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Jean-Bernard Chatelain, Kirsten Ralf

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Abstract

This article presents an algorithm that extends Ljungqvist and Sargent's (2012) dynamic Stackelberg game to the case of dynamic stochastic general equilibrium models including forcing variables. Its first step is the solution of the discounted augmented linear quadratic regulator as in Hansen and Sargent (2007). It then computes the optimal initial anchor of "jump" variables such as inflation. We demonstrate that it is of no use to compute non-observable Lagrange multipliers for all periods in order to obtain impulse response functions and welfare. The algorithm presented, however, enables the computation of a history-dependent representation of a Ramsey policy rule that can be implemented by policy makers and estimated within a vector auto-regressive model. The policy instruments depend on the lagged values of the policy instruments and of the private sector's predetermined and "jump" variables. The algorithm is applied on the new-Keynesian Phillips curve as a monetary policy transmission mechanism.

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1 Introduction

In a Ramsey optimal-policy model, the ability of the policy maker to stabilize the economy depends on his credibility to anchor the path of the private sector's endogenous variables in an optimal way (Ljungqvist and Sargent (2012, chapter 19)). The private sector's intertemporal optimal behavior implies that its decision variables, such as consumption and inflation, are free to jump.

Since Miller and Salmon (1985), the existing algorithms of Ramsey optimal policy require the computation of Lagrange multipliers in order to compute impulse response functions and welfare. The policy rule is written as a linear function of observable private-sector's predetermined variables, such as the stock of debt or of wealth, and of non-observable Lagrange multipliers. Because Lagrange multipliers are non-observable time

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series, such a representation of a Ramsey-optimal policy rule is useless, both for policy makers wishing to implement optimal policy and for econometricians aiming to evaluate stabilization policy.

Following Smets and Wouters (2007), all estimated dynamic stochastic general equilibrium (DSGE) models now include auto-regressive forcing variables that are non-observable time series, neither by policy makers, nor by econometricians. Each forcing variable corresponds to one of the private sector's "jump variables". These non-observable forcing variables are not included in Ljungqvist and Sargent's (2012, chapter 19) algorithm. Because these auto-regressive forcing variables are non-observable time series, the same statement as for the Lagrange multipliers can be made if those variables appear in the representation of the Ramsey optimal policy rule. Such a representation is useless both for policy makers and for econometricians.

This note makes three contributions:

Firstly, it solves the optimal initial anchor of the private sector's "jump variables" on the forcing variables in the linear quadratic case.

Secondly, it demonstrates that, in order to compute impulse response functions and welfare, it is of no use to compute Lagrange multipliers that cannot be observed by policy makers or econometricians. This allows to skip step two and step three in Ljungqvist and Sargent's (2012) algorithm. The numerical complexity of the algorithm can be reduced by providing formulas that substitute Lagrange multipliers in some of the equations. This is advantageous in larger models or in a structural estimation, when the model has to be solved repeatedly.

Thirdly, the algorithm computes "history-dependent" vector auto-regressive (VAR) representations of the stabilization policy-transmission mechanism and of the policy rule as a function of lags of observable time-series only: policy instruments, jump variables (consumption flow and prices) and predetermined variables (the stock of debt or the stock of wealth). This is done by substitution of *forcing variables which are not observable.* These representations are valid under two assumptions. Firstly, it assumes Smets and Wouters' (2007) hypothesis that the number of auto-regressive forcing variables is equal to the number of observable time series depending on forward-looking variables. Secondly, it assumes that a certain matrix is invertible. This matrix corresponds to the reduced-form parameters of the response of the policy instruments to auto-regressive forcing variables derived from the solution of a Sylvester equation.

The implication of the findings for the implementation and the evaluation of stabilization policy is that, firstly, the algorithm provides a representation of the optimal Ramsey policy rule which can be implemented by policy makers. Policy makers can decide the current value of the policy instrument knowing its lagged values and the lags of the private sector's predetermined and jump variables. Secondly, the proposed VAR representation of the optimal policy-rule and of the transmission mechanism can be directly tested by econometricians, because the disturbances of this VAR are white noise, instead of being auto-regressive forcing variables.

2 Ramsey optimal policy

To derive Ramsey optimal policy a Stackelberg leader-follower model is analyzed where the government is the leader and the private sector is the follower. Let $\mathbf{k}_t$ be an $n_k \times 1$ vector of controllable predetermined state variables with initial conditions $\mathbf{k}_0$ given, $\mathbf{x}_t$ an

$n_x \times 1$ vector of endogenous variables free to jump at t without a given initial condition for $\mathbf{x}_0$ (our first addition to Hansen and Sargent (2007)), put together in the $(n_k + n_x) \times$ 1vector $\mathbf{y}_t = (\mathbf{k}_t^T, \mathbf{x}_t^T)^T$. The $n_u \times 1$ vector $\mathbf{u}_t$ denotes government policy instruments.

Our second addition to Sargent and Ljungqvist's (2012) Stackelberg problem is to include an $n_z \times 1$ vector of non-controllable, exogenous forcing state-variables $\mathbf{z}_t$, such as auto-regressive shocks. All variables are expressed as absolute or proportional deviations from a steady state.

