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A DIRECT APPROACH TO THE DUALITY OF GRAND AND SMALL LEBESGUE SPACES

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Abstract. In this paper we show, by elementary methods, that the quasinorms of the grand and small Lebesgue spaces

$$\|f\|_{L^{p)}} \approx \sup_{0<t<1} (1-\log t)^{-\frac{1}{p}} \left(\int_t^1 [f^*(s)]^p ds \right)^{\frac{1}{p}}, \quad 1<p<\infty$$

$$\|f\|_{L^{(q}} \approx \int_0^1 (1-\log t)^{-\frac{1}{q}} \left(\int_0^t f^*(s)^q ds \right)^{\frac{1}{q}} \frac{dt}{t}, \quad q = \frac{p}{p-1}$$

found by Fiorenza and Karadzhov in [8], by using deeply extrapolation-interpolation techniques, are associate each other. In other terms, the sharp Hölder's type inequality:

$$\int_0^1 fg dx \le c(p) \|f\|_{L^{(q}} \|g\|_{L^{p)}}$$

is proven, where the sharpness means that $\|\cdot\|_{L^{(q}}$ is the smallest quasinorm (up to equivalences) such that the inequality holds. The method is based entirely on integral estimates, makes use of asymptotic properties of the Euler's Gamma function, and gives an explicit estimate of the constant $c(p)$. All the results are expressed in terms of the more general spaces $L^{p),\theta}$ and $L^{(q,\theta}$, $\theta > 0$.

1. Introduction and preliminaries

In [17] the authors introduced the *grand Lebesgue spaces*, in connection with the study of the integrability properties of the Jacobian determinant. Such spaces are Banach Function Spaces (see e.g. [2], [18] for the definition), and play a key role in PDE theory, as shown by various papers ([3], [9], [13], [15], [19], [20]). Moreover, their abstract characterization given in [8] suggested a reasonable introduction of the notion of grand Orlicz spaces, which again play a role in questions about the integrability of the Jacobian determinant (see [5]).

Recently a special attention was given to the associate spaces to the grand Lebesgue spaces, called *small Lebesgue spaces*. They were introduced by the second author in [7], and the first properties which follow from their definition, along with some applications, are in [10] and [4] (see also references therein). Their role in Calculus of Variations (see [11]) stimulated the introduction of a more general class of spaces, the $G\Gamma$ spaces (see [12]). Moreover, for a quite general class of operators they look

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as the appropriate spaces to be considered in the extrapolation process of families of inequalities, see [6].

Let $\mathcal{M}_0$ be the set of all Lebesgue measurable functions in $(0,1) \subset \mathbb{R}$, whose values lie in $[-\infty, +\infty]$, finite a.e. in $(0,1)$. Also, let $\mathcal{M}_0^+$ be the class of functions in $\mathcal{M}_0$ whose values lie in $[0, +\infty]$.

Let $1 < p < \infty$. The *grand Lebesgue spaces* $L^{p)}$ are defined (see [17]) by

$$L^{p)} = \{f \in \mathcal{M}_0 : ||f||_{p)} := \sup_{0<\epsilon<p-1} \left(\epsilon \int_0^1 |f|^{p-\epsilon} dx\right)^{\frac{1}{p-\epsilon}} < \infty\}$$

It is easy to check that the expression $||\cdot||_{p)}$ is a function norm through which $L^{p)}$ becomes a Banach Function Space. It is called the *grand* L^p because of its evident property to contain L^p (continuous embedding).

The associate spaces of the grand Lebesgue spaces $L^{p)}$, denoted by $L^{(q}$, $q = p/(p-1)$, are Banach Function Spaces, defined through the function norm

$$\|g\|_{(q} = \sup\left\{\int_0^1 fg dx : f \in \mathcal{M}_0^+, \|f\|_{p)} \leq 1\right\}$$

They are called *small Lebesgue spaces*, are evidently continuously embedded in L^q and were first studied by the second author in [7], who found the following explicit expression of the norm (see also [4]):

$$\|g\|_{(q} = \inf_{g = \sum_{k=1}^{\infty} g_k} \left\{\sum_{k=1}^{\infty} \inf_{0<\epsilon<p-1} \epsilon^{-\frac{1}{p-\epsilon}} \left(\int_0^1 |g_k|^{\frac{p-\epsilon}{p-\epsilon-1}} dx\right)^{\frac{1}{(p-\epsilon)'}}\right\}$$

the functions g_k, $k \in \mathbb{N}$, being in $\mathcal{M}_0$. For digressions about embedding properties of such spaces into standard Banach spaces, see e.g. [14, 16, 4].

In [8] the following much more simple equivalent expression for the norm was found:

$$\|f\|_{(q} \approx [\![f]\!]_{(q} := \int_0^1 (1 - \log t)^{-\frac{1}{q}} \left(\int_0^t f^*(s)^q ds\right)^{\frac{1}{q}} \frac{dt}{t} \tag{1.1}$$

where by f^* we mean the *decreasing rearrangement* of f, defined by (for details on this well known operator see e.g. [2])

$$f^*(t) = \inf\{\lambda > 0 : |\{x \in (0,1) : |f(x)| > \lambda\}| \leq t\} \quad \forall t \in [0,1]$$

(here the convention $\inf \emptyset = +\infty$ is adopted; the symbol $|E|$ denotes, for any measurable set $E \subset (0,1)$, its Lebesgue measure). It is quite easy to prove the following result (see [12]):

Proposition 1.1. *The functional* $[\![\cdot]\!]_{(q}$ *is equivalent to a function norm.*

Notice that the expression $[\![f]\!]_{(q}$ reveals immediately the rearrangement invariant nature of such spaces. The symbol $\approx$ in (1.1) means, as usual, that there exist two positive constants such that

$$c_1\|f\|_{(q} \leq [\![f]\!]_{(q} \leq c_2\|f\|_{(q}$$

In the same paper [8] the associate expression of such norm (which therefore coincides with a quasinorm for the grand Lebesgue spaces) was determined, and it was found to be:

$$\|f\|_{p)} = \sup_{0<t<1} (1-\log t)^{-\frac{1}{p}} \left(\int_t^1 f^*(s)^p ds \right)^{\frac{1}{p}} , \quad 1 < p < \infty. \tag{1.2}$$

The expressions in (1.1) and (1.2) are not exactly function norms, but, exactly like in the case of Lorentz spaces, they become function norms replacying f^* by f^{**} (for the definition and details see [12]).

