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Abstract: This paper deals with a fluid-flow modeling under compartmental representation of a network. The motivating example is a communication network made up of buffers and transmission lines where densities and flows of packets are viewed as macroscopic variables respecting the conservation laws. The main contribution lies in the resulting model. It is a coupled linear hyperbolic partial differential equations (PDEs) with an ordinary differential equation (ODEs) along with a dynamic boundary condition. Input-to-state stability of an optimal equilibrium is analyzed using Lyapunov techniques.

Keywords: Compartmental networks, Fluid-flow modeling, Hyperbolic systems of conservation laws, Lyapunov techniques.

1. INTRODUCTION

Fluid-flow modeling of physical networks is a way of describing flow of matter through elements of the network consisting of a finite collection of nodes communicating to each other and links that connect them. One of the most important features of the fluid-flow modeling is the conservation of mass. Several models, in which this property is preserved, may result in a macroscopic description. The most traditional and well-studied example is about the evolution of vehicular traffic in roads (Garavello and Piccoli (2009)), whose macroscopic variables are mainly the density of cars and the averaged velocity. The models in that framework are then given by one dimensional partial differential equations (PDEs) of conservation laws (Coclite et al. (2005); Treiber and Kesting (2013)). Furthermore, inspired by traffic flow on road networks, several studies deal with flow of information on telecommunication networks made up of transmission lines and nodes (see e.g. D'Apice et al. (2006)). One common issue on both road traffic and communication networks under PDE setting is related to the congestion. For both applications, there exist critical densities that split the operation of the network in two zones (according to the so-called *fundamental diagram of flow-density*).

In addition to macroscopic models, compartmental systems are also known as suitable framework to describe conservation laws in networks. See for instance Jacquez and Simon (2002) for the study of the dynamics of compartmental systems, even with lags representing delays which turn out to be represented by linear transport equations or linear conservation laws. In Bastin and Guffens

(2006), congestion control of compartmental networks is studied, still under fluid-flow modeling, in which they exploit properties of positive systems to establish stability of the network and prevent congestion by means of nonlinear controls. In those studies, each node represents a compartment which contains a variable quantity, e.g. information being processed. Conceptually, a compartment is a kind of storage device and is typically modeled by ordinary differential equations (ODEs). Related works to Bastin and Guffens (2006), one can also find nonlinear continuous-time model using fluid-flow approach, as introduced in Malrait et al. (2009) or conservation law-based fluid-flow for network congestion control as in Briat et al. (2012).

Networks of conservation laws are also studied in Bastin et al. (2007), using linear hyperbolic equations. Moreover, the stability analysis of such linear hyperbolic equations is carried out by means of Lyapunov analysis. The network of conservation laws is represented as a compartmental network with the particularity that the accumulated quantities are distributed on space. It is worth pointing out that hyperbolic equations have important applications in the modeling and control of physical networks. Besides road traffic networks (Coclite et al. (2005)), as previously discussed, one can highlight other examples such as hydraulic networks Bastin et al. (2008) and gas pipeline networks Gugat et al. (2011).

This paper deals with stability analysis of a communication network which is modeled using the fluid-flow and compartmental setting, combining hyperbolic conservation laws with ordinary differential equations. More precisely, we establish that a compartment is made up of two sub-compartments: one for the fluid dynamics of servers, composed mainly by buffers (modeled by ODEs), the other to represent delays in transmission (if there are), or simply

a transmission line modeled by hyperbolic PDEs. Since we combine both dynamics, the coupling is defined at the boundaries of the hyperbolic PDEs. It results in what we call, coupled PDE-ODE with *dynamic boundary condition.* We aim then at studying the stability of such systems under the assumption that one wants the system to operate in *free-flow* zone to avoid congestion. The main contribution of this work is the modeling and the study of input-to-state stability properties of a communication network when operating at some optimal equilibrium point.

This paper is organized as follows. In Section 2 we introduce the proposed model. Section 3 contains the well-posedness and stability analysis, including the steady-state characterization. Section 4 provides a numerical example to illustrate the results. Finally, conclusions are given in Section 5.

Preliminary definitions and notation.

