

$^4$Here we see $\lambda$ has an element of $\mathfrak{t}$, through the identification $\mathfrak{g}^* \simeq \mathfrak{g}$. 

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A small computation gives that the differential of the $T$-character $\Lambda_{\max}^C V$ is equal to $i$ times $2\rho(\lambda)$. In other words, $\Lambda_{\max}^C V = \mathbb{C}_{2\rho(\lambda)}$.

In the next Lemma we see that for the regular elliptic orbits, the notion of admissible orbits is equivalent to the notion of spin$^c$-prequantized orbits.

**Lemma 3.4** Let $O = G\lambda$ be a regular elliptic coadjoint orbit. Then $O$ is spin$^c$-prequantized if and only if $\lambda - \rho(\lambda) \in \Lambda$.

**Proof.** Any $G$-equivariant spin$^c$-bundle on $O$ is of the form $S_\phi = S_\phi \otimes L_\phi$ where $L_\phi = G \times_T \mathbb{C}_\phi$ is a line bundle associated to a character $e^X \mapsto e^{i\langle \phi, X \rangle}$ of the group $T$. Then we have

$$\det(S_\phi) = \det(S_\phi) \otimes L_\phi^\otimes = G \times_T \mathbb{C}_{2\phi + 2\rho(\lambda)}.$$ 

By $G$-invariance we know that the only Kostant-Souriau line bundle on $(G\lambda, 2\Omega_{G\lambda})$ is the line bundle $G \times_T \mathbb{C}_{2\lambda}$. Finally we see that $G\lambda$ is spin$^c$-prequantized by $S_\phi$ if and only if $\phi = \lambda - \rho(\lambda)$. □

If $O$ is a regular admissible elliptic coadjoint orbit, we denote $S_O := S_\phi \otimes L_{\lambda - \rho(\lambda)}$ the corresponding spin$^c$ bundle. Here we use the grading $S_O = S_O^+ \oplus S_O^-$ induced by the symplectic orientation.

### 3.3 Spin$^c$ quantization of compact manifolds

Let us consider a compact Hamiltonian $K$-manifold $(M, \Omega, \Phi_K)$ which is spin$^c$-prequantized by a spin$^c$-bundle $S$. The (symplectic) orientation induces a decomposition $S = S^+ \oplus S^-$, and the corresponding spin$^c$ Dirac operator is a first order elliptic operator [3]

$$D_S : \Gamma(M, S^+) \rightarrow \Gamma(M, S^-).$$

Its principal symbol is the bundle map $\sigma(M, S) \in \Gamma(T^*M, \text{hom}(p^*S^+, p^*S^-))$ defined by the relation

$$\sigma(M, S)(m, \nu) = c_{S|m}(\tilde{\nu}) : S^+_m \longrightarrow S^-_m.$$ 

Here $\nu \in T^*M \mapsto \tilde{\nu} \in TM$ is the identification defined by an invariant Riemannian structure.

**Definition 3.5** The spin$^c$ quantization of a compact Hamiltonian $K$-manifold $(M, \Omega, \Phi_K)$ is the equivariant index of the elliptic operator $D_S$ and is denoted $Q_{\text{spin}}^K(M) \in R(K)$. 

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3.4 Quantization commutes with reduction

Now we will explain how the multiplicities of $Q^K_{\text{spin}}(M) \in R(K)$ can be computed geometrically.

Recall that the dual $\hat{K}$ is parametrized by the regular admissible coadjoint orbits. They are those of the form $P = K\mu$ where $\mu - p_\rho \in \Lambda \cap t^*_\mathfrak{g}$. After Lemma 3.4, we know that any regular admissible coadjoint orbit $P \in \hat{K}$ is spin$^c$-prequantized by a spin$^c$ bundle $S_P$ and a small computation shows that $Q^K_{\text{spin}}(P) = \pi^K_P$ (see [32]).

For any $P \in \hat{K}$, we define the symplectic reduced space

$$M/P := \Phi^{-1}_K(P)/K.$$ 

If $M/P \neq \emptyset$, then any $m \in \Phi^{-1}_K(P)$ has abelian infinitesimal stabilizer. It implies then that the generic infinitesimal stabilizer for the $K$-action on $M$ is abelian. Let us explain how we can quantize these symplectic reduced spaces (for more details see [25, 28, 33]).

**Proposition 3.6** Suppose that the generic infinitesimal stabilizer for the $K$-action on $M$ is abelian.

- If $P \in \hat{K}$ belongs to the set of regular values of $\Phi_K : M \to \mathfrak{t}^*$, then $M/P$ is a compact symplectic orbifold which is spin$^c$-prequantized. We denote $Q^{\text{spin}}(M/P) \in \mathbb{Z}$ the index of the corresponding spin$^c$ Dirac operator [12].

- In general, if $P = K\lambda$ with $\lambda \in \mathfrak{t}^*$, we consider the orbits $P_\epsilon = K(\lambda + \epsilon)$ for generic small elements $\epsilon \in \mathfrak{t}^*$ so that $M/P_\epsilon$ is a compact symplectic orbifold with a peculiar spin$^c$-structure. Let $Q^{\text{spin}}(M/P_\epsilon) \in \mathbb{Z}$ be the index of the corresponding spin$^c$ Dirac operator. The crucial fact is that the quantity $Q^{\text{spin}}(M/P_\epsilon)$ does not depends on the choice of generic and small enough $\epsilon$.

Then we take

$$Q^{\text{spin}}(M/P) := Q^{\text{spin}}(M/P_\epsilon)$$

for generic and small enough $\epsilon$.

The following theorem is proved in [25].

**Theorem 3.7** Let $(M, \Omega, \Phi_K)$ be a spin$^c$-prequantized compact Hamiltonian $K$-manifold. Suppose that the generic infinitesimal stabilizer for the $K$-action on $M$ is abelian. Then the following relation holds in $R(K)$:

$$Q^K_{\text{spin}}(M) = \sum_{P \in \hat{K}} Q^{\text{spin}}(M/P) \pi^K_P.$$
Remark 3.8 Identity 3.7 admits generalisations when we do not have conditions on the generic stabilizer [28] and also when we allow the 2-form $\Omega$ to be degenerate [33]. In this article, we do not need such generalizations.

