Local stability of energy estimates for the Navier–Stokes equations

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Abstract. We study the regularity of the weak limit of a sequence of dissipative solutions to the Navier–Stokes equations when no assumptions is made on the behavior of the pressures.

1. Local weak solutions.

In this paper, we are interested in local properties (regularity, local energy estimates) of weak solutions of Navier–Stokes equations.

Definition 1.1 (**Local weak solutions**). Let Ω be a domain in $\mathbb{R}\times\mathbb{R}^3$ and $\vec f\in L^2_{\mathrm{loc}}(\Omega)$ a divergence-free time-dependent vector field. A vector field $\vec u$ will be said to be a *local weak solution* of the Navier–Stokes equations on Ω (associated to the force $\vec f$) if, for each cylinder $Q=I\times O$ (where I is an open interval in $\mathbb{R}$ and O an open subset of $\mathbb{R}^3$) such that $\bar Q$ is a compact subset of Ω, we have $\vec u\in L^\infty_t L^2_x(Q)\cap L^2_t H^1_x(Q)$, $\vec u$ is divergence-free and, for every smooth compactly supported divergence-free vector field $\vec\phi\in\mathcal{D}(Q)$ we have

$$\iint_Q \vec u\cdot(\partial_t\vec\phi+\Delta\vec\phi)+\vec u\cdot(\vec u\cdot\vec\nabla\vec\phi)+\vec f\cdot\vec\phi\,dt\,dx=0. \tag{1}$$

More precisely, we shall address the behavior of a weak limit of regular solutions.

Definition 1.2 (**Regular local solutions**). Let Ω be a domain in $\mathbb{R}\times\mathbb{R}^3$ and $\vec f\in L^2_{\mathrm{loc}}(\Omega)$ a divergence-free time-dependent vector field and $\vec u$ a local weak solution of the Navier–Stokes equations on Ω (associated to the force $\vec f$).

A) $\vec u$ is a *regular local solution* if, for each cylinder $Q\subset\subset\Omega$, we have $\vec u\in L^\infty_{t,x}(Q)$,
B) The set $R(\vec u)$ of *regular points* of $\vec u$ is the largest open subset of Ω on which $\vec u$ is a regular solution. The set $\Sigma(\vec u)$ of *singular points* is the complement of $R(\vec u)$: $\Sigma(\vec u)=\Omega\setminus R(\vec u)$.

Our result is then the following one [**M**] :

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THEOREM 1.3 (**Singular points of a weak limit.**). *Let Ω be a domain in $\mathbb{R}\times\mathbb{R}^3$. Assume that we have sequences $\vec{f}_n$ of divergence-free time-dependent vector fields and $\vec{u}_n$ of local weak solutions of the Navier–Stokes equations on Ω (associated to the forces $\vec{f}_n$) such that, for each cylinder $Q \subset\subset \Omega$, we have*

- *$\vec{f}_n \in L^2_t H^1_x(Q)$ and $\vec{f}_n$ converges weakly in $L^2_t H^1_x$ to a limit $\vec{f}$,*
- *the sequence $\vec{u}_n$ is bounded in $L^\infty_t L^2_x(Q) \cap L^2_t H^1_x(Q)$ and converges weakly in $L^2_t H^1_x(Q)$ to a limit $\vec{u}$,*
- *for every n, $\vec{u}_n$ is bounded on Q.*

Then the limit $\vec{u}$ is a local weak solution on Ω of the Navier–Stokes equations associated to the force $\vec{f}$, and its set $\Sigma(\vec{u})$ has parabolic one-dimensional Hausdorff measure equal to 0.

As we shall see, the main tool of the proof is an extension of the Caffarelli–Kohn–Nirenberg theory [**Ca**] to the case where we have no control on the pressure (i.e. the case of generalized suitable solutions [**W**] or dissipative solutions [**Ch**]).

2. Pressure.

Equations (1) can classically be rewritten as an equation involving a pressure term. See for instance [**W**]. In the following, we shall only need the pressure inside spherical cylinders $Q = I \times B$ (where I is an open interval in $\mathbb{R}$ and B an open ball of $\mathbb{R}^3$). In that case, it is very easy to define a pressure p such that

$$\partial_t \vec{u} = \Delta \vec{u} - \vec{u}\cdot\vec{\nabla}\vec{u} - \vec{\nabla} p + \vec{f} \quad \text{in } \mathcal{D}'(Q). \tag{2}$$

Indeed, let Q, $Q^{\#}$, and Q^* be three relatively compact cylinders in Ω with $\overline{Q} \subset Q^{\#}$ and $\overline{Q^{\#}} \subset Q^*$ and ψ a cut off smooth function supported in Q^* and equal to 1 on a neighboorhood of $\overline{Q^{\#}}$. The function

$$p_0 = -\frac{1}{\Delta}\left(\sum_{i=1}^{3}\sum_{j=1}^{3} \partial_i\partial_j(\psi u_i u_j)\right)$$

belongs to $L^2_t L^{3/2}_x$ and, on $Q^{\#}$, the distribution

$$\vec{T} = \partial_t \vec{u} - \Delta\vec{u} + \vec{u}\cdot\vec{\nabla}\vec{u} + \vec{\nabla} p_0 - \vec{f}$$

satisfies

$$\text{curl } \vec{T} = 0 \text{ and div } \vec{T} = 0.$$

Moreover, $\vec{T}_0 = \vec{T} - \partial_t\vec{u}$ belongs to $L^2_t H^{-2}_x(Q^{\#})$. Picking $t_0 \in I$, we define $\vec{S} = \vec{u} + \int_{t_0}^{t} \vec{T}_0(s,.)\, ds$. We have $\vec{S} \in L^\infty_t H^{-2}_x(Q^{\#})$. Moreover, we have ∂_t curl $\vec{S} = 0$ and ∂_t div $\vec{S} = 0$. Thus, if $\alpha \in \mathcal{D}(I)$ with $\int \alpha\, dt = 1$, we find that

$$\vec{S}_0 = \vec{S} - \int_I \alpha(s)\vec{S}(s,.)\, ds$$

satisfies

$$\partial_t \vec{S}_0 = \vec{T}, \quad \text{curl } \vec{S}_0 = 0 \text{ and } \text{ div } \vec{S}_0 = 0.$$

In particular,

$$\Delta\vec{S}_0 = \vec{\nabla}(\text{ div } \vec{S}_0) - \vec{\nabla}\wedge(\text{ curl } \vec{S}_0) = 0.$$

Thus, we get that $\vec{S}_0$ is smooth in the space variable; in particular $\vec{S}_0 \in L^\infty_t W^{1,\infty}_x(Q)$. If $x_0 \in B$ and if we define

$$\varpi(t,x) = \int_0^1 \vec{S}_0(t,(1-\theta)x_0+\theta x)\cdot(x-x_0)\,d\theta,$$

we find that $\varpi \in L^\infty_{t,x}(Q)$ and $\vec{\nabla}\varpi = \vec{S}_0$. Defining $p = p_0 - \partial_t \varpi$, we find the equality (2).

