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Syzygies among reduction operators

Cyrille Chenavier

Abstract

We introduce the notion of syzygy for a set of reduction operators and relate it to the notion of syzygy for presentations of algebras. We give a method for constructing a linear basis of the space of syzygies for a set of reduction operators. We interpret these syzygies in terms of the confluence property from rewriting theory. This enables us to optimise the completion procedure for reduction operators based on a criterion for detecting useless reductions. We illustrate this criterion with an example of construction of commutative Gröbner basis.

Keywords: reduction operators, syzygies, completion procedures, commutative Gröbner bases.

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1 Introduction

Description and computation of syzygies for presentations of algebraic structures has been investigated by methods from homological algebra, Koszul duality and Gröbner bases theory. In homological algebra, the constructive methods using syzygies are initiated in the works of Koszul [13] and Tate [20] who describe free resolutions by mean of higher-order syzygies. Koszul duality, introduced by Priddy [18] and extended by Berger [2], is inspired by these works: for homogeneous associative algebras, a candidate for the space of syzygies, that is for constructing a minimal resolution, is the Koszul dual.

For commutative algebras, methods for computing syzygies are based on Gröbner bases: the module of syzygies for a Gröbner basis is spanned by $S$-polynomials of critical pairs [19], that is the overlapping of two reductions, also called rewriting rules, on a term. Conversely, a critical pair whose $S$-polynomial reduces into zero leads to a syzygy. This correspondence between syzygies and critical pairs has applications in two directions: improvements of Buchberger’s completion algorithm are based on the computation of syzygies [9] [16] and construction of free resolutions of commutative algebras are based on the computation of a Gröbner basis [15]. The construction of free resolutions using rewriting theory

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for computing syzygies also appear for other algebraic structures, such as associative algebras [11] or monoids [10] [11] [22].

In this paper, we give a method based on the lattice of reduction operators for computing syzygies for rewriting systems whose underlying set of terms is a vector space. This method enables us to deduce a lattice criterion for detecting useless reductions during the completion procedure. As pointed out by Lazard [13], the completion procedure is interpreted as Gaussian elimination, which leads to use linear algebra techniques for studying completion. In particular, the \( F_1 \) and \( F_5 \) algorithms [6, 7] are based on such techniques and adaptations of Buchberger, \( F_4 \) or \( F_5 \) algorithms to various algebraic contexts were introduced, such as associative algebras [17, 22], invariant rings [8], tropical Gröbner bases [21] or operads [4], for instance.

We consider a vector space \( V \) equipped with a well-ordered basis \((G, <)\). For instance, if \( V \) is a polynomial algebra (respectively a tensor algebra, an invariant ring or an operad), \( G \) is a set of monomials (respectively words, orbit sums of monomials or trees) and \(<\) is an admissible order on \( G \).

In our examples, we consider the case where \( V \) is finite-dimensional and \((G, <)\) is a totally ordered basis of \( V \).

**Reduction operators.** In this work, we describe linear rewriting systems by reduction operators. A reduction operator relative to \((G, <)\) is an idempotent linear endomorphism \( T \) of \( V \) such that for every \( g \notin \text{im}(T) \), \( T(g) \) is a linear combination of elements of \( G \) strictly smaller than \( g \). We denote by \( \text{RO}(G, <) \) the set of reduction operators relative to \((G, <)\).

Recall from [3, Proposition 2.1.14] that the kernel map induces a bijection between \( \text{RO}(G, <) \) and subspaces of \( V \). Hence, \( \text{RO}(G, <) \) admits a lattice structure, where the order \( \leq \), the lower-bound \( \wedge \) and the upper-bound \( \vee \) are defined by

- \( T_1 \leq T_2 \) if \( \ker(T_2) \subseteq \ker(T_1) \),
- \( T_1 \wedge T_2 = \ker^{-1}(\ker(T_1) + \ker(T_2)) \),
- \( T_1 \vee T_2 = \ker^{-1}(\ker(T_1) \cap \ker(T_2)) \).

Given a subset \( F \) of \( \text{RO}(G, <) \), we denote by \( \wedge F \) the lower-bound of \( F \), that is the reduction operator whose kernel is the sums of kernels of elements of \( F \). We have the following lattice formulation of confluence: a subset \( F \) of \( \text{RO}(G, <) \) is said to be confluent if and only if the rewrite relation on \( V \) defined by \( v \rightarrow T(v) \), for every \( T \in F \) and every \( v \notin \text{im}(T) \), is confluent.

**Upper-bound of reduction operators and syzygies.** In [22, 23] we define the syzygies for a finite set \( F = \{T_1, \ldots, T_n\} \) of reduction operators as being the elements of the kernel of the application \( \pi_F : \ker(T_1) \times \cdots \times \ker(T_n) \rightarrow \ker(\wedge F) \), mapping \((v_1, \ldots, v_n)\) to \( v_1 + \cdots + v_n \). The set of syzygies for \( F \) is denoted by \( \text{syz}(F) \). In [24, 10] we interpret syzygies for presentations of algebras in terms of syzygies for a set of reduction operators.

In Lemma [22, 6] we show that for every integer \( 2 \leq i \leq n \), \( \text{syz}(T_1 \wedge \cdots \wedge T_{i-1}, T_i) \) is isomorphic to a supplement of \( \text{syz}(T_1, \cdots, T_{i-1}) \) in \( \text{syz}(T_1, \cdots, T_i) \). In Proposition [22, 8], we give an explicit description of this supplement using the operator \((T_1 \wedge \cdots \wedge T_{i-1}) \vee T_i \). Using these two intermediate results, we obtain a procedure for constructing a basis of \( \text{syz}(F) \): we construct inductively bases of \( \text{syz}(T_1, \cdots, T_i) \) using the supplement of \( \text{syz}(T_1, \cdots, T_{i-1}) \) defined from \((T_1 \wedge \cdots \wedge T_{i-1}) \vee T_i \). The correctness of this procedure is proven in Theorem [22, 7].