For $\mathcal{P} \in \hat{K}$, we denote $\mathcal{P}^-$ the coadjoint orbit with $\mathcal{P}$ with opposite symplectic structure. The corresponding spin$^c$ bundle is $\mathcal{S}_{\mathcal{P}^-}$. It is not difficult to see that $Q^K_{\text{spin}}(\mathcal{P}^-) = (\pi^K_{\mathcal{P}})^*$ (see [32]). The shifting trick tell us then that the multiplicity of $\pi^K_{\mathcal{P}} \circ \kappa$ in $Q^K_{\text{spin}}(M)$ is equal to $[Q^K_{\text{spin}}(M \times \mathcal{P}^-)]^K$. If we suppose furthermore that the generic infinitesimal stabilizer is abelian we obtain the useful relation

$$Q^{\text{spin}}(M/\mathcal{P}) := [Q^K_{\text{spin}}(M \times \mathcal{P}^-)]^K.$$

Let $\gamma$ that belongs to the center of $K$: it acts trivially on the orbits $\mathcal{P} \in \hat{K}$. Suppose now that $\gamma$ acts also trivially on the manifolds $M$. We are interested by the action of $\gamma$ on the fibers of the spin$^c$-bundle $\mathcal{S} \boxtimes \mathcal{S}_{\mathcal{P}^-}$. We denote $[\mathcal{S} \boxtimes \mathcal{S}_{\mathcal{P}^-}]^\gamma$ the subbundle where $\gamma$ acts trivially.

Lemma 3.9 If $[\mathcal{S} \boxtimes \mathcal{S}_{\mathcal{P}^-}]^\gamma = 0$ then $Q^{\text{spin}}(M/\mathcal{P}) = 0$.

Proof. Let $D$ be the Dirac operator on $M \times \mathcal{P}^-$ associated to the spin$^c$ bundle $\mathcal{S} \boxtimes \mathcal{S}_{\mathcal{P}^-}$. Then

$$[Q^K_{\text{spin}}(M \times \mathcal{P}^-)]^K = [\ker(D)]^K - [\text{coker}(D)]^K.$$

Obviously $[\ker(D)]^K \subset [\ker(D)]^\gamma$ and $[\ker(D)]^\gamma$ is contained in the set of smooth section of the bundle $[\mathcal{S} \boxtimes \mathcal{S}_{\mathcal{P}^-}]^\gamma$. The same result holds for $[\text{coker}(D)]^K$. Finally, if $[\mathcal{S} \boxtimes \mathcal{S}_{\mathcal{P}^-}]^\gamma = 0$, then $[\ker(D)]^K$ and $[\text{coker}(D)]^K$ are reduced to 0. □

4 Spin$^c$ quantization of non-compact Hamiltonian manifolds

In this section our Hamiltonian $K$-manifold $(M, \Omega, \Phi_K)$ is not necessarily compact, but the moment map $\Phi_K$ is supposed to be proper. We assume that $(M, \Omega, \Phi_K)$ is spin$^c$-prequantized by a spin$^c$-bundle $\mathcal{S}$.

In the next section, we will explain how to quantize the data $(M, \Omega, \Phi_K, \mathcal{S})$. 

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4.1 Formal geometric quantization: definition

We choose an invariant scalar product in $\mathfrak{k} \simeq \mathfrak{k}^\ast$ that provides an identification $\mathfrak{k} \cong \mathfrak{k}^\ast$.

**Definition 4.1** • The **Kirwan vector field** associated to $\Phi_K$ is defined by

\[
\kappa(m) = -\Phi_K(m) \cdot m, \quad m \in M.
\]

We denote by $Z_M$ the set of zeroes of $\kappa$. It is not difficult to see that $Z_M$ corresponds to the set of critical points of the function $\|\Phi_K\|^2 : M \to \mathbb{R}$.

The set $Z_M$, which is not necessarily smooth, admits the following description. Choose a Weyl chamber $\mathfrak{t}^\ast \subset \mathfrak{k}^\ast$ in the dual of the Lie algebra of a maximal torus $T$ of $K$. We see that

\[
Z_M = \bigcap_{\beta \in \mathcal{B}} Z_{\beta}
\]

where $Z_{\beta}$ corresponds to the compact set $K(M^\beta \cap \Phi_K^{-1}(\beta))$, and $\mathcal{B} = \Phi_K(Z_M) \cap \mathfrak{t}^\ast$. The properness of $\Phi_K$ insures that for any compact subset $C \subset \mathfrak{t}^\ast$ the intersection $\mathcal{B} \cap C$ is finite.

The principal symbol of the Dirac operator $D_S$ is the bundle map $\sigma(M, S) \in \Gamma(T^*M, \text{hom}(S^+, S^-))$ defined by the Clifford action

\[
\sigma(M, S)(m, \nu) = c_m(\tilde{\nu}) : S|_m^+ \to S|_m^-,
\]

where $\nu \in T^*M \simeq \tilde{\nu} \in T^*M$ is an identification associated to an invariant Riemannian metric on $M$.

**Definition 4.2** The symbol $\sigma(M, S, \Phi_K)$ shifted by the vector field $\kappa$ is the symbol on $M$ defined by

\[
\sigma(M, S, \Phi_K)(m, \nu) = \sigma(M, S)(m, \tilde{\nu} - \kappa(m))
\]

for any $(m, \nu) \in T^*M$.

For any $K$-invariant open subset $\mathcal{U} \subset M$ such that $\mathcal{U} \cap Z_M$ is compact in $M$, we see that the restriction $\sigma(M, S, \Phi_K)|_{\mathcal{U}}$ is a transversally elliptic symbol on $\mathcal{U}$, and so its equivariant index is a well defined element in $\overline{R}(K)$ (see [1, 31]).