(1) $\mathcal{I}_n$ is the set of the number of compartments, numbered from 1 to n.
(2) $\mathcal{I}_{in} \subset \mathcal{I}_n$ is the index set of input compartments.
(3) $\mathcal{I}_{out} \subset \mathcal{I}_n$ is the index set of outputs compartments.
(4) $\mathcal{D}_i \subset \mathcal{I}_n$ is the index set of downstream compartments connected directly to compartment i (i.e. those compartments receiving flow from compartment i).
(5) $\mathcal{U}_i \subset \mathcal{I}_n$ is the index set of upstream compartments connected directly to compartment i (i.e. those compartments sending flow to compartment i).
(6) $\mathcal{R} \subset \mathcal{I}_n$ is the index set of routing compartments (i.e. those compartments sending flow to two or more downstream compartments).

Due to space limitation, all proofs are omitted.

2. FLUID-FLOW MODELING

In this section, we present a model of communication networks under compartmental fluid-flow dynamics using both partial differential equations and ordinary differential equations. Highly inspired by Guffens (2005), Bastin and
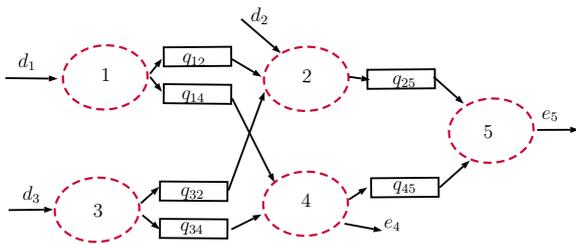

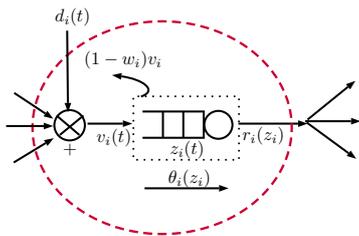Fig. 1. Example of a compartmental network.

Fig. 2. Compartment: buffer.

Guffens (2006) and Bastin et al. (2007), let us consider

an example of a general network depicted in Figure 1 where each node is illustrated in Figure 2. It is a server in which a buffer stores information to be processed. The other compartments considered in this network are the transmission lines when delays in time may exist. The fact of considering delays in transmission (not buffer delays) allows to enrich the model introduced in Guffens (2005) where delays are not taken into account. Actually, they assume that the flow transfer is instantaneous between buffers. In this work we will consider rather that traffic flow takes a while in traveling from one buffer to another. We assume that the network is a directed graph in which the directed arcs $i \rightarrow j$ of the network represent instantaneous mass transfers between compartments (more precicely, between servers compartments and transmission line compartments). As already mentioned, if there are time delays, they can be enclosed into the compartment representing the transmission line by a transport equation with positive velocity. We assume that traffic flow may be routed to different compartments of the network until reaching the destination. That routing mechanism is assumed to be located the end of the server compartments.

Under the proposed general topology of compartments, transmission lines are modeled by the following nonlinear conservation laws (D'Apice et al. (2006, 2008))

$$\partial_t \rho_{ij}(t,x) + \partial_x f_{ij}(\rho_{ij}(t,x)) = 0, \quad i \in \mathcal{I}_n, j \in \mathcal{D}_i \quad (1)$$

for all $x \in [0,1], t \in \mathbb{R}^+$, where ρ_{ij} and $f_{ij}(\rho_{ij})$ are the density and flow of packets respectively and σ_{ij} is a critical density closely related to the probability of loosing packets.

$$f_{ij}(\rho_{ij}) = \begin{cases} \lambda_{ij}\rho_{ij}, & \text{if } 0 \le \rho_{ij} \le \sigma_{ij} \\ \lambda_{ij}(2\sigma_{ij} - \rho_{ij}), & \text{if } \sigma_{ij} \le \rho_{ij} \le \rho_{ij}^{max} \end{cases} \quad (2)$$

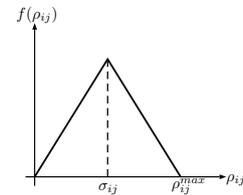Figure 3 shows the so-called *fundamental diagram of flow-density.*

Fig. 3. Fundamental triangular diagram of flow-density

For $\rho_{ij} \le \sigma_{ij}$, the network is said to be in *free-flow*. For $\rho_{ij} \ge \sigma_{ij}$, the network is said to be congested. Note that this fundamental diagram is quite similar to the one used in road traffic networks (see e.g. (Treiber and Kesting, 2013, Chapter 8) for CTM models). In communication networks, the averaged velocity of packets is supposed to be constant. The macroscopic model (1) has been validated in D'Apice. et al. (2012).

