A generalization of the fundamental theorem of Brown for fine ferromagnetic particles

G. Di Fratta, C. Serpico, M. d'Aquino

A generalization of the fundamental theorem of Brown for fine ferromagnetic particles

G. Di Fratta[a], C. Serpico[a], M. d'Aquino[b]

a. Department of Engineering, University of Naples Federico II, via Claudio, 21 I-80125 Naples, Italy

b. Department of Technology, University of Napoli "Parthenope", I-80143 Napoli, Italy

Abstract. In this paper we extend the Brown's fundamental theorem on fine ferromagnetic particles to the case of a general ellipsoid. By means of Poincaré inequality for the Sobolev space $H^1(\Omega, \mathbb{R}^3)$, and some properties of the induced magnetic field operator, it is rigorously proven that for an ellipsoidal particle, with diameter d, there exists a critical size (diameter) d_c such that for $d < d_c$ the uniform magnetization states are the only global minimizers of the Gibbs-Landau free energy functional $\mathcal{G}_{\mathcal{L}}$. A lower bound for d_c is then given in terms of the demagnetizing factors.

Keywords: Micromagnetics, Single Domain Particles, Poincaré inequality

Table of contents

1. Introduction

Theoretical discussions of the coercivity of magnetic materials make considerable use of the following idea [Brown]: "*whereas a ferromagnetic material in bulk* (*in zero applied field*) *possesses a domain structure, the same material in the form of a sufficiently fine* *particle is uniformly magnetized to (very near) the saturation value, or in other words consists of a single domain*". On the other hand, as Brown points out in [Brown]: "*the idea as thus expresses, scarcely is to be called a theorem, for it is not a proved proposition nor a strictly true one*".

The first rigorous formulation of this idea is due to Brown himself who, in his fundamental paper [Brown] rigorously proved for spherical particles what is known as *Brown's fundamental theorem of the theory of fine ferromagnetic particles*. This fundamental theorem states the existence of a *critical radius* r_c of the spherical particle such that for $r < r_c$ and zero applied field the state of lowest free energy (the ground state) is one of uniform magnetization.

The physical importance of Brown's fundamental theorem is that it formally explains, although in the case of spherical particles, the high coercivity that fine particles materials have, compared with the same material in bulk [Brown]. In fact, if the particles are fine enough to be single domain, and magnetic interactions between particles have a negligible effect, each particle can reverse its magnetization only by rigid rotation of the magnetization vector of the particle as a whole, a process requiring a large reversed field (rather than by domain wall displacement, which is the predominant process in bulk materials at small fields) [Brown]. The main limitation of the theorem is that it applies to spherical particles whereas, real particles are most of the time elongated [Aharoni]. Motivated by this, Aharoni [Aharoni], by using the same mathematical reasoning as Brown, was able to extend the Fundamental Theorem to the case of a *prolate spheroid*.

The main objective of this paper is to extend, using the Poincaré inequality for the Sobolev space $H^1(\Omega, \mathbb{R}^3)$ [Payne, Bebe] and some properties of the magnetostatic self-energy [Brown2, Brown3, Friedman, Aharoni3], the fundamental theorem of Brown to the case of a *general ellipsoid*. In the sequel, it is rigorously proven that for an *ellipsoidal* particle, with diameter d, there exists a *critical size* (diameter) d_c such that for $d < d_c$ the uniform magnetization states are the only global minimizers of the micromagnetic free energy functional. A lower bound for d_c is then given in terms of the demagnetizing tensor eigenvalues [DeSim] (the so called *demagnetizing factors* [Osborn]), which completely characterize the induced magnetic field inside ellipsoidal particles, thanks to Payne and Weinberger result on the best Poincaré constant [Payne, Bebe].

2. Formal theory of micromagnetic equilibria

We start our discussion by recalling basic facts about micromagnetic theory. According to micromagnetics, the local state of magnetization of the matter is described by a vector field, the magnetization $\boldsymbol{m}$, defined over Ω which is the region occupied by the body.