Thus we can define the following localized equivariant indices.

**Definition 4.3** • A closed invariant subset $Z \subset Z_M$ is called a component of $Z_M$ if it is a union of connected components of $Z_M$. 

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• If \( Z \) is a compact component of \( \mathcal{Z}_M \), we denote by

\[ Q^{\text{spin}}_K(M, Z) \in \hat{R}(K) \]

the equivariant index of \( \sigma(M, S, \Phi_K)|_{\mathcal{U}} \) where \( \mathcal{U} \) is an invariant neighbourhood of \( Z \) so that \( \mathcal{U} \cap \mathcal{Z}_M = Z \).

By definition, \( Z = \emptyset \) is a component of \( \mathcal{Z}_M \) and \( Q^{\text{spin}}_K(M, \emptyset) = 0 \). For any \( \beta \in \mathcal{B}, Z_\beta \) is a compact component of \( \mathcal{Z}_M \).

When the manifold \( M \) is compact, the set \( \mathcal{B} \) is finite and we have the decomposition

\[ Q^{\text{spin}}_K(M) = \sum_{\beta \in \mathcal{B}} Q^{\text{spin}}_K(M, Z_\beta) \in \hat{R}(K). \]

See [24, 31]. When the manifold \( M \) is not compact, but the moment map \( \Phi_K \) is proper, we can define

\[ \hat{Q}^{\text{spin}}_K(M) := \sum_{\beta \in \mathcal{B}} Q^{\text{spin}}_K(M, Z_\beta) \in \hat{R}(K). \]

The sum of the right hand side is not necessarily finite but it converges in \( \hat{R}(K) \) (see [27, 21, 10]).

**Definition 4.4** We call \( \hat{Q}^{\text{spin}}_K(M) \in \hat{R}(K) \) the spin\(^c\) formal geometric quantization of the Hamiltonian manifold \((M, \Omega, \Phi_K)\).

We end up this section with the example of the coadjoint orbits that parametrize the discrete series representations. We have seen in Lemma 3.4 that any \( \mathcal{O} \in \hat{G}_d \) is spin\(^c\)-prequantized. Moreover, if we look at the \( K \)-action on \( \mathcal{O} \), we know also that the moment map \( \Phi_K : \mathcal{O} \to \mathfrak{k}^* \) is proper. The element \( \hat{Q}^{\text{spin}}_K(\mathcal{O}) \in \hat{R}(K) \) is then well-defined.

The following result can be understood as a geometric interpretation of the Blattner formula.

**Proposition 4.5 ([25])** For any \( \mathcal{O} \in \hat{G}_d \) we have the following equality in \( \hat{R}(K) \):

\[ \hat{Q}^{\text{spin}}_K(\mathcal{O}) = \pi^G_{\mathcal{O}|K}. \]
4.2 Formal geometric quantization: main properties

In this section, we recall two important functorial properties of the formal geometric quantization process $Q^{\text{spin}}$.

We start with the following result of Hochs and Song.

**Theorem 4.6 ([10])** Let $(M, \Omega, \Phi_K)$ be a $\text{spin}^c$ prequantized Hamiltonian $K$-manifold. Assume that the moment map $\Phi_K$ is proper and that the generic infinitesimal stabilizer for the $K$-action on $M$ is abelian. Then the following relation holds in $\hat{R}(K)$:

\begin{equation}
\hat{Q}^{\text{spin}}_K(M) = \sum_{P \in \hat{K}} Q^{\text{spin}}(M/\mathcal{P}) \pi^K_{\mathcal{P}}.
\end{equation}

**Remark 4.7** Identity (4.11) admits generalizations when we do not have conditions on the generic stabilizer and also when we allow the 2-form $\Omega$ to be degenerate (see [10]).

Like in the compact setting, consider an element $\gamma$ belonging to the center of $K$ that acts trivially on the manifold $M$. Let $\mathcal{P} \in \hat{K}$ and let $\mathcal{P}^-$ be the orbit $\mathcal{P}$ with opposite symplectic structure. We are interested by the action of $\gamma$ on the fibers of the $\text{spin}^c$-bundle $S \boxtimes S_{\mathcal{P}^-}$. We denote $[S \boxtimes S_{\mathcal{P}^-}]^\gamma$ the subbundle where $\gamma$ acts trivially.

Lemma 3.9 extends to the non-compact setting.

**Lemma 4.8** If $[S \boxtimes S_{\mathcal{P}^-}]^\gamma = 0$ then $Q^{\text{spin}}(M/\mathcal{P}) = 0$.

**Proof.** The multiplicative property proved by Hochs and Song [10] tells us that the shifting trick still holds in the non compact setting: the multiplicity of $\pi^K_{\mathcal{P}}$ in $\hat{Q}^{\text{spin}}_K(M)$ is equal to $[\hat{Q}^{\text{spin}}_K(M \times \mathcal{P}^-)]^K$. If we suppose furthermore that the generic infinitesimal stabilizer is abelian we obtain

\begin{equation}
Q^{\text{spin}}(M/\mathcal{P}) = \left[\hat{Q}^{\text{spin}}_K(M \times \mathcal{P}^-)\right]^K = \left[Q^{\text{spin}}_K(M \times \mathcal{P}^-, Z_0)\right]^K
\end{equation}

where $Z_0 \subset M \times \mathcal{P}^-$ is the compact set $\{(m, \xi) \in M \times \mathcal{P}^-, \Phi_K(m) = \xi\}$.