In this work, we assume the specific case when each flow is a static monotonic increasing function of the density ρ_{ij}. Under this assumption, the flow $f_{ij}(\rho_{ij})$ equals $\lambda_{ij}\rho_{ij}$, for $0 \le \rho_{ij} \le \sigma_{ij}$, where λ_{ij} is the average velocity of packets among buffers traveling through the transmission line. Let us denote the flow $f_{ij}(\rho_{ij}) := q_{ij}$. We will focus on the case in which the network operates in free-flow and we will study Lyapunov stability properties on the network when operating in this zone. This case implies that, according to (2), $q_i = \lambda_{ij}\rho_{ij}$ for $0 \le \rho_i \le \sigma_{ij}$. Therefore

$$\partial_t \rho_{ij}(t,x) = \frac{1}{\lambda_{ij}} \partial_t q_{ij}(t,x).$$

Replacing it in (1), we obtain the linear hyperbolic system as *Kinematic wave equations* (as in Bastin et al. (2007)), that is

$$\partial_t q_{ij}(t,x) + \lambda_{ij} \partial_x q_{ij}(t,x) = 0, \quad i \in \mathcal{I}_n, \quad j \in \mathcal{D}_i \quad (3)$$

Concerning the modeling of buffers, let us consider the balance equation for each buffer $i \in \mathcal{I}_n$ as follows:

$$\dot{z}_i(t) = v_i(t) - r_i(z_i(t)) \quad (4)$$

where v_i is the sum of all input flows getting into the buffer and r_i is the output flow of the buffer. It will be characterized later as the *processing rate function*. We assume that there is only an outgoing flow r_i from buffer i getting into the transmission line instantaneously. Furthermore, with $v_i(t) = d_i(t) + \sum_{\substack{k \neq i \\ k \in \mathcal{U}_i}} q_{ki}(t,1)$, where

- $d_i(t)$ is the external input flow demands. Note that $d_i \equiv 0$ as long as $i \notin \mathcal{I}_{in}$;
- $\sum_{\substack{k \neq i \\ k \in \mathcal{U}_i}} q_{ki}(t,1)$ are the flows coming from the transmission line connecting upstreams compartments,

we obtain

$$\dot{z}_i(t) = d_i(t) + \sum_{\substack{k \neq i \\ k \in \mathcal{U}_i}} q_{ki}(t,1) - r_i(z_i(t)) \quad (5)$$

It represents the rate of accumulation of quantity z_i. The flows can be modulated by suitable actuators. On one hand, we have control actions $u_{ij}(\cdot)$ devoted to route the flow of information through different paths of the network. In this work, they are time-varying and represent continuous time control values, that we call *routing splitting controls*. The routing takes place at the output of the server. On the other hand, we have control actions $w_i(\cdot)$ devoted to reject packets (traffic flow) before they enter to the buffers. Combining both control actions, we have, on one hand, that the buffer model given in (5) can be expressed as follows:

$$\dot{z}_i(t) = w_i(t)d_i(t) + \sum_{\substack{k \neq i \\ k \in \mathcal{U}_i}} w_i(t)q_{ki}(t,1) - r_i(z_i(t))$$

where $d_i(t)$ is the actual input flow demand, then $w_i d_i$ is a fraction of such a demand and $w_i(t)q_{ki}(t,1)$ is a fraction of the incoming flow entering to the buffer i, $0 \leq w_i(t) \leq 1$. The output flow $r_i(z_i)$ (processing rate function) of each buffer is considered as the ratio between the quantity z_i and the *residence time*. It is given as follows (see (Guffens, 2005, Chapter 2))

$$r_i(z_i) = \frac{z_i}{\theta_i(z_i)}$$

The *residence time* is the averaged time at which packets stay in the server when being processed.