**Application to completion.** A completion of a set \( F = \{T_1, \cdots, T_n\} \) of reduction operators is a confluent set \( F' \) containing \( F \). In Section 3 we present a procedure for completing \( F \) taking into account useless reductions. For that, we first remark that the vector space \( \ker(T_1) \times \cdots \times \ker(T_n) \) admits as a basis the set of all \( e_{i,g} = (0, \cdots, 0, g - T_i(g), 0, \cdots, 0) \), where \( 1 \leq i \leq n \),
$g \notin \text{im}(T_i)$ and $g - T_i(g)$ is at position $i$. Using a well-order $\sqsubset$ on this basis, we consider the set $\tilde{F} = \{\tilde{T}_1, \ldots, \tilde{T}_n\}$ of reduction operators obtaining from $F$ removing the reductions

$$g \xrightarrow{F} T_i(g),$$

(1)

where $e_{i,g}$ is the leading term of an element of $\text{syz}(F)$ for the order $\sqsubset$. Formally, the operators $\tilde{T}_i$ are defined in the following way:

$$\tilde{T}_i(g) = \begin{cases} g, & \text{if } e_{i,g} \text{ is a leading term of an element of } \text{syz}(F) \\ T_i(g), & \text{otherwise.} \end{cases}$$

We call the set $\tilde{F}$, the *reduction* of $F$. In Section 3.2.1 we construct inductively a set $C = \{C_2, \ldots, C_n\}$ of reduction operators which leads to a completion of $\tilde{F}$. We call the set $C$ the *incremental completion* of $\tilde{F}$. In Theorem 3.2.4 we show that the reductions $g \xrightarrow{F} T_i(g)$, where $g \notin \text{im}((T_1 \land \cdots \land T_{i-1}) \lor T_i)$, are useless reductions.

**Useless reductions and construction of commutative Gröbner bases.** In Section 3.3 we relate the confluence property and the completion procedure for reduction operators to the construction of commutative Gröbner bases. We consider a set $X$ of variables as well as an ideal $I$ of $K[X]$ spanned by a set of polynomials $R = \{f_1, \ldots, f_n\}$. Given an admissible order on the set of monomials, we consider the reduction operator $T_i$ whose kernel is the ideal spanned by $f_i$. In Proposition 3.3.3 we show that $R$ is a Gröbner basis of $I$ if and only if the set $F_R = \{T_1, \ldots, T_n\}$ of reduction operators associated to $R$ is confluent. This characterisation of Gröbner bases enables us to interpret the completion of a set of reduction operators as a procedure for constructing commutative Gröbner bases. Hence, the criterion of Section 3.2 enables us to detect useless reductions during the construction of commutative Gröbner bases. In Example 3.3.6 we illustrate with an example how to use this criterion.

**Organisation.** In Section 2.1 we recall the definition and the lattice structure of reduction operators. We interpret the upper-bound of two reduction operators in terms of syzygies. In Section 2.2 we construct a basis of syzygies using the lattice structure of reduction operators. In particular, we characterise leading terms of syzygies using the lattice structure. In Section 2.3 we illustrate how our basis is constructed. In Section 3.1 we recall how works the completion in terms of reduction operators. In Section 3.2 we exploit the relationship between syzygies and useless reductions as well as our construction of a basis of syzygies to provide a lattice criterion for rejecting useless reductions during a completion procedure. In Section 3.3 we show how to use this criterion during the construction of commutative Gröbner bases.

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## 2 Computation of syzygies

In this section, we define syzygies for a set of reduction operators and we compute these syzygies using the lattice structure of reduction operators.
2.1 Syzygies for a set of reduction operators

We fix a commutative field $\mathbb{K}$ as well as a well-ordered set $(G, <)$. We denote by $\mathbb{K}G$ the vector space spanned by $G$.

2.1.1. Support, leading term and leading coefficient. For every $v \in \mathbb{K}G \setminus \{0\}$, we denote by $\text{supp}(v)$ the support of $v$, that is, the set of elements of $G$ which belongs to the decomposition of $v$. The greatest element of $\text{supp}(v)$ is denoted by $\text{lt}(v)$ and the coefficient of $\text{lt}(v)$ in $v$ is denoted by $\text{lc}(v)$. The notations $\text{lt}(v)$ and $\text{lc}(v)$ are the abbreviations of leading term and leading coefficient of $v$, respectively. Given a subset $E$ of $\mathbb{K}G$, we denote by $\text{lt}(E)$ the set of leading terms of elements of $E$: $\text{lt}(E) = \{ \text{lt}(v) \mid v \in E \}$. We extend the order $<$ on $G$ into a partial order on $\mathbb{K}G$ in the following way: we have $u < v$ if $u = 0$ and $v \neq 0$ or if $\text{lt}(u) < \text{lt}(v)$.

2.1.2. Reduced bases. Let $V$ be a subspace of $\mathbb{K}G$. A reduced basis of $V$ is a basis $\mathcal{B}$ of $V$ such that the following two conditions are fulfilled:

i. for every $e \in \mathcal{B}$, $\text{lc}(e)$ is equal to 1,

ii. given two different elements $e$ and $e'$ of $\mathcal{B}$, $\text{lt}(e')$ does not belong to the support of $e$.

Recall from [3, Theorem 2.1.13] that $V$ admits a unique reduced basis.

2.1.3. Reduction operators. A reduction operator relative to $(G, <)$ is an idempotent endomorphism $T$ of $\mathbb{K}G$ such that for every $g \in G$, we have $T(g) \leq g$. We denote by $\text{RO}(G, <)$ the set of reduction operators relative to $(G, <)$. Given $T \in \text{RO}(G, <)$, a term $g$ is said to be a $T$-normal form or $T$-reducible according to $T(g) = g$ or $T(g) \neq g$, respectively. We denote by $\text{NF}(T)$ the set of $T$-normal forms and by $\text{Red}(T)$ the set of $T$-reducible terms.

2.1.4. $T$-decompositions. Let $T \in \text{RO}(G, <)$. The kernel of $T$ admits as a basis the set of elements $g - T(g)$, where $g$ belongs to $\text{Red}(T)$. Hence, every $v \in \ker(T)$ admits a unique decomposition

$$v = \sum \lambda_g \left(g - T(g)\right).$$

(2)

The decomposition (2) is called the $T$-decomposition of $v$.

2.1.5. Kernels of reduction operators. Let $\mathcal{L}(\mathbb{K}G)$ be the set of subspaces of $\mathbb{K}G$. Recall from [3, Proposition 2.1.14] that the kernel map induces a bijection between $\text{RO}(G, <)$ and $\mathcal{L}(\mathbb{K}G)$. The inverse map is denoted by $\ker^{-1}$. Explicitly, for every $V \in \mathcal{L}(\mathbb{K}G)$, let $\mathcal{B}$ be the unique reduced basis of $V$. Then, $T = \ker^{-1}(V)$ is defined on the basis $G$ by:

$$T(g) = \begin{cases} g - e_g, & \text{if } g \in \text{lt}(\mathcal{B}) \\ g, & \text{otherwise,} \end{cases}$$

where $e_g$ is the unique element of $\mathcal{B}$ with leading term $g$.