The policy maker maximizes the following quadratic function (minimizes the quadratic loss) subject to an initial condition for $\mathbf{k}_0$ and $\mathbf{z}_0$, but not for $\mathbf{x}_0$:

$$-\frac{1}{2}\sum_{t=0}^{+\infty}\beta^t \left(\mathbf{y}_t^T \mathbf{Q}_{yy}\mathbf{y}_t + 2\mathbf{y}_t^T \mathbf{Q}_{yz}\mathbf{z}_t + \mathbf{u}_t^T \mathbf{R}\mathbf{u}_t\right) \tag{1}$$

where β is the policy maker's discount factor and her policy preferences are the relative weights included in the matrices $\mathbf{Q}$ and $\mathbf{R}$. $\mathbf{Q}_{yy} \geq \mathbf{0}$ is a $(n_k + n_x) \times (n_k + n_x)$ positive symmetric semi-definite matrix, $\mathbf{R} > \mathbf{0}$ is a $p \times p$ *strictly* positive symmetric definite matrix so that the policy maker has at least a very small concern for the volatility of policy instruments. The cross-product of controllable policy targets with non-controllable forcing variables $\mathbf{y}_t^T \mathbf{Q}_{yz}\mathbf{z}_t$ is introduced by Hansen and Sargent (2007, section 4.5). To our knowledge, it has always been set to zero $\mathbf{Q}_{yz} = \mathbf{0}$ so far in models of Ramsey optimal policy. This simplifies the Sylvester equation in step 3.

The policy transmission mechanism of the private sector's behavior is summarized by this system of equations:

$$\begin{pmatrix} E_t\mathbf{y}_{t+1} \\ \mathbf{z}_{t+1} \end{pmatrix} = \begin{pmatrix} \mathbf{A}_{yy} & \mathbf{A}_{yz} \\ \mathbf{0}_{zy} & \mathbf{A}_{zz} \end{pmatrix} \begin{pmatrix} \mathbf{y}_t \\ \mathbf{z}_t \end{pmatrix} + \begin{pmatrix} \mathbf{B}_y \\ \mathbf{0}_z \end{pmatrix} \mathbf{u}_t, \tag{2}$$

where $\mathbf{A}$ is an $(n_k + n_x + n_z) \times (n_k + n_x + n_z)$ matrix and $\mathbf{B}$ is the $(n_k + n_x + n_z) \times p$ matrix of marginal effects of policy instruments $\mathbf{u}_t$ on next period policy targets $\mathbf{y}_{t+1}$.

The government chooses sequences $\{\mathbf{u}_t, \mathbf{x}_t, \mathbf{k}_{t+1}\}_{t=0}^{+\infty}$ taking into account the policy transmission mechanism (2) and $2(n_x + n_k + n_z)$ boundary conditions detailed below.

The certainty equivalence principle of the linear quadratic regulator allows us to work with a non-stochastic model. "*We would attain the same decision rule if we were to replace $\mathbf{x}_{t+1}$ with the forecast $E_t\mathbf{x}_{t+1}$ and to add a shock process $C\varepsilon_{t+1}$ to the right hand side of the private sector policy transmission mechanism, where ε_{t+1} is an i.i.d. random vector with mean of zero and identity covariance matrix.*" (Ljungqvist and Sargent, 2012 p.767).

The policy maker's choice can be solved with Lagrange multipliers using Bellman's method (Ljungqvist and Sargent (2012)). It is practical (but not necessary) to rewrite the objective function of the policy maker by adding the constraints of the private sector's policy transmission mechanisms multiplied by their respective Lagrange multipliers $2\beta^{t+1}\mu_{t+1}$:

$$-\frac{1}{2}\sum_{t=0}^{+\infty} \begin{array}{l} \beta^t \left[\mathbf{y}_t^T \mathbf{Q}_{yy}\mathbf{y}_t + 2\mathbf{y}_t^T \mathbf{Q}_{yz}\mathbf{z}_t + \mathbf{u}_t^T \mathbf{R}\mathbf{u}_t\right] + \\ 2\beta^{t+1}\mu_{t+1}\left[\mathbf{A}_{yy}\mathbf{y}_t + \mathbf{B}_y u_t - \mathbf{y}_{t+1}\right]. \end{array} \tag{3}$$

The dynamics of non-controllable variables can be excluded from the Lagrangian (Hansen and Sargent (2007, section 4.5)). It is important to partition the Lagrange multipliers μ_t consistent with our partition of $\mathbf{y}_t = \begin{bmatrix} \mathbf{k}_t \\ \mathbf{x}_t \end{bmatrix}$, so that $\mu_t = \begin{bmatrix} \mu_{k,t} \\ \mu_{x,t} \end{bmatrix}$, where $\mu_{x,t}$ is an $n_x \times 1$

vector of Lagrange multipliers of forward-looking variables.

The first order conditions with respect to the policy transmission mechanism lead to the linear Hamiltonian system of the discrete-time linear quadratic regulator (Hansen and Sargent (2007, section 4.5)). The $2(n_x+n_k+n_z)$ boundary conditions determine the policy maker's Lagrangian system with $2(n_x+n_k+n_z)$ variables $(\mathbf{y}_t, \mu_t, \mathbf{z}_t)$ where μ_t are the policy maker's Lagrange multipliers related to each of the controllable variables $\mathbf{y}_t$.

Essential boundary conditions are the initial conditions of predetermined variables $\mathbf{k}_0$ and $\mathbf{z}_0$ which are given. Natural boundary conditions are chosen by the policy maker to anchor the unique optimal initial values of the private sector's forward-looking variables. The policy maker's Lagrange multipliers of the private sector's forward (Lagrange multipliers) variables are *predetermined at the value zero:* $\frac{\partial L}{\partial \mathbf{x}_0} = \mu_{\mathbf{x},t=0} = 0$ in order to determine the unique optimal initial value $\mathbf{x}_0 = \mathbf{x}_0^*$ of the private sector's forward variables.