$$\theta_i(z_i) = \frac{1 + z_i}{\epsilon_i} \quad (6)$$

with $\epsilon_i > 0$ as the maximal processing capacity of each server. Hence, the processing rate function is given by

$$r_i(z_i) = \frac{\epsilon_i z_i}{1 + z_i} \quad (7)$$

It can be can noticed that $r_i(z_i)$ is a positive bounded function of the quantity z_i $(0 \leq r_i(z_i) < \epsilon_i)$.

On the other hand, regarding the routing splitting control, the boundary condition for the linear hyperbolic system (3) is as follows:

$$q_{ij}(t,0) = u_{ij}(t)r_i(z_i(t)) \quad (8)$$

with $0 \leq u_{ji}(t) \leq 1$, $j \in \mathcal{D}_i, i \in \mathcal{I}_n$. In fact $i \in \mathcal{R}$. Note that the left boundary condition (8) depends on the state variable z_i, being this one a solution to the ODE system (5). In that sense, we shall consider in the sequel that the boundary condition of the linear hyperbolic PDE is a dynamic boundary one. The left-boundary condition itself is enough to be considered in the whole model for the study of well-posedness and stability issues.

Finally, the output function for each output compartment $i \in \mathcal{I}_{out}$ is given by

$$e_i(t) = u_i(t)r_i(z_i(t)) \quad (9)$$

with $\sum_{\substack{i \neq j \\ j \in \mathcal{D}_i}} u_{ij}(t) + u_i(t) = 1$ $(u_i(t) \equiv 0$ if $i \notin \mathcal{I}_{out})$. The complete model, for the network as depicted in Figures 1 and 2, is then:

$$\begin{cases} \partial_t q_{ij}(t,x) + \lambda_{ij}\partial_x q_{ij}(t,x) = 0, \quad i \in \mathcal{I}_n, \quad j \in \mathcal{D}_i \\ \dot{z}_i(t) = w_i(t)d_i(t) + \sum_{\substack{k \neq i \\ k \in \mathcal{U}_i}} w_i(t)q_{ki}(t,1) - r_i(z_i(t)) \end{cases}$$
$$(10)$$

with dynamic boundary condition,

$$q_{ij}(t,0) = u_{ij}(t)r_i(z_i(t)), \quad r_i \geq 0 \quad (11)$$

output function,

$$e_i(t) = u_i(t)r_i(z_i(t)) \quad (12)$$

and initial conditions

$$\begin{cases} q_{ij}(0,x) = q_{ij}^0(x), \quad x \in [0,1] \\ z_i(0) = z_i^0. \end{cases} \quad (13)$$

3. STABILITY ANALYSIS AND WELL-POSEDNESS OF THE NETWORK

We want the network to operate at some equilibrium point, to be precise, at some free-flow steady-state. Since we are going to deal with input flow demands, we aim at studying the influence of those inputs over the stability of the network. Therefore, Input-to-state stability (ISS) property with respect to those inputs by using Lyapunov analysis will be addressed for the corresponding linearized system.

3.1 Free-flow steady-state characterization

For a given constant input flow demand d_i^*, system (10)-(13) has infinitely many equilibrium points $\{q_{ki}^*, z_i^*, u_{ij}^*, u_i^*, w_i^*, e_i^*\}$ with q_{ij}^* time and space invariant. They satisfy the following algebraic equations:

$$\begin{cases} w_i^* d_i^* + \sum_{\substack{k \neq i \\ h \in \mathcal{U}_i}} w_i^* q_{ki}^* - r_i(z_i^*) = 0 \\ q_{ij}^* = u_{ij}^* r_i(z_i^*) \\ e_i^* = u_i^* r_i(z_i^*). \end{cases} \quad (14)$$