In Section 2.2 we need the following lemma:

2.1.6. Lemma. Let $V$ be a subspace of $\mathbb{K}G$. We have an isomorphism:

$$\mathbb{K}G/V \simeq \mathbb{K}\left\{ g \in G \mid g \notin \text{lt}(V) \right\}.$$

Proof. Let $T = \ker^{-1}(V)$. The operator $T$ being a projector, we have an isomorphism between $\mathbb{K}G/V = \mathbb{K}G/\ker(T)$ and $\text{im}(T) = \mathbb{K}\text{NF}(T)$. The latter is equal to $\mathbb{K}\left\{ g \in G \mid g \notin \text{lt}(V) \right\}$, which proves Lemma 2.1.6 □
2.1.7. Lattice structure. We deduce from the bijection induced by the kernel map that RO(G, <) admits a lattice structure, where the order \( \preceq \), the lower-bound \( \land \) and the upper-bound \( \lor \) are defined by:

i. \( T_1 \preceq T_2 \) if \( \ker (T_1) \subseteq \ker (T_2) \).

ii. \( T_1 \land T_2 = \ker^{-1} (\ker (T_1) + \ker (T_2)) \).

iii. \( T_1 \lor T_2 = \ker^{-1} (\ker (T_1) \cap \ker (T_2)) \).

Given a subset \( F \) of RO(G, <), the lower-bound of \( F \) is written \( \land F \):

\[
\land F = \ker^{-1} \left( \sum_{F} \ker (T) \right).
\]

Moreover, recall from Lemma 2.1.18] that \( T_1 \preceq T_2 \) implies that NF(T_1) is included in NF(T_2). Passing to the complement, we obtain

\[
T_1 \preceq T_2 \text{ implies } \text{Red}(T_2) \subseteq \text{Red}(T_1).
\]

2.1.8. Syzygies. Let \( F = \{T_1, \ldots, T_n\} \) be a finite subset of RO(G, <). The vector space \( \ker (T_1) \times \cdots \times \ker (T_n) \) is denoted \( \ker (F) \). We consider the linear map \( \pi_F : \ker (F) \rightarrow \ker (\land F) \) defined by

\[
\pi_F (v_1, \ldots, v_n) = \sum_{i=1}^{n} v_i,
\]

for every \((v_1, \ldots, v_n) \in \ker (F)\). The elements of \( \ker (\pi_F) \) are called the syzygies for \( F \), and the set of syzygies for \( F \) is denoted by \( \text{syz} (F) \).

In Section 2.2, we construct a basis of \( \text{syz} (F) \). This construction requires to relate syzygies to the upper-bound of reduction operators. This link is given by the following proposition:

2.1.9. Proposition. Let \( P = \{T_1, T_2\} \) be a pair of reduction operators. We have an isomorphism:

\[
\ker (T_1 \lor T_2) \overset{\sim}{\longrightarrow} \text{syz} (P).
\]

Proof. Since \( \ker (T_1 \lor T_2) = \ker (T_1) \cap \ker (T_2) \), the map \( \pi_F \) is well-defined. Moreover, it is injective since \((0, 0)\) is equal to \( (0, 0) \) and only if \( v \) is equal to \( 0 \). Finally, it is surjective since \((v_1, v_2) \) belongs to \( \text{syz} (P) \) if and only if \( v_2 = -v_1 \) and in this case, \( v_2 \) belongs to \( \ker (T_1) \cap \ker (T_2) \).

2.1.10. Syzygies for presentations of algebras. We end this section by briefly relating syzygies for a set of reduction operators to syzygies for a presentation \((X \mid R)\) of a commutative or noncommutative algebra \( A \). We denote by \( \mathbb{K}[X] \) and \( T(X) \) the polynomial algebra and the tensor algebra over \( X \), respectively. Let \( G \) be the set of commutative or noncommutative monomials over \( X \), according to \( A \) is commutative or not, and let \( < \) be an admissible order on \( G \). We let \( R = \{f_1, \ldots, f_n\} \) and for every integer \( 1 \leq i \leq n \), we denote by \( T_i \in \text{RO}(G, <) \) the reduction operator whose kernel is the ideal of \( \mathbb{K}[X] \) or the two-sided ideal of \( T(X) \) spanned by \( f_i \), according to \( A \) is commutative or not. Then, the syzygies for the presentation \((X \mid R)\) are the syzygies for \((T_1, \ldots, T_n)\).

2.2 Construction of a basis of syzygies

Throughout the section, we fix a set \( F = \{T_1, \ldots, T_n\} \) of reduction operators.
2.2.1. Well-ordered basis of \( \ker(F) \). For every \( 1 \leq i \leq n \) and for every \( g \in \text{Red}(T_i) \), we denote by

\[
e_{i,g} = (0, \cdots, 0, g - T_i(g), 0, \cdots, 0),
\]

where \( g - T_i(g) \) is at position \( i \). The set of all \( e_{i,g} \)'s is a basis of \( \ker(F) \). Moreover, we let \( e_{i,g} \sqsubset e_{i',g'} \) if \( i < i' \) or if \( i = i' \) and \( g < g' \). Such defined, \( \sqsubset \) is a well-order, so that \( \ker(F) \) is a vector space equipped with a well-ordered basis.

2.2.2. Canonical decompositions. By definition of syzygies, we have an isomorphism of vector spaces \( \ker(F)/\text{syz}(F) \simeq \ker(\wedge F) \). From Lemma 2.1.6, \( \ker(\wedge F) \) admits as a basis the set

\[
\left\{ \pi_F(e_{i,g}) \mid e_{i,g} \notin \text{lt}(\text{syz}(F)) \right\},
\]

where \( \text{lt}(\text{syz}(F)) \) is the set of leading terms of elements of \( \text{syz}(F) \) for the order \( \sqsubset \). Hence, every \( v \in \ker(\wedge F) \) admits a unique decomposition

\[
v = \sum_{i,g} \lambda_{i,g} \pi_F(e_{i,g}) = \sum_{i,g} \lambda_{i,g}(g - T_i(g)),
\]

where, for every index \((i, g)\) in the sum, \( g \) belongs to \( \text{Red}(T_i) \). The decomposition \( \boxed{[5]} \) in called the **canonical decomposition** of \( v \) with respect to \( F \).