It is clear that the expressions $|| \cdot ||_{p)}$ and (1.2) must be equivalent, but such statement relies upon the duality between $||\cdot||_{p)}$ and $||\cdot||_{(q}$ shown in [7] and upon the results in [8], whose proofs use deeply extrapolation-interpolation techniques. The first main result of this paper is a direct proof of such equivalence, shown through Theorems 3.1 and 3.2. As a byproduct, one more equivalent, new expression for the norm of the grand Lebesgue spaces is obtained (see Proposition 3.1). The second main result is Theorem 4.4, where we prove the Hölder's type inequality:

$$\int_0^1 fg dx \le c(p) \|f\|_{p)} \|g\|_{(q} \ , \quad q = \frac{p}{p-1} \tag{1.3}$$

We stress that our arguments do not cover the very general context of the paper [8], where the duality between grand and small Lebesgue spaces is shown in the framework of the duality between grand and small abstract spaces. In fact, the logarithm in the factor $(1-\log t)^{-\frac{1}{p}}$ appearing in $\|\cdot\|_{p)}$ seems crucial in our estimates. Moreover, for the same reason it seems that our arguments cannot be easily generalized in the context of the $G\Gamma$ spaces studied in [12]. The problem to characterize the dual of such spaces remains open. An attempt in this direction is made in the same paper [12], where only partial estimates appear.

On the other hand, our proof is completely sharp and new. The sharpness here is the fact that $\|\cdot\|_{(q}$ is the smallest quasinorm (up to equivalences) such that the inequality (1.3) holds. This is an immediate consequence of Proposition 5.1 and the Hölder's inequality proven in [7].

The novelty of our approach is not only the fact that the result is direct and based entirely on integral estimates, but also that at our knowledge for the first time the asymptotic behaviour at infinity of the Euler's Gamma function plays a role in the study of this kind of function spaces. Moreover, our proof of (1.3) provides an estimate of the constant $c(p)$, missing in [8]. We believe that this new technique may stimulate new ideas in the study of the grand and small Lebesgue spaces.

All the results of this paper are stated and proven in the slight more general case when the factor $(1-\log t)^{-\frac{1}{p}}$ appearing in $\|\cdot\|_{p)}$ is replaced by $(1-\log t)^{-\frac{\theta}{p}}$, where $\theta > 0$ is a parameter. The corresponding quasinorm is denoted adding the letter θ in the subscript. In Example 5.1 of [8] the corresponding quasinorm for the small Lebesgue spaces has been computed, so that in the end we will work with the following positions (the model case can be obtained setting $\theta = 1$):

$$\|f\|_{p),\theta} = \sup_{0<t<1} (1-\log t)^{-\frac{\theta}{p}} \left(\int_t^1 f^*(s)^p ds \right)^{\frac{1}{p}}, \quad 1<p<\infty$$

$$\|f\|_{(q,\theta} := \int_0^1 (1-\log t)^{\frac{\theta}{p}-1} \left(\int_0^t f^*(s)^q ds \right)^{\frac{1}{q}} \frac{dt}{t}, \quad q = \frac{p}{p-1}.$$

2. A few technical lemmas

The first two lemmas are already known (see respectively [1], 5.1.8 p. 228 and 6.1.39 p. 257), therefore we state them without proof.

Lemma 2.1. *Let $1<p<\infty$, $0<\beta<1$. For every $\epsilon>0$ it is*

$$\int_1^{+\infty} t^{\left\lceil (\theta+\beta)\frac{p-\epsilon}{\epsilon} \right\rceil} e^{-\frac{p-1}{\epsilon}t} dt$$

$$= \left\lceil (\theta+\beta)\frac{p-\epsilon}{\epsilon} \right\rceil ! \left(\frac{\epsilon}{p-1} \right)^{\left\lceil (\theta+\beta)\frac{p-\epsilon}{\epsilon} \right\rceil + 1} e^{-\frac{p-1}{\epsilon}} \sum_{i=0}^{\left\lceil (\theta+\beta)\frac{p-\epsilon}{\epsilon} \right\rceil} \left(\frac{p-1}{\epsilon} \right)^i \frac{1}{i!}$$

where for every $x \in \mathbb{R}$ the symbol $\lceil x \rceil$ denotes the smallest integer greater or equal than x.

Lemma 2.2. *Let $1<p<\infty$, $0<\beta<1$, and let Γ be the classical Euler's Gamma function, defined by*

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt \quad \forall x > 0$$

For $\epsilon > 0$ sufficiently small it is

$$\Gamma\left((\theta+1-\beta)\frac{p-\epsilon}{\epsilon} + 2 \right) \approx \sqrt{2\pi} e^{-(\theta+1-\beta)\frac{p}{\epsilon}} \left((\theta+1-\beta)\frac{p}{\epsilon} \right)^{(\theta+1-\beta)\frac{p-\epsilon}{\epsilon}+\frac{3}{2}}$$

Lemma 2.3. *If $0<\beta<1$ and $0<\epsilon<\min\left(p-1, p\frac{(\theta+\beta)}{(\theta+\beta)+1}\right)$, then*

$$\left(\int_1^{+\infty} t^{(\theta+\beta)\frac{p-\epsilon}{\epsilon}} e^{-\frac{p-1}{\epsilon}t} dt \right)^{\frac{\epsilon}{p}} \le \left(\Gamma\left((\theta+\beta)\frac{p-\epsilon}{\epsilon} + 2 \right) \right)^{\frac{\epsilon}{p}} \left(\frac{\epsilon}{p-1} \right)^{(\theta+\beta)\frac{p-\epsilon}{p}+\frac{\epsilon}{p}}$$

Proof. For every $t>1$ it is

$$t^{(\theta+\beta)\frac{p-\epsilon}{\epsilon}} \le t^{\left\lceil (\theta+\beta)\frac{p-\epsilon}{\epsilon} \right\rceil}$$

and therefore

$$\int_1^{+\infty} t^{(\theta+\beta)\frac{p-\epsilon}{\epsilon}} e^{-\frac{p-1}{\epsilon}t} dt \le \int_1^{+\infty} t^{\left\lceil (\theta+\beta)\frac{p-\epsilon}{\epsilon} \right\rceil} e^{-\frac{p-1}{\epsilon}t} dt$$

By Lemma 2.1, taking into account that $0 < \epsilon < p-1$, $\Gamma(n+1) = n!$ for all $n \in \mathbb{N}$ and that $x \leq \lceil x \rceil$ for all $x \in \mathbb{R}$, we get

$$\begin{aligned}\int_1^{+\infty} t^{(\theta+\beta)\frac{p-\epsilon}{\epsilon}} e^{-\frac{p-1}{\epsilon}t} dt &\leq \left\lceil (\theta+\beta)\frac{p-\epsilon}{\epsilon} \right\rceil! \left(\frac{\epsilon}{p-1}\right)^{\left\lceil (\theta+\beta)\frac{p-\epsilon}{\epsilon}\right\rceil+1} e^{-\frac{p-1}{\epsilon}} e^{\frac{p-1}{\epsilon}} \\ &= \left\lceil (\theta+\beta)\frac{p-\epsilon}{\epsilon} \right\rceil! \left(\frac{\epsilon}{p-1}\right)^{\left\lceil (\theta+\beta)\frac{p-\epsilon}{\epsilon}\right\rceil+1} \\ &\leq \Gamma\left(\left\lceil (\theta+\beta)\frac{p-\epsilon}{\epsilon} \right\rceil + 1\right) \left(\frac{\epsilon}{p-1}\right)^{(\theta+\beta)\frac{p-\epsilon}{\epsilon}+1} \qquad (2.1)\end{aligned}$$