We assume then that the system admits a *free-flow steady-state*. Among all possible equilibrium points, we choose the free-flow steady-state that meets some performance criterion for the network. Inspired by road traffic networks where two usual performance metrics such as the total travel time (TTT) and total travel distance (TTD) are considered (see e.g. Treiber and Kesting (2013)), here we focus on a particular static case:

1) Maximizing the *total output flow rate* of the network. The first optimization objective is as follows:

$$\text{maximize} \quad J_1 = \sum_{i \in \mathcal{I}_{out}} e_i^*$$

2) Minimizing the *total mean travel time* (T_{MTT}). In each compartment $i \in \mathcal{I}_n$, information is processed and it takes some time according to the residence time $\theta_i(z_i)$ before it is sent through the transmission line. Besides this, there is a time propagation given by $\frac{1}{\lambda_{ij}}$ due to the transport equation. Let us denote the total travel time in each compartment i by $T_i = \theta_i(z_i^*)$. We do not give any explicit formula of the T_{MTT} because of the complexity that the network topology might have, but we explain the approach to compute it. The first issue worth remarking is that in this framework, there are no cycles in the network. It implies that there is a finite number of possible paths that the information flow can follow from input compartments until output compartments. Therefore, looking at each input-output path, the sum of total times T_i of compartments involved in along with time propagation between compartments $\frac{1}{\lambda_{ij}}$, is weighted by the effective output flow which travels through them. In order to homogenize, the result is divided by the sum of the output flows of the whole network. Repeating the same procedure with every input-output path and adding the obtained weighted average value, the total mean travel time can be deduced accordingly. In Section 4, a specific example is provided to better illustrate the idea. Hence, the second optimization objective is as follows:

$$\text{minimize} \quad J_2 = T_{MTT}$$

Let us call $J = \alpha(-J_1) + (1 - \alpha)J_2$ the cost function with weighting coefficient $\alpha \in [0, 1]$. J is a nonlinear function to be minimized subject to (14) along with the following constraints related to:
(1) free-flow conditions over the linear hyperbolic system:

$$q_{ij}^* \leq \sigma_{ij}\lambda_{ij}$$

(2) maximal processing rate capacity:

$$r_i(z_i^*) < \epsilon_i$$

(3) control variables:

$$0 \leq u_{ij}^* \leq 1, \quad 0 \leq w_i^* \leq 1, \quad 0 \leq u_i^* \leq 1, \quad \sum_{\substack{i \neq j \\ j \in \mathcal{D}_i}} u_{ij}^* + u_i^* = 1$$

3.2 Linearization around the free-flow steady-state

Defining the deviations $y_{ij} = q_{ij} - q_{ij}^*, Z_i = z_i - z_i^*, U_{ij} = u_{ij} - u_{ij}^*, W_i = w_i - w_i^*$ and $\tilde{d}_i = d_i - d_i^*, \tilde{e}_i(t) = e_i(t) - e_i^*$ the linearization of the coupled PDE-ODE system in (10) with dynamic boundary condition (11) and initial condition (13) around the optimal free-flow equilibrium is given by

$$\begin{cases} \partial_t y_{ij}(t,x) + \lambda_{ij}\partial_x y_{ij}(t,x) = 0, \quad i \in \mathcal{I}_n, \quad j \in \mathcal{D}_i \\ \dot{Z}_i(t) = d_i^* W_i(t) + w_i^* \tilde{d}_i(t) + \sum_{\substack{k \neq i \\ k \in \mathcal{U}_i}} q_{ki}^* W_i(t) \\ \qquad\quad + \sum_{\substack{k \neq i \\ k \in \mathcal{U}_i}} w_i^* y_{ki}(t,1) - r_i'(z_i^*)Z_i(t) \end{cases}$$

(15)

with dynamic boundary condition

$$y_{ij}(t,0) = u_{ij}^* r_i'(z_i^*)Z_i(t) + r_i(z_i^*)U_{ij}(t), \quad i \in \mathcal{I}_n, \quad j \in \mathcal{D}_i$$
(16)

with output function

$$\tilde{e}_i(t) = u_i^* r_i'(z_i^*)Z_i(t) + r_i(z_i^*)U_i(t) \tag{17}$$

and initial conditions

$$\begin{cases} y_{ij}(0,x) = y_{ij}^0(x), \quad x \in [0,1] \\ Z_i(0) = Z_i^0. \end{cases} \tag{18}$$