2.2.3. Procedure for constructing a basis of \( \text{syz}(F) \). For every integer \( i \) such that \( 2 \leq i \leq n \), we consider the reduction operator

\[
U_{i-1} = T_1 \wedge \cdots \wedge T_{i-1}.
\]

For every \( g_0 \in \text{Red}(U_{i-1} \vee T_i) \), we denote by

\[
v_{i,g_0} = g_0 - (U_{i-1} \vee T_i)(g_0).
\]

The vector \( v_{i,g_0} \) belongs to \( \ker(U_{i-1}) = \ker(T_1) + \cdots + \ker(T_{i-1}) \) and to \( \ker(T_i) \), so that it admits a canonical decomposition relative to \( \{T_1, \cdots, T_{i-1}\} \) as well as a \( T_i \)-decomposition. Let

\[
\sum_{j,g'} \lambda_{j,g'} (g' - T_j(g')) \quad \text{and} \quad \sum_{i} \lambda_{i,g} (g - T_i(g)),
\]

be these two decompositions. We let:

\[
\boxed{[6]} \quad s_{i,g_0} = \sum_{i} \lambda_{i,g} e_{i,g} - \sum_{j,g'} \lambda_{j,g'} e_{j,g'}.
\]

We define by inductions sets \( B_1, \cdots, B_n \) in the following way: \( B_1 = \emptyset \) and for every \( 2 \leq i \leq n \),

\[
B_i = B_{i-1} \cup \left\{ s_{i,g_0} \mid g_0 \in \text{Red}(U_{i-1} \vee T_i) \right\}.
\]

2.2.4. Correctness. The correctness of the procedure \( \boxed{[5]} \) proven in Theorem \( \boxed{[2.2.7]} \) is a consequence of Proposition \( \boxed{[2.2.6]} \), which we prove using intermediate results of Lemma \( \boxed{[2.2.8]} \).

Until the end of the section, we use notations that we present now. For every integer \( i \) such that \( 2 \leq i \leq n \), we define \( U_{i-1}, v_{i,g_0} \) and \( s_{i,g_0} \) such as in \( \boxed{[7]} \), \( \boxed{[8]} \) and \( \boxed{[9]} \) of \( \boxed{[2.2.3]} \) respectively and we consider the following maps:

\[
i_i : \text{syz}(T_1, \cdots, T_{i-1}) \longrightarrow \text{syz}(T_1, \cdots, T_i), (v_1, \cdots, v_{i-1}) \longmapsto (v_1, \cdots, v_{i-1}, 0),
\]

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ii. \( \pi_i : \ker(T_1) \times \cdots \times \ker(T_i) \to \ker(U_{i-1}) \times \ker(T_i), \ (v_1, \ldots, v_i) \mapsto (v_1 + \cdots + v_{i-1}, v_i) \),

iii. \( \tilde{\pi}_i : \text{syz}(T_1, \ldots, T_i) \to \text{syz}(U_{i-1}, T_i), \ (v_1, \ldots, v_i) \mapsto (v_1 + \cdots + v_{i-1}, v_i) \).

Moreover, we abuse notations in the following ways:

i. given two integers \( i \) and \( j \) such that \( 2 \leq j \leq i \leq n \), we still denote by \( e_{j,g} \) and \( s_{j,g} \) their images by the natural projection of \( \ker(F) \) on \( \ker(T_1) \times \cdots \times \ker(T_i) \).

ii. using the injection \( \iota_j \), we consider that we have \( \text{syz}(T_1, \cdots, T_{i-1}) \subseteq \text{syz}(T_1, \cdots, T_i) \), for every integer \( i \) such that \( 2 \leq i \leq n \).

2.2.5. Lemma. Let \( i \) be an integer such that \( 2 \leq i \leq n \).

i. We have \( \text{im}(\iota_i) = \ker(\tilde{\pi}_i) \).

ii. For every \( g_0 \in \text{Red}(U_{i-1} \lor T_i) \), we have

\[
\pi_i(s_{i,g_0}) = (-v_{i,g_0}, v_{i,g_0}).
\]

Proof. First, we show i. An element \( v = (v_1, \ldots, v_i) \in \text{syz}(T_1, \ldots, T_i) \) belongs to the kernel of \( \tilde{\pi}_i \) if and only if \( v_i = -(v_1 + \cdots + v_{i-1}) \) is equal to 0. Hence, \( v \) belongs to the the kernel of \( \tilde{\pi}_i \) if and only if it belongs to the image of \( \iota_i \).

Let us show ii. Let

\[
\sum_{j,g'} \lambda_{j,g'}(g' - T_j(g'))
\]

be the canonical decomposition of \( v_{i,g_0} \) with respect to \( \{T_1, \cdots, T_i\} \). Every index \( j \) of the sum is strictly smaller than \( i \), so that we have

\[
\pi_i \left( \sum_{j,g'} \lambda_{j,g'}e_{j,g'} \right) = \left( \sum_{j,g'} \lambda_{j,g'}(g' - T_j(g')) , 0 \right).
\]

Moreover, letting \( \sum_g \lambda_g(g - T_i(g)) \) the canonical the \( T_i \)-decomposition of \( v_{i,g_0} \), we have

\[
\pi_i \left( \sum_g \lambda_g e_{i,g} \right) = \left( 0, \sum_g \lambda_g(g - T_i(g)) \right).
\]

Hence, we have

\[
\pi_i(s_{i,g_0}) = \pi_i \left( \sum_g \lambda_g e_{i,g} - \sum_{j,g'} \lambda_{j,g'}e_{j,g'} \right)
\]

\[
= \left( 0, \sum_g \lambda_g(g - T_i(g)) \right) - \left( \sum_{j,g'} \lambda_{j,g'}(g' - T_j(g')) , 0 \right)
\]

\[
= (-v_{i,g_0}, v_{i,g_0}).
\]

\[\square\]
2.2.6. Proposition. Let \( i \) be an integer such that \( 2 \leq i \leq n \). We have the following direct sum decomposition:

\[
\text{syz} (T_1, \ldots, T_i) = \text{im} (t_i) \oplus \mathbb{K}\{ s_{i, g_0} \mid g_0 \in \text{Red} (U_{i-1} \lor T_i) \}.
\]

Proof. The set of all \( v_{i, g_0} \), where \( g_0 \) belongs to \( \text{Red} (U_{i-1} \lor T_i) \), is a basis of \( \ker (U_{i-1} \lor T_i) \), so that the set of pairs \( (-v_{i, g_0}, v_{i, g_0}) \), where \( g_0 \) belongs to \( \text{syz} (U_{i-1}, T_i) \) from Proposition 2.2.8. The morphism \( \tilde{\pi} \) from ii. of Lemma 2.2.5 \( \tilde{\pi} \) induces an isomorphism between the vector space \( V_i \) spanned by elements \( s_{i, g_0} \), where \( g_0 \) belongs to \( \text{Red} (U_{i-1} \lor T_i) \), and \( \text{im} (\tilde{\pi}_i) \). In particular, \( V_i \) is a supplement of \( \ker (\tilde{\pi}_i) \) in \( \text{syz} (T_1, \ldots, T_i) \). From i. of Lemma 2.2.5 \( \ker (\tilde{\pi}_i) \) is equal to \( \text{im} (t_i) \), which proves Proposition 2.2.6.