Of course, the pressure may be singular in time (as $\partial_t\varpi$ is only the derivative of a bounded function). We shall comment further on this in Sections 3 and 5.

3. Energy balance.

This section is devoted to the study of $\partial_t|\vec{u}|^2$, as it is the main tool to estimate the partial regularity of $\vec{u}$. If $\vec{u}$ and the pressure p were regular, we could write from equality (2)

$$\partial_t|\vec{u}|^2 = 2\vec{u}\cdot\partial_t\vec{u} = 2\vec{u}\cdot\Delta\vec{u} - 2\vec{u}\cdot(\vec{u}\cdot\vec{\nabla}\vec{u}+\vec{\nabla}p) + 2\vec{u}\cdot\vec{f}$$

and rewrite

$$2\vec{u}\cdot\Delta\vec{u} = \Delta(|\vec{u}|^2) - 2|\vec{\nabla}\otimes\vec{u}|^2$$

and, since $\operatorname{div}\vec{u}=0$,

$$2\vec{u}\cdot(\vec{u}\cdot\vec{\nabla}\vec{u}+\vec{\nabla}p) = \operatorname{div}\,((|\vec{u}|^2+2p)\vec{u}).$$

This would give the following local energy balance in Q

$$\partial_t|\vec{u}|^2 = \Delta(|\vec{u}|^2) - 2|\vec{\nabla}\otimes\vec{u}|^2 - \operatorname{div}\,((|\vec{u}|^2+2p)\vec{u}) + 2\vec{u}\cdot\vec{f}. \tag{3}$$

However, local weak solutions (and their associates pressures) are not regular enough to allow those computations : the problem lies in the fact that the terms $\vec{u}\cdot(\vec{u}\cdot\nabla\vec{u})$ and $\vec{u}\cdot\vec{\nabla}p$ are not well defined in $\mathcal{D}'$. If the pressure is regular enough (for instance, $p\in L^{3/2}_{t,x}(Q)$) then one first smoothens $\vec{u}$ with a mollifier $\varphi_\epsilon = \frac{1}{\epsilon^3}\varphi(\frac{x}{\epsilon})$, defining $\vec{u}_\epsilon = \varphi_\epsilon * \vec{u}$. One then finds

$$\partial_t|\vec{u}_\epsilon|^2 = \Delta(|\vec{u}_\epsilon|^2) - 2|\vec{\nabla}\otimes\vec{u}_\epsilon|^2 - 2\vec{u}_\epsilon\cdot\varphi_\epsilon*(\vec{u}.\vec{\nabla}\vec{u}) - 2\operatorname{div}\,((p*\varphi_\epsilon)\vec{u}_\epsilon) + 2\vec{u}_\epsilon\cdot(\varphi_\epsilon*\vec{f}).$$

The limit $\epsilon\to 0$ gives then

$$\partial_t|\vec{u}|^2 = \Delta(|\vec{u}|^2) - 2|\vec{\nabla}\otimes\vec{u}|^2 - 2\lim_{\epsilon\to 0}\vec{u}_\epsilon\cdot\varphi_\epsilon*(\vec{u}\cdot\vec{\nabla}\vec{u}) - 2\operatorname{div}\,(p\vec{u}) + 2\vec{u}\cdot\vec{f}.$$

In order to compare this expression with (3), we define

$$M_\epsilon(\vec{u}) = -\operatorname{div}\,(|\vec{u}|^2\vec{u}) + 2\vec{u}_\epsilon\cdot\varphi_\epsilon*(\vec{u}\cdot\vec{\nabla}\vec{u})$$

and write

$$\partial_t|\vec{u}|^2 = \Delta(|\vec{u}|^2) - 2|\vec{\nabla}\otimes\vec{u}|^2 - \operatorname{div}\,((|\vec{u}|^2+2p)\vec{u}) + 2\vec{u}\cdot\vec{f} - \lim_{\epsilon\to 0}M_\epsilon(\vec{u}).$$

However, our assumptions on weak solutions don't allow us to make all those computations, as the pressure we can define on Q has no regularity with respect to the time variable, so that $p\vec{u}$ is not well defined in $\mathcal{D}'$. Thus, one must smoothens as well $\vec{u}$ with respect to the time variable, with a mollifier $\psi_\eta(t) = \frac{1}{\eta}\psi(\frac{t}{\eta})$. Defining $\vec{u}_{\epsilon,\eta} = \psi_\eta *_t \varphi_\epsilon *_x \vec{u} = \xi_{\eta,\epsilon} *_{t,x}\vec{u}$, one finds

$$\begin{aligned}\partial_t|\vec{u}_{\epsilon,\eta}|^2 \;&=\; \Delta(|\vec{u}_{\epsilon,\eta}|^2) - 2|\vec{\nabla}\otimes\vec{u}_{\epsilon,\eta}|^2 - 2\vec{u}_{\epsilon,\eta}\cdot\xi_{\eta,\epsilon}*(\vec{u}\cdot\vec{\nabla}\vec{u})\\ &\quad -2\operatorname{div}\,((p*\xi_{\eta,\epsilon})\vec{u}_{\epsilon,\eta}) + 2\vec{u}_{\epsilon,\eta}\cdot(\xi_{\eta,\epsilon}*\vec{f}).\end{aligned}$$