Recall that $\sum_{\substack{i \neq j \\ j \in \mathcal{D}_i}} u_{ij}(t) + u_i(t) = 1$ and $\sum_{\substack{i \neq j \\ j \in \mathcal{D}_i}} U_{ij}(t) + U_i(t) = 0$. We remark that since $u_i(t) \equiv 0, u_i^* \equiv 0$ if $i \notin \mathcal{I}_{out}$, thus, $U_i(t) \equiv 0$ as well.

The system (15)-(18), can be written in matrix form as

$$\begin{cases} \partial_t y(t,x) + \Lambda \partial_x y(t,x) = 0 \\ \dot{Z}(t) = AZ(t) + G_y y(t,1) + B_w W(t) + D\tilde{d}(t) \end{cases} \tag{19}$$

with boundary condition

$$y(t,0) = G_z Z(t) + B_u U(t) \tag{20}$$

and initial condition

$$\begin{cases} y(0,x) = y^0(x), \quad x \in [0,1] \\ Z(0) = Z^0. \end{cases} \tag{21}$$

where $y : \mathbb{R}^+ \times [0,1] \to \mathbb{R}^m$ with m is given by

$$m := \sum_{i \in \mathcal{I}_n} card(\mathcal{D}_i) \tag{22}$$

For the system (19)-(21) in this new matrix formulation, Λ is a diagonal positive matrix in $\mathbb{R}^{m \times m}$ such that $\Lambda = \text{diag}(\lambda_{ij}), i \in \mathcal{I}_n, j \in \mathcal{D}_i$. $W : \mathbb{R}^+ \to \mathbb{R}^n, Z : \mathbb{R}^+ \to \mathbb{R}^n$ and

- $A = \text{diag}(-r_i'(z_i^*)) \in \mathbb{R}^{n \times n}$;
- $G_y \in \mathbb{R}^{n \times m}$ with $G_y[i,j] = w_i^*$ if $j \in \mathcal{U}_i$ or $G_y[i,j] = 0$ otherwise;
- $B_w = \text{diag}(d_i^* + \sum_{\substack{k \neq i \\ k \in \mathcal{U}_i}} q_{ki}^*)$ in $\mathbb{R}^{n \times n}$ ($d_i \equiv 0$ as long as $i \notin \mathcal{I}_{in}$);
- D is a diagonal matrix in $\mathbb{R}^{n \times n}$ whose diagonal entries are w_i^* if $i \in \mathcal{I}_{in}$ or 0 otherwise.

In addition, $U : \mathbb{R}^+ \to \mathbb{R}^l$ where l is given by $l := \sum_{i \in \mathcal{R}}(card(\mathcal{D}_i) - 1) + \sum_{i \in \mathcal{R} \cap \mathcal{I}_{out}} card(\mathcal{D}_i)$. In order to characterize B_u, we define for $i \in \mathcal{R}$, $\tilde{B}_{ui} = \begin{bmatrix} diag(r_i(z_i^*)) \\ -r_i(z_i^*) \end{bmatrix} \in \mathbb{R}^{card(\mathcal{D}_i) \times (card(\mathcal{D}_i-1))}$. Similarly, in order to characterize G_z, we define, for $i \in \mathcal{R}$, $\tilde{G}_{zi} = [u_{ij}^* r_i'(z_i^*)] \in \mathbb{R}^{card(\mathcal{D}_i)}$. Note that $\tilde{y}_i(t,0) = \tilde{G}_{zi}Z_i(t) + \tilde{B}_{ui}\tilde{U}_i(t)$, for $i \in \mathcal{R}$. With all these ingredients, we can finally build up the matrix B_u as a diagonal block matrix of $\tilde{B}_{ui}$'s, that is $B_u = diag(\tilde{B}_{ui}) \in \mathbb{R}^{m \times l}$ and the matrix G_z as block matrix made up of G_{zi}'s, that is $G_z = [diag(\tilde{G}_{zi}) \ 0_{m \times 1}] \in \mathbb{R}^{m \times n}$. To finish stating the system (19)-(21), it remains to say that $\tilde{d}(t)$ is the input flow that can be viewed in the sequel as an input disturbance. We assume that $\tilde{d}$ is in $\mathcal{C}_{pw}(\mathbb{R}^+; \mathbb{R}^n)$. Finally, $y(0,x) = y^0(x) \in L^2([0,1]; \mathbb{R}^m)$ and $Z(0) = Z^0 \in \mathbb{R}^n$.