Hansen and Sargent (2007) assume a bounded discounted quadratic loss function:

$$E\left(\sum_{t=0}^{+\infty}\beta^t\left(\mathbf{y}_t^T\mathbf{y}_t+\mathbf{z}_t^T\mathbf{z}_t+\mathbf{u}_t^T\mathbf{u}_t\right)\right)<+\infty \tag{4}$$

which implies

$$\begin{aligned}
&\lim_{t\to+\infty}\beta^t\mathbf{z}_t=\mathbf{z}^*=\mathbf{0},\ \mathbf{z}_t \text{ bounded},\\
&\lim_{t\to+\infty}\beta^t\mathbf{y}_t=\mathbf{y}^*=\mathbf{0}\Leftrightarrow\lim_{t\to+\infty}\frac{\partial L}{\partial\mathbf{y}_t}=\mathbf{0}=\lim_{t\to+\infty}\beta^t\mu_t,\ \mu_t \text{ bounded}.
\end{aligned}$$

This implies a stability criterion for eigenvalues of the dynamic system such that $\left|\left(\beta\lambda_i^2\right)^t\right| < |\beta\lambda_i^2| < 1$, so that stable eigenvalues are such that $|\lambda_i| < 1/\sqrt{\beta} < 1/\beta$. A preliminary step is to multiply matrices by $\sqrt{\beta}$as follows $\sqrt{\beta}\mathbf{A}_{yy}$ $\sqrt{\beta}\mathbf{B}_y$ in order to apply formulas of Riccati and Sylvester equations for the non-discounted augmented linear quadratic regulator (Hansen and Sargent (2007)).

Assumption 1: The matrix pair $(\sqrt{\beta}\mathbf{A}_{yy}\ \sqrt{\beta}\mathbf{B}_y)$ is Kalman controllable if the controllability matrix has full rank:

$$\text{rank}\ \left(\sqrt{\beta}\mathbf{B}_y\ \ \beta\mathbf{A}_{yy}\mathbf{B}_y\ \ \beta^{\frac{3}{2}}\mathbf{A}_{yy}^2\mathbf{B}_y\ \ ...\ \ \beta^{\frac{n_k+n_x}{2}}\mathbf{A}_{yy}^{n_k+n_x-1}\mathbf{B}_y\right)=n_k+n_x. \tag{5}$$

Economic interpretation: Kalman controllability implies that policy instruments can be chosen in a way that they have a direct non-zero effect ($\mathbf{B}_y$) in the first period or an indirect non-zero effect ($\mathbf{A}_{yy}^k\mathbf{B}_y$) in the subsequent periods (period 2 to n_k+n_x-1) on all (n_k+n_x) policy targets. An indirect effect is obtained for example when a policy instrument has an effect on the future value of a first policy target in the next period, and that this first policy target has an effect on the next period's value of a second policy target. Assumption 1 is always assumed in all models of the transmission mechanism of macroeconomic stabilization. Else, at least one of the policy targets is not controllable and the stabilization of all policy targets is impossible. If ever a variable is not controllable, then assumption 2 is necessary to ensure that stabilization is possible.

Assumption 2: The system is can be stabilized when the transition matrix $\mathbf{A}_{zz}$ for the non-controllable variables has stable eigenvalues, such that $|\lambda_i| < 1/\sqrt{\beta}$.

Economic interpretation: For variables that cannot be controlled by the policy instruments stabilization is impossible, if the dynamics of the block of these exogenous variables is diverging (i.e. it includes at least one unstable eigenvalue); the overall system including the dynamics of exogenous variables is never stable. Assumption 2 is always assumed in all models of the transmission mechanism of macroeconomic stabilization, else stabilization is impossible. Typical non-controllable variables are auto-regressive shocks in DSGE models (Smets and Wouters (2007)) with auto-correlation strictly below one, the eigenvalues of the matrix $\mathbf{A}_{zz}$.

The algorithm proceeds in 4 steps:

Step 1: Stabilizing solution

A stabilizing solution of the augmented linear quadratic regulator satisfies (Hansen and Sargent (2007, section 4.5)):

$$\mu_t = \mathbf{P}_y \mathbf{y}_t + \mathbf{P}_z \mathbf{z}_t. \tag{6}$$

The optimal rule of the augmented linear quadratic regulator is:

$$\mathbf{u}_t = \mathbf{F}_y \mathbf{y}_t + \mathbf{F}_z \mathbf{z}_t, \tag{7}$$

where $\mathbf{P}_y$ solves the matrix Riccati equation:

$$\mathbf{P}_y = \mathbf{Q}_y + \beta \mathbf{A}_{yy}^{'} \mathbf{P}_y \mathbf{A}_{yy} - \beta^{'} \mathbf{A}_{yy}^{'} \mathbf{P}_y \mathbf{B}_y \left(\mathbf{R} + \beta \mathbf{B}_y^{'} \mathbf{P}_y \mathbf{B}_y \right)^{-1} \beta \mathbf{B}_y^{'} \mathbf{P}_y \mathbf{A}_{yy}, \tag{8}$$

where $\mathbf{F}_y$ is computed knowing $\mathbf{P}_y$:

$$\mathbf{F}_y = - \left(\mathbf{R} + \beta \mathbf{B}_y^{'} \mathbf{P}_y \mathbf{B}_y \right)^{-1} \beta \mathbf{B}_y^{'} \mathbf{P}_y \mathbf{A}_{yy}, \tag{9}$$

where $\mathbf{P}_z$ solves the matrix Sylvester equation knowing $\mathbf{P}_y$ and $\mathbf{F}_y$:

$$\mathbf{P}_z = \mathbf{Q}_{yz} + \beta \left(\mathbf{A}_{yy} + \mathbf{B}_y \mathbf{F}_y \right)^{\prime} \mathbf{P}_y \mathbf{A}_{yz} + \beta \left(\mathbf{A}_{yy} + \mathbf{B}_y \mathbf{F}_y \right)^{\prime} \mathbf{P}_z \mathbf{A}_{zz}, \tag{10}$$

where $\mathbf{F}_z$ is computed knowing $\mathbf{P}_z$ and $\mathbf{P}_y$:

$$\mathbf{F}_z = \left(\mathbf{R} + \beta \mathbf{B}_y^{'} \mathbf{P}_y \mathbf{B}_y \right)^{-1} \beta \mathbf{B}_y^{\prime} \left(\mathbf{P}_y \mathbf{A}_{yz} + \mathbf{P}_z \mathbf{A}_{zz} \right). \tag{11}$$

For step 4, this matrix $\mathbf{F}_z$ is assumed to be invertible, which amounts to assume that $\mathbf{B}_y^{\prime} \left(\mathbf{P}_y \mathbf{A}_{yz} + \mathbf{P}_z \mathbf{A}_{zz} \right)$ is invertible.

Step 2: Solve for x_0, the optimal initial anchor of forward-looking variables

Proposition 1 *The optimal initial anchor adds the term $P_{y,x}^{-1} P_{z,x} z_0$ with respect to Ljungqvist and Sargent's (2012) algorithm (step 4) missing exogenous forcing variables z_0:*

$$\mathbf{x}_0 = \mathbf{P}_{y,x}^{-1} \mathbf{P}_{y,kx} \mathbf{k}_0 + \mathbf{P}_{y,x}^{-1} \mathbf{P}_{z,x} \mathbf{z}_0 \tag{12}$$

Proof. The policy maker's Lagrange multipliers on private sector forward-looking variables are such that $\mu_{0,x} = \mathbf{0}$, at the initial date. The optimal stabilizing condition is:

$$\begin{pmatrix} \mu_{0,k} \\ \mu_{0,x} \end{pmatrix} = \begin{pmatrix} \mathbf{P}_{y,k} & \mathbf{P}_{y,kx} \\ \mathbf{P}_{y,kx} & \mathbf{P}_{y,x} \end{pmatrix} \begin{pmatrix} \mathbf{k}_0 \\ \mathbf{x}_0 \end{pmatrix} + \begin{pmatrix} \mathbf{P}_{z,k} \\ \mathbf{P}_{z,x} \end{pmatrix} \mathbf{z}_0 = \begin{pmatrix} \mu_{0,k} \\ \mathbf{0} \end{pmatrix}. \tag{13}$$

This implies:

$$\mathbf{P}_{y,kx}\mathbf{k}_0 + \mathbf{P}_{y,x}\mathbf{x}_0 + \mathbf{P}_{z,x}\mathbf{z}_0 = \mathbf{0} \tag{14}$$

Which provides the optimal initial anchor:

$$\mathbf{x}_0 = -\mathbf{P}_{y,x}^{-1}\mathbf{P}_{y,kx}\mathbf{k}_0 - \mathbf{P}_{y,x}^{-1}\mathbf{P}_{z,x}\mathbf{z}_0 \tag{15}$$

■

Proposition 1 generalizes the solution of Ljungqvist and Sargent ((2012), chapter 19, part 2) Matlab code related to their specific example of a large firm with a competitive fringe and a single forcing variable. There is no reference to the solution $\mathbf{P}_z$ of Sylvester equation in their formal algorithm of Ramsey optimal policy in the general linear quadratic case (Ljungqvist and Sargent ((2012), chapter 19) part 1).

Step 3: Impulse response functions and optimal loss function

Proposition 2 *Impulse response functions and welfare can be computed without computing all the values over time of all policy-makers Lagrange multipliers μ_t. F_y and F_z provide a reduced form of the optimal policy rule. P_y and P_z provide the missing initial conditions according to proposition 1.*

$$\begin{pmatrix} E_t\mathbf{y}_{t+1} \\ \mathbf{z}_{t+1} \end{pmatrix} = \begin{pmatrix} \mathbf{A}_{yy} & \mathbf{A}_{yz} \\ \mathbf{0}_{zy} & \mathbf{A}_{zz} \end{pmatrix} \begin{pmatrix} \mathbf{y}_t \\ \mathbf{z}_t \end{pmatrix} + \begin{pmatrix} \mathbf{B}_y \\ \mathbf{0}_z \end{pmatrix} \mathbf{u}_t,$$
$$\mathbf{u}_t = \mathbf{F}_y\mathbf{y}_t + \mathbf{F}_z\mathbf{z}_t,$$
$$\mathbf{x}_0 = \mathbf{P}_{y,x}^{-1}\mathbf{P}_{y,kx}\mathbf{k}_0 + \mathbf{P}_{y,x}^{-1}\mathbf{P}_{z,x}\mathbf{z}_0,\ \mathbf{k}_0 \textit{ and } \mathbf{z}_0 \textit{ given.}$$