The stable equilibrium states of magnetization are the *minimizers* of the so called Gibbs-Landau free energy functional associated with the magnetic body. In dimensionless form, and for zero applied field, this functional can be written as [Brown, DeSim, Aharoni2, Serpico]:

$$\mathcal{G}_{\mathcal{L}}(\boldsymbol{m},\Omega)=\frac{1}{|\Omega|}\int_{\Omega}\left(\frac{\ell_{\rm ex}^2}{2}|\nabla\boldsymbol{m}|^2-\frac{1}{2}\mathbf{h}_d[\boldsymbol{m}]\cdot\boldsymbol{m}\right)\mathrm{d}\tau, \tag{1}$$

where $\boldsymbol{m}\colon\Omega\to\mathbf{S}^2$ is a vector field taking values on the unit sphere $\mathbf{S}^2$ of $\mathbb{R}^3$, and $|\Omega|$ denotes the volume of the region Ω, and $\ell_{\rm ex}^2$ is a positive material constant.

The constraint on the image of $\boldsymbol{m}$ is due to the following fundamental assumption of the micromagnetic theory: a ferromagnetic body well below the Curie temperature is always locally saturated. This means that the following constraint is satisfied:

$$|\boldsymbol{m}|=1 \quad \text{a.e. in } \Omega. \tag{2}$$

Global micromagnetic minimizers correspond to vector fields which minimize the Gibbs-Landau energy functional (1) in the class of vector fields which take values on the unit sphere $\mathbf{S}^2$.

2.1. The variational formulation for the demagnetizing field

The energy functional $\mathcal{G}_{\mathcal{L}}$ given by (1) is the sum of two terms: the exchange energy and the Maxwellian magnetostatic self-energy (the second term).

The *magnetostatic self-energy* is the energy due to the (dipolar) magnetic field $\mathbf{h}_d[\boldsymbol{m}]$ generated by $\boldsymbol{m}$. From the mathematical point of view, assuming Ω to be open, bounded and with a Lipschitz boundary, a given magnetization $\boldsymbol{m}\in L^2(\mathbb{R}^3,\mathbb{R}^3)$ generates the stray field $\boldsymbol{h}_{\rm d}[\boldsymbol{m}]=\nabla u_{\boldsymbol{m}}$ where the magnetostatic potential $u_{\boldsymbol{m}}$ solves:

$$\Delta u_{\boldsymbol{m}}=-\mathrm{div}\,(\boldsymbol{m}) \ \text{ in } \ \mathcal{S}'(\mathbb{R}^3). \tag{3}$$

A straightforward application of Lax-Milgram lemma guarantees that equation (3) has a unique solution into the Beppo-Levi space (cf. [Dautray-Lions])

$$W^1(\mathbb{R}^3)=\{u\in\mathcal{S}'(\mathbb{R}^3): u\omega\in L^2(\mathbb{R}^3), \nabla u\in L^2(\mathbb{R}^3,\mathbb{R}^3)\}, \text{ with } \quad \omega(x)=\frac{1}{\sqrt{1+|x|^2}}, \tag{4}$$

which is a Hilbert space when endowed with the norm $\|u\|^2_{W^1(\mathbb{R}^3)}= \ \|\nabla u\|^2$.

The quantity $\mathbf{h}_d[\boldsymbol{m}]:=\nabla u_{\boldsymbol{m}}$ is what is referred to as the demagnetizing field, and it is a linear and continuous operator from $L^2(\mathbb{R}^3,\mathbb{R}^3)$ into $L^2(\mathbb{R}^3,\mathbb{R}^3)$. In particular, $\boldsymbol{m}\chi_\Omega\in L^2(\mathbb{R}^3)$ for every $\boldsymbol{m}\in L^2(\Omega)$ and therefore $\mathbf{h}_d$ is a bounded linear operator also from $L^2(\Omega,\mathbb{R}^3)$ into $L^2(\mathbb{R}^3,\mathbb{R}^3)$. It is straightforward to check that the operator $-\mathbf{h}_d$ is self-adjoint and positive semidefinite:

$$(\mathbf{h}_d[\boldsymbol{m}],\boldsymbol{u})_\Omega=(\boldsymbol{m},\mathbf{h}_d[\boldsymbol{u}])_\Omega, \quad -(\mathbf{h}_d[\boldsymbol{m}],\boldsymbol{m})_\Omega=\|\mathbf{h}_d[\boldsymbol{m}]\|^2_\Omega\geq 0. \tag{5}$$

Obviously, the semidefinite positiveness of the induced magnetic field assures the positiveness of the Gibbs-Landau free energy functional.

Finally let us recall the following Brown lower bound to the magnetostatic self-energy [Brown, Brown2, Brown3] as reported by Brown in [Brown]: Consider an arbitrary irrotational vector field $\boldsymbol{h}$ which is defined over the whole space $\mathbb{R}^3$ and is regular at infinity. Under these assumptions, Brown proved that:

$$-\int_{\Omega}\boldsymbol{h}\cdot\boldsymbol{m}\,\mathrm{d}\tau-\frac{1}{2}\int_{\mathbb{R}^3}|\boldsymbol{h}|^2\,\mathrm{d}\tau\leq-\frac{1}{2}\int_{\Omega}\mathbf{h}_d[\boldsymbol{m}]\cdot\boldsymbol{m}\,\mathrm{d}\tau, \tag{6}$$

the equality holding if and only if $\boldsymbol{h}=\mathbf{h}_d[\boldsymbol{m}]$. In other terms, for every irrotational and regular at infinity vector field $\boldsymbol{h}\colon\mathbb{R}^3\to\mathbb{R}^3$, the left hand side of (6) does not exceed the magnetostatic self-energy and becomes equal to it only when $\boldsymbol{h}$ is everywhere equal to $\mathbf{h}_d[\boldsymbol{m}]$. It is worthwhile emphasizing that the vector field $\boldsymbol{h}$ in this inequality needs not be related in any way to $\boldsymbol{m}$ [Aharoni].

A very useful particular case of this lower bound can be obtained by letting $\boldsymbol{h}=\mathbf{h}_d[\boldsymbol{u}]$ with $u\in L^2(\Omega,\mathbb{R}^3)$. In this way we arrive at the following form of the Brown lower bound which we state here as a lemma:

Lemma 1. *Let $\Omega\subseteq\mathbb{R}^3$ be open, bounded and with Lipchitz boundary. For every $\boldsymbol{u}$, $\boldsymbol{m}\in L^2(\Omega,\mathbb{R}^3)$:*

$$-(\mathbf{h}_d[\boldsymbol{u}],\boldsymbol{m})_\Omega+\frac{1}{2}(\mathbf{h}_d[\boldsymbol{u}],\boldsymbol{u})_\Omega\leq-\frac{1}{2}(\mathbf{h}_d[\boldsymbol{m}],\boldsymbol{m})_\Omega, \tag{7}$$

with equality if and only if $\boldsymbol{u}=\boldsymbol{m}$.

3. The case of ellipsoidal geometry. Demagnetizing tensor

Since $\mathbf{h}_d$ is a linear operator, the restriction of $\mathbf{h}_d$ to the subspace $U(\Omega,\mathbb{R}^3)$ of constant in space vector fields can be identified with a second order tensor known as the *effective demagnetizing tensor* of Ω and defined by [DeSim, Osborn]:

$$N_{\mathrm{eff}}[\boldsymbol{m}]=-\int_{\Omega}\mathbf{h}_d[\boldsymbol{m}]\,\mathrm{d}\tau=-|\Omega|\langle\mathbf{h}_d[\boldsymbol{m}]\rangle_\Omega, \tag{8}$$