Let us now study the well-posedness and the stability of the system (19)-(21) in open-loop, i.e. $U(t) = 0$ and $W(t) = 0$.

3.3 Well-posedness of the system in open-loop

Proposition 1. *[Well-posedness in open-loop] Let $\tilde{d}$ be in $\mathcal{C}_{pw}(\mathbb{R}^+;\mathbb{R}^n)$. For every $Z^0 \in \mathbb{R}^n$ and $y^0 \in L^2([0,1];\mathbb{R}^m)$, there exists a unique solution to the system (19)-(21). Moreover, $y \in \mathcal{C}^0(\mathbb{R}^+;L^2([0,1];\mathbb{R}^m))$ and $Z \in \mathcal{C}^0(\mathbb{R}^+;\mathbb{R}^n)$.*

3.4 Stability results in open-loop

Let us define the notion of input-to-state stability for the system (19)-(21).

Definition 1. *[Input-to-state stability ISS] The system (19)-(21) is input-to-state stable (ISS) with respect to $\tilde{d} \in \mathcal{C}_{pw}(\mathbb{R}^+;\mathbb{R}^n)$, if there exist $\nu > 0$, $C_1 > 0$ and $C_2 > 0$ such that, for every $Z^0 \in \mathbb{R}^n$, $y^0 \in L^2([0,1];\mathbb{R}^m)$, the solution satisfies, for all $t \in \mathbb{R}^+$,*

$$
(\|Z(t)\|^2 + \|y(t,\cdot)\|^2_{L^2([0,1],\mathbb{R}^m)}) \le
$$
$$
C_1 e^{-2\nu t}(\|Z^0\|^2 + \|y^0\|^2_{L^2([0,1];\mathbb{R}^m)}) + C_2 \sup_{0 \le s \le t} \|\tilde{d}(s)\|^2
$$
(23)

C_2 is called the asymptotic gain.

Theorem 1. *[Input-to-state stability analysis in open-loop] Let $\underline{\lambda} = \min\{\lambda_{ij}\}_{\substack{i \in \mathcal{I}_n \\ j \in \mathcal{D}_i}}$. Assume that there exist $\mu, \gamma > 0$, a symmetric positive definite matrix $P \ge I \in \mathbb{R}^{n \times n}$ and a diagonal positive matrix $Q \ge I \in \mathbb{R}^{m \times m}$ such that the following matrix inequality is satisfied:*

$$
\begin{bmatrix} A^T P + PA + G_z^T Q\Lambda G_z + 2\mu\underline{\lambda}P & PG_y & PD \\ \star & -e^{-2\mu}Q\Lambda & 0 \\ \star & \star & -\gamma I \end{bmatrix} \le 0
$$
(24)

Then, the system (19)-(21) is input-to-state stable (ISS) with respect to inputs $\tilde{d} \in \mathcal{C}_{pw}(\mathbb{R}^+;\mathbb{R}^n)$, and the asymptotic gain satisfies

$$
C_2 \le \frac{\gamma}{2\mu\underline{\lambda}} e^{2\mu}.
$$
(25)

4. NUMERICAL SIMULATIONS

Let us consider a network under compartmental setting as represented in Figure 4 which is made up of 4 buffers along with 5 transmission lines.