Since $\epsilon < p\frac{(\theta+\beta)}{(\theta+\beta)+1}$ implies that $\lceil(\theta+\beta)\frac{p-\epsilon}{\epsilon}\rceil + 1 > 2$, and since Γ is increasing in $[2, \infty[$, it is

$$\Gamma\left(\left\lceil (\theta+\beta)\frac{p-\epsilon}{\epsilon} \right\rceil + 1\right) \left(\frac{\epsilon}{p-1}\right)^{(\theta+\beta)\frac{p-\epsilon}{\epsilon}+1} \leq \Gamma\left((\theta+\beta)\frac{p-\epsilon}{\epsilon} + 2\right) \left(\frac{\epsilon}{p-1}\right)^{(\theta+\beta)\frac{p-\epsilon}{\epsilon}+1}$$

from which

$$\left(\int_1^{+\infty} t^{(\theta+\beta)\frac{p-\epsilon}{\epsilon}} e^{-\frac{p-1}{\epsilon}t} dt\right)^{\frac{\epsilon}{p}} \leq \left(\Gamma\left((\theta+\beta)\frac{p-\epsilon}{\epsilon} + 2\right)\right)^{\frac{\epsilon}{p}} \left(\frac{\epsilon}{p-1}\right)^{(\theta+\beta)\frac{p-\epsilon}{p}+\frac{\epsilon}{p}}$$

□

3. The equivalence $\|\cdot\|_{p),\theta} \approx [\![\cdot]\!]_{p),\theta}$

Let $1 < p < \infty$, $\theta > 0$. For all $f \in \mathcal{M}_0$ we set

$$[\![f]\!]_{p),\theta} = \sup_{0<t<1} (1-\log t)^{-\frac{\theta}{p}} \left(\int_t^1 f^*(s)^p ds\right)^{\frac{1}{p}}$$

$$\|f\|_{p),\theta} = \sup_{0<\epsilon<p-1} \left(\epsilon^\theta \int_0^1 |f(x)|^{p-\epsilon} dx\right)^{\frac{1}{p-\epsilon}}$$

The following result, which is a simple remark, provides a new way to express the quasinorm $[\![\cdot]\!]_{p),\theta}$ and is the heart of our approach.

Proposition 3.1. *Let υ be the function defined by*

$$\upsilon\colon \epsilon \in \,]0, p-1[\, \mapsto \upsilon(\epsilon) = e^{1-\frac{p-1}{\epsilon}}$$

The following holds:

$$[\![f]\!]_{p),\theta} = \sup_{0<\epsilon<p-1} \left[\left(\frac{\epsilon}{p-1}\right)^\theta \int_{\upsilon(\epsilon)}^1 f^*(s)^p ds\right]^{\frac{1}{p}} \qquad \forall f \in \mathcal{M}_0^+$$

Proof. Observe that $\upsilon(0^+) = 0$, $\upsilon((p-1)^-) = 1$ and that υ is strictly increasing. Setting $t = e^{1-\frac{p-1}{\epsilon}}$, it is

$$(1-\log t)^{-\frac{\theta}{p}} = \left(\frac{p-1}{\epsilon}\right)^{-\frac{\theta}{p}} = \left(\frac{\epsilon}{p-1}\right)^{\frac{\theta}{p}}$$

and therefore the assertion follows. □

We begin with the following

Theorem 3.1. *There exists $c = c(p,\theta) > 0$ such that*

$$\|f\|_{p),\theta} \leq c\,\|\!|f|\!\|_{p),\theta} \qquad \forall f \in \mathcal{M}_0^+$$

Proof. An immediate consequence of Proposition 3.1 is the following inequality:

$$\left(\frac{\epsilon}{p-1}\right)^{\theta} \int_{\upsilon(\epsilon)}^{1} f^*(s)^p\,ds \leq \|\!|f|\!\|_{p),\theta}^{p} \quad \forall \epsilon \in]0, p-1[\,. \tag{3.1}$$

Let us fix now $0 < \beta < 1$. From (3.1) we get

$$\left(\frac{\epsilon}{p-1}\right)^{\theta-\beta} \int_{\upsilon(\epsilon)}^{1} f^*(s)^p\,ds \leq \left(\frac{\epsilon}{p-1}\right)^{-\beta} \|\!|f|\!\|_{p),\theta}^{p} \quad \forall \epsilon \in]0, p-1[$$

and therefore, for any t, $0 < t < p-1$, it is

$$\left(\frac{\epsilon}{p-1}\right)^{\theta-\beta} \int_{\upsilon(\epsilon)}^{\upsilon(t)} f^*(s)^p\,ds \leq \left(\frac{\epsilon}{p-1}\right)^{-\beta} \|\!|f|\!\|_{p),\theta}^{p} \quad \forall \epsilon \in]0, t[$$

Integrating in $d\epsilon$ we obtain

$$\int_0^t d\epsilon \int_{\upsilon(\epsilon)}^{\upsilon(t)} \left(\frac{\epsilon}{p-1}\right)^{\theta-\beta} f^*(s)^p\,ds \leq (p-1)^{\beta}\frac{t^{1-\beta}}{1-\beta}\,\|\!|f|\!\|_{p),\theta}^{p}$$

and therefore, since the inverse function of υ is $w(s) =: \frac{p-1}{1-\log s}$, $0 < s < 1$,

$$\begin{aligned}
\int_0^{\upsilon(t)} \frac{f^*(s)^p}{(1-\log s)^{\theta-\beta+1}}\,ds &= \frac{\theta-\beta+1}{p-1}\int_0^{\upsilon(t)} \frac{f^*(s)^p}{(p-1)^{\theta-\beta}(\theta-\beta+1)}\left(\frac{p-1}{1-\log s}\right)^{\theta-\beta+1} ds \\
&= \frac{\theta-\beta+1}{p-1}\int_0^{\upsilon(t)} f^*(s)^p\,ds \int_0^{\frac{p-1}{1-\log s}} \left(\frac{\epsilon}{p-1}\right)^{\theta-\beta} d\epsilon \\
&= \frac{\theta-\beta+1}{p-1}\int_0^{\upsilon(t)} ds \int_0^{\frac{p-1}{1-\log s}} \left(\frac{\epsilon}{p-1}\right)^{\theta-\beta} f^*(s)^p\,d\epsilon \\
&= \frac{\theta-\beta+1}{p-1}\int_0^{t} d\epsilon \int_{\upsilon(\epsilon)}^{\upsilon(t)} \left(\frac{\epsilon}{p-1}\right)^{\theta-\beta} f^*(s)^p\,ds \\
&= \frac{\theta-\beta+1}{p-1}\int_0^{t} \left(\frac{\epsilon}{p-1}\right)^{-\beta}\left[\left(\frac{\epsilon}{p-1}\right)^{\theta}\int_{\upsilon(\epsilon)}^{\upsilon(t)} f^*(s)^p\,ds\right] d\epsilon \\
&\leq (\theta-\beta+1)(p-1)^{\beta-1}\int_0^t \epsilon^{-\beta}\|\!|f|\!\|_{p),\theta}^{p}\,d\epsilon \\
&= (\theta-\beta+1)(p-1)^{\beta-1}\frac{t^{1-\beta}}{1-\beta}\|\!|f|\!\|_{p),\theta}^{p}
\end{aligned}$$