where $\boldsymbol{m}\in U(\Omega,\mathbb{R}^3)$ and for all $\boldsymbol{u}\in L^2(\Omega,\mathbb{R}^3)$ we have denoted by

$$\langle\boldsymbol{u}\rangle_\Omega=\frac{1}{|\Omega|}\int_{\Omega}\boldsymbol{u}\,\mathrm{d}\tau \tag{9}$$

the average of $\boldsymbol{u}$ over Ω. The tensor N_{eff} is known in the literature as the *effective demagnetizing tensor* of Ω, where the qualifier effective is used as a reminder of the fact that N_{eff} is related to the average of $\mathbf{h}_d[\boldsymbol{m}]$ over Ω [DeSim, Osborn].

In addition to that, a well known result of potential theory, states that when Ω is an ellipsoid and $\boldsymbol{m}\in U(\Omega,\mathbb{R}^3)$ also $\mathbf{h}_d[\boldsymbol{m}]\in U(\Omega,\mathbb{R}^3)$; i.e. if Ω is an ellipsoid and $\boldsymbol{m}$ is constant, then $\mathbf{h}_d[\boldsymbol{m}]$ is also constant in Ω.

In physical terms, this means that uniformly magnetized ellipsoids induce uniform magnetic fields in their interiors. In this case, the effective demagnetizing tensor N_{eff} is pointwise related to $\boldsymbol{m}$ since the relation (8) becomes:

$$N_{\text{eff}}[\boldsymbol{m}] = -\mathbf{h}_d[\boldsymbol{m}]. \tag{10}$$

In the rest of the present paper, we will indicate with N_d the demagnetizing tensor associated to an ellipsoidal particle Ω.

Obviuosly, from (5) we get that the quadratic form $Q_d(\boldsymbol{m}) = N_d[\boldsymbol{m}] \cdot \boldsymbol{m}$ is a definite positive quadratic form We will indicate with

$$\mu^2 = \inf_{u \in \mathbb{R}^3 - \{0\}} \frac{Q_d(\boldsymbol{u})}{|\boldsymbol{u}|^2} \tag{11}$$

the first eigenvalue associated to this quadratic form, i.e. the *minimum demagnetizing factor* for the ellipsoid Ω. This quantity can be expressed analytically in terms of elliptic integrals [Osborn].

It is important to stress that the eigenvalues of the quadratic form Q_d are shape-dependent but not size-dependent so that, when the volume $|\Omega|$ is changed by preserving the shape of the ellipsoid, μ^2 does not change.

4. The exchange energy and the Poincaré inequality. Null average micromagnetic minimizers

The *exchange energy* (the first term in eq. (1)), energetically penalize spatially non-uniform magnetization states: it takes into account the presence of the microscopic exchange interactions which tends to align the atomic magnetic moments.

A natural space in which to look for minimizers of the Gibbs-Landau functional is one in which the energy (1) is finite. Since the induced magnetic field operator $\mathbf{h}_d$ has a meaning in $L^2(\Omega, \mathbb{R}^3)$, and the exchange energy has a meaning in the Sobolev space $H^1(\Omega, \mathbb{R}^3)$ we will assume $\boldsymbol{m} \in H^1(\Omega, \mathbb{R}^3)$ and we will write $\boldsymbol{m} \in H^1(\Omega, \mathbf{S}^2)$ to emphasize that the magnetization field satisfies the local saturation constraint given by $|\boldsymbol{m}| = 1$ a.e. in Ω.