Proof. This information is sufficient to compute impulse response functions (the optimal path of the expected values of variables $\mathbf{y}_t$ $\mathbf{z}_t$ and $\mathbf{u}_t$) and to sum up over time their value in the the discounted loss function. ■

Step 4: The estimation of an implementable history-dependent policy rule

Policymakers cannot implement a Ramsey optimal policy rule where policy instruments respond to non-observable forcing variables $\mathbf{z}_t$ or to Lagrange multipliers μ_t that are not observable. They can implement an observationally equivalent representation of the Ramsey optimal policy rule where policy instruments respond to observable variables $\mathbf{y}_t$ and their lags and the lags of the policy instruments $\mathbf{u}_{t-1}$ ("history-dependent rule"). In many DSGE models, the observable and controllable predetermined variables $\mathbf{k}_t$ are set to zero at all periods (Smets and Wouters (2007)), so that all predetermined variables are *non-observable forcing variables* $\mathbf{z}_t$. Sargent and Ljunqgvist's (2012) $\mathbf{u}_t = f(\mathbf{u}_{t-1}, \mathbf{k}_t, \mathbf{k}_{t-1})$ history-dependent rule for *observable and controllable* predetermined variables $\mathbf{k}_t$ should be changed when dealing with *non-controllable predetermined forcing variables* $\mathbf{z}_t$ which are not observable time-series for the econometrician.

Proposition 3 *If all the predetermined variables are non-observable forcing variables z_t, if their number n_z is identical to the number n_y of forward-looking variables y_t as in Smets and Wouters (2007), if the matrix F_z is invertible, one can compute a history-dependent rule depending on lagged values of observable variables y_t and lagged values of policy instruments u_{t-1} in the following way:*

$$
(H)\left\{\begin{array}{c}
\begin{pmatrix} E_t\mathbf{y}_{t+1} \\ \mathbf{z}_{t+1} \end{pmatrix} = \begin{pmatrix} \mathbf{A}_{yy}+\mathbf{B}_y\mathbf{F}_y & \mathbf{A}_{yz}+\mathbf{B}_y\mathbf{F}_z \\ \mathbf{0}_{zy} & \mathbf{A}_{zz} \end{pmatrix} \begin{pmatrix} \mathbf{y}_t \\ \mathbf{z}_t \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \varepsilon_t \\
\mathbf{u}_t = \mathbf{F}_y\mathbf{y}_t + \mathbf{F}_z\mathbf{z}_t \\
\mathbf{x}_0 = \mathbf{P}_{y,x}^{-1}\mathbf{P}_{y,kx}\mathbf{k}_0 + \mathbf{P}_{y,x}^{-1}\mathbf{P}_{z,x}\mathbf{z}_0,\ \mathbf{k}_0 \text{ and } \mathbf{z}_0 \text{ given}
\end{array}\right.
$$

$$
\Leftrightarrow \left\{\begin{array}{c}
\begin{pmatrix} E_t\mathbf{y}_{t+1} \\ \mathbf{u}_{t+1} \end{pmatrix} = \mathbf{M}^{-1}\left(\mathbf{A}+\mathbf{BF}\right)\mathbf{M}\begin{pmatrix} \mathbf{y}_t \\ \mathbf{u}_t \end{pmatrix} + \mathbf{M}^{-1}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \varepsilon_t \\
\mathbf{z}_t = \mathbf{F}_z^{-1}\mathbf{u}_t - \mathbf{F}_z^{-1}\mathbf{F}_y\mathbf{y}_t \\
\mathbf{x}_0 = \mathbf{P}_{y,x}^{-1}\mathbf{P}_{y,kx}\mathbf{k}_0 + \mathbf{P}_{y,x}^{-1}\mathbf{P}_{z,x}\mathbf{z}_0,\ \mathbf{k}_0 \text{ and } \mathbf{z}_0 \text{ given.}
\end{array}\right.
$$

Proof.

$$
\mathbf{A}+\mathbf{BF} = \begin{pmatrix} \mathbf{A}_{yy}+\mathbf{B}_y\mathbf{F}_y & \mathbf{A}_{yz}+\mathbf{B}_y\mathbf{F}_z \\ \mathbf{0}_{zy} & \mathbf{A}_{zz} \end{pmatrix}
$$

$$
\begin{pmatrix} \mathbf{y}_t \\ \mathbf{u}_t \end{pmatrix} = \mathbf{M}^{-1}\begin{pmatrix} \mathbf{y}_t \\ \mathbf{z}_t \end{pmatrix} \text{ with } \mathbf{M}^{-1} = \begin{pmatrix} \mathbf{I}_{n_y} & \mathbf{0} \\ \mathbf{F}_y & \mathbf{F}_z \end{pmatrix}
$$

$$
\text{and } \mathbf{M} = \begin{pmatrix} \mathbf{I}_{n_y} & \mathbf{0} \\ -\mathbf{F}_z^{-1}\mathbf{F}_y & \mathbf{F}_z^{-1} \end{pmatrix}
$$

There are as many auto-regressive forcing variables as controllable forward-looking variables. If the number of policy instrument is equal to the number of controllable forward-looking policy targets, $\mathbf{F}_z$ is a square matrix which can be inverted. One eliminates forcing variables $\mathbf{z}_t$ and replaces them by policy instruments $\mathbf{u}_t$ in the recursive equation, doing a change of the vector basis. ■

In what follows we will apply the algorithm to the New-Keynesian Phillips curve.