The quantity $Q^{\text{spin}}_K(M \times \mathcal{P}^-, Z_0) \in \hat{R}(K)$ is computed as an index of a $K$-transversally elliptic operator $D_0$ acting on the sections of $S \boxtimes S_{\mathcal{P}^-}$. The argument used in the compact setting still work (see Lemma 1.3 in [31]): if $[S \boxtimes S_{\mathcal{P}^-}]^\gamma = 0$ then $[\ker(D_0)]^K$ and $[\text{coker}(D_0)]^K$ are reduced to 0. \qed
Another important property of the formal geometric quantization procedure is the functoriality relatively to restriction to subgroup. Let $H \subset K$ be a closed connected subgroup. We denote $\Phi_H : M \to \mathfrak{h}^*$ the moment map relative to the $H$-action: it is equal to the composition of $\Phi_K$ with the projection $\mathfrak{t}^* \to \mathfrak{h}_H$.

**Theorem 4.9** ([30]) Let $(M, \Omega, \Phi_K)$ be a spin$^c$ prequantized Hamiltonian $K$-manifold. Assume that the moment map $\Phi_H$ is a proper. Then the element $\hat{Q}_{spin}^K(M) \in \hat{R}(K)$ is $H$-admissible and we have

$$\hat{Q}_{spin}^H(M)|_H = \hat{Q}_{spin}^H(M).$$

If we apply the previous Theorem to the spin$^c$-prequantized coadjoint orbits $\mathcal{O} \in \hat{G}_d$, we obtain the following extension of Proposition 4.5. This result was obtained by other means by Duflo-Vergne [7].

**Corollary 4.10** Let $\mathcal{O} \in \hat{G}_d$, and $H \subset K$ a closed connected subgroup such that $\Phi_H : \mathcal{O} \to \mathfrak{h}^*$ is proper. Then $\pi^{G}_{\mathcal{O}}$ is $H$-admissible and

$$\hat{Q}_{spin}^{G}_{H}(\mathcal{O}) = \pi^{G}_{\mathcal{O}|H}.$$

## 5 Spin$^c$ quantization of $G$-Hamiltonian manifolds

In this section $G$ denotes a connected semi-simple Lie group, and we consider a symplectic manifold $(M, \Omega)$ equipped with an Hamiltonian action of $G$: we denote $\Phi_G : M \to \mathfrak{g}^*$ the corresponding moment map.

### 5.1 Proper$^2$ Hamiltonian $G$-manifolds

In this section we suppose that:

1. the moment map $\Phi_G$ is proper,
2. the $G$-action on $M$ is proper.

For simplicity, we says that $(M, \Omega, \Phi_G)$ is a proper$^2$ Hamiltonian $G$-manifold.

Following Weinstein [38], we consider the $G$-invariant open subset

$$g_{se}^* = \{ \xi \in \mathfrak{g}^* \mid G_\xi \text{ is compact} \}$$

of strongly elliptic elements. It is non-empty if and only if the groups $G$ and $K$ have the same rank: real semi-simple Lie groups with this property are
the ones admitting discrete series. If we denote $t^*_se := g^*_se \cap t^*$, we see that $g^*se = G \cdot t^*_se$. In other words, any coadjoint orbit contained in $g^*_se$ is elliptic.

First we recall the geometric properties associated to proper Hamiltonian $G$-manifolds. We denote $K$ a maximal compact subgroup of $G$ and we denote $\Phi_K : G \rightarrow G\hat{\sim}$ the moment map relative to the $K$-action on $(M, \Omega)$.  

**Proposition 5.1 ([29])** Let $(M, \Omega, \Phi_G)$ be a proper Hamiltonian $G$-manifold. Then:

1. the map $\Phi_K$ is proper,
2. the set $g^*_se$ is non-empty,
3. the image of $\Phi_G$ is contained in $g^*_se$,
4. the set $N := \Phi^{-1}_G(t^*)$ is a smooth $K$-submanifold of $M$,
5. the restriction of $\Omega$ on $N$ defines a symplectic form $\Omega_N$,
6. the map $[g, n] \mapsto gn$ defines a diffeomorphism $G \times K N \simeq M$.

Let $T$ be a maximal torus in $K$, and let $t^*_+$ be a Weyl chamber. Since any coadjoint orbit in $g^*_se$ is elliptic, the coadjoint orbits belonging to the image of $\Phi_G : N \rightarrow g^*$ are parametrized by the set
\begin{equation}
\Delta_G(M) = \Phi_G(M) \cap t^*_+.
\end{equation}

We remark that $t^*_+ \cap g^*_se$ is equal to $(t^*_+)se := \{\xi \in t^*_+, (\xi, \alpha) \neq 0, \forall \alpha \in \mathfrak{r}_n\}$. The connected component $(t^*_+)se$ are called chambers and if $C$ is a chamber, we denote $\hat{G}_d(C)$ the set of regular admissible elliptic orbits intersecting $C$ (see Definition 2.2).

The following fact was first noticed by Weinstein [38].

**Proposition 5.2** $\Delta_G(M)$ is a convex polyhedral set contained in a unique chamber $C_M \subset (t^*_+)se$.

**Proof.** We denote $\Phi^N_K : N \rightarrow t^*$ the restriction of the map $\Phi_G$ on the sub-manifold $N$. It corresponds to the moment map relative to the $K$-action on $(N, \Omega_N)$: notice that $\Phi^N_K$ is a proper map.

The diffeomorphism $G \times K N \simeq M$ shows that the set $\Delta_G(M)$ is equal to $\Delta_K(N) := \text{Image}(\Phi^N_K) \cap t^*_+$, and the Convexity Theorem [14, 20] asserts that $\Delta_K(N)$ is a convex polyhedral subset of the Weyl chamber. Finally since $\Delta_K(N)$ is connected and contained in $(t^*_+)se$, it must belongs to a unique chamber $C_M$. $\square$
5.2 Spin\(^c\)-quantization of proper\(^2\) Hamiltonian \(G\)-manifolds

Now we assume that our proper\(^2\) Hamiltonian \(G\)-manifold \((M, \Omega, \Phi_G)\) is spin\(^c\)-prequantized by a \(G\)-equivariant spin\(^c\)-bundle \(S\).