We recall that $H^1(\Omega, \mathbb{R}^3)$ is the space of square summable vector fields $\boldsymbol{m} \in L^2(\Omega, \mathbb{R}^3)$ whose first order weak partial derivatives $\partial_i \boldsymbol{m}$ belong to $L^2(\Omega, \mathbb{R}^3)$. We also recall that in the Sobolev space $H^1(\Omega, \mathbb{R}^3)$ the following Poincaré inequality holds [Payne, Bebe]:

LEMMA 2. *Let Ω be a bounded connected open subset of $\mathbb{R}^3$ with a Lipschitz boundary. Then there exists a constant C_P (the so called* Poincaré constant*), depending only on Ω, such that for every vector field $m \in H^1(\Omega, \mathbb{R}^3)$:*

$$\|\boldsymbol{m} - \langle \boldsymbol{m} \rangle_\Omega\|_\Omega \leq C_p \|\nabla \boldsymbol{m}\|_\Omega \tag{12}$$

where $\langle \boldsymbol{m} \rangle_\Omega$ denotes the spatial average of $\boldsymbol{m}$ over Ω (see eq. (9)*).*

For practical purposes is important to know an explicit expression for the Poincaré constant. The main result is this direction concerns the special case of a convex domain [Payne, Bebe].

Lemma 3. *Let* Ω *be a convex domain with diameter* $\mathrm{diam}(\Omega)$. *Then for every vector field* $\boldsymbol{m}\in H^1(\Omega,\mathbb{R}^3)$*:*

$$\|\boldsymbol{m}-\langle\boldsymbol{m}\rangle_\Omega\|_\Omega \leq \frac{\mathrm{diam}(\Omega)}{\pi}\|\nabla\boldsymbol{m}\|_\Omega. \tag{13}$$

In terms of the $L^2(\Omega,\mathbb{R}^3)$ norm and scalar product the Gibbs-Landau functional (1) reads as:

$$\mathcal{G}_{\mathcal{L}}(\boldsymbol{m},\Omega)=\frac{\ell_{\mathrm{ex}}^2}{2|\Omega|}\|\nabla\boldsymbol{m}\|_\Omega^2-\frac{1}{2|\Omega|}(\mathbf{h}_d[\boldsymbol{m}],\boldsymbol{m})_\Omega. \tag{14}$$

We now observe that if $\boldsymbol{m}_0\in H^1(\Omega,\,\mathbf{S}^2)$ is a global minimizer of the Gibbs-Landau energy functional (14) then for every $\boldsymbol{u}\in U(\Omega,\mathbb{R}^3)$ such that $|\boldsymbol{u}|=1$ a.e. in Ω, we have $\|\nabla\boldsymbol{u}\|_\Omega^2=0$. Thus

$$\mathcal{G}_{\mathcal{L}}(\boldsymbol{m}_0)\leq -\frac{1}{2|\Omega|}(\mathbf{h}_d[\boldsymbol{u}],\boldsymbol{u})_\Omega, \tag{15}$$

and hence:

$$\mathcal{G}_{\mathcal{L}}(\boldsymbol{m}_0)\leq\frac{1}{2}\inf_{|\boldsymbol{u}|=1}Q_d(\boldsymbol{u})=\frac{1}{2}\,\mu^2. \tag{16}$$

From this simple observation and the use of Poincaré inequality (12) we get that if $\boldsymbol{m}_0$ is a *null average* magnetization state, then

$$\mu^2\geq 2\,\mathcal{G}_{\mathcal{L}}(\boldsymbol{m}_0)\geq\frac{\ell_{\mathrm{ex}}^2}{C_P^2} \tag{17}$$

and therefore $C_P\geq\ell_{\mathrm{ex}}\,\mu^{-1}$. Thus we proved the following lemma:

Lemma 4. *Let* $\Omega\subseteq\mathbb{R}$ *be an ellipsoid and let* $\boldsymbol{m}_0\in H^1(\Omega,\mathbf{S}^2)$ *be a global minimizer of the Gibbs-Landau energy functional* (1). *If* $\langle\boldsymbol{m}_0\rangle_\Omega=0$ *then*

$$\mathrm{diam}(\Omega)\geq\pi\,\ell_{\mathrm{ex}}\,\mu^{-1} \tag{18}$$

where we have indicated with $\mathrm{diam}(\Omega)$ *the diameter of the ellipsoid* Ω.