The limit $\eta \to 0$ gives then

$$\partial_t|\vec{u}_\epsilon|^2 = \Delta(|\vec{u}_\epsilon|^2) - 2|\vec{\nabla}\otimes\vec{u}_\epsilon|^2 - 2\vec{u}_\epsilon\cdot\varphi_\epsilon*(\vec{u}\cdot\vec{\nabla}\vec{u}) - 2\lim_{\eta\to 0} \text{ div } ((p*\xi_{\eta,\epsilon})\vec{u}_{\epsilon,\eta}) + 2\vec{u}_\epsilon\cdot(\varphi_\epsilon*\vec{f}).$$

The limit $\epsilon \to 0$ gives finally

$$\begin{aligned}\partial_t|\vec{u}|^2 &= \Delta(|\vec{u}|^2) - 2|\vec{\nabla}\otimes\vec{u}|^2 \\ &\quad - 2\lim_{\epsilon\to 0}\left(\vec{u}_\epsilon\cdot\varphi_\epsilon*(\vec{u}\cdot\vec{\nabla}\vec{u}) + \lim_{\eta\to 0} \text{ div } ((p*\xi_{\eta,\epsilon})\vec{u}_{\epsilon,\eta})\right) + 2\vec{u}\cdot\vec{f}.\end{aligned} \tag{4}$$

In order to circumvene the problems of lack of regularity for the pressure, we introduce the notion of harmonic correction :

Definition 3.1 (**Harmonic corrections**). Let Ω be a domain in $\mathbb{R}\times\mathbb{R}^3$, $\vec{f} \in L^2_{\text{loc}}(\Omega)$ a divergence-free time-dependent vector field and $\vec{u}$ a local weak solution of the Navier–Stokes equations on Ω (associated to the force $\vec{f}$). A *harmonic correction* $\vec{H}$ on a cylinder $Q \subset\subset \Omega$ is a vector field such that

- div $\vec{H} = 0$ and $\Delta\vec{H} = 0$,
- $\vec{H} \in L^\infty_{t,x}(Q)$ and $\partial_i\vec{H} \in L^\infty_{t,x}(Q)$ for $i = 1, 2, 3$,
- there exists $\vec{F} \in L^2_{t,x}(Q)$ and $P \in L^{3/2}_{t,x}(Q)$ such that the vector field $\vec{U} = \vec{u} + \vec{H}$ satisfies
$$\partial_t\vec{U} = \Delta\vec{U} - \vec{U}\cdot\vec{\nabla}\vec{U} - \vec{\nabla}P + \vec{F}.$$

In the literature, one can find at least two such harmonic corrections for local weak solutions :

Lemma 3.2. *Let Ω be a domain in $\mathbb{R}\times\mathbb{R}^3$, $\vec{f} \in L^2_{\text{loc}}(\Omega)$ a divergence-free time-dependent vector field and $\vec{u}$ a local weak solution of the Navier–Stokes equations on Ω (associated to the force $\vec{f}$). Let Q be a spherical cylinder in Ω. Then:*

A) *the decomposition of the pressure p as $p = p_0 - \partial_t\varpi$ described in Section 1 provides a harmonic correction $\vec{H} = -\vec{\nabla}\varpi$ of $\vec{u}$ on Q,*

B) *Let $\psi(t,x) = \alpha(t)\beta(x)$ be a smooth cut-off function supported by a cylinder $Q^* \subset\subset \Omega$ and equal to 1 on a neighborhood of Q. Then $\vec{U} = -\frac{1}{\Delta}\vec{\nabla}\wedge(\psi\vec{\nabla}\vec{u})$ is such that $\vec{H} = \vec{U} - \vec{u}$ is a harmonic correction of $\vec{u}$ on Q.*

Proof. The case of $\vec{H} = -\vec{\nabla}\varpi$ has been discussed by Wolf [**W**]. For $\vec{U} = \vec{u} - \vec{\nabla}\varpi$, we have $\vec{\nabla}\wedge\vec{U} = \vec{\nabla}\wedge\vec{u}$ and $\Delta\vec{U} = \Delta\vec{u}$, so that

$$\begin{aligned}\partial_t\vec{U} - \Delta\vec{U} + \vec{U}\cdot\vec{\nabla}\vec{U} &= \partial_t\vec{u} - \partial_t\vec{\nabla}\varpi - \Delta\vec{u} + (\vec{\nabla}\wedge\vec{u})\wedge(\vec{u} - \vec{\nabla}\varpi) + \vec{\nabla}(\frac{|\vec{U}|^2}{2}) \\ &= \vec{\nabla}(\frac{|\vec{U}|^2}{2} - \frac{|\vec{u}|^2}{2} - p_0) + \vec{f} - (\vec{\nabla}\wedge\vec{u})\wedge\vec{\nabla}\varpi\end{aligned}$$

We may then decompose $(\vec{\nabla}\wedge\vec{u})\wedge\vec{\nabla}\varpi \in L^2_tL^2_x(Q)$ into $\vec{f_1} + \vec{\nabla}p_1$ with $\vec{f_1} \in L^2_tL^2_x$ and div $\vec{f_1} = 0$ and $p_1 \in L^2_tL^6_x(Q)$ (for instance, by extending $(\vec{\nabla}\wedge\vec{u})\wedge\vec{\nabla}\varpi$ by 0 outside Q and then using the Leray projection operator). We thus find

$$P = p_0 + \frac{|\vec{u}|^2}{2} - \frac{|\vec{U}|^2}{2} + p_1 \text{ and } \vec{F} = \vec{f} - \vec{f_1}.$$

The case of $\vec{U} = -\frac{1}{\Delta}\vec{\nabla} \wedge (\psi \vec{\nabla} \vec{u})$ has been discussed by Chamorro, Lemarié-Rieusset and Mayoufi in [**Ch, Le**]. It is worth noticing that the pressure P they obtain belongs to $L^2_t L^q_x(Q)$ for every $q < 3/2$.

