Syzygies among reduction operators

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Abstract

We introduce the notion of syzygy for a set of reduction operators and relate it to the notion of syzygy for presentations of algebras. We give a method for constructing a linear basis of the space of syzygies for a set of reduction operators. We interpret these syzygies in terms of the confluence property from rewriting theory. This enables us to optimise the completion procedure for reduction operators based on a criterion for detecting useless reductions. We illustrate this criterion with an example of construction of commutative Gröbner basis.

Keywords: reduction operators, syzygies, completion procedures, commutative Gröbner bases.

Contents

1 Introduction

Description and computation of syzygies for presentations of algebraic structures has been investigated by methods from homological algebra, Koszul duality and Gröbner bases theory. In homological algebra, the constructive methods using syzygies are initiated in the works of Koszul [18] and Tate [25] who describe free resolutions by mean of higher-order syzygies. Koszul duality, introduced by Priddy [23] and extended by Berger [5], is inspired by these works: for homogeneous associative algebras, a candidate for the space of syzygies, that is for constructing a minimal resolution, is the Koszul dual.

For commutative algebras, methods for computing syzygies are based on Gröbner bases: the module of syzygies for a Gröbner basis is spanned by S-polynomials of critical pairs [24], that is the overlapping of two reductions, also called rewriting rules, on a term. Conversely, a critical pair whose S-polynomial reduces into zero leads to a syzygy. This correspondence between syzygies and critical pairs has applications in two directions: improvements of Buchberger's completion algorithm are based on the computation of syzygies [14, 21] and construction of free resolutions of commutative algebras are based on the computation of a Gröbner basis [20]. The construction of free resolutions using rewriting theory

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for computing syzygies also appear for other algebraic structures, such as associative algebras [1, 8] or monoids [15, 16, 17].

In this paper, we give a method based on the lattice of reduction operators for computing syzygies for rewriting systems whose underlying set of terms is a vector space. Description of rewriting systems by mean of reduction operators was initiated in the works of Bergman [6] for noncommutative Gröbner bases and exploited by Berger for studying homological properties of quadratic algebras [2, 3, 4]. Using reduction operators enables us to deduce a lattice criterion for detecting useless reductions during the completion procedure. As pointed out by Lazard [19], the completion procedure is interpreted as Gaussian elimination, which leads to use linear algebra techniques for studying completion. In particular, the F_4 and F_5 algorithms [11, 12] are based on such techniques and adaptations of Buchberger, F_4 or F_5 algorithms to various algebraic contexts were introduced, such as associative algebras [22, 27], invariant rings [13], tropical Gröbner bases [26] or operads [9], for instance.

We consider a vector space V equipped with a well-ordered basis $(G, <)$. For instance, if V is a polynomial algebra (respectively a tensor algebra, an invariant ring or an operad), G is a set of monomials (respectively words, orbit sums of monomials or trees) and $<$ is an admissible order on G. In our examples, we consider the case where V is finite-dimensional and $(G, <)$ is a totally ordered basis of V.

Reduction operators. In this work, we describe linear rewriting systems by *reduction operators*. A reduction operator relative to $(G, <)$ is an idempotent linear endomorphism T of V such that for every $g \notin \operatorname{im}(T)$, $T(g)$ is a linear combination of elements of G strictly smaller than g. We denote by $\mathbf{RO}(G, <)$ the set of reduction operators relative to $(G, <)$.

Recall from [7, Proposition 2.1.14] that the kernel map induces a bijection between $\mathbf{RO}(G, <)$ and subspaces of V. Hence, $\mathbf{RO}(G, <)$ admits a lattice structure, where the order $\preceq$, the lower-bound $\wedge$ and the upper-bound $\vee$ are defined by

- $T_1 \preceq T_2$ if $\ker(T_2) \subseteq \ker(T_1)$,
- $T_1 \wedge T_2 = \ker^{-1}(\ker(T_1) + \ker(T_2))$,
- $T_1 \vee T_2 = \ker^{-1}(\ker(T_1) \cap \ker(T_2))$.

Given a subset F of $\mathbf{RO}(G, <)$, we denote by $\wedge F$ the lower-bound of F, that is the reduction operator whose kernel is the sums of kernels of elements of F. We have the following lattice formulation of confluence: a subset F of $\mathbf{RO}(G, <)$ is said to be *confluent* if the image of $\wedge F$ is equal to the intersection of images of elements of F. Recall from [7, Corollary 2.3.9] that F is confluent if and only if the rewrite relation on V defined by $v \longrightarrow T(v)$, for every $T \in F$ and every $v \notin \operatorname{im}(T)$, is confluent.

Upper-bound of reduction operators and syzygies. In 2.1.3, we define the *syzygies* for a finite set $F = \{T_1, \cdots, T_n\}$ of reduction operators as being the elements of the kernel of the application $\pi_F : \ker(T_1) \times \cdots \times \ker(T_n) \longrightarrow \ker(\wedge F)$, mapping $(v_1, \cdots, v_n)$ to $v_1 + \cdots + v_n$. The set of syzygies for F is denoted by $\mathbf{syz}(F)$. In 3.3, we interpret syzygies for presentations of algebras in terms of syzygies for a set of reduction operators.

In Lemma 2.2.3, we show that for every integer $2 \leq i \leq n$, $\mathbf{syz}(T_1 \wedge \cdots \wedge T_{i-1}, T_i)$ is isomorphic to a supplement of $\mathbf{syz}(T_1, \cdots, T_{i-1})$ in $\mathbf{syz}(T_1, \cdots, T_i)$. In Proposition 2.2.4, we give an explicit description of this supplement using the operator $(T_i \wedge \cdots \wedge T_{i-1}) \vee T_i$. Using these two intermediate results, we obtain a procedure for constructing a basis of $\mathbf{syz}(F)$: we construct inductively bases of $\mathbf{syz}(T_1, \cdots, T_i)$ using the supplement of $\mathbf{syz}(T_1, \cdots, T_{i-1})$ defined from $(T_1 \wedge \cdots \wedge T_{i-1}) \vee T_i$. The correctness of this procedure is proven in Theorem 2.2.2.

Application to completion. A *completion* of a set $F = \{T_1, \cdots, T_n\}$ of reduction operators is a confluent set F' containing F. In Section 3, we present a procedure for completing F taking into account useless reductions, that is the reductions which do not change the final result of a completion procedure. This notion is formally defined in Definition 3.1.1.