Note that \(p\) is even dimensional since the groups \(G\) and \(K\) have the same rank. Recall that the morphism \(K \to \text{SO}(p)\) lifts to a morphism \(\tilde{K} \to \text{Spin}(p)\), where \(\tilde{K} \to K\) is either an isomorphism or a two-fold cover (see Section 2.2). We start with the

**Lemma 5.3**

- The \(G\)-equivariant spin\(^c\) bundle \(S\) on \(M\) induces a \(\tilde{K}\)-equivariant spin\(^c\) bundle \(S_N\) on \(N\) such that \(\det(S_N) = \det(S)|_N\).
- The \(\tilde{K}\)-Hamiltonian manifold \((N, \Omega_N, \Phi_K^N)\) is spin\(^c\)-prequantized by \(S_N\).

**Proof.** By definition we have \(TM|_N = p \oplus TN\). The manifolds \(M\) and \(N\) are oriented by their symplectic forms. The vector space \(p\) inherits an orientation \(o(p, N)\) satisfying the relation \(o(M) = o(p, N)o(N)\). The orientation \(o(p, N)\) can be computed also as follows: takes any \(\xi \in \text{Image}(\Phi_K^N)\), then \(o(p, N) = o(\xi)\) (see Example 2.6).

Let \(S_p\) be the spinor representation that we see as a \(\tilde{K}\)-module. The orientation \(o(p) := o(p, N)\) determines a decomposition \(S_p = S_p^{+, o(p)} \oplus S_p^{-, o(p)}\) and we denote
\[
S_p^{o(p)} := S_p^{+, o(p)} \ominus S_p^{-, o(p)} \in R(\tilde{K}).
\]

Let \(S_N\) be the unique spin\(^c\)-bundle, \(\tilde{K}\)-equivariant on \(N\) defined by the relation
\[
S|_N = S_p^{o(p)} \boxtimes S_N.
\]

Since \(\det(S_p^{o(p)})\) is trivial (as \(\tilde{K}\)-module), we have the relation \(\det(S_N) = \det(S)|_N\) that implies the second point. \(\square\)

For \(\mathcal{O} \in \hat{G}_d\), we consider the symplectic reduced space
\[
M/\mathcal{O} := \Phi^{-1}_G(\mathcal{O})/G.
\]

Notice that \(M/\mathcal{O} = \emptyset\) when \(\mathcal{O}\) does not belong to \(\hat{G}_d(\mathcal{C}_M)\). Moreover the diffeomorphism \(G \times_K N \simeq M\) shows that \(M/\mathcal{O}\) is equal to the reduced space
\[
N/\mathcal{O}_K := (\Phi_K^N)^{-1}(\mathcal{O}_K)/K.
\]
with $\mathcal{O}_K = \mathcal{O} \cap t^*$. Here $N/\mathcal{O}_K$ should be understood as the symplectic reduction of the $\tilde{K}$-manifold $N$ relative to the $\tilde{K}$-admissible coadjoint orbit $\mathcal{O}_K \in \tilde{K}$. Hence the quantization $Q^{\text{spin}}(N/\mathcal{O}_K) \in \mathbb{Z}$ of the reduced space $N/\mathcal{O}_K$ is well defined (see Proposition 3.6).

**Definition 5.4** For any $\mathcal{O} \in \hat{G}_d$, we take $Q^{\text{spin}}(M/\mathcal{O}) := Q^{\text{spin}}(N/\mathcal{O}_K)$.

The main tool to prove Theorem 1.2 is the comparison of the formal geometric quantization of three different geometric data: we work here in the setting where the $G$-action on $M$ has abelian infinitesimal stabilizers.

1. The formal geometric quantization of the $G$-action on $(M, \Omega, \Phi_G, \mathcal{S})$ is the element $\hat{Q}^{\text{spin}}_G(M) \in \hat{R}(G, d)$ defined by the relation
   \[ Q^{\text{spin}}_G(M) := \sum_{\mathcal{O} \in \hat{G}} Q^{\text{spin}}(M/\mathcal{O}) \pi^\mathcal{O}_G. \]

2. The formal geometric quantization of the $K$-action on $(M, \Omega, \Phi_K, \mathcal{S})$ is the element $\hat{Q}^{\text{spin}}_K(M) \in \hat{R}(K)$ (see Definition 4.4). As the $K$-action on $M$ has abelian infinitesimal stabilizers, we have the decomposition
   \[ \hat{Q}^{\text{spin}}_K(M) = \sum_{\mathcal{P} \in \hat{K}} Q^{\text{spin}}(M/\mathcal{P}) \pi^{K}_{\mathcal{P}}. \]

3. The formal geometric quantization of the $\tilde{K}$-action on $(N, \Omega_N, \Phi^{\text{N}}_{\tilde{K}}, \mathcal{S}_N)$ is the element $\hat{Q}^{\text{spin}}_{\tilde{K}}(N) \in \hat{R}(\tilde{K})$. As the $\tilde{K}$-action on $N$ has abelian infinitesimal stabilizers, we have the decomposition
   \[ \hat{Q}^{\text{spin}}_{\tilde{K}}(N) = \sum_{\bar{\mathcal{P}} \in \hat{\tilde{K}}} Q^{\text{spin}}(N/\bar{\mathcal{P}}) \pi^{\tilde{K}}_{\bar{\mathcal{P}}}. \]

In the next section we explain the link between these three elements.

### 5.3 Spin$^c$-quantization: main results

Let $\mathcal{C}_M \subset t^*_+$ be the chamber containing $\Phi_G(M) \cap t^*_+$.

**Definition 5.5** We define the orientation $o^+$ and $o^-$ on $\mathfrak{p}$ as follows. Take $\lambda \in \mathcal{C}_M$, then $o^+ := o(\lambda)$ and $o^- := o(-\lambda)$ (see Example 2.6).
We denote \( S^o_\mathbb{p}, S^o_\mathbb{p}' \) the virtual representations of \( \hat{K} \) associated to the spinor representation of Spin(\( \mathbb{p} \)) and the orientations \( o^+ \) and \( o^- \). We denote \( S^o_\mathbb{p} \) the \( \hat{K} \)-module with opposite complex structure. Remark that \( S^o_\mathbb{p} \cong S^o_\mathbb{p}' \).