Note that, in both cases, even if $\vec{f}$ is assumed to be more regular, we cannot get a better regularity for $\vec{F}$ than $L^2_t L^2_x$, because of the contribution of $(\vec{\nabla} \wedge \vec{u}) \wedge \vec{H}$ to the force. □

An important result of Chamorro, Lemarié-Rieusset and Mayoufi is the following one [**Ch, Le**] :

THEOREM 3.3 (**Energy balance**). *Let Ω be a domain in $\mathbb{R} \times \mathbb{R}^3$, $\vec{f} \in L^2_{\rm loc}(\Omega)$ a divergence-free time-dependent vector field and $\vec{u}$ a local weak solution of the Navier–Stokes equations on Ω (associated to the force $\vec{f}$). Let Q be a spherical cylinder in Ω and p the pressure associated to $\vec{u}$ on Q. Then:*

A) *The quantities*

$$M(\vec{u}) = \lim_{\epsilon \to 0} \Big(- \text{ div } (|\vec{u}|^2 \vec{u}) + 2\vec{u}_\epsilon \cdot \varphi_\epsilon * (\vec{u} \cdot \vec{\nabla} \vec{u}) \Big)$$

and

$$<< \text{div } (p\vec{u}) >> = \lim_{\epsilon \to 0} \lim_{\eta \to 0} \text{div } ((p * \xi_{\eta,\epsilon}) \vec{u}_{\epsilon,\eta})$$

are well defined in $\mathcal{D}'(Q)$.

B) *We have the energy balance on Q :*

$$\partial_t |\vec{u}|^2 = \Delta(|\vec{u}|^2) - 2|\vec{\nabla} \otimes \vec{u}|^2 - \text{ div } (|\vec{u}|^2 \vec{u}) - 2 << \text{div } (p\vec{u}) >> + 2\vec{u} \cdot \vec{f} - M(\vec{u}).$$

C) *$M(\vec{u})$ can be computed as a defect of regularity. More precisely, we have, for*

$$A_{k,\epsilon}(\vec{u}) = \frac{(u_k(t, x-y) - u_k(t,x))(\vec{u}(t, x-y) - \vec{u}(t,x)) \cdot \int \varphi_\epsilon(z)(\vec{u}(t, x-z) - \vec{u}(t,x))\, dz}{\epsilon}$$

and

$$B_{k,\epsilon}(\vec{u}) = \frac{(u_k(t, x-y) - u_k(t,x))|\vec{u}(t, x-y) - \vec{u}(t,x)|^2}{\epsilon},$$

the identity

$$M_\epsilon(\vec{u}) = \sum_{k=1}^{3} \int \frac{1}{\epsilon^3} \partial_k \varphi(\frac{y}{\epsilon})(2A_{k,\epsilon}(\vec{u}) - B_{k,\epsilon}(\vec{u}))\, dy - C_\epsilon(\vec{u}) \tag{5}$$

where $\lim_{\epsilon \to 0} C_\epsilon(\vec{u}) = 0$ *in* $\mathcal{D}'(Q)$.

D) *If $\vec{U} = \vec{u} + \vec{H}$ where $\vec{H}$ is a harmonic correction of $\vec{u}$, then $M(\vec{U}) = M(\vec{u})$.*

PROOF. The key tool is identity (5) which has been described by Duchon and Robert [**D**] for any divergence-free vector field $\vec{u}$ in $L^\infty_t L^2_x(Q) \cap L^2_t H^1_x(Q)$. Let us remark that if w_1 and w_2 belong to $L^\infty_t L^2_x(Q) \cap L^2_t H^1_x(Q)$ and w_3 to $L^\infty_t \text{Lip}_x(Q)$ then we have obviously

$$\lim_{\epsilon \to 0} \int \frac{1}{\epsilon^3} \partial_k \varphi(\frac{y}{\epsilon}) \frac{(w_1(x-y) - w_1(x))(w_2(x-y) - w_2(x))(w_3(x-y) - w_3(x))}{\epsilon}\, dy = 0.$$

Thus, if $\vec{H}$ is a harmonic correction of $\vec{u}$, we have $\lim_{\epsilon \to 0} M_\epsilon(\vec{u} + \vec{H}) - M_\epsilon(\vec{u}) = 0$. Since the limits $\lim_{\epsilon \to 0} M_\epsilon(\vec{u} + \vec{H})$ and $\lim_{\epsilon \to 0}(M_\epsilon(\vec{u}) + 2\lim_{\eta \to 0} \text{div } ((p * \xi_{\eta,\epsilon})\vec{u}_{\epsilon,\eta}))$

are well defined in $\mathcal{D}'(Q)$, we find that $M(\vec{u})$ and $<<$ div $(p\vec{u}) >>$ are well defined and that $M(\vec{u}) = M(\vec{u} + \vec{H})$. $\square$

Of course, if $\vec{u}$ is regular enough, we have $M(\vec{u}) = 0$. Due to formula (5), Duchon and Robert [**D**] could see that when $\vec{u}$ belongs locally to $L^3_t(B^{1/3}_{3,q})_x$ with $q < +\infty$, then $M(\vec{u}) = 0$. This is the case when the classical criterion $\vec{u} \in L^4_{t,x}(\Omega)$ is fulfilled, since $L^4_t L^4_x \cap L^2_t H^1_x \subset L^3_t(B^{1/3}_{3,3})_x$. In particular, the support of the distribution $M(\vec{u})$ is a subset of the set $\Sigma(\vec{u})$ of singular points.

4. Dissipativity and partial regularity.

The best result we know about (partial) regularity of weak solutions has been given in 1982 by Caffarelli, Kohn and Nirenberg [**Ca, La**]. Their result is based on the notion of suitable solutions (due to Scheffer [**Sc**]):

Definition 4.1 (**Suitable solutions**). Let $\vec{u}$ be a local weak solutions of the Navier–Stokes solutions on a domain $\Omega \subset \mathbb{R} \times \mathbb{R}^3$. Then $\vec{u}$ is suitable if if satisfies the following two conditions :

- the pressure p is locally in $L^{3/2}_{t,x}$,
- $M(\vec{u}) \geq 0$ (i.e. $M(\vec{u})$ is a non-negative locally finite Borel measure).

Let us define now the parabolic metric $\rho((t,x),(s,y)) = \max(\sqrt{|t-s|}, |x-y|^2)$ and the parabolic cylinders $Q_r(t,x) = \{(s,y) :\ \rho((t,x),(s,y)) < r\}$.