We first remark that the vector space $\ker(T_1) \times \cdots \times \ker(T_n)$ admits as a basis the set of all $e_{i,g} = \big(0, \cdots, 0, g - T_i(g), 0, \cdots, 0\big)$, where $1 \leq i \leq n$, $g \notin \mathrm{im}(T_i)$ and $g - T_i(g)$ is at position i. Using a well-order $\sqsubset$ on this basis, we consider the set $\tilde{F} = \left\{\tilde{T}_1, \cdots, \tilde{T}_n\right\}$ of reduction operators obtaining from F removing the reductions

$$g \underset{F}{\longrightarrow} T_i(g), \tag{1}$$

where $e_{i,g}$ is the leading term of an element of $\mathbf{syz}(F)$ for the order $\sqsubset$. Formally, the operators $\tilde{T}_i$ are defined in the following way:

$$\tilde{T}_i(g) = \begin{cases} g, & \text{if } e_{i,g} \text{ is a leading term of an element of } \mathbf{syz}(F) \\ T_i(g), & \text{otherwise.} \end{cases}$$

We call the set $\tilde{F}$, the *reduction* of F. In 3.2.4, we construct inductively a set $C = \{C_2, \cdots, C_n\}$ of reduction operators which leads to a completion of $\tilde{F}$. We call the set C the *incremental completion* of $\tilde{F}$. In Theorem 3.2.5, we show that the reductions (1) are useless in the sense that C completes F:

Theorem 3.2.5. *Let F be a set of reduction operators, let $\tilde{F}$ be the reduction of F and let C be the incremental completion of $\tilde{F}$. Then, $F \cup C$ is a completion of F.*

Moreover, a consequence of our method for constructing the basis of $\mathbf{syz}(F)$ is that its leading terms are the elements $e_{i,g}$ such that g does not belong to the image of $(T_1 \wedge \cdots \wedge T_{i-1}) \vee T_i$. Hence, we obtain the following lattice criterion: the reductions $g \underset{F}{\longrightarrow} T_i(g)$, where $g \notin \mathrm{im}((T_1 \wedge \cdots \wedge T_{i-1}) \vee T_i)$, are useless reductions.

Useless reductions and construction of commutative Gröbner bases. In Section 3.3, we relate the confluence property and the completion procedure for reduction operators to the construction of commutative Gröbner bases. We consider a set X of variables as well as an ideal I of $\mathbb{K}[X]$ spanned by a set of polynomials $R = \{f_1, \cdots, f_n\}$. Given an admissible order on the set of monomials, we consider the reduction operator T_i whose kernel is the ideal spanned by f_i. In Proposition 3.3.4, we show that R is a Gröbner basis of I if and only if the set $F_R = \{T_1, \cdots, T_n\}$ of reduction operators associated to R is confluent. This characterisation of Gröbner bases enables us to interpret the completion of a set of reduction operators as a procedure for constructing commutative Gröbner bases. Hence, the criterion of Section 3.2 enables us to detect useless reductions during the construction of commutative Gröbner bases. In Example 3.3.6, we illustrate with an example how to use this criterion.

Organisation. In Section 2.1 we recall the definition and the lattice structure of reduction operators. We interpret the upper-bound of two reduction operators in terms of syzygies. In Section 2.2, we construct a basis of syzygies using the lattice structure of reduction operators. In particular, we characterise leading terms of syzygies using the lattice structure. In Section 2.3, we illustrate how our basis is constructed. In Section 3.1, we recall how works the completion in terms of reduction operators. In Section 3.2, we exploit the relationship between syzygies and useless reductions as well as our construction of a basis of syzygies to provide a lattice criterion for rejecting useless reductions during a completion procedure. In Section 3.3, we show how to use this criterion during the construction of commutative Gröbner bases.

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2 Computation of syzygies

In this section, we define syzygies for a set of reduction operators and we compute these syzygies using the lattice structure of reduction operators.

2.1 Syzygies for a set of reduction operators

Conventions and notations. We fix a commutative field $\mathbb{K}$ as well as a well-ordered set $(G, <)$. We denote by $\mathbb{K}G$ the vector space spanned by G.

For every $v \in \mathbb{K}G \setminus \{0\}$, we denote by $\mathrm{supp}(v)$ the *support* of v, that is the set of elements of G which belongs to the decomposition of v. The greatest element of $\mathrm{supp}(v)$ is denoted by $\mathrm{lt}(v)$ and the coefficient of $\mathrm{lt}(v)$ in v is denoted by $\mathrm{lc}(v)$. The notations $\mathrm{lt}(v)$ and $\mathrm{lc}(v)$ are the abbreviations of *leading term* and *leading coefficient* of v, respectively. Given a subset E of $\mathbb{K}G$, we denote by $\mathrm{lt}(E)$ the set of leading terms of elements of E: $\mathrm{lt}(E) = \Big\{\mathrm{lt}(v) \mid v \in E\Big\}$. We extend the order $<$ on G into a partial order on $\mathbb{K}G$ in the following way: we have $u < v$ if $u = 0$ and $v \neq 0$ or if $\mathrm{lt}(u) < \mathrm{lt}(v)$.

Let V be a subspace of $\mathbb{K}G$. A *reduced basis* of V is a basis $\mathscr{B}$ of V such that the following two conditions are fulfilled:

i. for every $e \in \mathscr{B}$, $\mathrm{lc}(e)$ is equal to 1,

ii. given two different elements e and e' of $\mathscr{B}$, $\mathrm{lt}(e')$ does not belong to the support of e.

Recall from [7, Theorem 2.1.13] that V admits a unique reduced basis.

Definition 2.1.1. A *reduction operator relative to* $(G, <)$ is an idempotent endomorphism T of $\mathbb{K}G$ such that for every $g \in G$, we have $T(g) \leq g$. We denote by $\mathbf{RO}(G, <)$ the set of reduction operators relative to $(G, <)$. Given $T \in \mathbf{RO}(G, <)$, a term g is said to be a *T-normal form* or *T-reducible* according to $T(g) = g$ or $T(g) \neq g$, respectively. We denote by $\mathrm{NF}(T)$ the set of T-normal forms and by $\mathrm{Red}(T)$ the set of T-reducible terms.

Kernels of reduction operators. Let $T \in \mathbf{RO}(G, <)$. The kernel of T admits as a basis the set of elements $g - T(g)$, where g belongs to $\mathrm{Red}(T)$. Hence, every $v \in \ker(T)$ admits a unique decomposition

$$v = \sum \lambda_g \Big(g - T(g)\Big), \tag{2}$$

The decomposition (2) is called the *T-decomposition* of v.

Let $\mathscr{L}(\mathbb{K}G)$ be the set of subspaces of $\mathbb{K}G$. Recall from [7, Proposition 2.1.14] that the kernel map induces a bijection between $\mathbf{RO}(G, <)$ and $\mathscr{L}(\mathbb{K}G)$. The inverse map is denoted by $\ker^{-1}$. Explicitly, for every $V \in \mathscr{L}(\mathbb{K}G)$, let $\mathscr{B}$ be the unique reduced basis of V. Then, $T = \ker^{-1}(V)$ is defined on the basis G by:

$$T(g) = \begin{cases} g - e_g, \text{ if } g \in \mathrm{lt}(\mathscr{B}) \\ g, \text{ otherwise,} \end{cases}$$

where e_g is the unique element of $\mathscr{B}$ with leading term g.