Hence, writing ϵ in the place of t,

$$\int_0^{\upsilon(\epsilon)} \frac{f^*(s)^p}{(1-\log s)^{\theta-\beta+1}}\,ds \le (\theta-\beta+1)(p-1)^{\beta-1}\frac{\epsilon^{1-\beta}}{1-\beta}\,\|f\|_{p),\theta}^p \quad \forall\epsilon\in]0,p-1[\tag{3.2}$$

We are now ready to estimate $\|\cdot\|_{p),\theta}$. We have

$$\int_0^{\upsilon(\epsilon)} f^*(x)^{p-\epsilon}\,dx = \int_0^{\upsilon(\epsilon)} \left[\frac{f^*(x)^p}{(1-\log x)^{\theta-\beta+1}}\right]^{\frac{p-\epsilon}{p}} (1-\log x)^{(\theta-\beta+1)\frac{p-\epsilon}{p}}\,dx$$

and by Hölder's inequality

$$\int_0^{\upsilon(\epsilon)} f^*(x)^{p-\epsilon}\,dx \le \left(\int_0^{\upsilon(\epsilon)}\frac{f^*(x)^p}{(1-\log x)^{\theta-\beta+1}}\,dx\right)^{\frac{p-\epsilon}{p}} \left(\int_0^{\upsilon(\epsilon)}(1-\log x)^{(\theta-\beta+1)\frac{p-\epsilon}{\epsilon}}\,dx\right)^{\frac{\epsilon}{p}}$$

By (3.2) we get

$$\begin{aligned}\epsilon^\theta\int_0^{\upsilon(\epsilon)} f^*(x)^{p-\epsilon}\,dx \le \left(\frac{\theta-\beta+1}{1-\beta}\right)^{\frac{p-\epsilon}{p}} \left(\frac{1}{p-1}\right)^{(1-\beta)\frac{p-\epsilon}{p}} \epsilon^{(1-\beta)\frac{p-\epsilon}{p}+\theta}\\ \cdot\left(\int_0^{\upsilon(\epsilon)}(1-\log x)^{(\theta-\beta+1)\frac{p-\epsilon}{\epsilon}}\,dx\right)^{\frac{\epsilon}{p}} \|f\|_{p),\theta}^{p-\epsilon}\end{aligned}$$

We will prove later that

$$\sup_{0<\epsilon<p-1} \epsilon^{(1-\beta)\frac{p-\epsilon}{p}+\theta}\left(\int_0^{\upsilon(\epsilon)}(1-\log x)^{(\theta-\beta+1)\frac{p-\epsilon}{\epsilon}}\,dx\right)^{\frac{\epsilon}{p}} = K(p,\theta,\beta)<+\infty \tag{3.3}$$

and therefore we can assert that

$$\epsilon^\theta\int_0^{\upsilon(\epsilon)} f^*(x)^{p-\epsilon}\,dx \le c(p,\theta,\beta)\cdot\|f\|_{p),\theta}^{p-\epsilon} \quad \forall\epsilon\in]0,p-1[\tag{3.4}$$

On the other hand we observe that by Hölder's inequality

$$\begin{aligned}\epsilon^\theta\int_{\upsilon(\epsilon)}^1 f^*(x)^{p-\epsilon}\,dx &\le \epsilon^\theta\left(\int_{\upsilon(\epsilon)}^1 f^*(x)^p\,dx\right)^{\frac{p-\epsilon}{p}}\left(\int_{\upsilon(\epsilon)}^1 dx\right)^{\frac{\epsilon}{p}}\\ &= \epsilon^\theta\cdot\epsilon^{-\theta\frac{p-\epsilon}{p}}\left(\epsilon^\theta\int_{\upsilon(\epsilon)}^1 f^*(x)^p\,dx\right)^{\frac{p-\epsilon}{p}}(1-\upsilon(\epsilon))^{\frac{\epsilon}{p}}\\ &\le \epsilon^{\theta\left(1-\frac{p-\epsilon}{p}\right)}\left(\epsilon^\theta\int_{\upsilon(\epsilon)}^1 f^*(x)^p\,dx\right)^{\frac{p-\epsilon}{p}}\end{aligned}$$

By (3.1) we can conclude that

$$\epsilon^\theta\int_{\upsilon(\epsilon)}^1 f^*(x)^{p-\epsilon}\,dx \le \epsilon^{\frac{\theta\epsilon}{p}}(p-1)^{\theta\frac{p-\epsilon}{p}}\|f\|_{p),\theta}^{p-\epsilon} \le c(p,\theta)\,\|f\|_{p),\theta}^{p-\epsilon} \tag{3.5}$$

Joining (3.4) and (3.5) we obtain

$$\begin{aligned}\epsilon^\theta \int_0^1 f(x)^{p-\epsilon} dx &= \epsilon^\theta \int_0^{\upsilon(\epsilon)} f(x)^{p-\epsilon} dx + \epsilon^\theta \int_{\upsilon(\epsilon)}^1 f(x)^{p-\epsilon} dx \\ &\leq c(p,\theta,\beta)\, \| f \|_{p),\theta}^{p-\epsilon} + c(p,\theta) \| f \|_{p),\theta}^{p-\epsilon} \\ &= c(p,\theta,\beta)\, \| f \|_{p),\theta}^{p-\epsilon}\end{aligned}$$

from which the assertion follows. In order to finish the proof we need to prove (3.3), i.e. that

$$\sup_{0<\epsilon<p-1} \left(\epsilon^{(1-\beta)\frac{p-\epsilon}{\epsilon}+\frac{p\theta}{\epsilon}} \int_0^{\upsilon(\epsilon)} (1-\log x)^{(\theta-\beta+1)\frac{p-\epsilon}{\epsilon}} dx \right)^{\frac{\epsilon}{p}} = K(p,\theta,\beta) < +\infty$$