3 Example: New-Keynesian Phillips curve

The new-Keynesian Phillips curve constitutes the monetary policy transmission mechanism:

$$
\pi_t = \beta E_t\left[\pi_{t+1}\right] + \kappa x_t + z_t \text{ where } \kappa > 0,\ 0 < \beta < 1, \tag{16}
$$

where x_t represents the output gap, i.e. the deviation between (log) output and its efficient level. π_t denotes the rate of inflation between periods $t-1$ and t and plays the role of the vector of forward-looking variables $\mathbf{x}_t$ in the above general case. β denotes the discount factor. E_t denotes the expectation operator. The cost push shock z_t includes an exogenous auto-regressive component:

$$
z_t = \rho z_{t-1} + \varepsilon_t \text{ where } 0 < \rho < 1 \text{ and } \varepsilon_t \text{ i.i.d. normal } N\left(0, \sigma_\varepsilon^2\right), \tag{17}
$$

where ρ denotes the auto-correlation parameter and ε_t is identically and independently distributed (i.i.d.) following a normal distribution with constant variance σ_ε^2.

The loss function is such that the policy target is inflation and the policy instrument is the output gap (Gali (2015), chapter 5):

$$\max -\frac{1}{2}E_0\sum_{t=0}^{t=+\infty}\beta^t\left(\pi_t^2+rx_t^2\right)=-\frac{1}{2}\sum_{t=0}^{+\infty}\beta^t\left(\pi_t^T\mathbf{Q}_{\pi\pi}\pi_t+2\pi_t^T\mathbf{Q}_{\pi z}z_t+u_t^T\mathbf{R}u_t\right),$$

where $\mathbf{Q}_{\pi\pi}=1$, $\mathbf{Q}_{\pi z}=0$ and $\mathbf{R}=r>0$.

The transmission mechanism can be written as:

$$\begin{pmatrix} E_t\pi_{t+1} \\ z_{t+1} \end{pmatrix}=\begin{pmatrix} \mathbf{A}_{yy}=\frac{1}{\beta} & \mathbf{A}_{yz}=-\frac{1}{\beta} \\ \mathbf{A}_{zy}=0 & \mathbf{A}_{zz}=\rho \end{pmatrix}\begin{pmatrix} \pi_t \\ z_t \end{pmatrix}+\begin{pmatrix} \mathbf{B}_y=-\frac{\kappa}{\beta} \\ \mathbf{B}_z=0 \end{pmatrix}x_t+\begin{pmatrix} 0_y \\ 1 \end{pmatrix}\varepsilon_t. \quad (18)$$

This linear system of two equations ($n=2$) does not satisfy the Kalman controllability condition:

$$\mathbf{AB}=\begin{pmatrix} \frac{1}{\beta} & -\frac{1}{\beta} \\ 0 & \rho \end{pmatrix}\begin{pmatrix} -\frac{\kappa}{\beta} \\ 0 \end{pmatrix}=\begin{pmatrix} -\frac{\kappa}{\beta^2} \\ 0 \end{pmatrix}$$

$$rank\left(\mathbf{B}|\mathbf{AB}\right)=rank\begin{pmatrix} -\frac{\kappa}{\beta} & -\frac{\kappa}{\beta^2} \\ 0 & 0 \end{pmatrix}=1<n=2$$

We will now look at the question whether the two assumptions are met. **Assumption 1**: When one considers only the first equation, there is only one policy target, $n=1$, the Kalman controllability condition is: $rank\left(\mathbf{B}_y\right)=rank\left(-\frac{\kappa}{\beta}\right)=n=1$, if $-\frac{\kappa}{\beta}\neq 0$. If we assume that the slope of the new-Keynesian Phillips curve is not equal to zero: $\kappa\neq 0$, then *inflation is controllable by the output gap*. There is a non-zero correlation between the policy instrument (output gap x_t) and the expected value of the policy target (inflation $E_t\pi_{t+1}$).

Assumption 2: When one considers only the second equation, the Kalman controllability condition is: $rank\left(\mathbf{B}_z\right)=rank\left(0\right)=0<1$. The cost-push shock is not controllable by the output gap. There is a zero correlation between the policy instrument (output gap x_t) and the future value of the cost-push forcing variable z_{t+1}. If we assume that the non-controllable cost-push forcing variable is stationary, $0<A_{zz}=\rho<1$, then *the dynamic system including both equations can be stabilized.*

The system is already written in Kalman canonical form which is defined such that the bottom left block matrix of $\mathbf{A}$ is equal to zero $\mathbf{A}_{zy}=0$ and the bottom block matrix of $\mathbf{B}$ is zero: $\mathbf{B}_z=0$.

After substitution of the optimal policy rule ($x_t=F_\pi\pi_t+F_zz_t$), we obtain the closed loop dynamic system:

$$\begin{pmatrix} E_t\pi_{t+1} \\ z_{t+1} \end{pmatrix}=\begin{pmatrix} \lambda=\frac{1-\kappa F_\pi}{\beta} & \frac{-1-\kappa F_z}{\beta} \\ 0 & \rho \end{pmatrix}\begin{pmatrix} \pi_t \\ z_t \end{pmatrix}+\varepsilon_t.$$

We denote $\lambda=\frac{1-\kappa F_\pi}{\beta}$ the "inflation eigenvalue" of the closed loop dynamics. We use Gali's (2015, chapter 5) numerical values, $\rho=0.8$, $\beta=0.99$, $\varepsilon=6$, $\kappa=0.1275$, $r=\kappa/\varepsilon=0.1275/6=0.02125$, when computing the algorithm.