In Section 2.2, we need the following lemma:

Lemma 2.1.2. *Let V be a subspace of $\mathbb{K}G$. We have an isomorphism:*

$$\mathbb{K}G/V \simeq \mathbb{K}\Big\{g \in G \mid g \notin \mathrm{lt}(V)\Big\}.$$

Proof. Let $T = \ker^{-1}(V)$. The operator T being a linear map, we have an isomorphism between $\mathbb{K}G/V = \mathbb{K}G/\ker(T)$. Moreover, it is also a projector, so that we have $\mathrm{im}(T) = \mathbb{K}\mathrm{NF}(T)$. The latter is equal to $\mathbb{K}\Big\{g \in G \mid g \notin \mathrm{lt}(V)\Big\}$, which proves Lemma 2.1.2. □

Lattice structure. We deduce from the bijection induced by the kernel map that $\mathbf{RO}(G, <)$ admits a lattice structure, where the order $\preceq$, the lower-bound $\wedge$ and the upper-bound $\vee$ are defined by

i. $T_1 \preceq T_2$ if $\ker(T_2) \subseteq \ker(T_1)$,

ii. $T_1 \wedge T_2 = \ker^{-1}(\ker(T_1) + \ker(T_2))$,

iii. $T_1 \vee T_2 \ = \ \ker^{-1}(\ker(T_1) \cap \ker(T_2))$.

Given a subset F of $\mathbf{RO}(G, <)$, the lower-bound of F is written $\wedge F$:

$$\wedge F \ = \ \ker^{-1}\left(\sum_{T \in F} \ker(T)\right).$$

Moreover, recall from [7, Lemma 2.1.18] that $T_1 \preceq T_2$ implies that $\mathrm{NF}(T_1)$ is included in $\mathrm{NF}(T_2)$. Passing to the complement, we obtain

$$T_1 \preceq T_2 \ \text{ implies } \ \mathrm{Red}(T_2) \subseteq \mathrm{Red}(T_1). \tag{3}$$

Notations. Let $F = \{T_1, \cdots, T_n\}$ be a finite subset of $\mathbf{RO}(G, <)$. The vector space $\ker(T_1) \times \cdots \times \ker(T_n)$ is denoted $\mathbf{ker}(F)$. We consider the linear map $\pi_F : \mathbf{ker}(F) \longrightarrow \ker(\wedge F)$ defined by

$$\pi_F(v_1, \cdots, v_n) \ = \ \sum_{i=1}^{n} v_i,$$

for every $(v_1, \cdots, v_n) \in \mathbf{ker}(F)$.

Definition 2.1.3. The elements of $\ker(\pi_F)$ are called the *syzygies* for F, and the set of syzygies for F is denoted by $\mathbf{syz}(F)$.

In Section 2.2, we construct a basis of $\mathbf{syz}(F)$. This construction requires to relate syzygies to the upper-bound of reduction operators. This link is given by the following proposition:

Proposition 2.1.4. *Let $P = \{T_1, T_2\}$ be a pair of reduction operators. We have an isomorphism:*

$$\begin{aligned} \ker(T_1 \vee T_2) &\xrightarrow{\ \sim\ } \boldsymbol{syz}(P). \\ v &\longmapsto (-v, v) \end{aligned} \tag{4}$$

Proof. Since $\ker(T_1 \vee T_2)$ is equal to $\ker(T_1) \cap \ker(T_2)$, the map (4) is well-defined. Moreover, it is injective since $(-v, v)$ is equal to $(0, 0)$ if and only if v is equal to 0. Finally, it is surjective since (v_1, v_2) belongs to $\mathbf{syz}(P)$ if and only if $v_2 = -v_1$ and in this case, v_2 belongs to $\ker(T_1) \cap \ker(T_2)$. □

2.2 Construction of a basis of syzygies

Throughout the section, we fix a set $F = \{T_1, \cdots, T_n\}$ of reduction operators.

For every $1 \leq i \leq n$ and for every $g \in \mathrm{Red}(T_i)$, we denote by

$$e_{i,g} \ = \ \big(0, \cdots, 0, g - T_i(g), 0, \cdots, 0\big),$$

where $g - T_i(g)$ is at position i. The set of all $e_{i,g}$'s is a basis of $\mathbf{ker}(F)$. Moreover, we let $e_{i,g} \sqsubset e_{i',g'}$ if $i < i'$ or if $i = i'$ and $g < g'$. Such defined, $\sqsubset$ is a well-order, so that $\mathbf{ker}(F)$ is a vector space equipped with a well-ordered basis.

Remark 2.2.1. By definition of syzygies, we have an isomorphism of vector spaces $\mathbf{ker}(F)/\mathbf{syz}(F) \simeq \ker(\wedge F)$. From Lemma 2.1.2, $\ker(\wedge F)$ admits as a basis the set

$$\Big\{\pi_F(e_{i,g}) \ \mid \ e_{i,g} \notin \mathrm{lt}(\mathbf{syz}(F))\Big\}, \tag{5}$$

where $\mathrm{lt}(\mathbf{syz}(F))$ is the set of leading terms of elements of $\mathbf{syz}(F)$ for the order $\sqsubset$. Hence, every $v \in \ker(\wedge F)$ admits a unique decomposition

$$\begin{aligned} v &= \sum_{i,g} \lambda_{i,g} \pi_F(e_{i,g}) \\ &= \sum_{i,g} \lambda_{i,g}\big(g - T_i(g)\big), \end{aligned} \tag{6}$$

where, for every index (i,g) in the sum, g belongs to $\mathrm{Red}(T_i)$. The decomposition (6) in called the *canonical decomposition* of v with respect to F.

Procedure for constructing a basis of syz (F). For every integer i such that $2 \leq i \leq n$, we consider the reduction operator

$$U_{i-1} = T_1 \wedge \cdots \wedge T_{i-1}. \tag{7}$$

For every $g_0 \in \mathrm{Red}(U_{i-1} \vee T_i)$, we denote by

$$v_{i,g_0} = g_0 - (U_{i-1} \vee T_i)(g_0). \tag{8}$$

The vector v_{i,g_0} belongs to $\ker(U_{i-1}) = \ker(T_1) + \cdots + \ker(T_{i-1})$ and to $\ker(T_i)$, so that it admits a canonical decomposition relative to $\{T_1, \cdots, T_{i-1}\}$ as well as well as a T_i-decomposition. Let

$$\sum_{j,g'} \lambda_{j,g'}(g' - T_j(g')) \quad \text{and} \quad \sum_{g} \lambda_g (g - T_i(g)),$$

be these two decompositions. We let:

$$s_{i,g_0} = \sum_{g} \lambda_g e_{i,g} - \sum_{j,g'} \lambda_{j,g'} e_{j,g'}. \tag{9}$$

We define by induction sets $B_1, \cdots, B_n$ in the following way: $B_1 = \emptyset$ and for every $2 \leq i \leq n$,

$$B_i = B_{i-1} \cup \left\{ s_{i,g_0} \mid g_0 \in \mathrm{Red}(U_{i-1} \vee T_i) \right\}. \tag{10}$$

Theorem 2.2.2. *With the previous notations, B_n is a basis of **syz** (F).*

The proof of Theorem 2.2.2 is done at the end of the section. This is a consequence of Proposition 2.2.4, which we prove using intermediate results of Lemma 2.2.3. For that, we need to fix some notations.