We recall that $\mathrm{diam}(\Omega)$ is defined as the largest distance between couples of points in Ω, and in the case of an ellipsoid it coincides with two times the largest semiaxis.

By letting $|\Omega|$ decrease by keeping the shape of ellipsoid invariant, so that μ is constant, we arrive to a violation of the the inequality (18) which implies that zero-average global minimizers cannot exist when the dimension of the particle is reduced below the critical diameter $\pi\,\ell_{\mathrm{ex}}\,\mu^{-1}$.

From the physical point of view, this result is interesting in its own right when one interprets zero-average global minimizers as the usual demagnetized states of a magnetic particle. The above Lemma implies that there is no unmagnetized ground state in fine particles.

5. The generalization of the fundamental theorem of Brown to the case of ellipsoidal particles

Consider a homogeneous ferromagnetic particle occupying the region of space Ω which is assumed to be a general ellipsoid in $\mathbb{R}^3$ and let $\boldsymbol{m}\in H^1(\Omega,\mathbf{S}^2)$. From (7) we have that for every constant in space vector field $\boldsymbol{u}\in U(\Omega,\mathbb{R}^3)$:

$$|\Omega|\, N_d[\boldsymbol{u}]\cdot\langle\boldsymbol{m}\rangle_\Omega-\frac{1}{2}|\Omega|\, Q_d(\boldsymbol{u})\leq-\frac{1}{2}(\mathbf{h}_d[\boldsymbol{m}],\boldsymbol{m})_\Omega. \tag{19}$$

In particular, letting $\boldsymbol{u}=\langle\boldsymbol{m}\rangle$ we get that for all $\boldsymbol{m}\in L^2(\Omega,\mathbb{R}^3)$:

$$|\Omega| Q_d(\langle\boldsymbol{m}\rangle_\Omega)\leq-(\boldsymbol{h}_d[\boldsymbol{m}],\boldsymbol{m})_\Omega. \tag{20}$$

From Lemma 4 we get that if $C_P<\ell_{\rm ex}\,\mu^{-1}$ then the global minimizer $\boldsymbol{m}_0$ cannot be null average ($\langle\boldsymbol{m}_0\rangle_\Omega\neq 0$) and so after dividing and multiplying the left hand side of (20) by $|\langle\boldsymbol{m}_0\rangle_\Omega|^2$, passing to the inf we get:

$$|\langle\boldsymbol{m}_0\rangle_\Omega|^2\,\mu^2\leq-(\mathbf{h}_d[\boldsymbol{m}],\boldsymbol{m})_\Omega. \tag{21}$$

From (16) and (21) we infer that if $\boldsymbol{m}_0$ is a global minimizer for $\mathcal{G}_\mathcal{L}$ then:

$$\mu^2\geq 2\,\mathcal{G}_\mathcal{L}(\boldsymbol{m}_0)\geq\frac{\ell_{\rm ex}^2}{C_P^2}(1-|\langle\boldsymbol{m}_0\rangle_\Omega|^2)+|\langle\boldsymbol{m}_0\rangle_\Omega|^2\,\mu^2, \tag{22}$$

where the first lower bound is due to Poincaré inequality (12).

Thus we arrive at the conclusion that if $\boldsymbol{m}_0$ is a global minimizer for $\mathcal{G}_L$ then:

$$(1-|\langle\boldsymbol{m}_0\rangle|^2)\left(\frac{\ell_{\rm ex}^2}{C_P^2}-\mu^2\right)\leq 0. \tag{23}$$

As a consequence, if $C_P<\ell_{\rm ex}\,\mu^{-1}$, then $|\langle\boldsymbol{m}_0\rangle|^2=1$ and hence $\boldsymbol{m}_0$ is contant a.e. in Ω.