Theorem 4.2 (**Caffarelli, Kohn and Nirenberg's regularity theorem**). *Let Ω be a domain in $\mathbb{R} \times \mathbb{R}^3$, $\vec{f} \in L^2_{\mathrm{loc}}(\Omega)$ a divergence-free time-dependent vector field and $\vec{u}$ a local weak solution of the Navier–Stokes equations on Ω (associated to the force $\vec{f}$). Assume that moreover*

- *$\vec{u}$ is suitable,*
- *the force $\vec{f}$ is regular : $\vec{f}$ belongs locally to $L^2_t H^1_x$,*

Then:

- *if $(t,x) \notin \Sigma(\vec{u})$, there exists a neighborhood of (t,x) on which $\vec{u}$ is Hölderian (with respect to the parabolic metric ρ) and we have*
$$\lim_{r\to 0} \frac{1}{r} \iint_{Q_r(t,x)} |\vec{\nabla} \otimes \vec{u}|^2 \, ds \, dy = 0.$$
- *if $(t,x) \in \Sigma(\vec{u})$,then*
$$\limsup_{r\to 0} \frac{1}{r} \iint_{Q_r(t,x)} |\vec{\nabla} \otimes \vec{u}|^2 \, ds \, dy > \epsilon^*,$$
where ϵ^* is a positive constant (which doesn't depend on $\vec{u}$, $\vec{f}$ nor Ω).

The size of $\Sigma(\vec{u})$ is then easily controlled with the following lemma :

Lemma 4.3 (**Parabolic Hausdorff dimension.**). *Let u belongs locally to $L^2_t H^1_x$ and let Σ be the set defined by*
$$(t,x) \in \Sigma \Leftrightarrow \limsup_{r\to 0} \frac{1}{r} \iint_{Q_r(t,x)} |\vec{\nabla} u|^2 \, ds \, dy > 0.$$
Then Σ has parabolic one-dimensional Hausdorff measure equal to 0.

Chamorro, Lemarié–Rieusset and Mayoufi [**Ch**] have considered the case where no integrability assumptions were made on the pressure p. This implies to change the definition of suitable solutions. Following [**D**], they introduced the notion of dissipative solutions :

DEFINITION 4.4 (**Dissipative solutions**). Let $\vec{u}$ be a local weak solutions of the Navier–Stokes solutions on a domain $\Omega \subset \mathbb{R} \times \mathbb{R}^3$. Then $\vec{u}$ is dissipative if $M(\vec{u}) \geq 0$.

A similar notion has been given by Wolf [**W**]. Indeed, if $\vec{u}$ is dissipative and if we use the harmonic correction $\vec{H} = -\vec{\nabla}\varpi$, we find, for $\vec{U} = \vec{u} + \vec{H}$:

$$\begin{aligned} M(\vec{U}) &= -\partial_t|\vec{U}|^2 + \Delta(|\vec{U}|^2) - 2|\vec{\nabla}\otimes\vec{U}|^2 - \text{ div } (|\vec{U}|^2\vec{U}) - 2 \text{ div } (P\vec{U}) + 2\vec{U}\cdot\vec{F} \\ &= -\partial_t|\vec{U}|^2 + \Delta(|\vec{U}|^2) - 2|\vec{\nabla}\otimes\vec{U}|^2 - \text{ div } ((|\vec{U}|^2 + 2p_0)\vec{U}) \\ &\quad +2\vec{U}\cdot\vec{f} - 2\vec{U}\cdot\vec{f_1} - 2 \text{ div } (p_1\vec{U}) \\ &= -\partial_t|\vec{U}|^2 + \Delta(|\vec{U}|^2) - 2|\vec{\nabla}\otimes\vec{U}|^2 - \text{ div } ((|\vec{U}|^2 + 2p_0)\vec{U}) \\ &\quad +2\vec{U}\cdot\vec{f} + 2\vec{U}\cdot(\vec{\nabla}\varpi \wedge (\vec{\nabla}\wedge\vec{U})). \end{aligned}$$

Writing $M(\vec{U}) \geq 0$ is exactly expressing that $\vec{u}$ is a generalized suitable solution, as defined by Wolf.

Another tool used by Chamorro, Lemarié–Rieusset and Mayoufi is the notion of parabolic Morey space :

DEFINITION 4.5 (**Parabolic Morrey spaces**). A function θ belongs to the parabolic Morrey space $\mathcal{M}^{s,\tau}(\Omega)$ if

$$\sup_{x_0,t_0,r} \frac{1}{r^{5(1-\frac{s}{\tau})}} \iint_\Omega 1_{|t-t_0|<r^2} 1_{|x-x_0|<r} |\theta(t,x)|^s \, dt \, dx < +\infty.$$

Parabolic Morrey spaces have been used by Kukavica [**K**] in a variant of Caffarelli, Kohn and Nirenberg's theorem [**Ca**], and by O'Leary [**O, Le**] in a variant of Serrin's regularity theorem [**Se**] :

THEOREM 4.6 (**Kukavica's theorem**). *There exists a positive constant* ϵ^* *such that the following holds : If* $\vec{U}$ *is a solution of the Navier–Stokes equations on a domain* Ω_1*, associated to a force* $\vec{F}$ *and a pressure* P *and if* x_0*,* t_0*,* $\vec{U}$*,* P *and* $\vec{F}$ *satisfy the following assumptions*

- $\vec{U}$ *belongs to* $L^\infty_t L^2_x \cap L^2_t H^1_x$,
- $P \in L^{3/2}_{t,x}(\Omega)$,
- *div* $\vec{F} = 0$ *and* $\vec{F} \in L^2_{t,x}(\Omega_1)$,
- $\vec{U}$ *is suitable,*
- $(t_0, x_0) \in \Omega_1$ *and*

$$\limsup_{r\to 0} \frac{1}{r} \iint_{(t_0-r^2,t_0+r^2)\times B(x_0,r)} |\vec{\nabla}\otimes\vec{U}|^2 \, ds \, dx < \epsilon^*,$$

then there exists $\tau > 5$ *and a neighborhood* Ω_2 *of* (t_0, x_0) *such that* $\vec{U} \in \mathcal{M}^{3,\tau}(\Omega_2)$.