Recall that the map \( V \rightarrow V|_K \) defines a morphism \( \hat{R}(G, d) \rightarrow \hat{R}(K) \). We have also the morphism \( r^o = \hat{R}(G, d) \rightarrow \hat{R}(\hat{K}) \) defined by \( r^o(V) = V|_K \otimes S^o_\mathbb{p} \).

We start with the following

**Theorem 5.6** If the \( G \)-action on \( M \) has abelian infinitesimal stabilizers then

\[
(5.15) \quad r^o \left( \hat{Q}^\text{spin}_G(M) \right) = \epsilon_M^o \hat{Q}^\text{spin}_K(N).
\]

Here \( \epsilon_M^o = \pm \) is equal to the ratio between \( o \) and \( o^- \).

**Proof.** If the \( G \)-action on \( M \) has abelian infinitesimal stabilizers, then the \( \hat{K} \)-action on \( N \) has also abelian infinitesimal stabilizers. It implies the following relation:

\[
\hat{Q}^\text{spin}_K(N) = \sum_{\hat{P} \in \hat{K}} Q^\text{spin}(N/\hat{P}) \pi^{\hat{K}}_{\hat{P}} \in \hat{R}(\hat{K}).
\]

Following the first point of Lemma 2.9, we consider the following subset

\[ \Gamma := \{ O_K := O \cap \mathfrak{t}^*, O \in \hat{G}_d \} \subset \hat{K}_{\text{out}} \subset \hat{K}. \]

Thanks to the second point of Lemma 2.9 we have

\[
r^o \left( \hat{Q}^\text{spin}_G(M) \right) = \sum_{O \in \hat{G}_d} Q^\text{spin}(M/O) \pi^{G}_{\hat{O}|_K} \otimes S^o_\mathbb{p}.
\]

\[
= \epsilon_M^o \sum_{O \in \hat{G}_d} Q^\text{spin}(N/O_K) \pi^{\hat{K}}_{\hat{O}_K}
\]

\[
= \epsilon_M^o \sum_{\hat{P} \in \Gamma} Q^\text{spin}(N/\hat{P}) \pi^{\hat{K}}_{\hat{P}}.
\]

Identity (5.15) is proved if we check that \( Q^\text{spin}(N/\hat{P}) = 0 \) for any \( \hat{P} \in \hat{K} \) which does not belong to \( \Gamma \).

Suppose first that \( \hat{K} \approx K \). In this case we have \( \hat{K} = \hat{K}_{\text{out}} = \hat{K} \) and a coadjoint orbit \( \hat{P} = K\mu \in \hat{K} \) does not belong to \( \Gamma \) if and only if \( \mu \) is not contained in \( \mathfrak{g}_{\text{sc}}^* \). But the image of \( \Phi_G \) is contained in \( \mathfrak{g}_{\text{sc}}^* \), so \( N/\hat{P} = \emptyset \) and then \( Q^\text{spin}(N/\hat{P}) = 0 \) if \( \hat{P} \notin \Gamma \).
Suppose now that $\tilde{K} \to K$ is a two-fold cover and let us denote by $\{ \pm 1_{\tilde{K}} \}$ the kernel of this morphism. Here $\gamma := -1_{\tilde{K}}$ acts trivially on $N$ and (5.14) shows that $\gamma$ acts by multiplication by $-1$ on the fibers of the spin$^c$ bundle $S_N$. The element $\gamma$ acts also trivially on the orbits $\tilde{P} \in \tilde{K}$:

- if $\tilde{P} \in \tilde{K}_{out}$, then $\gamma$ acts by multiplication by $-1$ on the fibers of the spin$^c$ bundle $S_{\tilde{P}}$,
- if $\tilde{P} \notin \tilde{K}_{out}$, then $\gamma$ acts trivially on the fibers of the spin$^c$ bundle $S_{\tilde{P}}$.

Our considerations show that $[S_N \boxtimes S_{\tilde{P}}]^{\gamma} = 0$ when $\tilde{P} \in \tilde{K}\backslash \tilde{K}_{out}$. Thanks to Lemma 4.8, it implies the vanishing of $Q_{\text{spin}}(N/\tilde{P})$ for any $\tilde{P} \in \tilde{K}\backslash \tilde{K}_{out}$.

Like in the previous case, when $\tilde{P} \in \tilde{K}_{out}\backslash \Gamma$, we have $Q_{\text{spin}}(N/\tilde{P}) = 0$ because $N/\tilde{P} = \emptyset$. □

We compare now the formal geometric quantizations of the $K$-manifolds $M$ and $N$.

**Theorem 5.7** We have the following relation

\[
Q_{\text{spin}}^\pi(K) \otimes S_p^o = Q_{\text{spin}}^\pi(N) \in R(\tilde{K}).
\]

When $M = O \in \tilde{G}_d$ the manifold $N$ is equal to $O_K := O \cap t^*$. We have $\hat{Q}_{\text{spin}}^\pi(K) = \pi^K_{O_K}$ and we know also that $\hat{Q}_{\text{spin}}^\pi(O) = \pi^G_O|_K$ (see Proposition 4.5). Here (5.16) becomes

\[
\pi^G_O|_K \otimes S_p^o = \pm \pi^K_{O_K}
\]

where the sign $\pm$ is the ratio between the orientations $o$ and $o^-$ of the vector space $p$.

If we use Theorems 5.6 and 5.7 we get the following

**Corollary 5.8** If the $G$-action on $M$ has abelian infinitesimal stabilizers, we have $r^o \left( \hat{Q}_{\text{spin}}^\pi(G) \right) = \hat{Q}_{\text{spin}}^\pi(K) \otimes S_p^o$.