Setting $x := e^{1-\frac{p-1}{\epsilon}t}$ in the integral, the thesis reduces to the boundedness of the following expression when $\epsilon > 0$ is sufficiently small:

$$e^{\frac{\epsilon}{p}} (p-1)^{(\theta+1-\beta)\frac{p-\epsilon}{p}+\frac{\epsilon}{p}} \epsilon^{(\theta-1)\frac{\epsilon}{p}} \left(\int_1^{+\infty} t^{(\theta+1-\beta)\frac{p-\epsilon}{\epsilon}} e^{-\frac{p-1}{\epsilon}t} dt \right)^{\frac{\epsilon}{p}}$$

which, by Lemma 2.3 applied replacing β by $1-\beta$, is bounded by

$$e^{\frac{\epsilon}{p}} (p-1)^{(\theta+1-\beta)\frac{p-\epsilon}{p}+\frac{\epsilon}{p}} \epsilon^{(\theta-1)\frac{\epsilon}{p}} \left(\Gamma\left((\theta+1-\beta)\frac{p-\epsilon}{\epsilon} + 2 \right) \right)^{\frac{\epsilon}{p}} \left(\frac{\epsilon}{p-1} \right)^{(\theta+1-\beta)\frac{p-\epsilon}{p}+\frac{\epsilon}{p}}$$

When $\epsilon > 0$ is sufficiently small, such quantity is equivalent to the following one, which in turn can be transformed by using Lemma 2.2 :

$$\begin{aligned}&\epsilon^{\theta+1-\beta} \left(\Gamma\left((\theta+1-\beta)\frac{p-\epsilon}{\epsilon} + 2 \right) \right)^{\frac{\epsilon}{p}} \\ &\approx \epsilon^{\theta+1-\beta} \left(\sqrt{2\pi}\, e^{-(\theta+1-\beta)\frac{p}{\epsilon}} \left((\theta+1-\beta)\frac{p}{\epsilon} \right)^{(\theta+1-\beta)\frac{p-\epsilon}{\epsilon}+\frac{3}{2}} \right)^{\frac{\epsilon}{p}} \\ &= \epsilon^{\theta+1-\beta} (\sqrt{2\pi})^{\frac{\epsilon}{p}} e^{-(\theta+1-\beta)} \left((\theta+1-\beta)\frac{p}{\epsilon} \right)^{(\theta+1-\beta)\frac{p-\epsilon}{p}+\frac{3}{2}\frac{\epsilon}{p}} \\ &\approx e^{-(\theta+1-\beta)} \left((\theta+1-\beta)p \right)^{\theta+1-\beta}\end{aligned}$$

□

We prove now the inequality in the opposite direction.

Theorem 3.2. *There exists $c = c(p,\theta) > 0$ such that*

$$\| f \|_{p),\theta} \leq c\, \| f \|_{p),\theta} \qquad \forall f \in \mathcal{M}_0^+$$

Proof. Setting, as before,

$$\upsilon \colon \epsilon \in [0, p-1] \mapsto \upsilon(\epsilon) = e^{1-\frac{p-1}{\epsilon}}$$

we show first that there exists $K_p > 0$ such that

$$f^*(\upsilon(\epsilon))^\epsilon \le K_p \left(\int_0^1 f^*(s)^{p-\epsilon} ds \right)^{\frac{\epsilon}{p-\epsilon}} \qquad \forall \epsilon \in]0, p-1[\tag{3.6}$$

From the monotonicity of f^* it follows that for every $0 < \epsilon < p-1$ it is

$$f^*(\upsilon(\epsilon)) \le f^*(s) \quad \forall s \in [\frac{\upsilon(\epsilon)}{2}, \upsilon(\epsilon)] \ ,$$

and by Hölder's inequality:

$$\frac{\upsilon(\epsilon)}{2} f^*(\upsilon(\epsilon))^\epsilon \le \int_{\frac{\upsilon(\epsilon)}{2}}^{\upsilon(\epsilon)} f^*(s)^\epsilon ds \le \left(\frac{\upsilon(\epsilon)}{2} \right)^{\frac{p-2\epsilon}{p-\epsilon}} \left(\int_0^1 f^*(s)^{p-\epsilon} ds \right)^{\frac{\epsilon}{p-\epsilon}}$$

from which (3.6) follows, where $K_p = \sup\limits_{0<\epsilon<p-1} \left(\frac{\upsilon(\epsilon)}{2} \right)^{-\frac{\epsilon}{p-\epsilon}} < \infty$.

At this point it is sufficient to observe that by Proposition 3.1

$$\begin{aligned}
[\![f]\!]_{p),\theta} &= (p-1)^{-\frac{\theta}{p}} \sup_{0<\epsilon<p-1} \left(\epsilon^\theta \int_{\upsilon(\epsilon)}^1 f^*(s)^p ds \right)^{\frac{1}{p}} \\
&= (p-1)^{-\frac{\theta}{p}} \sup_{0<\epsilon<p-1} \left(\epsilon^\theta \int_{\upsilon(\epsilon)}^1 f^*(s)^\epsilon f^*(s)^{p-\epsilon} ds \right)^{\frac{1}{p}} \\
&\le (p-1)^{-\frac{\theta}{p}} \sup_{0<\epsilon<p-1} \left(\epsilon^\theta f^*(\upsilon(\epsilon))^\epsilon \int_{\upsilon(\epsilon)}^1 f^*(s)^{p-\epsilon} ds \right)^{\frac{1}{p}}
\end{aligned}$$

and by (3.6)

$$\begin{aligned}
&\le (p-1)^{-\frac{\theta}{p}} \sup_{0<\epsilon<p-1} \left[\epsilon^\theta K_p \left(\int_0^1 f^*(s)^{p-\epsilon} ds \right)^{\frac{\epsilon}{p-\epsilon}} \int_{\upsilon(\epsilon)}^1 f^*(s)^{p-\epsilon} ds \right]^{\frac{1}{p}} \\
&\le c(p,\theta) \sup_{0<\epsilon<p-1} \left(\epsilon^\theta \int_0^1 f^*(s)^{p-\epsilon} ds \right)^{\frac{1}{p-\epsilon}} \\
&= c(p,\theta) \, \| f \|_{p),\theta}
\end{aligned}$$

□

4. The Hölder's type inequality

We begin with three lemmas, useful for the proof of the Hölder's type inequality (see next Theorem 4.4). The letter p will denote a fixed exponent in the interval $]1, \infty[$, $q = p/(p-1)$ will denote its Hölder's conjugate, θ will be a positive parameter and υ, w will denote, as in the Theorem 3.1, the functions, inverse each to the other,

$$\upsilon \colon \epsilon \in]0, p-1[\mapsto \upsilon(\epsilon) = e^{1-\frac{p-1}{\epsilon}}$$

and

$$w\colon t\in]0,1[\mapsto w(t)=\frac{p-1}{1-\log t}$$

Observe that with this notation it is

$$\| g \|_{(q,\theta}=(p-1)^{\frac{\theta}{p}-1}\int_0^1 w(t)^{1-\frac{\theta}{p}}\left(\int_0^t g^*(x)^q\,dx\right)^{\frac{1}{q}}\frac{dt}{t}\quad \forall g\in\mathcal{M}_0^+ \tag{4.1}$$