THEOREM 4.7 (**O'Leary's theorem**). *If* $\vec{u}$ *is a solution of the Navier–Stokes equations on a domain* Ω_2*, associated to a force* $\vec{f}$ *and if* $\vec{u}$ *and* $\vec{f}$ *satisfy the following assumptions*

- *$\vec u$ belongs to $L^\infty_t L^2_x \cap L^2_t H^1_x$,*
- *div $\vec f = 0$ and $\vec f \in L^2_t H^k_x(\Omega_2)$ for some $k \in \mathbb{N}$,*
- *$\vec u \in \mathcal{M}^{s,\tau}(\Omega_2)$ with $\tau > 5$ and $2 < s \le \tau$,*

then, for every subdomain Ω_3 which is relatively compact in Ω_2, we have

$$\vec u \in L^\infty_t H^{k+1}_x \cap L^2_t H^{k+2}_x(\Omega_3).$$

Using those theorems, Chamorro, Lemarié–Rieusset and Mayoufi [**Ch**] could prove the following theorem (which is essentially the result proved previously by Wolf [**W**]) :

THEOREM 4.8 (**Wolf's theorem**). *Let Ω be a domain in $\mathbb{R}\times\mathbb{R}^3$, $\vec f \in L^2_{\rm loc}(\Omega)$ a divergence-free time-dependent vector field and $\vec u$ a local weak solution of the Navier–Stokes equations on Ω (associated to the force $\vec f$). Assume that moreover*

- *$\vec u$ is dissipative,*
- *the force $\vec f$ is regular : $\vec f$ belongs locally to $L^2_t H^1_x$,*

Then:

- *if $(t,x) \notin \Sigma(\vec u)$, then*
$$\lim_{r\to 0} \frac{1}{r} \iint_{Q_r(t,x)} |\vec\nabla \otimes \vec u|^2 \, ds\, dy = 0.$$
- *if $(t,x) \in \Sigma(\vec u)$, then*
$$\limsup_{r\to 0} \frac{1}{r} \iint_{Q_r(t,x)} |\vec\nabla \otimes \vec u|^2 \, ds\, dy \ge \epsilon^*$$
where ϵ^* is a positive constant (which doesn't depend on $\vec u$, $\vec f$ nor Ω).

PROOF. We sketch the proof given in [**Ch, Le**]. Let ϵ^* be the constant in Kukavica's theorem. Let $(x_0,t_0) \in \Omega$ with

$$\limsup_{r\to 0} \frac{1}{r} \iint_{Q_r(t_0,x_0)} |\vec\nabla \otimes \vec u|^2 \, ds\, dy < \epsilon^*.$$

We introduce a harmonic correction $\vec H$ on a cylindric neighborhood of (x_0,t_0) and consider the vector field $\vec U = \vec u + \vec H$. If $\vec u$ is dissipative, then $\vec U$ is suitable, associated to a force $\vec F \in L^2_t L^2_x(Q)$ and a pressure $P \in L^{3/2}_t L^{3/2}_x(Q)$. Moreover,

$$\limsup_{r\to 0} \frac{1}{r} \iint_{Q_r(t_0,x_0)} |\vec\nabla \otimes \vec U|^2 \, ds\, dy = \limsup_{r\to 0} \frac{1}{r} \iint_{Q_r(t_0,x_0)} |\vec\nabla \otimes \vec u|^2 \, ds\, dy < \epsilon^*.$$

Thus, by Kukavica's theorem, there exists $\tau > 5$ and a neighborhood $\Omega_2 \subset Q$ of (t_0,x_0) such that $\vec U \in \mathcal{M}^{3,\tau}(\Omega_2)$. As $\vec u = \vec U - \vec H$, we see that we have as well $\vec u \in \mathcal{M}^{3,\tau}(\Omega_2)$. As $\vec f \in L^2_t H^1_x$, we may use O'Leary's theorem and find that, on a cylindric neighborhood Ω_3 of (t_0,x_0), we have $\vec u \in L^\infty_t H^2_x(\Omega_3) \subset L^\infty_{t,x}(\Omega_3)$. Thus, $(t_0,x_0) \notin \Sigma(\vec u)$. □

5. Weak convergence of local weak solutions.

In this final section, we prove Theorem 1.3. Recall that we consider a sequence $(\vec f_n)_{n\in\mathbb{N}}$ of divergence-free time-dependent vector fields on a domain $\Omega \subset \mathbb{R}\times\mathbb{R}^3$ and a sequence $(\vec u_n)_{n\in\mathbb{N}}$ of local weak solutions of the Navier–Stokes equations on Ω (associated to the forces $\vec f_n$) such that, for each cylinder $Q \subset\subset \Omega$, we have

- $\vec{f}_n \in L^2_t H^1_x(Q)$ and $\vec{f}_n$ converges weakly in $L^2_t H^1_x$ to a limit $\vec{f}$,
- the sequence $\vec{u}_n$ is bounded in $L^\infty_t L^2_x(Q) \cap L^2_t H^1_x(Q)$ and converges weakly in $L^2_t H^1_x(Q)$ to a limit $\vec{u}$,
- for every n, $\vec{u}_n$ is bounded on Q (the bound depending on n).

We know that we may define a pressure p_n on Q and that we have the energy equality

$$M(\vec{u}_n) = 0,$$

where

$$\begin{aligned} M(\vec{u}_n) \quad = \quad & -\partial_t |\vec{u}_n|^2 + \Delta(|\vec{u}_n|^2) - 2|\vec{\nabla} \otimes \vec{u}_n|^2 - \text{ div } (|\vec{u}_n|^2 \vec{u}_n) \\ & -2 << \text{div } (p_n \vec{u}_n) >> +2\vec{u}_n \cdot \vec{f}_n. \end{aligned}$$

Our aim is then to prove that the limit $\vec{u}$ is a solution to the Navier–Stokes aquations associated to the limit $\vec{f}$ and that this solution is dissipative :

$$M(\vec{u}) \geq 0.$$

We cannot give a direct proof, as it is possible that no term in the definition of $M(\vec{u}_n)$ converge to the corresponding term in $M(\vec{u})$: p is not the limit in $\mathcal{D}'$ of p_n and $|\vec{u}|^2$ is not the limit in $\mathcal{D}'$ of $|\vec{u}_n|^2 \dots$ It is easy to find an example of such a bad behavior by studying Serrin's example of smooth in space and singular in time solution to the Navier–Stokes equations [**Se**] :