The following conjecture says that the functorial property of $\hat{Q}_{\text{spin}}$ relative to restrictions (see Theorem 4.9) should also holds for non-compact groups.
**Conjecture 5.9** If the $G$-action on $M$ has abelian infinitesimal stabilizers then the following relation

$$\hat{Q}_G^{\text{spin}}(M)|_K = \hat{Q}_K^{\text{spin}}(M)$$

holds in $\hat{R}(K)$.

The remaining part of this section is devoted to the proof of Theorem 5.7.

We work with the manifold $M := G \times_K N$. We denote $\Phi^N_K : N \to \mathfrak{t}^*$ the restriction of $\Phi^N_G : M \to \mathfrak{g}^*$ to the submanifold $N$. We will use the $K$-equivariant isomorphism $p \times N \simeq M$ defined by $(X, n) \mapsto [e^X, n]$.

The maps $\Phi^N_G, \Phi^N_K$ are related through the relations $\Phi^N_G(X, n) = e^X \cdot \Phi^N_K(n)$ and $^5 \Phi_K(X, n) = p_{\mathfrak{t}_G}(e^X \cdot \Phi^N_K(n))$.

We consider the Kirwan vector fields on $N$ and $M$

$$\kappa_N(n) = -\Phi^N_K(n) \cdot n, \quad \kappa_M(m) = -\Phi_K(m) \cdot m.$$

The following result is proved in [29][Section 2.2].

**Lemma 5.10** An element $(X, n) \in p \times N$ belongs to $Z_M := \{\kappa_M = 0\}$ if and only if $X = 0$ and $n \in Z_N := \{\kappa_N = 0\}$.

Let us recall how are defined the characters $\hat{Q}_K^{\text{spin}}(M)$ and $\hat{Q}_K^{\text{spin}}(N)$. We start with the decomposition $Z_N = \bigsqcup_{\beta \in \mathcal{B}} Z_\beta$ where $Z_\beta = K(N^\beta \cap (\Phi^N_K)^{-1}(\beta))$, and $\mathcal{B} = \Phi^N_K(Z_N) \cap \mathfrak{t}^*_+$. Thanks to Lemma 5.10 the corresponding decomposition on $M$ is $Z_M := \bigsqcup_{\beta \in \mathcal{B}} \{0\} \times Z_\beta$.

By definition we have

$$\hat{Q}_K^{\text{spin}}(N) := \sum_{\beta \in \mathcal{B}} Q_K^{\text{spin}}(N, Z_\beta) \in \hat{R}(\tilde{K})$$

and $\hat{Q}_K^{\text{spin}}(M) = \hat{Q}_K^{\text{spin}}(p \times N) := \sum_{\beta \in \mathcal{B}} Q_K^{\text{spin}}(p \times N, \{0\} \times Z_\beta) \in \hat{R}(K)$. The proof of Theorem 5.7 is completed if we show that for any $\beta \in \mathcal{B}$ we have

$$Q_K^{\text{spin}}(p \times N, \{0\} \times Z_\beta) \otimes S^\omega_{\mathfrak{t}_G} = Q_K^{\text{spin}}(N, Z_\beta) \in R(\tilde{K}).$$

Let $\mathcal{S}$ be the $G$-equivariant spin$^c$-bundle on $M$. The $K$-equivariant diffeomorphism $M \simeq p \times N$ induces a $\tilde{K}$-equivariant isomorphism at the level of spin$^c$ bundles:

$$\mathcal{S} \simeq S^\omega_{\mathfrak{t}_G} \otimes S_N.$$

$^5p_{\mathfrak{t}_G} : \mathfrak{g}^* \to \mathfrak{t}^*$ is the canonical projection.
We denote $\text{cl}_p : p \to \text{End}(S_p)$ the Clifford action associated to the Clifford module $S_p$. Any $X \in p$ determines an odd linear map $\text{cl}_p(X) : S_p \to S_p$.

For $n \in N$, we denote $\text{cl}_n : T_nN \to \text{End}(S_N|_n)$ the Clifford action associated to the spin$^c$ bundle $S_N$. Any $v \in T_nN$ determines an odd linear map $\text{cl}_n(v) : S_N|_n \to S_N|_n$.

**Lemma 5.11** Let $U_\beta \subset N$ be a small invariant neighborhood of $Z_\beta$ such that $Z_N \cap \overline{U_\beta} = Z_\beta$.

- The character $Q_{Kp}^{\text{spin}}(N, Z_\beta)$ is equal to the index of the $K$-transversally elliptic symbol
  \[ \sigma_1^p(v) : S_N^+|_n \to S_N^-|_n, \quad v \in T_nU_\beta \]
  defined by $\sigma_1^p(v) = \text{cl}_n(v + \Phi^N_K(n) \cdot n)$.

- The character $Q_{Kp}^{\text{spin}}(p \times N, \{0\} \times Z_\beta)$ is equal to the index of the $K$-transversally elliptic symbol
  \[ \sigma_2^{(A,n)}(X, v) : (S_p^{\sigma^+} \otimes S_N|_n)^+ \to (S_p^{\sigma^+} \otimes S_N|_n)^- \]
  defined by $\sigma_2^{(A,n)}(X, v) = \text{cl}_p(X + [\Phi^N_K(n), A]) \otimes \text{cl}_n(v + \Phi^N_K(n) \cdot n)$ for $(X, v) \in T_{(A,n)}(p \times U_\beta)$.