\[ \square \]

2.2.7. Theorem. The procedure 2.2.3 computes a basis of \( \text{syz} (F) \).

Proof. We show by induction that for every integer \( i \) such that \( 1 \leq i \leq n \), the set \( B_i \) obtained in 10 of the procedure is a basis of \( \text{syz} (T_1, \ldots, T_i) \). If \( i \) is equal to 1, there is nothing to prove since \( \text{syz} (T_1) \) is reduced to \( \{0\} \). Let \( i \) be an integer such that \( 2 \leq i \leq n \) and assume by induction hypothesis that \( B_{i-1} \) is a basis of \( \text{syz} (T_1, \ldots, T_{i-1}) \). From Proposition 2.2.6

\[ B_i = B_{i-1} \cup \{ s_{i, g_0} \mid g_0 \in \text{Red} (U_{i-1} \lor T_i) \}, \]

is a basis of \( \text{syz} (T_1, \ldots, T_i) \). Hence, \( B_n \) is a basis of \( \text{syz} (T_1, \ldots, T_n) = \text{syz} (F) \).

\[ \square \]

We deduce the following lattice description of the set of leading terms of syzygies:

2.2.8. Proposition. Let \( F = \{ T_1, \ldots, T_n \} \) be a finite set of reduction operators. We have

\[ \text{lt} (\text{syz} (F)) = \{ e_{i, g_0} \mid 2 \leq i \leq n \text{ and } g_0 \in \text{Red} (U_{i-1} \lor T_i) \}. \]

Proof. By definition, for every \( 2 \leq i \leq n \) and for every \( g_0 \in \text{Red} (U_{i-1} \lor T_i) \), \( \text{lt} (s_{i, g_0}) \) is equal to \( e_{i, g_0} \). Hence, the leading terms of the elements of \( B_n \) are pairwise distinct, so that we have

\[ \text{lt} (KB_n) = \text{lt} (B_n) = \{ e_{i, g_0} \mid 2 \leq i \leq n \text{ and } g_0 \in \text{Red} (U_{i-1} \lor T_i) \}. \]

From Theorem 2.2.7 \( B_n \) is a basis of \( \text{syz} (F) \), so that Proposition 2.2.8 holds.

\[ \square \]

2.2.9. Remark. In 2.1.10 we gave the relationship between syzygies for a set of reduction operators and syzygies for presentations of algebras. Hence, the set \( B_n \) constructed in 2.2.3 is a basis of syzygies for such presentations. In this context of presentations of algebras, the set of terms is a set of monomials, so that it is an infinite set and the construction of \( B_n \) is not an algorithm.

2.3 Illustration

In this section we illustrate the construction of \( B_n \) with an example. For that, we use the implementation of the lattice structure of reduction operators available online.

\[ \text{https://pastebin.com/Ds5haArR} \]
### 2.3.1. Example. We consider $G = \{g_1 < g_2 < g_3 < g_4 < g_5\}$. We let $F = \{T_1, T_2, T_3, T_4, T_5\}$, where the operators $T_i$ are defined by their matrices with respect to the basis $G$:

$$
T_1 = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix},
T_2 = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix},
T_3 = \begin{pmatrix}
1 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix},
T_4 = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix},
T_5 = \begin{pmatrix}
1 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix}.
$$

The vector space $\ker(F)$ is spanned by the following eight vectors:

$$
e_{1,g_5} = (g_5 - g_3, 0, 0, 0, 0),
\quad e_{2,g_5} = (0, g_3 - g_2, 0, 0, 0),
\quad e_{2,g_5} = (0, g_5 - g_2, 0, 0, 0),
\quad e_{3,g_5} = (0, 0, g_5 - g_1, 0, 0),
\quad e_{4,g_5} = (0, 0, 0, g_4 - g_3, 0),
\quad e_{5,g_5} = (0, 0, 0, g_4 - g_1).
$$

### 2.3.2. Notations. We simplify notations:

$$
e_1 = e_{1,g_5},
\quad e_2 = e_{2,g_3},
\quad e_3 = e_{2,g_5},
\quad e_4 = e_{3,g_5},
\quad e_5 = e_{4,g_4},
\quad e_6 = e_{5,g_4}.
$$

In particular, we have $e_1 < e_2 < \cdots < e_6$. Moreover, as done in the previous section, we let $U_{i-1} = T_1 \wedge \cdots \wedge T_{i-1}$, for $2 \leq i \leq 5$.

### 2.3.3. Step 1. We have $B_1 = \emptyset$.

### 2.3.4. Step 2. We have

$$
U_1 \vee T_2 = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}.
$$

The set $\text{Red}(T_1 \vee T_2)$ is reduced to $\{g_5\}$ and $g_5 - (T_1 \vee T_2)(g_5)$ is equal to $g_5 - g_3$. We have

$$
g_5 - g_3 = (g_5 - T_1(g_5)),
$$
and its $T_5$-decomposition is

$$
g_5 - g_3 = (g_5 - g_2) - (g_3 - g_2)
= (g_5 - T_2(g_5)) - (g_3 - T_2(g_3)).
$$

Hence, we get $B_2 = \{e_3 - e_2 - e_1\}$.
2.3.5. Step 3. The operator $U_2 \lor T_3$ is equal to the identity of $\mathbb{K}G$, so that we have $B_3 = B_2$.

2.3.6. Step 4. The operator $U_3 \lor T_4$ is equal to the identity of $\mathbb{K}G$, so that we have $B_4 = B_3$.

2.3.7. Step 5. We have

$$U_4 \lor T_5 = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$ 

The set $\text{Red}(U_4 \lor T_5)$ is reduced to $\{g_4\}$ and $g_4 - (U_4 \lor T_5)(g_4)$ is equal to $g_4 - g_1$. The canonical decomposition of $g_4 - g_1$ with respect to $\{T_1, T_2, T_3, T_4\}$ is equal to

$$g_4 - g_1 = (g_4 - g_3) - (g_5 - g_3) + (g_5 - g_1)$$

$$= (g_4 - T_4(g_4)) - (g_5 - T_1(g_5)) + (g_5 - T_3(g_5)),$$

and

$$g_4 - g_1 = (g_4 - T_5(g_4)).$$

Hence, we get $B_5 = \{e_3 - e_2 + e_1, e_6 - e_5 - e_4 + e_1\}$.