Example 5.1 (**Serrin's example**). Let ψ be defined on a neighborhood of $B(x_0, r_0)$ and be harmonic, $\Delta\psi = 0$, and let $\vec{f} = 0$ and

$$\vec{u} = \alpha(t) \vec{\nabla} \psi(x),$$

where $\alpha \in L^\infty((a,b))$. Then $\vec{u}$ is a local weak solution of the Navier–Stokes equations on $(a,b) \times B(x_0, r_0)$:

$$\partial_t \vec{u} = \Delta \vec{u} - \vec{u} \cdot \vec{\nabla} \vec{u} - \vec{\nabla}(-\dot{\alpha}\psi - \frac{|\vec{u}|^2}{2}) + \vec{f}.$$

Clearly, if α is not regular, the pressure p has no integrability in the time variable (because of the presence of the singular term $\dot{\alpha}(t)$)and $\vec{u}$ has no regularity in the time variable. Thus, $\vec{u}$ is dissipative (as a matter of fact, $M(\vec{u}) = 0$) but not suitable, as it violates both assumptions and conclusions of the Caffarelli, Kohn and Nirenberg theorem.

Let us adapt this example to our problem. We define

$$\vec{u}_n(t,x) = \cos(nt) \begin{pmatrix} x_1 \\ -x_2 \\ 0 \end{pmatrix}$$

- $\vec{u}_n$ is a solution on $\mathbb{R} \times \mathbb{R}^3$ of

$$\begin{cases} \partial_t \vec{u}_n = \Delta \vec{u}_n - (\vec{u}_n \cdot \vec{\nabla})\vec{u}_n - \vec{\nabla} p_n \\ \text{div} \vec{u}_n = 0 \end{cases}$$

- In this example, we have for a bounded domain Ω_0

$$\vec{u}_n \rightharpoonup 0$$

in $L^2_t H^1_x(\Omega_0)$ and

$$(\vec{u}_n \cdot \vec{\nabla})\vec{u}_n \rightharpoonup \frac{1}{2}\begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix} \neq 0,$$

in $\mathcal{D}'(\Omega_0)$.

In order to circumvene this problem of non-convergence, we shall use two tools : equations on vorticities $\vec{\omega}_n = \vec{\nabla} \wedge \vec{u}_n$ and on harmonic corrections $\vec{U}_n = \vec{u}_n + \vec{H}_n = -\frac{1}{\Delta}\vec{\nabla} \wedge \Big(\psi(\vec{\nabla} \wedge \vec{u}_n)\Big)$.

Step 1 : Vorticities.

On a cylinder $Q \subset\subset \Omega$, we may write the Navier–Stokes equations on the divergence-free vector field $\vec{u}_n$ in many ways. The first one is given by equation (1) : for every smooth compactly supported divergence-free vector field $\vec{\phi} \in \mathcal{D}(Q)$ we have

$$\iint_Q \vec{u}_n \cdot (\partial_t \vec{\phi} + \Delta \vec{\phi}) + \vec{u}_n \cdot (\vec{u}_n \cdot \vec{\nabla}\vec{\phi}) + \vec{f}_n \cdot \vec{\phi}\, dt\, dx = 0.$$

We may rewrite this equation as:

$$\partial_t \vec{u}_n = \Delta \vec{u}_n - \vec{u}_n \cdot \vec{\nabla}\vec{u}_n + \vec{f}_n \text{ in } (\mathcal{D}_\sigma(Q))'$$

where $\mathcal{D}_\sigma(Q)$ is the space of smooth compactly supported divergence-free vector fields on Q.

The second one is given by equations (2): for a distribution $p_n \in \mathcal{D}'(Q)$, we have

$$\partial_t \vec{u}_n = \Delta \vec{u}_n - \vec{u}_n \cdot \vec{\nabla}\vec{u}_n - \vec{\nabla} p_n + \vec{f}_n \quad \text{ in } \mathcal{D}'(Q).$$

The next one is based on the identity

$$\vec{u}_n \cdot \vec{\nabla}\vec{u}_n = \vec{\omega}_n \wedge \vec{u}_n + \vec{\nabla}(\frac{|\vec{u}_n|^2}{2})$$

from which we get

$$\partial_t \vec{u}_n = \Delta \vec{u}_n - \vec{\omega}_n \wedge \vec{u}_n + \vec{f}_n \text{ in } (\mathcal{D}_\sigma(Q))'.$$

We have seen that, in some cases, we don't have the convergence of $(\vec{u}_n \cdot \vec{\nabla})\vec{u}_n$ to $\vec{u} \cdot \vec{\nabla}\vec{u}$ in $\mathcal{D}'(\Omega_0)$. But we shall prove the following lemma :

Lemma 5.2 (**Convergence of the non-linear term**). *We have the following convergence results :*

$$\vec{\omega}_n \wedge \vec{u}_n \rightharpoonup \vec{\omega} \wedge \vec{u} \textit{ in } \mathcal{D}'(Q)$$

so that

$$(\vec{u}_n \cdot \vec{\nabla})\vec{u}_n \rightharpoonup \vec{u} \cdot \vec{\nabla}\vec{u} \textit{ in } (\mathcal{D}_\sigma(Q))'.$$

Thus, this lemma will prove the first half of Theorem 1.3: the limit $\vec{u}$ is a local weak solution on Ω of the Navier–Stokes equations associated to the force $\vec{f}$. The proof of the lemma is based on the following variant of the classical Rellich lemma [**Le, M**] :

Lemma 5.3 (**Rellich's lemma**). *Let $-\infty < \sigma_1 < \sigma_2 < +\infty$. Let Ω be a domain in $\mathbb{R} \times \mathbb{R}^3$. If a sequence of distribution T_n is weakly convergent to a distribution T in $(L^2_t H^{\sigma_2}_x)_{\rm loc}$ and if the sequence $(\partial_t T_n)$ is bounded in $(L^2_t H^{\sigma_1}_x)_{\rm loc}$, then T_n is strongly convergent in $(L^2_t H^{\sigma}_x)_{\rm loc}$ for every $\sigma < \sigma_2$.*

We apply Rellich's lemma to $\vec{\omega}_n$. We have

$$\partial_t \vec{\omega}_n = \Delta \vec{\omega}_n - \text{ div } (\vec{u}_n \otimes \vec{\omega}_n - \vec{\omega}_n \otimes \vec{u}_n) - \vec{\nabla} \wedge \vec{f}_n,$$

so that the sequence $(\partial_t \vec{\omega}_n)$ is bounded in $(L^2_t H^{\sigma_1}_x)_{\text{loc}}$ for all $\sigma_1 < -5/2$. Moreover, $\vec{\omega}_n$ is weakly convergent to $\vec{\omega}$ in $(L^2_t L^2_x)_{\text{loc}}$. Thus, $\vec{\omega}_n$ is strongly convergent in $(L^2_t H^{-1}_x)_{\text{loc}}$. As $\vec{u}_n$ is weakly convergent to $\vec{u}$ in $(L^2_t H^1_x)_{\text{loc}}$, we find that $\vec{\omega}_n \wedge \vec{u}_n$ is weakly convergent to $\vec{\omega} \wedge \vec{u}$ in $\mathcal{D}'(\Omega)$.