Lemma 4.1. *The following inequality holds:*

$$\| f \|_{p),\theta}\geq(\theta+1)^{-\frac{1}{p}}(p-1)^{-\frac{\theta}{p}}\sup_{0<t<1}\left(\frac{1}{w(t)}\int_0^t f^*(x)^p w(x)^{\theta+1}\,dx\right)^{\frac{1}{p}}\quad \forall f\in\mathcal{M}_0^+$$

Proof. By Proposition 3.1 it is

$$\| f \|_{p),\theta}=\sup_{0<\epsilon<p-1}\left[\left(\frac{\epsilon}{p-1}\right)^{\theta}\int_{\upsilon(\epsilon)}^1 f^*(s)^p\,ds\right]^{\frac{1}{p}}$$

therefore

$$\left(\frac{\epsilon}{p-1}\right)^{\theta}\int_{\upsilon(\epsilon)}^1 f^*(x)^p\,dx\leq \| f \|_{p),\theta}^p\quad \forall\epsilon\in]0,p-1[.$$

Integrating in $d\epsilon$ on the interval $(0,t)$, $0<t<p-1$, we get

$$\int_0^t\left(\frac{\epsilon}{p-1}\right)^{\theta}\int_{\upsilon(\epsilon)}^1 f^*(x)^p\,dx\,d\epsilon\leq t\| f \|_{p),\theta}^p$$

from which

$$\begin{aligned} t\cdot\| f \|_{p),\theta}^p &\geq \int_0^{\upsilon(t)}\left[\int_0^{w(x)}\left(\frac{\epsilon}{p-1}\right)^{\theta} f^*(x)^p\,d\epsilon\right]dx+\int_{\upsilon(t)}^1\left[\int_0^t\left(\frac{\epsilon}{p-1}\right)^{\theta} f^*(x)^p\,d\epsilon\right]dx\\ &=\frac{1}{(\theta+1)(p-1)^{\theta}}\int_0^{\upsilon(t)} f^*(x)^p w(x)^{\theta+1}\,dx\;+\;\frac{t^{\theta+1}}{(\theta+1)(p-1)^{\theta}}\int_{\upsilon(t)}^1 f^*(x)^p\,dx\,. \end{aligned}$$

Hence

$$\| f \|_{p),\theta}^p\geq\frac{1}{(\theta+1)(p-1)^{\theta}t}\int_0^{\upsilon(t)} f^*(x)^p w(x)^{\theta+1}\,dx$$

therefore

$$\| f \|_{p),\theta}\geq(\theta+1)^{-\frac{1}{p}}(p-1)^{-\frac{\theta}{p}}\left(\frac{1}{t}\int_0^{\upsilon(t)} f^*(x)^p w(x)^{\theta+1}\,dx\right)^{\frac{1}{p}}.$$

Passing to the supremum in the right hand side we get

$$\| f \|_{p),\theta}\geq\sup_{0<t<p-1}(\theta+1)^{-\frac{1}{p}}(p-1)^{-\frac{\theta}{p}}\left(\frac{1}{t}\int_0^{\upsilon(t)} f^*(x)^p w(x)^{\theta+1}\,dx\right)^{\frac{1}{p}}$$

and setting $t := w(t)$

$$\|f\|_{p),\theta} \geq \sup_{0<t<1} (\theta+1)^{-\frac{1}{p}}(p-1)^{-\frac{\theta}{p}} \left(\frac{1}{w(t)}\int_0^t f^*(x)^p w(x)^{\theta+1}\,dx\right)^{\frac{1}{p}}$$

from which the assertion follows. □

Lemma 4.2. *The following inequality holds:*

$$\|g\|_{(q,\theta} \geq (p-1)^{\frac{\theta}{p}-1} \int_0^1 w(t)^{1-\theta-\frac{1}{q}} \left(\int_0^t g^*(x)^q w(x)^{\theta+1}\,dx\right)^{\frac{1}{q}} \frac{dt}{t} \quad \forall g \in \mathcal{M}_0^+$$

Proof. By (4.1) we have

$$\begin{aligned}\|g\|_{(q,\theta} &= (p-1)^{\frac{\theta}{p}-1} \int_0^1 w(t)^{1-\frac{\theta}{p}} \left(\int_0^t g^*(x)^q\,dx\right)^{\frac{1}{q}} \frac{dt}{t} \\ &= (p-1)^{\frac{\theta}{p}-1} \int_0^1 w(t)^{1-\frac{\theta}{p}} \left(\int_0^t g^*(x)^q \frac{w(x)^{\theta+1}}{w(x)^{\theta+1}}\,dx\right)^{\frac{1}{q}} \frac{dt}{t}\end{aligned}$$

and therefore, using the fact that $w^{-(\theta+1)}$ is decreasing:

$$\geq (p-1)^{\frac{\theta}{p}-1} \int_0^1 w(t)^{1-\frac{\theta}{p}} \cdot \frac{1}{w(t)^{\frac{\theta+1}{q}}} \left(\int_0^t g^*(x)^q w(x)^{\theta+1}\,dx\right)^{\frac{1}{q}} \frac{dt}{t}$$

from which the assertion follows.

□

Lemma 4.3. *The following inequality holds:*

$$\|f\|_{p),\theta} \geq (p-1)^{-\frac{\theta}{p}} 2^{-\frac{\theta+1}{p}} \sup_{0<t<1} \left[t\, w(t)^\theta\, f^*(t)^p\right]^{\frac{1}{p}} \quad \forall f \in \mathcal{M}_0^+$$