Notations. For every integer i such that $2 \leq i \leq n$, we define U_{i-1}, v_{i,g_0} and s_{i,g_0} such as in (7), (8) and (9), respectively and we consider the following maps:

i. $\iota_i : \mathbf{syz}(T_1, \cdots, T_{i-1}) \longrightarrow \mathbf{syz}(T_1, \cdots, T_i), \ (v_1, \cdots, v_{i-1}) \longmapsto (v_1, \cdots, v_{i-1}, 0)$,

ii. $\pi_i : \ker(T_1) \times \cdots \times \ker(T_i) \longrightarrow \ker(U_{i-1}) \times \ker(T_i), \ (v_1, \cdots, v_i) \longmapsto (v_1 + \cdots + v_{i-1}, v_i)$,

iii. $\tilde{\pi}_i : \mathbf{syz}(T_1, \cdots, T_i) \longrightarrow \mathbf{syz}(U_{i-1}, T_i), \ (v_1, \cdots, v_i) \longmapsto (v_1 + \cdots + v_{i-1}, v_i)$.

Moreover, we abuse notations in the following ways:

i. given two integers i and j such that $2 \leq j \leq i \leq n$, we still denote by $e_{j,g}$ and $s_{j,g}$ their images by the natural projection of $\mathbf{ker}(F)$ on $\ker(T_1) \times \cdots \times \ker(T_i)$,

ii. using the injection ι_i, we consider that we have $\mathbf{syz}(T_1, \cdots, T_{i-1}) \subseteq \mathbf{syz}(T_1, \cdots, T_i)$, for every integer i such that $2 \leq i \leq n$.

Lemma 2.2.3. *Let i be an integer such that $2 \leq i \leq n$.*

i. *We have $\mathrm{im}(\iota_i) = \ker(\tilde{\pi}_i)$.*

ii. *For every $g_0 \in \mathrm{Red}(U_{i-1} \vee T_i)$, we have*

$$\pi_i(s_{i,g_0}) = (-v_{i,g_0}, v_{i,g_0}).$$

Proof. First, we show **i.** An element $\mathbf{v} = (v_1, \cdots, v_i) \in \mathbf{syz}(T_1, \cdots, T_i)$ belongs to the kernel of $\tilde{\pi}_i$ if and only if $v_i = -(v_1 + \cdots + v_{i-1})$ is equal to 0. Hence, $\mathbf{v}$ belongs to the the kernel of $\tilde{\pi}_i$ if and only if it belongs to the image of ι_i.

Let us show **ii.** Let

$$\sum_{j,g'} \lambda_{j,g'} (g' - T_j(g')), \tag{11}$$

be the canonical decomposition of v_{i,g_0} with respect to $\{T_1, \cdots, T_i\}$. Every index j of the sum (11) is strictly smaller than i, so that we have

$$\pi_i \left(\sum_{j,g'} \lambda_{j,g'} e_{j,g'} \right) = \left(\sum_{j,g'} \lambda_{j,g'} (g' - T_j(g')), 0 \right).$$

Moreover, letting $\sum_g \lambda_g (g - T_i(g))$ the canonical the T_i-decomposition of v_{i,g_0}, we have

$$\pi_i \left(\sum_g \lambda_g e_{i,g} \right) = \left(0, \sum_g \lambda_g (g - T_i(g)) \right).$$

Hence, we have

$$\begin{aligned} \pi_i(s_{i,g_0}) &= \pi_i \left(\sum_g \lambda_g e_{i,g} - \sum_{j,g'} \lambda_{j,g'} e_{j,g'} \right) \\ &= \left(0, \sum_g \lambda_g (g - T_i(g)) \right) - \left(\sum_{j,g'} \lambda_{j,g'} (g' - T_j(g')), 0 \right) \\ &= (-v_{i,g_0}, v_{i,g_0}). \end{aligned}$$

□

Proposition 2.2.4. *Let i be an integer such that $2 \leq i \leq n$. We have the following direct sum decomposition:*

$$\boldsymbol{syz}(T_1, \cdots, T_i) = \operatorname{im}(\iota_i) \oplus \mathbb{K}\Big\{ s_{i,g_0} \mid g_0 \in \operatorname{Red}(U_{i-1} \vee T_i) \Big\}.$$

Proof. The set of all v_{i,g_0}, where g_0 belongs to $\operatorname{Red}(U_{i-1} \vee T_i)$, is a basis of $\ker(U_{i-1} \vee T_i)$, so that the set of pairs $(-v_{i,g_0}, v_{i,g_0})$, where g_0 belongs to $\operatorname{Red}(U_{i-1} \vee T_i)$, is a basis of $\mathbf{syz}(U_{i-1}, T_i)$ from Proposition 2.1.4. The morphism $\tilde{\pi}_i$ is surjective, so that we have $\operatorname{im}(\tilde{\pi}_i) = \mathbf{syz}(U_{i-1}, T_i)$. Hence, from **ii.** of Lemma 2.2.3, $\tilde{\pi}_i$ induces an isomorphism between the vector space V_i spanned by elements s_{i,g_0}, where g_0 belongs to $\operatorname{Red}(U_{i-1} \vee T_i)$, and $\operatorname{im}(\tilde{\pi}_i)$. In particular, V_i is a supplement of $\ker(\tilde{\pi}_i)$ in $\mathbf{syz}(T_1, \cdots, T_i)$. From **i.** of Lemma 2.2.3, $\ker(\tilde{\pi}_i)$ is equal to $\operatorname{im}(\iota_i)$, which proves Proposition 2.2.4. □

Now, we can show Theorem 2.2.2.

Proof of theorem 2.2.2. We show by induction that for every integer i such that $1 \leq i \leq n$, the set B_i obtained in 10 of the procedure is a basis of $\mathbf{syz}(T_1, \cdots, T_i)$. If i is equal to 1, there is nothing to prove since $\mathbf{syz}(T_1)$ is reduced to $\{0\}$. Let i be an integer such that $2 \leq i \leq n$ and assume by induction hypothesis that B_{i-1} is a basis of $\mathbf{syz}(T_1, \cdots, T_{i-1})$. From Proposition 2.2.4

$$B_i = B_{i-1} \cup \Big\{ s_{i,g_0} \mid g_0 \in \operatorname{Red}(U_{i-1} \vee T_i) \Big\},$$

is a basis of $\mathbf{syz}(T_1, \cdots, T_i)$. Hence, B_n is a basis of $\mathbf{syz}(T_1, \cdots, T_n) = \mathbf{syz}(F)$.