We have in this way proved the following generalization of Brown's fundamental theorem for fine ferromagnetic particles:

THEOREM 5. *Let $\Omega\subseteq\mathbb{R}$ be an ellipsoid and let $\boldsymbol{m}_0\in H^1(\Omega,\mathbf{S}^2)$ be a global minimizer of the Gibbs-Landau energy functional* (1)*. If $C_P<\ell_{\rm ex}\,\mu^{-1}$ then $\boldsymbol{m}_0\in U(\Omega,\mathbb{R}^3)$, i.e. $\boldsymbol{m}_0$ is constant a.e. in Ω. Thus a sufficient condition for $\boldsymbol{m}_0$ to be constant is that*

$$\operatorname{diam}(\Omega)<\pi\,\ell_{\rm ex}\,\mu^{-1} \tag{24}$$

where $\operatorname{diam}(\Omega)$ *is twice the largest semi-axis of the ellipsoid Ω.*

The inequality (24) means that if we consider particles of given ellipsoidal shape (given ratio of semi-axes) with decreasing volume, there is a critical dimension below which the global minimizers (ground states) are uniform.

It is interesting to consider the case of very slender ellipsoid, i.e. an ellipsoid with semiaxis $a \gg b \geq c$). In this case, the asymptotic behavior of μ^2 is given by [Osborn]:

$$\mu^2 \approx \frac{b\,c}{a^2}\left[\log\left(\frac{4\,a}{b+c}\right) - 1\right]. \tag{25}$$

Now, by using the notation $\beta = b/a$ and $\gamma = c/a$, and the fact that $\mathrm{diam}(\Omega) = 2a$, the inequality (24) can be read as

$$a < a_c = \frac{\pi}{2}\,\ell_{\mathrm{ex}}\frac{1}{\sqrt{\beta\gamma}}\left[\log\left(\frac{4}{\beta+\gamma}\right) - 1\right]^{-1/2}, \tag{26}$$

which provides a more explicit lower bound for the critical size to have spatially uniform ground in ellipsoidal particles.

6. Some remarks on the value of the critical size. The best Poincaré constant in the case of a spherical particle

It is well known that the best Poincaré constant in $H^1(\Omega,\ \mathbb{R}^3)$, in the class of all convex domains having the same diameter, is given by $C_P = \mathrm{diam}(\Omega)/\pi$ [Payne, Bebe]. However, it is also well known that once fixed the domain Ω (not just the diameter), the best Poincaré constant is given by $C_P = \lambda_1^{-1}$ where λ_1 is the smallest positive eigenvalue associated with the following Neumann problem for the Helmholtz equation [Saloff]:

$$\begin{cases} -\Delta\,\varphi = \lambda\,\varphi \ \text{ in } \Omega \\ \partial_n\,\varphi = 0 \ \text{ on } \partial\,\Omega \end{cases}. \tag{27}$$

Thus a better estimate of (24) can be obtained by solving equations (27) when the geometry of Ω is that of the general ellipsoid under consideration.

For the case of a spherical particle (a ball of radius r) the first eigenvalue of (27) is given by $\lambda_1 = \frac{x_{11}}{r}$, where x_{11} is the first positive root of the equation:

$$2\,x\,J'_{1+\frac{1}{2}}(x) - J_{1+\frac{1}{2}}(x) = 0, \tag{28}$$

and where we have indicated with J_α the Bessel functions of the first kind [Polyanin, Lizorkin]. Equivalently the factor x_{11} can be found computing the first positive root of the equation $j_1'(x) = 0$ where we have indicated with j_1 the spherical Bessel function [Lizorkin], related to J_α by the equation:

$$j_1(x) = \frac{1}{\sqrt{2x/\pi}}\,J_{1+\frac{1}{2}}(x). \tag{29}$$

A numerical computation gives for this first positive root the following approximated value $x_{11} \approx 2.0816$. Thus recalling that in the case of a sphere [Brown2, DeSim2]:

$$\mu^2 = \inf_{|u|=1} Q_d(\boldsymbol{u}) = \frac{1}{3}, \tag{30}$$

we get, from theorem 5, that m_0 is constant in space when $C_P = \frac{r}{x_{11}} < \ell_{\text{ex}}\, \mu^{-1}$, and this inequality holds if and only if:

$$r < r_c \quad , \quad r_c \approx 3.6055\, \ell_{\text{ex}}. \tag{31}$$

Thus, for the special case of a spherical particle, we arrive at the same estimate found by Brown in [Brown].