**Proof.** The first point corresponds to the definition of the character $Q_{Kp}^{\text{spin}}(N, Z_\beta)$.

By definition, $Q_{Kp}^{\text{spin}}(p \times N, \{0\} \times Z_\beta)$ is equal to the index of the $K$-transversally elliptic symbol
\[ \tau_{(A,n)}(X, v) = \text{cl}_p(X + [\Phi_K(X, n), A]) \otimes \text{cl}_n(v + \Phi_K(X, n) \cdot n). \]

It is not difficult to see that
\[ \tau_{(A,n)}^t(X, v) = \text{cl}_p(X + [\Phi_K(tX, n), A]) \otimes \text{cl}_n(v + \Phi_K(tX, n) \cdot n), \quad 0 \leq t \leq 1, \]
defines an homotopy of transversally elliptic symbols between $\sigma^2 = \tau^0$ and $\tau = \tau^1$: like in Lemma 5.10, we use the fact that $[\Phi_K(0, A), A] = 0$ only if $A = 0$. It proves the second point. □

We can now finish the proof of (5.18). We use here the following isomorphism of Clifford modules for the vector space $p \times p$:
\[ S_p^{\sigma^+} \otimes \overline{S_p^{\sigma^+}} \simeq \bigwedge C p, \]
where the Clifford action \((X, Y) \in \mathfrak{p} \times \mathfrak{p}\) on the left is \(\text{cl}_\mathfrak{p}(X) \otimes \text{cl}_\mathfrak{p}(Y)\) and on the right is \(\text{cl}_{\mathfrak{p}C}(X + iY)\).

The product \(\sigma^2 \otimes S^0_\mathfrak{p} \) corresponds to the symbol

\[
\text{cl}_\mathfrak{p}(X + [\Phi_K(X, n), A]) \otimes \text{cl}_\mathfrak{p}(0) \otimes \text{cl}_n(v + \Phi_N^K(n) \cdot n)
\]

which is homotopic to

\[
\text{cl}_\mathfrak{p}(X + [\Phi_K(X, n), A]) \otimes \text{cl}_2(A) \otimes \text{cl}_n(v + \Phi_N^K(n) \cdot n),
\]

and is also homotopic to

\[
\sigma^3 := \text{cl}_\mathfrak{p}(X) \otimes \text{cl}_2(A) \otimes \text{cl}_n(v + \Phi_N^K(n) \cdot n).
\]

We have then proved that the \(K\)-equivariant index of \(\sigma^2 \) times \(S^0_\mathfrak{p}\) is equal to the \(\tilde{K}\)-equivariant index of \(\sigma^3\) (that we denote \(\text{Index}^{p \times U_\beta}_K(\sigma^3)\)). The multiplicative property of the equivariant index \([1]\) tells us that

\[
\text{Index}^{p \times U_\beta}_K(\sigma^3) = \text{Index}^{\mathfrak{p}C}_K(\text{cl}_{\mathfrak{p}C}(X + iA)) \cdot \text{Index}^{U_\beta}_K(\sigma^1).
\]

But \(\text{cl}_{\mathfrak{p}C}(X + iA) : \bigwedge^+_C \mathfrak{p}_C \to \bigwedge^_C \mathfrak{p}_C, (X, A) \in T\mathfrak{p},\) is the Bott symbol and its index is equal to the trivial 1-dimensional representation of \(\tilde{K}\). We have finally proved that the \(K\)-equivariant index of \(\sigma^2 \) times \(S^0_\mathfrak{p}\) is equal to the \(\tilde{K}\)-equivariant index of \(\sigma^1\). The proof of (5.18) is complete. \(\square\)

### 5.4 Proof of the main Theorem

Let \(G\) be a connected semi-simple subgroup of \(G'\) with finite center, and let \(\mathcal{O}' \in \mathcal{G}'_d\). We suppose that the representation \(\pi^{G'}_{\mathcal{O}'}\) is \(G\)-admissible. Then we have a decomposition

\[
\pi_{\mathcal{O}'}^{G'}|_G = \sum_{\mathcal{O} \in \mathcal{G}'_d} m_{\mathcal{O}} \pi^{\mathcal{G}}_{\mathcal{O}}.
\]

Let \(\Phi_G : \mathcal{O}' \to \mathfrak{g}^*\) be the moment map relative to the \(G\)-action on \(\mathcal{O}'\). We have proved in Theorem 2.10, that the \(G\)-admissibility of \(\pi_{\mathcal{O}'}^{G'}\) implies the properness of \(\Phi_G\). Moreover, since \(\mathcal{O}'\) is a regular orbit, the \(G\)-action on it is proper. Finally we see that \(\mathcal{O}'\) is a spin\(^c\) prequantized proper\(^2\) Hamiltonian \(G\)-manifold. We can consider its formal spin\(^c\) quantization \(\hat{Q}_G^{\text{spin}}(\mathcal{O}') \in \hat{R}(G, d)\), which is defined by the relation

\[
\hat{Q}_G^{\text{spin}}(\mathcal{O}') := \sum_{\mathcal{O} \in \mathcal{G}'_d} Q^{\text{spin}}(\mathcal{O}' / \mathcal{O}) \pi^{\mathcal{G}}_{\mathcal{O}}.
\]
Theorem 1.2 is proved if we show that $\pi_{G'}|G$ and $\hat{Q}^\text{spin}_G(O')$ are equal in $\hat{R}(G, d)$. Since the morphism $r^o : \hat{R}(G, d) \to \hat{R}(K)$ is one to one, it is sufficient to prove that

$$r^o \left( \pi_{G'}|G \right) = r^o \left( \hat{Q}^\text{spin}_G(O') \right).$$

(5.19)

On one hand, the element $r^o \left( \pi_{G'}|G \right)$ is equal to $\pi_{G'}|K \otimes S_p^o$. The restriction $\pi_{G'}|K \in \hat{R}(K)$, which is well defined since the moment map $\Phi_K : O' \to \mathfrak{t}^*_{\mathbb{R}}$ is proper, is equal to $\hat{Q}^\text{spin}_K(O')$ (see Corollary 4.10). So we get

$$r^o \left( \pi_{G'}|G \right) = \hat{Q}^\text{spin}_K(O') \otimes S_p^o.$$

On the other hand, Corollary 4.10 tells us that

$$r^o \left( \hat{Q}^\text{spin}_G(O') \right) = \hat{Q}^\text{spin}_K(O') \otimes S_p^o.$$

Hence we obtain Equality (5.19). The proof of Theorem 1.2 is completed.

References