3 Useless reductions for the completion procedure

In this section, we interpret leading terms of syzygies as useless reductions during a completion procedure in rewriting theory. We deduce a lattice criterion for rejecting useless critical pairs for the construction of commutative Gröbner bases.

3.1 Reduction operators and completion

In this section, we recall from Section 2.3 the basic notions from rewriting theory used in the sequel and how reduction operators are related to abstract rewriting theory, confluence and completion.

3.1.1. Abstract rewriting systems. An abstract rewriting system is a pair $(A, \rightarrow)$, where $A$ is a set and $\rightarrow$ is a binary relation on $A$, called rewrite relation. An element of $\rightarrow$ is called a reduction and we write $a \rightarrow b$ instead of $(a, b) \in \rightarrow$ such a reduction. We denote by $\xrightarrow{*}$ the reflexive transitive closure of $\rightarrow$. If we have $a \xrightarrow{*} b$, we say that $a$ rewrites into $b$.

3.1.2. Confluence. Let $(A, \rightarrow)$ be an abstract rewriting system. We say that the rewrite relation $\rightarrow$ is confluent if for every $a, a_2, a_3 \in A$ such that $a_1 \xrightarrow{*} a_2$ and $a_1 \xrightarrow{*} a_3$, there exists $a_4 \in A$ such that $a_2 \xrightarrow{*} a_4$ and $a_3 \xrightarrow{*} a_4$: 

![Diagram of confluence]

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3.1.3. **Obstructions and confluence for reduction operators.** Let $F$ be a subset of $\text{RO}(G, \prec)$. We let:

$$\text{NF}(F) = \bigcap_{T \in F} \text{NF}(T).$$

For every $T \in F$, we have $\wedge F \preceq T$, so that $\text{NF}(\wedge F)$ is included in $\text{NF}(T)$ from \([3]\). Hence, $\text{NF}(\wedge F)$ is included in $\text{NF}(F)$ and we let $\text{Obs}^F = \text{NF}(F) \setminus \text{NF}(\wedge F)$. We say that $F$ is confluent if $\text{Obs}^F$ is equal to the empty set.

3.1.4. **Reduction operators and abstract rewriting.** Given a subset $F$ of $\text{RO}(G, \prec)$, we consider the abstract rewriting system \((K_G, \xrightarrow{F})\) defined by $v \xrightarrow{F} T(v)$, for every $T \in F$ and for every $v /\in K_{\text{NF}}(T)$. Recall from \([3, \text{Corollary 2.3.9}]\) that $F$ is confluent if and only if $\xrightarrow{F}$ is confluent.

3.1.5. **Example.** We consider the example of Section 2.3: $G = \{g_1 < g_2 < g_3 < g_4 < g_5\}$ and $F = \{T_1, T_2, T_3, T_4, T_5\}$, where

$$T_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad T_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad T_3 = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$T_4 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad T_5 = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

We have

$$\wedge F = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

We have $\text{NF}(\wedge F) = \{g_1\}$ and $\text{NF}(F) = \{g_1, g_2\}$, so that we have $\text{Obs}^F = \{g_2\}$, that is $F$ is not confluent. We check that the rewrite relation induced by $F$ is not confluent, since we have

Indeed, $g_4$ and $g_5$ rewrite into $g_2$ and $g_1$, but there is no reduction between $g_2$ and $g_1$. 
3.1.6. Completion. Let \( F \) be a subset of \( \mathsf{RO}(G, <) \). A completion of \( F \) is a subset \( F' \) of \( \mathsf{RO}(G, <) \) such that

i. \( F' \) is confluent,

ii. \( F \subseteq F' \) and \( \bigwedge F' = \bigwedge F \).

We denote by \( C^F \) the reduction operator \((\bigwedge F) \lor (\lor F)\), where \( \lor F \) is equal to \( \text{ker}^{-1}(\mathbb{K}N\mathsf{F}(F)) \).

Recall from [3, Theorem 3.2.6] that the set \( F \cup \{C^F\} \) is a completion of \( F \).

3.1.7. Example. Consider Example 3.1.5. We have:

\[
C^F = \begin{pmatrix}
1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix}.
\]

We check that \( F \cup \{C^F\} \) is a completion of \( F \) by the following diagram:

![Diagram]

3.1.8. Ambiguities. Given a subset \( F \) of \( \mathsf{RO}(G, <) \), an ambiguity of \( F \) is a triple \((g_0, T, T')\) such that \( g_0 \) belongs to \( \text{Red}(T) \cap \text{Red}(T') \). The possible obstructions to confluence come from these ambiguities, as it is the case in Example 3.1.5 since we have the following non confluent diagrams

![Non-confluent diagrams]

We see that among the three ambiguities \((g_5, T_1, T_3)\), \((g_5, T_2, T_3)\) and \((g_4, T_4, T_5)\), two can be avoided during the completion procedure since they are completed using a single reduction: \( g_2 \rightarrow g_1 \).

In the next section, we introduce a lattice criterion for detecting these useless ambiguities. In order to prove the correctness of this criterion, we need Proposition 3.1.9.

3.1.9. Proposition. Let \( F \) and \( C \) be two subsets of \( \mathsf{RO}(G, <) \). Then, \( F \cup C \) is a completion of \( F \) if and only if

\[
\text{Obs}^F \subseteq \bigcup_{T \in C} \text{Red}(T) \quad \text{and} \quad \bigwedge F \preceq \bigwedge C.
\]

Proof. We denote by \( \text{Red}(C) \) the union of the sets \( \text{Red}(T) \), where \( T \) belongs to \( C \).

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The relation $\wedge F \preceq \wedge C$ is equivalent to $(\wedge F) \wedge (\wedge C) = \wedge F$, that is it is equivalent to the relation $\wedge (F \cup C) = \wedge F$. Hence, we have to show that given a set $C$ of reduction operators such that $\wedge F \preceq \wedge C$, $F \cup C$ is confluent if and only if $\text{Obs}^F$ is included in $\text{Red}(C)$.