□

We deduce the following lattice description of the set of leading terms of syzygies:

Proposition 2.2.5. *Let* $F = \{T_1, \cdots, T_n\}$ *be a finite set of reduction operators. We have*

$$\mathrm{lt}\left(\boldsymbol{syz}(F)\right) = \left\{ e_{i,g_0} \mid 2 \leq i \leq n \text{ and } g_0 \in \mathrm{Red}\left(U_{i-1} \vee T_i\right) \right\}.$$

Proof. By definition, for every $2 \leq i \leq n$ and for every $g_0 \in \mathrm{Red}\left(U_{i-1} \vee T_i\right)$, $\mathrm{lt}\left(s_{i,g_0}\right)$ is equal to e_{i,g_0}. Hence, the leading terms of the elements of B_n are pairwise distinct, so that we have

$$\begin{aligned} \mathrm{lt}\left(\mathbb{K}B_n\right) &= \mathrm{lt}\left(B_n\right) \\ &= \left\{ e_{i,g_0} \mid 2 \leq i \leq n \text{ and } g_0 \in \mathrm{Red}\left(U_{i-1} \vee T_i\right) \right\}. \end{aligned}$$

From Theorem 2.2.2, B_n is a basis of $\mathbf{syz}\left(F\right)$, so that Proposition 2.2.5 holds. ∎

2.3 Illustration

In this section we illustrate the construction of B_n with an example. For that, we use the implementation of the lattice structure of reduction operators available online[1].

Notations. We consider $G = \{g_1 < g_2 < g_3 < g_4 < g_5\}$. We let $F = \{T_1, T_2, T_3, T_4, T_5\}$, where the operators T_i are defined by their matrices with respect to the basis G:

$$T_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad T_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad T_3 = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$T_4 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad T_5 = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

The vector space $\mathbf{ker}(F)$ is spanned by the following eight vectors:

$$e_{1,g_5} = (g_5 - g_3, 0, 0, 0, 0), \ e_{2,g_3} = (0, g_3 - g_2, 0, 0, 0), \ e_{2,g_5} = (0, g_5 - g_2, 0, 0, 0)$$

$$e_{3,g_5} = (0, 0, g_5 - g_1, 0, 0), \ e_{4,g_4} = (0, 0, 0, g_4 - g_3, 0), \ e_{5,g_4} = (0, 0, 0, 0, g_4 - g_1).$$

We simplify notations:

$$e_1 = e_{1,g_5}, \ e_2 = e_{2,g_3}, \ e_3 = e_{2,g_5}$$

$$e_4 = e_{3,g_5}, \ e_5 = e_{4,g_4}, \ e_6 = e_{5,g_4}.$$

In particular, we have $e_1 < e_2 < \cdots < e_6$. Moreover, as done in the previous section, we let $U_{i-1} = T_1 \wedge \cdots \wedge T_{i-1}$, for $2 \leq i \leq 5$.

Step 1. We have $B_1 = \emptyset$.

[1] `https://pastebin.com/Ds5haArH`

Step 2. We have

$$U_1 \vee T_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The set $\operatorname{Red}(T_1 \vee T_2)$ is reduced to $\{g_5\}$ and $g_5 - (T_1 \vee T_2)(g_5)$ is equal to $g_5 - g_3$. We have

$$g_5 - g_3 = \Big(g_5 - T_1(g_5)\Big),$$

and its T_2-decomposition is

$$\begin{aligned} g_5 - g_3 &= \Big(g_5 - g_2\Big) - \Big(g_3 - g_2\Big) \\ &= \Big(g_5 - T_2(g_5)\Big) - \Big(g_3 - T_2(g_3)\Big). \end{aligned}$$

Hence, we get $B_2 = \Big\{e_3 - e_2 - e_1\Big\}$.

Step 3. The operator $U_2 \vee T_3$ is equal to the identity of $\mathbb{K}G$, so that we have $B_3 = B_2$.

Step 4. The operator $U_3 \vee T_4$ is equal to the identity of $\mathbb{K}G$, so that we have $B_4 = B_3$.

Step 5. We have

$$U_4 \vee T_5 = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

The set $\operatorname{Red}(U_4 \vee T_5)$ is reduced to $\{g_4\}$ and $g_4 - (U_4 \vee T_5)(g_4)$ is equal to $g_4 - g_1$. The canonical decomposition of $g_4 - g_1$ with respect to $\{T_1, T_2, T_3, T_4\}$ is equal to

$$\begin{aligned} g_4 - g_1 &= \Big(g_4 - g_3\Big) - \Big(g_5 - g_3\Big) + \Big(g_5 - g_1\Big) \\ &= \Big(g_4 - T_4(g_4)\Big) - \Big(g_5 - T_1(g_5)\Big) + \Big(g_5 - T_3(g_5)\Big), \end{aligned}$$

and

$$g_4 - g_1 = \Big(g_4 - T_5(g_4)\Big).$$

Hence, we get $B_5 = \Big\{e_3 - e_2 + e_1,\ e_6 - e_5 - e_4 + e_1\Big\}$.

3 Useless reductions for the completion procedure

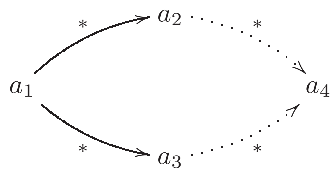
In this section, we interpret leading terms of syzygies as useless reductions during a completion procedure in rewriting theory. We apply this criterion to the construction of commutative Gröbner bases.

3.1 Reduction operators and completion

In this section, we recall from [7, Section 2.3] the basic notions from rewriting theory used in the sequel and how reduction operators are related to abstract rewriting theory, confluence and completion.

Abstract rewriting systems, confluence and completion. An *abstract rewriting system* is a pair $(A, \longrightarrow)$, where A is a set and $\longrightarrow$ is a binary relation on A, called *rewrite relation*. An element of $\longrightarrow$ is called a *reduction* and we write $a \longrightarrow b$ instead of $(a, b) \in \longrightarrow$ such a reduction. We denote by $\overset{*}{\longrightarrow}$ the reflexive transitive closure of $\longrightarrow$. If we have $a \overset{*}{\longrightarrow} b$, we say that *a rewrites into b*.

Let $(A, \longrightarrow)$ be an abstract rewriting system. We say that the rewrite relation $\longrightarrow$ is *confluent* if for every $a_1, a_2, a_3 \in A$ such that $a_1 \overset{*}{\longrightarrow} a_2$ and $a_1 \overset{*}{\longrightarrow} a_3$, there exists $a_4 \in A$ such that $a_2 \overset{*}{\longrightarrow} a_4$ and $a_3 \overset{*}{\longrightarrow} a_4$:



A *completion* of an abstract rewriting system $(A, \longrightarrow)$ is an abstract rewriting system $(A', \longrightarrow')$ such that

i. $A \subseteq A'$,

ii. the relation $\longrightarrow'$ is confluent,

iii. the residual sets obtained by taking the quotients of A and A' by the equivalence relations induced by $\longrightarrow$ and $\longrightarrow'$, respectively are equal.

In Section 3.2, we introduce a lattice criterion for detecting *useless reductions* during completion. Let us define formally the notion of useless reduction:

Definition 3.1.1. Let $(A, \longrightarrow)$ be an abstract rewriting system. A reduction $a \longrightarrow b$ is said to be *useless* if a completion of $(A, \longrightarrow')$, where $\longrightarrow'$ is $\longrightarrow$ without the reduction $a \longrightarrow b$, leads to a completion of $(A, \longrightarrow)$.