Step 1: P_π solves the matrix Riccati equation:

$$
\begin{aligned}
\mathbf{0} &= -\mathbf{P}_y + \mathbf{Q}_y + \beta \mathbf{A}_{yy}^{'} \mathbf{P}_y \mathbf{A}_{yy} - \beta^{'} \mathbf{A}_{yy}^{'} \mathbf{P}_y \mathbf{B}_y \left(\mathbf{R} + \beta \mathbf{B}_y^{'} \mathbf{P}_y \mathbf{B}_y \right)^{-1} \beta \mathbf{B}_y^{'} \mathbf{P}_y \mathbf{A}_{yy}. \\
0 &= -P_y + 1 + \frac{1}{\beta} P_y + P_y \frac{\kappa}{\beta} \left(r + \frac{\kappa^2}{\beta} P_y \right)^{-1} \left(-\frac{\kappa}{\beta} \right) P_y, \\
0 &= P_y^2 - \left(1 + \frac{r}{\kappa^2} - \beta \frac{r}{\kappa^2} \right) P_y - \beta \frac{r}{\kappa^2}, \\
0 &= P_y^2 - \left(1 + \frac{0.02125}{0.1275^2} - 0.99 \frac{0.02125}{0.1275^2} \right) P_y - 0.99 \frac{0.02125}{0.1275^2}, \\
P_y &= \frac{1}{2} \left[1 + \frac{r}{\kappa^2} - \beta \frac{r}{\kappa^2} + \sqrt{\left(1 + \frac{r}{\kappa^2} - \beta \frac{r}{\kappa^2} \right)^2 + 4 \beta \frac{r}{\kappa^2}} \right] = 1.751\,8 \ ,
\end{aligned}
$$

where F_π is computed knowing P_π.

$$
\begin{aligned}
\mathbf{F}_y &= -\left(\mathbf{R} + \beta \mathbf{B}_y^{'} \mathbf{P}_y \mathbf{B}_y \right)^{-1} \beta \mathbf{B}_y^{'} \mathbf{P}_y \mathbf{A}_{yy}. \\
F_\pi &= \frac{-\frac{\kappa}{\beta} P_\pi}{r + \frac{\kappa^2}{\beta} P_\pi} = \frac{-\frac{0.1275}{0.99} 1.751\,8}{0.02125 + \frac{0.1275^2}{0.99} 1.751\,8} = 4.\,510\,8.
\end{aligned}
$$

In the scalar case, other formulas are available:

$$
F_\pi = \frac{\kappa}{r} \left(P_\pi - 1 \right) = \frac{1 - \beta \lambda}{\kappa} = \frac{\kappa}{r} \left(\frac{\lambda}{1 - \lambda} \right) = 4.\,510\,8,
$$

where λ is equal to:

$$
\begin{aligned}
\lambda &= \frac{1 - \kappa F_\pi}{\beta} = \frac{1 - 0.1275 \cdot 4.\,510\,8}{0.99} = 0.42916 \\
\lambda &= 1 - \frac{1}{P_\pi} = \frac{1}{2} \left(1 + \frac{1}{\beta} + \frac{\kappa^2}{\beta r} \right) - \sqrt{\frac{1}{4} \left(1 + \frac{1}{\beta} \frac{\kappa^2}{\beta r} \right)^2 - \frac{1}{\beta}}
\end{aligned}
$$

P_z solves the matrix Sylvester equation knowing P_π and F_π or P_π and λ:

$$
\begin{aligned}
\mathbf{0} &= -\mathbf{P}_z + \mathbf{Q}_{yz} + \beta \left(\mathbf{A}_{yy} + \mathbf{B}_y \mathbf{F}_y \right)' \mathbf{P}_y \mathbf{A}_{yz} + \beta \left(\mathbf{A}_{yy} + \mathbf{B}_y \mathbf{F}_y \right)' \mathbf{P}_z \mathbf{A}_{zz}. \\
0 &= -P_u + \beta \left(\frac{1}{\beta} - \frac{\kappa}{\beta} F_\pi \right) P_\pi \left(-\frac{1}{\beta} \right) + \beta \left(\frac{1}{\beta} - \frac{\kappa}{\beta} F_\pi \right) P_u \rho, \\
P_\pi^{-1} P_z &= \frac{-\frac{1 - \kappa F_\pi}{\beta}}{1 - \left(1 - \kappa F_\pi \right) \rho} = \frac{-\lambda}{1 - \beta \rho \lambda} = \frac{-0.42916}{1 - 0.99 \cdot 0.8 \cdot 0.42916} = -0.650\,14.
\end{aligned}
$$

F_z is computed knowing $P_\pi^{-1} P_z$ and F_π or λ and F_π:

$$\mathbf{F}_z = \left(\mathbf{R}+\beta\mathbf{B}_y^{'}\mathbf{P}_y\mathbf{B}_y\right)^{-1}\beta\mathbf{B}_y^{\prime}\left(\mathbf{P}_y\mathbf{A}_{yz}+\mathbf{P}_z\mathbf{A}_{zz}\right).$$

$$F_z = \frac{P_\pi\frac{-1}{\beta}+\rho P_z}{P_\pi\frac{1}{\beta}}F_\pi = \left(-1+\beta\rho P_\pi^{-1}P_z\right)F_\pi = \frac{-1}{1-\beta\rho\lambda}F_\pi,$$

$$F_z = \frac{-1}{1-0.99\cdot 0.8\cdot 0.42916}F_\pi = -1.5149\cdot 4.5108 = -6.8334.$$