Let $C \subset \text{RO}(G, <)$ such that $\wedge F \preceq \wedge C$, that is $(\wedge F) \wedge (\wedge C) = \wedge F$. The set $F \cup C$ is confluent if and only if $\text{NF}(F \cup C) = \text{NF}(\wedge (F \cup C))$, that is $F \cup C$ is confluent if and only if $\text{NF}(F) \cap \text{NF}(C)$ is equal to $\text{NF}(\wedge F)$. By definition of $\text{Obs}^F$, we have

$$\text{NF}(F) \cap \text{NF}(C) = \left(\text{NF}(\wedge F) \cap \text{NF}(C)\right) \cup \left(\text{Obs}^F \cap \text{NF}(C)\right).$$

From [2.1.7] the inequality $\wedge F \preceq \wedge C$ implies that $\text{NF}(\wedge F)$ is included in $\text{NF}(\wedge C)$, which is included in $\text{NF}(C)$. Hence, we have

$$\text{NF}(F) \cap \text{NF}(C) = \text{NF}(\wedge F) \cup \left(\text{Obs}^F \cap \text{NF}(C)\right).$$

Hence, $F \cup C$ is confluent if and only if $\text{Obs}^F \cap \text{NF}(C)$ is empty, that is if and only if $\text{Obs}^F$ is included in the complement of $\text{NF}(C)$. The latter is equal to $\text{Red}(C)$, which concludes the proof.

\[\square\]

### 3.2 Completion procedure using syzygies

In this section we introduce a lattice criterion for detecting useless reductions during an incremental completion procedure of a set of reduction operators. We fix a finite subset $F = \{T_1, \cdots, T_n\}$ of $\text{RO}(G, <)$.

**3.2.1. Incremental completion.** We define by induction subsets $F_1, \cdots, F_n$ of $\text{RO}(G, <)$ in the following way: $F_1 = \{T_1\}$ and for every $2 \leq i \leq n$,

$$F_i = F_{i-1} \cup \{T_i, C_i\},$$

where $C_i = C^{F_{i-1} \cup \{T_1\}}$. The set $C = \{C_2, \cdots, C_n\}$ is called the incremental completion of $F$. The terminology comes from the fact that $F_i$ is equal to $\{T_1, \cdots, T_i, C_2, \cdots, C_i\}$ and it is confluent for every integer $i$ such that $1 \leq i \leq n$.

**3.2.2. Reduction of $F$.** For every integer $i$ such that $1 \leq i \leq n$, let $\tilde{T}_i$ be the reduction operator defined by

$$\tilde{T}_i(g) = \begin{cases} g, & \text{if } e_{i,g} \in \text{lt}(\text{syz}(F)) \\ T_i(g), & \text{otherwise,} \end{cases}$$

for every $g \in G$. The set $\tilde{F} = \{\tilde{T}_1, \cdots, \tilde{T}_n\}$ is called the reduction of $F$.

In Theorem [3.2.3] we show that the incremental completion procedure of $\tilde{F}$ leads to a completion of $F$. This is a consequence of the following proposition:

**3.2.3. Proposition.** We have $\wedge \tilde{F} = \wedge F$ and $\text{Obs}^F \subseteq \text{Obs}^\tilde{F}$.

**Proof.** First we prove that $\wedge \tilde{F} = \wedge F$. Let $S$ be the set of pairs $(i,g)$ such that $e_{i,g}$ belongs to $\text{lt}(\text{syz}(F))$. For every pair $(i,g)$ such that $1 \leq i \leq n$ and $g \in \text{Red}(T_i)$, we let

$$u_{i,g} = g - T_i(g).$$

We have

$$\ker(\wedge F) = \sum_{(j,g') \in S} \mathbb{K}_{u_{j,g'}} + \sum_{(i,g) \in S} \mathbb{K}_{u_{i,g}}.$$
and
\[ \ker(\wedge \tilde{F}) = \sum_{(i,g') \notin S} \mathbb{K}u_{i,g'} . \]

Hence, in order to prove that $\wedge \tilde{F} = \wedge F$, it is sufficient to show that each $u_{i,g}$ such that $(i,g) \in S$ belongs to the vector space spanned by $u_{i,g}$’s such that $(j,g') \notin S$.

Let $\mathcal{B}$ be the reduced basis of $\text{syz}(F)$. From Proposition 3.2.5, $\text{lt}(\mathcal{B})$ is equal to the set of $e_{i,g}$’s such that $(i,g) \in S$. Let
\[ b_{i,g} = e_{i,g} - \sum_{(j,g') \notin S} \lambda_{j,g'}e_{j,g'}, \]
be the element of $\mathcal{B}$ such that $\text{lt}(b_{i,g})$ is equal to $e_{i,g}$. The element $b_{i,g}$ being a syzygy, we have
\[ u_{i,g} = g - T_{i}(g) = \sum_{j,g' \notin S} \lambda_{j,g'}(g' - T_{j}(g')), \]
which proves that $\wedge \tilde{F} = \wedge F$.

Let us show that $\text{Obs}^{F} \subseteq \text{Obs}^{\tilde{F}}$. For every integer $i$ such that $1 \leq i \leq n$, $\text{NF}(T_{i})$ is included in $\text{NF}(\tilde{T}_{i})$, so that $\text{NF}(F)$ is included in $\text{NF}(\tilde{F})$. Moreover, we have $\wedge \tilde{F} = \wedge F$, so that $\text{Obs}^{F} = \text{NF}(F) \setminus \text{NF}(\wedge F)$ is included in $\text{Obs}^{\tilde{F}} = \text{NF}(\tilde{F}) \setminus \text{NF}(\wedge F)$.

\[ \square \]

3.2.4. Theorem. Let $F$ be a set of reduction operators, let $\tilde{F}$ be the reduction of $F$ and let $C$ be the incremental completion of $\tilde{F}$. Then, $F \cup C$ is a completion of $F$.

Proof. By construction, $\tilde{F} \cup C$ is a completion of $\tilde{F}$. From Proposition 3.1.9, $\text{Obs}^{\tilde{F}}$ is included in the union $\text{Red}(C)$ of the sets $\text{Red}(C)$ and $\wedge \tilde{F}$ is smaller than $\wedge C$. From Proposition 3.2.3, $\text{Obs}^{\tilde{F}}$ is included in $\text{Red}(C)$ and $\wedge F$ is smaller than $\wedge C$ for $\prec$. Using again Proposition 3.1.9, $F \cup C$ is a completion of $F$.

\[ \square \]

3.2.5. Lattice criterion for detecting useless reductions. Combining Theorem 3.2.4 and Theorem 3.2.5, we deduce a lattice criterion for detecting useless reductions during a completion procedure: they are the reductions $g \rightarrow T_{i}(g)$, where $g$ belongs to the kernel of $U_{i-1} \cup T_{i}$.