Reduction operators and abstract rewriting. Let F be a subset of $\mathbf{RO}(G, <)$. We let:

$$\mathrm{NF}(F) = \bigcap_{T \in F} \mathrm{NF}(T).$$

For every $T \in F$, we have $\wedge F \preceq T$, so that $\mathrm{NF}(\wedge F)$ is included in $\mathrm{NF}(T)$ from (3). Hence, $\mathrm{NF}(\wedge F)$ is included in $\mathrm{NF}(F)$ and we let $\mathrm{Obs}^F = \mathrm{NF}(F) \setminus \mathrm{NF}(\wedge F)$. We say that F is *confluent* if Obs^F is equal to the empty set.

Given a subset F of $\mathbf{RO}(G, <)$, we consider the abstract rewriting system $\left(\mathbb{K}G, \underset{F}{\longrightarrow}\right)$ defined by $v \underset{F}{\longrightarrow} T(v)$, for every $T \in F$ and for every $v \notin \mathbb{K}\mathrm{NF}(T)$. Recall from [7, Corollary 2.3.9] that F is confluent if and only if $\underset{F}{\longrightarrow}$ is confluent.

Example 3.1.2. We consider the example of Section 2.3: $G = \{g_1 < g_2 < g_3 < g_4 < g_5\}$ and $F = \{T_1, T_2, T_3, T_4, T_5\}$, where

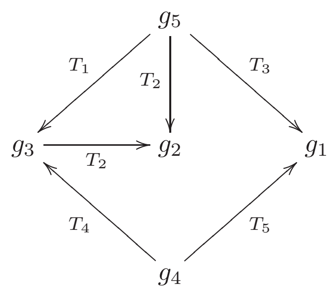
$$T_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad T_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad T_3 = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$T_4 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad T_5 = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

We have

$$\wedge F = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

We have $\mathrm{NF}(\wedge F) = \{g_1\}$ and $\mathrm{NF}(F) = \{g_1, g_2\}$, so that we have $\mathrm{Obs}^F = \{g_2\}$, that is F is not confluent. We check that the rewrite relation induced by F is not confluent, since we have



Indeed, g_4 and g_5 rewrite into g_2 and g_1, but there is no reduction between g_2 and g_1.

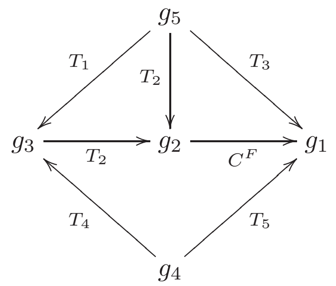
Definition 3.1.3. The completion procedure in terms of reduction operators is formalised as follows:

- **i.** Let F be a subset of $\mathbf{RO}(G, <)$. A *completion of* F is a subset F' of $\mathbf{RO}(G, <)$ such that
  - **i.** F' is confluent,
  - **ii.** $F \subseteq F'$ and $\wedge F' = \wedge F$.
- **ii.** We define the reduction operator C^F by $C^F = (\wedge F) \vee \big(\vee \overline{F}\big)$, where $\vee \overline{F}$ is equal to $\ker^{-1}(\mathbb{K}\mathrm{NF}(F))$. Recall from [7, Theorem 3.2.6] that the set $F \cup \{C^F\}$ is a completion of F.

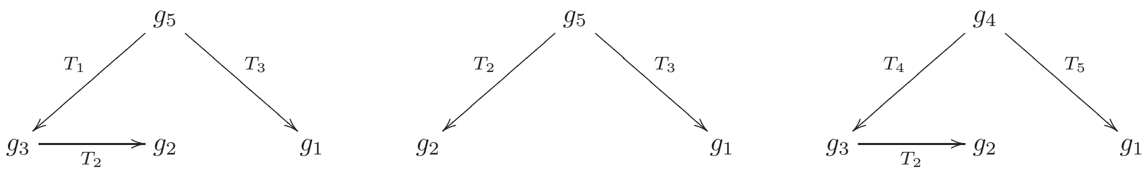
Example 3.1.4. Consider Example 3.1.2. We have:

$$C^F = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

We check that $F \cup \{C^F\}$ is a completion of F by the following diagram:



Remark 3.1.5. Given a subset F of $\mathbf{RO}(G, <)$, an *ambiguity* of F is a triple (g_0, T, T') such that g_0 belongs to $\mathrm{Red}(T) \cap \mathrm{Red}(T')$. The possible obstructions to confluence come from these ambiguities, as it is the case in Example 3.1.2 since we have the following non confluent diagrams



We see that among the three ambiguities (g_5, T_1, T_2), (g_5, T_2, T_3) and (g_4, T_4, T_5), two can be avoided during the completion procedure since they are completed using a single reduction: $g_2 \longrightarrow g_1$. In particular, detecting useless reductions enables us to remove ambiguities.

3.2 Completion procedure using syzygies

In this section, we define formally incremental completion procedures for reduction operators (see Definition 3.2.4) and we introduce a lattice criterion for detecting useless reductions during this procedure. This lattice criterion comes from the fact that leading terms of syzygies provide useless reductions as we will see in the sequel.

We fix a finite subset $F = \{T_1, \cdots, T_n\}$ of $\mathbf{RO}(G, <)$.

Definition 3.2.1. For every integer i such that $1 \leq i \leq n$, let $\tilde{T}_i$ be the reduction operator defined by

$$\tilde{T}_i(g) = \begin{cases} g, & \text{if } g \in \mathrm{Red}(T_i) \text{ and } e_{i,g} \in \mathrm{lt}(\mathbf{syz}(F)) \\ T_i(g), & \text{otherwise,} \end{cases}$$

for every $g \in G$. The set $\tilde{F} = \{\tilde{T}_1, \cdots, \tilde{T}_n\}$ is called the *reduction* of F.

In Theorem 3.2.5 we show that a completion of $\tilde{F}$ leads to a completion of F. This is a consequence of the following two propositions:

Proposition 3.2.2. *We have* $\wedge\tilde{F} = \wedge F$ *and* $\mathrm{Obs}^F \subseteq \mathrm{Obs}^{\tilde{F}}$.

Proof. First we prove that $\wedge\tilde{F} = \wedge F$. Let S be the set of pairs (i, g) such that $e_{i,g}$ belongs to $\mathrm{lt}(\mathbf{syz}(F))$. For every pair (i, g) such that $1 \leq i \leq n$ and $g \in \mathrm{Red}(T_i)$, we let

$$u_{i,g} = g - T_i(g).$$

We have

$$\ker(\wedge F) = \sum_{(j,g') \notin S} \mathbb{K}u_{j,g'} + \sum_{(i,g) \in S} \mathbb{K}u_{i,g},$$

and

$$\ker\left(\wedge\tilde{F}\right) = \sum_{(j,g')\,\notin\, S} \mathbb{K}u_{j,g'}.$$

Hence, in order to prove that $\wedge\tilde{F} = \wedge F$, it is sufficient to show that each $u_{i,g}$ such that $(i,g) \in S$ belongs to the vector space spanned by $u_{j,g'}$'s such that $(j,g') \notin S$.