Step 2 : Harmonic corrections.

We now end the proof of Theorem 1.3 by checking the dissipativity of the limit $\vec{u}$. We restate the theorem as a result of stability for dissipativity :

THEOREM 5.4 (**Dissipative limits**). *Let Ω be a domain in $\mathbb{R} \times \mathbb{R}^3$. Assume that we have sequences $\vec{f}_n$ of divergence-free time-dependent vector fields and $\vec{u}_n$ of local weak solutions of the Navier–Stokes equations on Ω (associated to the forces $\vec{f}_n$) such that, for each cylinder $Q \subset\subset \Omega$, we have*

- *$\vec{f}_n \in L^2_t H^1_x(Q)$ and $\vec{f}_n$ converges weakly in $L^2_t H^1_x$ to a limit $\vec{f}$,*
- *the sequence $\vec{u}_n$ is bounded in $L^\infty_t L^2_x(Q) \cap L^2_t H^1_x(Q)$ and converges weakly in $L^2_t H^1_x(Q)$ to a limit $\vec{u}$,*
- *for every n, $\vec{u}_n$ is dissipative.*

Then the limit $\vec{u}$ is a dissipative local weak solution on Ω of the Navier–Stokes equations associated to the force $\vec{f}$.

PROOF. We already know that $\vec{u}$ is a local weak solution on Ω of the Navier–Stokes equations associated to the force $\vec{f}$. We have to prove its dissipativity.

Let $Q \subset\subset \Omega$ be a cylinder and $\psi \in \mathcal{D}(\Omega)$ be a cut-off function which is equal to 1 on a neighborhood of Q. In order to prove that $\vec{u}$ is dissipative, we shall prove that the harmonic correction $\vec{U} = \vec{H} + \vec{u} = -\frac{1}{\Delta} \vec{\nabla} \wedge \left(\psi(\vec{\nabla} \wedge \vec{u}) \right)$ is suitable.

We define as well $\vec{U}_n = -\frac{1}{\Delta} \vec{\nabla} \wedge \left(\psi(\vec{\nabla} \wedge \vec{u}_n) \right)$. The weak convergence of $\vec{u}_n$ in $(L^2_t H^1_x)_{\text{loc}}(\Omega)$ implies the weak convergence of $\vec{U}_n$ to $\vec{U}$ in $L^2_t H^1_x(Q)$. Moreover, the uniform boundedness of the sequence $(\vec{u}_n)_{n \in \mathbb{N}}$ in $(L^2_t H^1_x \cap L^\infty_t L^2_x)_{\text{loc}}(\Omega)$ and of the sequence $(\vec{f}_n)_{n \in \mathbb{N}}$ in $(L^2_t H^1_x)_{\text{loc}}(\Omega)$ implies that the sequences of pressure P_n and of forces $\vec{F}_n$ associated to $\vec{U}_n$ are uniformly bounded (respectively in $L^{3/2}_t L^{3/2}_x(Q) \cap L^2_t L^{6/5}_x(Q)$ and in $L^2_t L^2_x(Q)$). Thus, $\partial_t \vec{U}_n$ is bounded in $L^2_t H^{-2}_x(Q)$ and Rellich's lemma gives us that $\vec{U}_n$ is strongly convergent to $\vec{U}$ in $(L^2_t L^2_x)_{\text{loc}}(Q)$ (and, since $\vec{U}_n$ is bounded in $L^{10/3}_t L^{10/3}_x(Q)$, we have strong convergence in $(L^3_t L^3_x)_{\text{loc}}(Q)$ as well).

Taking subsequences, we may assume that the bounded sequences P_n (in $L^{3/2}_t L^{3/2}_x(Q)$), $\vec{F}_n$ (in $L^2_t L^2_x(Q)$) and $|\vec{\nabla} U_n|^2$ (in $L^1_t L^1_x(Q)$) converge weakly in $\mathcal{D}'$ to limits $P_\infty \in L^{3/2}_t L^{3/2}_x(Q)$, $\vec{F}_\infty \in L^2_t L^2_x(Q)$ and ν_∞ (a non-negative finite measure on Q). In particular, we have enough convergence to see that every term in the right-hand side of equality

$$M(\vec{u}_n) = -\partial_t |\vec{U}_n|^2 + \Delta(|\vec{U}_n|^2) - 2|\vec{\nabla} \otimes \vec{U}_n|^2 - \text{ div } (|\vec{U}_n|^2 \vec{U}_n) - 2 \text{ div } (P_n \vec{U}_n) + 2\vec{U}_n \cdot \vec{U}_n$$

has a limit, so that $\nu_1 = \lim_{n \to +\infty} M(\vec{U}_n)$ exists and

$$\nu_1 = -\partial_t |\vec{U}|^2 + \Delta(|\vec{U}|^2) - 2\nu_\infty - \text{ div } (|\vec{U}|^2 \vec{U}) - 2 \text{ div } (P_\infty \vec{U}) + 2\vec{U} \cdot \vec{F}_\infty.$$

As $M(\vec{U}_n) \geq 0$, we find that $\nu_1 \geq 0$. Moreover, by the Banach–Steinhaus theorem, we find that $\nu_2 = \nu_\infty - |\vec{\nabla} \otimes \vec{U}|^2 \geq 0$. As $M(\vec{U}) = \nu_1 + 2\nu_2$, we have $M(\vec{U}) \geq 0$. Hence, $\vec{U}$ is suitable and $\vec{u}$ is dissipative. $\square$

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