3.2.6. Example. We consider Example 3.1.5. For that, we use the basis of syzygies constructed in Section 2.3. We have two elements in $\text{lt}(\text{syz}(F))$: $e_{2,g_{3}}$ and $e_{5,g_{4}}$. In particular, we have $\tilde{T}_{i} = T_{i}$ for $i = 1, 2, 3$, and for $i = 2$ or 5, we have
\[ \tilde{T}_{2} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \tilde{T}_{5} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} . \]

We have $C_{i} = \text{Id}_{\mathbb{K}G}$ for $i \neq 3$ and
\[ C_{3} = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} . \]
3.3 Useless reductions and commutative Gröbner bases

In this section, we relate the completion of a set of reduction operators to the construction of commutative Gröbner bases. Let $X$ be a set of variables and let us denote by $[X]$ and $\mathbb{K}[X]$ the set of monomials and the polynomial algebra over $X$, respectively. We fix a set $R = \{f_1, \cdots, f_n\}$ of polynomials as well as an admissible order $<$. We associate to $R$ the set $F_R = \{T_1, \cdots, T_n\}$ of reduction operators with respect to $([X], <)$, where the kernel of $T_i$ is the ideal of $\mathbb{K}[X]$ spanned by $f_i$, for every integer $i$ such that $1 \leq i \leq n$.

3.3.1. Reduction operators associated to $F$. We associate to $R$ the set $F_R = \{T_1, \cdots, T_n\}$ of reduction operators with respect to $([X], <)$, where the kernel of $T_i$ is the ideal of $\mathbb{K}[X]$ spanned by $f_i$, for every integer $i$ such that $1 \leq i \leq n$.

3.3.2. Remark. For every integer $1 \leq i \leq n$ and for every monomial $m$, $T_i(m)$ satisfies one of the following two conditions:

i. if $m$ is equal to $\text{ltx}(f_i)m'$ for a monomial $m'$, then we have $T_i(m) = 1/\text{lc}(f_i)(r(f_i)m')$, where $r(f_i) = \text{lc}(f_i)\text{ltx}(f_i) - f_i$;

ii. if $m$ is not divisible by $\text{ltx}(f_i)$, then $T_i(m) = m$.

In particular, $NF(T_i)$ is the set of monomials which are not divisible by $\text{ltx}(f_i)$, so that $NF(F)$ is the set of monomials which do not belong to the monomial ideal spanned by $\text{ltx}(F)$.

3.3.3. Proposition. Let $I$ be an ideal of $\mathbb{K}[X]$. A generating set $R$ of I is a Gröbner basis of I if and only if the set $F_R$ of reduction operators associated to $R$ is confluent.

Proof. The kernel of $\land F$ is the sum of the kernels of the operators $T_1, \cdots, T_n$, that is it is equal to $I$. Hence, $\text{Red}(\land F)$ is equal to $\text{ltx}(I)$. Moreover, $F$ is confluent if and only if $NF(F) = NF(\land F)$, that is if and only if the complements of $NF(F)$ and $NF(\land F)$ in $[X]$ are equal. Hence, $F$ is confluent if and only if the monomial ideal spanned by $\text{ltx}(F)$ is equal to $\text{ltx}(I)$, that is if and only if $R$ is a Gröbner basis of $I$.

3.3.4. Corollary. Let $I$ be an ideal of $\mathbb{K}[X]$, let $R$ be a generating set of $I$ and let $F_R$ be the reduction of the set $F_R$ of reduction operators associated to $R$. Let $R' \subset \mathbb{K}[X]$ such that $F_R \cup F_{R'}$ is confluent. Then, $R \cup R'$ is a Gröbner basis of $I$.

Proof. This is a consequence of Proposition 3.3.3 and Theorem 3.2.4.

3.3.5. Useless reductions. From Theorem 3.3.4 we deduce the following criterion for detecting useless reductions during the construction of Gröbner bases: they are the reductions induced by $mf_i$, where $m$ is a monomial such that $\text{mlt}(f_i)$ is reducible for $(T_1 \land \cdots \land T_{i-1}) \lor T_i$. We illustrate this criterion with the following example:

3.3.6. Example. Consider the example from [5 Example 4.3.4]: let $X = \{x, y, z, t\}$, let $< $ be the DRL-order induced by $t < z < y < x$ and let $R = \{f_1, f_2, f_3\}$, where $f_1 = y^2 - xz$, $f_2 = x^2 - yz$ and $f_3 = xyz - y^2z$. We denote by $T_i$ the reduction operator whose kernel is the ideal spanned by $f_i$. There is no critical pair between $f_1$ and $f_2$, so that $\{f_1, f_2\}$ is a Gröbner of the ideal spanned by $f_1$ and $f_2$. When considering $f_3$, there are two critical pairs:

i. $xy^2z$ is reducible both by $f_1$ and $f_3$,
ii. $x^2yz$ is reducible both by $f_2$ and $f_3$.

The polynomial $g = x^2yz - y^3z + xyz^2 - y^2z^2$ belongs to the kernel of $(T_1 \land T_2) \lor T_3$ since we have:

\[ g = xzf_1 + (yz + z^2)f_2 = (x + y + z)f_3. \]

Hence, the reduction induced by $xf_3$ is a useless reduction so that we can reject the second critical pair. Moreover, when reducing the $S$-polynomial of the first critical pair, we get the new polynomial $f_4 = xz^3 - yz^3$. We obtain two new critical pairs:

i. $x^2z^3$ is reducible both by $f_2$ and $f_4$,

ii. $xyz^3$ is reducible both by $f_3$ and $f_4$.

The polynomials $x^2z^3 - y^2z^3 + xz^4 - yz^4$ and $xyz^3 - y^3z^3$ belong to the kernel of $(T_1 \land T_2 \land T_3) \lor T_4$. Indeed, we have:

\( (x + y + z)f_4 = z^3(f_2 - f_1) \) and $yf_4 = z^2f_3$.

Hence, the reductions induced by $xf_4$ and $yf_4$ are useless reductions, so that we can reject the two critical pairs. Hence, \{f_1, f_2, f_3, f_4\} is a Gröbner basis of the ideal spanned by \{f_1, f_2, f_3\}.

**Conclusion.** We presented a method based on lattice constructions for constructing a basis of the space of syzygies for a set of reduction operators. Using the relationship between syzygies and useless reductions during the completion procedure, we deduced a lattice criterion for detecting these reductions and thus for avoiding useless critical pairs during the construction of commutative Gröbner bases. When syzygies are infinite dimensional, our method does not lead to an algorithm since infinite computations are necessary. However, this work was motivated by computation of syzygies for richer structures than vector spaces. Hence, a further work is to exploit these structures for obtaining an algorithm.

**References**