Let $\mathscr{B}$ be the reduced basis of $\mathbf{syz}(F)$. From Proposition 2.2.5, $\mathrm{lt}(\mathscr{B})$ is equal to the set of $e_{i,g}$'s such that $(i,g) \in S$. Let

$$b_{i,g} = e_{i,g} - \sum_{(j,g')\,\notin\, S} \lambda_{j,g'}e_{j,g'},$$

be the element of $\mathscr{B}$ such that $\mathrm{lt}(b_{i,g})$ is equal to $e_{i,g}$. The element $b_{i,g}$ being a syzygy, we have

$$\begin{aligned} u_{i,g} &= g - T_i(g) \\ &= \sum_{j,g'\,\notin\, S} \lambda_{j,g'}\Big(g' - T_j(g')\Big), \end{aligned}$$

which proves that $\wedge\tilde{F} = \wedge F$.

Let us show that $\mathrm{Obs}^F \subseteq \mathrm{Obs}^{\tilde{F}}$. For every integer i such that $1 \le i \le n$, $\mathrm{NF}(T_i)$ is included in $\mathrm{NF}\left(\tilde{T}_i\right)$, so that $\mathrm{NF}(F)$ is included in $\mathrm{NF}\left(\tilde{F}\right)$. Moreover, we have $\wedge\tilde{F} = \wedge F$, so that $\mathrm{Obs}^F = \mathrm{NF}(F) \setminus \mathrm{NF}(\wedge F)$ is included in $\mathrm{Obs}^{\tilde{F}} = \mathrm{NF}\left(\tilde{F}\right) \setminus \mathrm{NF}(\wedge F)$.

□

Proposition 3.2.3. *Let C be a subset of $\mathbf{RO}(G, <)$. Then, $F \cup C$ is a completion of F if and only if*

$$\mathrm{Obs}^F \subseteq \bigcup_{T\,\in\, C} \mathrm{Red}(T) \quad \textit{and} \quad \wedge F \preceq \wedge C.$$

Proof. We denote by $\mathrm{Red}(C)$ the union of the sets $\mathrm{Red}(T)$, where T belongs to C.

The relation $\wedge F \preceq \wedge C$ is equivalent to $(\wedge F) \wedge (\wedge C) = \wedge F$, that is it is equivalent to the relation $\wedge(F \cup C) = \wedge F$. Hence, we have to show that given a set C of reduction operators such that $\wedge F \preceq \wedge C$, $F \cup C$ is confluent if and only if Obs^F is included in $\mathrm{Red}(C)$.

Let $C \subset \mathbf{RO}(G, <)$ such that $\wedge F \preceq \wedge C$, that is $\wedge(F \cup C) = \wedge F$. The set $F \cup C$ is confluent if and only if $\mathrm{NF}(F \cup C) = \mathrm{NF}(\wedge(F \cup C))$, that is $F \cup C$ is confluent if and only if $\mathrm{NF}(F) \cap \mathrm{NF}(C)$ is equal to $\mathrm{NF}(\wedge F)$. By definition of Obs^F, we have

$$\mathrm{NF}(F) \cap \mathrm{NF}(C) = \Big(\mathrm{NF}(\wedge F) \cap \mathrm{NF}(C)\Big) \bigsqcup \Big(\mathrm{Obs}^F \cap \mathrm{NF}(C)\Big).$$

From (3), the inequality $\wedge F \preceq \wedge C$ implies that $\mathrm{NF}(\wedge F)$ is included in $\mathrm{NF}(\wedge C)$, which is included in $\mathrm{NF}(C)$. Hence, we have

$$\mathrm{NF}(F) \cap \mathrm{NF}(C) = \mathrm{NF}(\wedge F) \bigsqcup \Big(\mathrm{Obs}^F \cap \mathrm{NF}(C)\Big).$$

Hence, $F \cup C$ is confluent if and only if $\mathrm{Obs}^F \cap \mathrm{NF}(C)$ is empty, that is if and only if Obs^F is included in the complement of $\mathrm{NF}(C)$. The latter is equal to $\mathrm{Red}(C)$, which concludes the proof.

□

We can now introduce incremental completion procedures and establish the main result of the section.

Definition 3.2.4. We define by induction subsets $F_1, \cdots, F_n$ of $\mathbf{RO}(G, <)$ in the following way: $F_1 = \{T_1\}$ and for every $2 \le i \le n$,

$$F_i = F_{i-1} \cup \{T_i, C_i\},$$

where $C_i = C^{F_{i-1}\cup\{T_i\}}$. The set $C = \{C_2, \cdots, C_n\}$ is called the *incremental completion* of F.

Theorem 3.2.5. *Let F be a set of reduction operators, let $\tilde{F}$ be the reduction of F and let C be the incremental completion of $\tilde{F}$. Then, $F \cup C$ is a completion of F.*

Proof. By construction, $\tilde{F} \cup C$ is a completion of $\tilde{F}$. From Proposition 3.2.3, $\mathrm{Obs}^{\tilde{F}}$ is included in the union $\mathrm{Red}\,(C)$ of the sets $\mathrm{Red}\,(C_i)$ and $\wedge\tilde{F}$ is smaller than $\wedge C$. From Proposition 3.2.2, Obs^{F} is included in $\mathrm{Red}\,(C)$ and $\wedge F$ is smaller than $\wedge C$ for $\preceq$. Using again Proposition 3.2.3, $F \cup C$ is a completion of F.

□

Lattice criterion for detecting useless reductions. Combining Theorem 2.2.2 and Theorem 3.2.5, we deduce a lattice criterion for detecting useless reductions during a completion procedure: they are the reductions $g \longrightarrow T_i(g)$, where g belongs to $\mathrm{Red}\,(U_{i-1} \vee T_i)$.

Example 3.2.6. We consider Example 3.1.2. For that, we use the basis of syzygies constructed in Section 2.3. The set $\mathrm{lt}\,(\mathbf{syz}\,(F))$ contains two elements: e_{2,g_3} and e_{5,g_4}. In particular, $\tilde{T}_i$ is equal to T_i for $i = 1, 2, 3$, and for $i = 2$ or 5, we have

$$\tilde{T}_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \tilde{T}_5 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

We have $C_i = \mathrm{Id}_{\mathbb{K}G}$ for $i \neq 3$ and

$$C_3 = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Hence, $F \cup \{C_3\}$ is a completion of F.

3.3 Useless reductions and commutative Gröbner bases

In this section, we relate syzygies for reduction operators to classical syzygies for presentations of algebras, and we illustrate how to use the lattice criterion introduced in 3.2 for constructing commutative Gröbner bases.