Proof. By Proposition 3.1 we have

$$\begin{aligned}\|f\|_{p),\theta} &= \sup_{0<\epsilon<p-1} \left[\left(\frac{\epsilon}{p-1}\right)^\theta \int_{\upsilon(\epsilon)}^1 f^*(s)^p\,ds\right]^{\frac{1}{p}} \\ &= (p-1)^{-\frac{\theta}{p}} \sup_{0<\epsilon<p-1} \left(\epsilon^\theta \int_{\upsilon(\epsilon)}^1 f^*(s)^p\,ds\right)^{\frac{1}{p}} \\ &= (p-1)^{-\frac{\theta}{p}} \sup_{0<t<2(p-1)} \left[\left(\frac{t}{2}\right)^\theta \int_{\upsilon(\frac{t}{2})}^1 f^*(x)^p\,dx\right]^{\frac{1}{p}} \\ &\geq (p-1)^{-\frac{\theta}{p}} \sup_{0<t<p-1} \left[\left(\frac{t}{2}\right)^\theta \int_{\upsilon(\frac{t}{2})}^1 f^*(x)^p\,dx\right]^{\frac{1}{p}} \\ &= (p-1)^{-\frac{\theta}{p}} \sup_{0<t<1} \left[\left(\frac{w(t)}{2}\right)^\theta \int_{\upsilon(\frac{w(t)}{2})}^1 f^*(x)^p\,dx\right]^{\frac{1}{p}}\end{aligned}$$

and since $v(\frac{w(t)}{2}) = \exp[1-2(1-\log t)] = \exp[-1+\log(t^2)] = \frac{t^2}{e} < \frac{t}{2}$,

$$\geq (p-1)^{-\frac{\theta}{p}} \sup_{0<t<1} \left[\left(\frac{w(t)}{2} \right)^{\theta} \int_{\frac{t}{2}}^{t} f^*(x)^p \, dx \right]^{\frac{1}{p}}$$

$$\geq (p-1)^{-\frac{\theta}{p}} \sup_{0<t<1} \left[\left(\frac{w(t)}{2} \right)^{\theta} \frac{t}{2} f^*(t)^p \right]^{\frac{1}{p}}$$

□

We prove now the main result of this Section.

Theorem 4.4. *Let $1 < p < \infty$, $q = p/(p-1)$. The following inequality holds:*

$$\int_0^1 f(x)g(x)\,dx \leq \left[2^{\frac{\theta+1}{p}} + (\theta+1)^{1+\frac{1}{p}} \right] \| f \|_{p),\theta} \, \| g \|_{(q,\theta} \quad \forall f, g \in \mathcal{M}_0$$

Proof. By Lemma 4.3 and (4.1), we have

$$2^{\frac{\theta+1}{p}} (p-1) \| f \|_{p),\theta} \, \| g \|_{(q,\theta} \geq \int_0^1 [t\, w^{\theta}(t)]^{\frac{1}{p}} f^*(t) w(t)^{1-\frac{\theta}{p}} \left(\int_0^t g^*(x)^q \, dx \right)^{\frac{1}{q}} \frac{dt}{t}$$

$$\geq \int_0^1 t^{\frac{1}{p}} f^*(t) w(t) t^{\frac{1}{q}} \left(\frac{1}{t} \int_0^t g^*(x)^q \, dx \right)^{\frac{1}{q}} \frac{dt}{t}$$

and since g^* is decreasing:

$$\geq \int_0^1 f^*(t) g^*(t) w(t) \, dt \, .$$

On the other hand, by Lemmas 4.1 and 4.2:

$$(\theta+1)^{1+\frac{1}{p}} (p-1) \| f \|_{p),\theta} \, \| g \|_{(q,\theta} \geq (\theta+1) \int_0^1 \left(\frac{1}{w(t)} \int_0^t f^*(x)^p w(x)^{\theta+1} \, dx \right)^{\frac{1}{p}}$$

$$\cdot\, w(t)^{1-\theta-\frac{1}{q}} \left(\int_0^t g^*(x)^q w(x)^{\theta+1} \, dx \right)^{\frac{1}{q}} \frac{dt}{t}$$

$$= (\theta+1) \int_0^1 \frac{w(t)^{-\theta}}{t} \left(\int_0^t f^*(x)^p w(x)^{\theta+1} \, dx \right)^{\frac{1}{p}}$$

$$\cdot \left(\int_0^t g^*(x)^q w(x)^{\theta+1} \, dx \right)^{\frac{1}{q}} dt$$

and by Hölder's inequality:

$$\geq (\theta+1) \int_0^1 \frac{w(t)^{-\theta}}{t} \int_0^t f^*(x) g^*(x) \, w(x)^{\theta+1} \, dx \, dt$$

$$= (\theta+1) \int_0^1 f^*(x) g^*(x) \, w(x)^{\theta+1} \int_x^1 \frac{w(t)^{-\theta}}{t} \, dt \, dx \, .$$

Recalling that the inner integral can be explicitly computed using the definition of w, we find:

$$
\begin{aligned}
&= (\theta+1)\int_0^1 f^*(x)g^*(x)\left(\frac{p-1}{1-\log x}\right)^{\theta+1}\int_x^1 \frac{(p-1)^{-\theta}(1-\log t)^{\theta}}{t}\,dt\,dx\\
&= \int_0^1 f^*(x)g^*(x)\left[p-1-\frac{p-1}{(1-\log x)^{\theta+1}}\right]dx\\
&\geq \int_0^1 f^*(x)g^*(x)\left[p-1-\frac{p-1}{1-\log x}\right]dx\\
&= \int_0^1 f^*(x)g^*(x)(p-1-w(x))\,dx
\end{aligned}
$$

hence, summing the two relations obtained,

$$\left[2^{\frac{\theta+1}{p}}+(\theta+1)^{1+\frac{1}{p}}\right]\|f\|_{p),\theta}\,\|g\|_{(q,\theta}\geq\int_0^1 f^*(x)g^*(x)\,dx$$

from which, by the well known Hardy-Littlewood's inequality on rearrangements (see e.g. [2])

$$\left[2^{\frac{\theta+1}{p}}+(\theta+1)^{1+\frac{1}{p}}\right]\|f\|_{p),\theta}\,\|g\|_{(q,\theta}\geq\int_0^1 f(x)g(x)\,dx\ .$$

The theorem is therefore proven. □

5. Comparison of quasinorms

In order to simplify the volume of some formulas, in this Section for any $1<r<\infty$ we will denote by r' the exponent Hölder conjugate of r, namely, $r'=r/(r-1)$. Therefore, according to the notation of the previous Section, it is $p'=q$.

Lemma 5.1. *Let $1<p<\infty$, $w(t)=\frac{p-1}{1-\log t}$, $t\in]0,1[$. There exists a constant $K_{p,\theta}$ depending only on p and θ such that for every $0<\epsilon<p-1$ the following inequality holds*

$$\int_0^1 \frac{w(t)^{1-\frac{\theta}{p}}}{t}t^{\frac{1}{p'}-\frac{1}{(p-\epsilon)'}}\,dt\leq K_{p,\theta}\,\epsilon^{-\frac{\theta}{p-\epsilon}}$$

Proof. Set

$$p_\epsilon=\frac{p(p-\epsilon)}{\epsilon}$$

so that

$$\frac{1}{p'}-\frac{1}{(p-\epsilon)'}=\frac{\epsilon}{p(p-\epsilon)}=\frac{1}{p_\epsilon}$$