7. Final considerations

We have extended the Brown's fundamental theorem on fine ferromagnetic particles to the case of a general ellipsoid, and given (by means of Poincaré inequality for the Sobolev space $H^1(\Omega, \mathbb{R}^3)$) an upper bound to the critical size (diameter) under which the uniform magnetization states are the only global minimizers of the Gibbs-Landau free energy functional $\mathcal{G}_{\mathcal{L}}$. Although for the sake of clarity we have neglected any anisotropy energy term in the expression of the Gibbs-Landau functional (1), it is straightforward to extend the result to the case when (for example) uniaxial anisotropy of the easy-axis type is present. The problem of local minimizers of the Gibbs-Landau functional is currently under investigation and will be presented in future publications.

Bibliography

[Aharoni] Aharoni A., Elongated single-domain ferromagnetic particles, J. Appl. Phys., **63**, (1988) 5879–5881.

[Aharoni3] Aharoni A., Useful Upper and Lower Bounds to the Magnetostatic Self-Energy, IEEE Trans. Magn., **27**, (1991) 4793–4795.

[Aharoni2] Aharoni A., Introduction to the Theory of Ferromagnetism, Oxford Press, New York, 1996.

[Bebe] Bebendorf M., A Note on the Poincaré Inequality for Convex Domains, J. Anal. Appl., **22**, (2003) 751–756.

[Brown] Brown W. F., Jr., The Fundamental Theorem of the Theory of Fine Ferromagnetic Particles, J. Appl. Phys., **39**, (1968) 993–1010.

[Brown2] Brown W. F., Jr., Magnetostatic Principles in Ferromagnetism, North-Holland Publishing Co., 1962.

[Brown3] Brown W. F., Jr., Approximate Calculation of Magnetostatic Energies, J. Phys. Soc. Japan, Suppl. B-I to **17**, (1962) 540–542.

[Dautray-Lions] Dautray, R. and Lions, J.L., Mathematical Analysis and Numerical Methods for Science and Technology: Volume 4 Integral Equations and Numerical Methods, Springer Berlin Heidelberg, (1999).

[DeSim] DeSimone A., Hysteresis and Imperfections Sensitivity in Small Ferromagnetic Particles, Mecca. **30** (1995) 591–603.

[DeSim2] DeSimone A., Energy minimizers for large ferromagnetic bodies, Arch. Rat. Mech. Anal. **125** (1993) 99–143.

[Friedman] Friedman M. J., Mathematical study of the nonlinear singular integral magnetic field equation. I, SIAM J. Appl. Math. **39** (1980) 14–20.

[Lizorkin] Lizorkin P. I., Bessel functions, in: Hazewinkel, M. (Ed), Encyclopedia of Mathematics, Springer Verlag, 2001.

[Osborn] Osborn J. A., Demagnetizing factors of the general ellipsoid, Phys. Rev., **67**, (1945) 351–356.

[Payne] Payne L. E., Weinberger H. F., An optimal Poincaré inequality for convex domains, Arch. Rat. Mech. Anal. **5**, (1960) 268–292.

[Polyanin] Polyanin A. D., Linear Partial Differential Equations for Engineers and Scientists, CRC Press, London, 2002.

[Saloff] Saloff-Coste L., Aspects of Sobolev-type inequalities, CUP, London, 2002.

[Serpico] Mayergoyz I. D., Bertotti G., Serpico C., Nonlinear magnetization dynamics in nanosystems, Elsevier, United Kingdom, 2008.