Syzygies for reduction operators and presentations of algebras. Consider a commutative or a noncommutative algebra $\mathbf{A}$. Given a generating set X of $\mathbf{A}$, we denote by $\mathbb{K}[X]$ and $\mathrm{T}\,(X)$ the polynomial algebra and the tensor algebra over X, respectively. Let G be the set of commutative or noncommutative monomials over X, according to $\mathbf{A}$ is commutative or not, and let $<$ be an admissible order on G. Let $R = \{f_1, \cdots, f_n\}$ be a a generating set of relations of $\mathbf{A}$: R is a subset of $\mathbb{K}[X]$ or $\mathrm{T}\,(X)$, according to $\mathbf{A}$ is commutative or not. For every integer $1 \leq i \leq n$, we denote by $T_i \in \mathbf{RO}\,(G, <)$ the reduction operator whose kernel is the ideal of $\mathbb{K}[X]$ or the two-sided ideal of $\mathrm{T}\,(X)$ spanned by f_i, according to $\mathbf{A}$ is commutative or not. Then, the syzygies for the presentation $\langle X \mid R\rangle$ are the syzygies for $(T_1, \cdots, T_n)$.

Remark 3.3.1. The set B_n constructed in Section 2.2 is a basis of syzygies for presentations of algebras. However, in this context of presentations of algebras, the set of terms is a set of monomials, so that it is an infinite set and the construction of B_n is not an algorithm.

Now, we relate the completion of a set of reduction operators to the construction of commutative Gröbner bases. Let X be a set of variables and let us denote by $[X]$ and $\mathbb{K}[X]$ the set of monomials and the polynomial algebra over X, respectively. We fix a set $R = \{f_1, \cdots, f_n\}$ of polynomials as well as an admissible order $<$ on $[X]$.

Definition 3.3.2. We associate to R the set $F_R = \{T_1, \cdots, T_n\}$ of reduction operators with respect to $([X], <)$, where the kernel of T_i is the ideal of $\mathbb{K}[X]$ spanned by f_i, for every integer i such that $1 \leq i \leq n$.

Remark 3.3.3. For every integer $1 \leq i \leq n$ and for every monomial m, $T_i(m)$ satisfies one of the following two conditions:

- **i.** if m is equal to $\mathrm{lt}(f_i)\, m'$ for a monomial m', then we have $T_i(m) = 1/\mathrm{lc}(f_i)\,(r(f_i)m')$, where $r(f_i) = \mathrm{lc}(f_i)\,\mathrm{lt}(f_i) - f_i$,
- **ii.** if m is not divisible by $\mathrm{lt}(f_i)$, then $T_i(m) = m$.

In particular, $\mathrm{NF}(T_i)$ is the set of monomials which are not divisible by $\mathrm{lt}(f_i)$, so that $\mathrm{NF}(F)$ is the set of monomials which do not belong to the monomial ideal spanned by $\mathrm{lt}(R)$.

Proposition 3.3.4. *Let I be an ideal of $\mathbb{K}[X]$. A generating set R of I is a Gröbner basis of I if and only if the set F_R of reduction operators associated to R is confluent.*

Proof. The kernel of $\wedge F$ is the sum of the kernels of the operators $T_1, \cdots, T_n$, that is it is equal to I. Hence, $\mathrm{Red}(\wedge F)$ is equal to $\mathrm{lt}(I)$. Moreover, F is confluent if and only if $\mathrm{NF}(F) = \mathrm{NF}(\wedge F)$, that is if and only if the complements of $\mathrm{NF}(F)$ and $\mathrm{NF}(\wedge F)$ in $[X]$ are equal. Hence, F is confluent if and only if the monomial ideal spanned by $\mathrm{lt}(R)$ is equal to $\mathrm{lt}(I)$, that is if and only if R is a Gröbner basis of I.

□

Corollary 3.3.5. *Let I be an ideal of $\mathbb{K}[X]$, let R be a generating set of I and let $\tilde{F}_R$ be the reduction of the set F_R of reduction operators associated to R. Let $R' \subset \mathbb{K}[X]$ be such that $\tilde{F}_R \cup F_{R'}$ is confluent. Then, $R \cup R'$ is a Gröbner basis of I.*

Proof. This is a consequence of Proposition 3.3.4, and Theorem 3.2.5.

□

Useless reductions. From Theorem 3.3.5, we deduce the following criterion for detecting useless reductions during the construction of Gröbner bases: they are the reductions induced by mf_i, where m is a monomial such that $m\mathrm{lt}(f_i)$ is reducible for $(T_1 \wedge \cdots \wedge T_{i-1}) \vee T_i$. We illustrate this criterion with the following example:

Example 3.3.6. Consider the example from [10, Example 4.3.4]: let $X = \{x, y, z, t\}$, let $<$ be the DRL-order induced by $t < z < y < x$ and let $R = \{f_1, f_2, f_3\}$, where $f_1 = y^2 - xz$, $f_2 = x^2 - yz$ and $f_3 = xyz - y^2z$. We denote by T_i the reduction operator whose kernel is the ideal spanned by f_i. There is no critical pair between f_1 and f_2, so that $\{f_1, f_2\}$ is a Gröbner of the ideal spanned by f_1 and f_2. When considering f_3, there are two critical pairs:

- **i.** xy^2z is reducible both by f_1 and f_3,
- **ii.** x^2yz is reducible both by f_2 and f_3.

The polynomial $g = x^2yz - y^3z + xyz^2 - y^2z^2$ belongs to the kernel of $(T_1 \wedge T_2) \vee T_3$ since we have:

$$\begin{aligned} g &= xzf_1 + (yz + z^2)f_2 \\ &= (x + y + z)f_3. \end{aligned}$$

Hence, the reduction induced by xf_3 is a useless reduction so that we can reject the second critical pair. Moreover, when reducing the S-polynomial of the first critical pair, we get the new polynomial $f_4 = xz^3 - yz^3$. We obtain two new critical pairs:

i. x^2z^3 is reducible both by f_2 and f_4,

ii. xyz^3 is reducible both by f_3 and f_4.

The polynomials $x^2z^3 - y^2z^3 + xz^4 - yz^4$ and $xyz^3 - y^3z^3$ belong to the kernel of $(T_1 \wedge T_2 \wedge T_3) \vee T_4$. Indeed, we have:

$$(x + y + z)\, f_4 = z^3\,(f_2 - f_1) \quad \text{and} \quad yf_4 = z^2 f_3.$$

Hence, the reductions induced by xf_4 and yf_4 are useless reductions, so that we can reject the two critical pairs. Hence, $\{f_1,\ f_2,\ f_3,\ f_4\}$ is a Gröbner basis of the ideal spanned by $\{f_1,\ f_2,\ f_3\}$.

Conclusion. We presented a method based on lattice constructions for constructing a basis of the space of syzygies for a set of reduction operators. Using the relationship between syzygies and useless reductions during the completion procedure, we deduced a lattice criterion for detecting these reductions and thus for avoiding useless critical pairs during the construction of commutative Gröbner bases. When syzygies are infinite dimensional, our method does not lead to an algorithm since infinite computations are necessary. However, this work was motivated by computation of syzygies for richer structures than vector spaces. Hence, a further work is to exploit these structures for obtaining an algorithm.

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