Step 2: The natural boundary condition $\gamma_0 = 0$ minimizes the loss function with respect to inflation at the initial date. The optimal initial anchor includes the term $-P_\pi^{-1}P_z z_0$:

$$\gamma_0 = P_\pi\pi_0 + P_z z_0 = 0 \Rightarrow \pi_0^* = -P_\pi^{-1}P_z z_0 = 0.65014\cdot z_0 = \frac{\lambda}{1-\beta\rho\lambda}z_0.$$

Taking into account the optimal policy rule at the initial date:

$$x_0^* = F_\pi\pi_0^* + F_z z_0 = \left(-F_\pi P_\pi^{-1}P_z + F_z\right)z_0 = -\frac{\kappa}{r}\frac{\lambda}{1-\beta\lambda\rho}z_0,$$

$$x_0^* = (4.5108\cdot 0.65014 - 6.8334)\,z_0 = -3.9007\cdot z_0.$$

One has also:

$$\pi_0^* = \frac{-P_\pi^{-1}P_z}{-F_\pi P_\pi^{-1}P_z+F_z}x_0^* = -\frac{r}{\kappa}x_0^* = -\frac{1}{6}x_0^*.$$

Step 3: Impulse response functions are given by:

$$\begin{pmatrix} E_t\pi_{t+1} \\ z_{t+1}\end{pmatrix} = \begin{pmatrix}\lambda & \frac{-1-\kappa F_z}{\beta} \\ 0 & \rho\end{pmatrix}\begin{pmatrix}\pi_t \\ z_t\end{pmatrix} + \begin{pmatrix}0\\1\end{pmatrix}\varepsilon_t,$$

$$x_t = F_\pi\pi_t + F_z z_t \text{ and } \pi_0 = \frac{\lambda}{1-\beta\rho\lambda}z_0,\ z_0 \text{ given.}$$

With numerical values for expected impulse response functions:

Impulse response functions following z_0	Policy rule
$\begin{pmatrix}\pi_t\\ z_t\end{pmatrix} = \begin{pmatrix}0.42916 & -0.13004\\ 0 & 0.8\end{pmatrix}^t\begin{pmatrix}0.65\\1\end{pmatrix}z_0 + \begin{pmatrix}0\\1\end{pmatrix}\varepsilon_t$	$x_t = 4.51\pi_t - 6.83z_t$

Step 4: The vector auto-regressive model of the two observable variables (inflation and output gap) is given by a change of the vector basis:

$$\left\{\begin{array}{c}
\begin{pmatrix}\pi_t\\ x_t\end{pmatrix} = \mathbf{M}^{-1}\begin{pmatrix}\pi_t\\ z_t\end{pmatrix} \text{ with } \mathbf{M}^{-1} = \begin{pmatrix}1 & 0\\ 4.5108 & -6.8334\end{pmatrix},\ \mathbf{M} = \begin{pmatrix}1 & 0\\ 0.66011 & -0.14634\end{pmatrix}\\
z_t = -F_z^{-1}F_\pi\pi_t + F_z^{-1}x_t = 0.66011\cdot\pi_t - 0.14634\cdot x_t\\
\begin{pmatrix}E_t\pi_{t+1}\\ x_{t+1}\end{pmatrix} = \mathbf{M}^{-1}\begin{pmatrix}\lambda & \frac{-1-\kappa F_z}{\beta}\\ 0 & \rho\end{pmatrix}\mathbf{M}\begin{pmatrix}\pi_t\\ x_t\end{pmatrix} + \mathbf{M}^{-1}\begin{pmatrix}0\\1\end{pmatrix}\varepsilon_t\\
\begin{pmatrix}E_t\pi_{t+1}\\ x_{t+1}\end{pmatrix} = \begin{pmatrix}0.34332 & 0.01903\\ -2.0600 & 0.88584\end{pmatrix}\begin{pmatrix}\pi_t\\ x_t\end{pmatrix} + \begin{pmatrix}0\\ -6.8334\end{pmatrix}\varepsilon_t\\
x_0^* = \left(-F_\pi P_\pi^{-1}P_z + F_z\right)z_0 = -3.9007\cdot z_0 \text{ with } z_0 \text{ given}
\end{array}\right.,$$

where the transition matrix is computed by:

$$\begin{pmatrix} 1 & 0 \\ 4.5108 & -6.8334 \end{pmatrix} \begin{pmatrix} 0.42916 & -0.13004 \\ 0 & 0.8 \end{pmatrix} \begin{pmatrix} 1.0 & 0 \\ 0.66011 & -0.14634 \end{pmatrix}..$$

Step 3 or step 4 provide identical numbers to the numbers used for making the diagrams of the impulse response functions of inflation, output gap and the cost-push shock by Gali (2015), chapter 5.

4 Conclusion

The present article presented an algorithm for solving Ramsey optimal policy that can be used to map models into a VAR representation which in itself can be utilized for the estimation and interpretation of the outcome of such an estimation as well as for the discussions of identification. Chatelain and Ralf (2017) estimate this vector autoregressive representation for a new-Keynesian Phillips curve transmission mechanism. Chatelain and Ralf (2019) use this algorithm for the new-Keynesian Phillips curve and the consumption Euler equation as a monetary policy transmission mechanism.

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