Substituting $t := e^{1-p_\epsilon t}$ we get

$$\begin{aligned}
\int_0^1 \frac{w(t)^{1-\frac{\theta}{p}}}{t}\, t^{\frac{1}{p'}-\frac{1}{(p-\epsilon)'}}\, dt &= \int_0^1 w(t)^{1-\frac{\theta}{p}}\, t^{\frac{1}{p_\epsilon}-1}\, dt \\
&= p_\epsilon \int_{\frac{1}{p_\epsilon}}^{+\infty} \left(\frac{p-1}{p_\epsilon t}\right)^{1-\frac{\theta}{p}} e^{(1-p_\epsilon t)(\frac{1}{p_\epsilon}-1)}\, e^{1-p_\epsilon t}\, dt \\
&= (p-1)^{1-\frac{\theta}{p}} p_\epsilon^{\frac{\theta}{p}} \int_{\frac{1}{p_\epsilon}}^{+\infty} t^{\frac{\theta}{p}-1}\, e^{(1-p_\epsilon t)\frac{1}{p_\epsilon}}\, dt \\
&= (p-1)^{1-\frac{\theta}{p}} p_\epsilon^{\frac{\theta}{p}} e^{\frac{1}{p_\epsilon}} \int_{\frac{1}{p_\epsilon}}^{+\infty} t^{\frac{\theta}{p}-1}\, e^{-t}\, dt \\
&= (p-1)^{1-\frac{\theta}{p}} p_\epsilon^{\frac{\theta}{p}} e^{\frac{1}{p_\epsilon}}\, \Gamma\left(\frac{\theta}{p}; \frac{1}{p_\epsilon}\right) \qquad (5.1)
\end{aligned}$$

where by $\Gamma(a; z)$ we denote the so-called incomplete Euler's Gamma function defined by:

$$\Gamma(a; z) = \int_z^{+\infty} t^{a-1}\, e^{-t}\, dt.$$

The term in (5.1) can be written as

$$(p-1)^{1-\frac{\theta}{p}} \left[\frac{p(p-\epsilon)}{\epsilon}\right]^{\frac{\theta}{p}} e^{\frac{\epsilon}{p(p-\epsilon)}}\, \Gamma\left(\frac{\theta}{p}; \frac{\epsilon}{p(p-\epsilon)}\right)$$

and when $\epsilon \to 0$ it behaves as

$$(p-1)^{1-\frac{\theta}{p}} p^{\frac{2\theta}{p}}\, \Gamma\left(\frac{\theta}{p}\right) \epsilon^{-\frac{\theta}{p}} \approx (p-1)^{1-\frac{\theta}{p}} p^{\frac{2\theta}{p}}\, \Gamma\left(\frac{\theta}{p}\right) \epsilon^{-\frac{\theta}{p-\epsilon}}$$

The Lemma is proven. □

Proposition 5.1. *Let $1 < p < \infty$. There exists a constant $J_{p,\theta}$ depending only on p and θ such that for every $g \in \mathcal{M}_0^+$ the following inequality holds*

$$\| g \|_{(p',\theta} \le J_{p,\theta} \inf_{g=\sum_{k=1}^{+\infty} g_k\, , g_k \in \mathcal{M}_0^+} \sum_{k=1}^{+\infty} \inf_{0<\epsilon<p-1} \epsilon^{-\frac{\theta}{p-\epsilon}} \left(\int_0^1 g_k^{(p-\epsilon)'}\, dx\right)^{\frac{1}{(p-\epsilon)'}}$$

Proof. Let $(g_k)_{k\in\mathbb{N}}$, $g_k \in \mathcal{M}_0^+$ be a decomposition (a.e. convergence) of g:

$$g = \sum_{k=1}^{+\infty} g_k.$$

Then, for every $0 < t < 1$, $0 < \epsilon < p-1$, $k \in \mathbb{N}$, we have

$$\int_0^t g_k^{p'}\, dx \le \left(\int_0^t g_k^{(p-\epsilon)'}\, dx\right)^{\frac{p'}{(p-\epsilon)'}} t^{1-\frac{p'}{(p-\epsilon)'}}$$

therefore, setting as usual $w(t)=\frac{p-1}{1-\log t}$, $t\in]0,1[$, by (4.1)

$$\|g_k\|_{(p',\theta}=(p-1)^{\frac{\theta}{p}-1}\int_0^1 w(t)^{1-\frac{\theta}{p}}\left(\int_0^t g_k^*(x)^{p'}\,dx\right)^{\frac{1}{p'}}\frac{dt}{t}$$

by Hölder's inequality

$$\begin{aligned}&\le (p-1)^{\frac{\theta}{p}-1}\int_0^1 \frac{w(t)^{1-\frac{\theta}{p}}}{t}\left(\int_0^1 g_k^{(p-\epsilon)'}\,dx\right)^{\frac{1}{(p-\epsilon)'}} t^{\frac{1}{p'}-\frac{1}{(p-\epsilon)'}}\,dt\\ &\le (p-1)^{\frac{\theta}{p}-1}\left(\int_0^1 g_k^{(p-\epsilon)'}\,dx\right)^{\frac{1}{(p-\epsilon)'}}\int_0^1 \frac{w(t)^{1-\frac{\theta}{p}}}{t} t^{\frac{1}{p'}-\frac{1}{(p-\epsilon)'}}\,dt\end{aligned}$$

and by Lemma 5.1

$$\le K_{p,\theta}\epsilon^{-\frac{\theta}{p-\epsilon}}\left(\int_0^1 g_k^{(p-\epsilon)'}\,dx\right)^{\frac{1}{(p-\epsilon)'}}$$

so that

$$\|g_k\|_{(p',\theta}\le K_{p,\theta}\inf_{0<\epsilon<p-1}\epsilon^{-\frac{\theta}{p-\epsilon}}\left(\int_0^1 g_k^{(p-\epsilon)'}\,dx\right)^{\frac{1}{(p-\epsilon)'}}.$$

The functional $\|\cdot\|_{(p',\theta}$ satisfies the Fatou property (this follows easily from the corresponding properties of Lebesgue spaces along with the monotonicity properties of the decreasing rearrangement, see [12]) and, by Proposition 1.1, up to a multiplicative constant, satisfies the triangular inequality. In conclusion, there exist constants $H_{p,\theta}$, $J_{p,\theta}$ depending only on p and θ such that:

$$\begin{aligned}\|g\|_{(p',\theta}=\|\sum_{k=1}^{+\infty}g_k\|_{(p',\theta}&\le H_{p,\theta}\sum_{k=1}^{+\infty}\|g_k\|_{(p',\theta}\\ &\le J_{p,\theta}\sum_{k=1}^{+\infty}\inf_{0<\epsilon<p-1}\epsilon^{-\frac{\theta}{p-\epsilon}}\left(\int_0^1 g_k^{(p-\epsilon)'}\,dx\right)^{\frac{1}{(p-\epsilon)'}}.\end{aligned}$$

Since (g_k) is arbitrary, we get the assertion. □

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