The index sets involved in the example are: $\mathcal{I}_n = \{1,2,3,4\}$, $\mathcal{I}_{in} = \{1\}$, $\mathcal{I}_{out} = \{4\}$, $\mathcal{U}_1 = \emptyset$, $\mathcal{U}_2 = \{1\}$; $\mathcal{U}_3 = \{1,2\}$, $\mathcal{U}_4 = \{2,3\}$, $\mathcal{D}_1 = \{2,3\}$, $\mathcal{D}_2 = \{3,4\}$, $\mathcal{D}_3 = \{4\}$, $\mathcal{D}_4 = \emptyset$. Consider then the model (10)-(12) introduced in Section 2. We assume that the system admits a *free-flow steady-state* satisfying (14) and according to the constrained optimization problem described in Subsection 3.1. Recall that the cost function to be minimized is $J = \alpha(-J_1) + (1-\alpha)J_2$ with α chosen according to some decision-maker criteria by means of the so-called Pareto fronts. Here, $J_1 = e_4^*$ according to (3.1). Following the procedure to compute $J_2 = T_{MTT}$ we have explained in Subsection 3.1, we have

$$
T_{MTT} = \frac{(T_4 + T_2 + T_1 + \frac{1}{\lambda_{24}} + \frac{1}{\lambda_{12}})(w_4^*(1-u_{23}^*)w_2^* u_{12}^* w_1^* d_1^*)}{e_4^*}
$$
$$
+ \frac{(T_4 + T_3 + T_1 + \frac{1}{\lambda_{34}} + \frac{1}{\lambda_{13}})(w_4^* w_3^*(1-u_{12}^*)w_1^* d_1^*)}{e_4^*}
$$
$$
+ \frac{(T_4 + T_3 + T_2 + T_1 + \frac{1}{\lambda_{34}} + \frac{1}{\lambda_{23}} + \frac{1}{\lambda_{12}})(w_4^* w_3^* u_{23}^* w_2^* u_{12}^* w_1^* d_1^*)}{e_4^*}
$$

with $T_i = \theta_i(z_i^*)$, $i \in \mathcal{I}_n$ (θ_i given by (6)) that can be obtained using the minimizers of J, i.e. $(w_1^*, w_2^*, w_3^*, u_{12}^*, u_{23}^*)$ and d_1^*. The linearized system (19) around the free-flow equilibrium has the following matrices: $\Lambda = diag(\lambda_{12}, \lambda_{13}, \lambda_{23}, \lambda_{24}, \lambda_{34})$, $A = diag(-r_i'(z_i^*))$ with $r_i'(z_i^*) = \frac{\epsilon_i}{(1+z_i)^2}$.

$$
G_z = \begin{pmatrix} u_{12}^* r_1'(z_1^*) & 0 & 0 & 0 \\ (1-u_{12}^*)r_1'(z_1^*) & 0 & 0 & 0 \\ 0 & u_{23}^* r_2'(z_2^*) & 0 & 0 \\ 0 & (1-u_{23}^*)r_2'(z_2^*) & 0 & 0 \\ 0 & 0 & r_3'(z_3^*) & 0 \end{pmatrix}
$$

$$
G_y = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ w_2^* & 0 & 0 & 0 & 0 \\ 0 & w_3^* & w_3^* & 0 & 0 \\ 0 & 0 & 0 & w_4^* & w_4^* \end{pmatrix}, B_w = \begin{pmatrix} d_1^* & 0 & 0 & 0 \\ 0 & q_{12}^* & 0 & 0 \\ 0 & 0 & q_{13}^* + q_{23}^* & 0 \\ 0 & 0 & 0 & q_{24}^* + q_{34}^* \end{pmatrix},
$$

$$
D = \begin{pmatrix} w_1^* & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \text{ and } B_u = \begin{pmatrix} w_1^* d_1^* & 0 \\ -w_1^* d_1^* & 0 \\ 0 & w_2^* u_{12}^* w_1^* d_1^* \\ 0 & -w_2^* u_{12}^* w_1^* d_1^* \\ 0 & 0 \end{pmatrix}.
$$

As initial conditions, we have taken $Z^0 = 0.1z^*$ and $y^0(x) = 0.1q^*$ for all $x \in [0,1]$. Let us consider the following network parameters: i) for the processing capacities; $\epsilon_1 = 100$, $\epsilon_2 = 50$, $\epsilon_3 = 80$ and $\epsilon_3 = 80$, ii) the transport velocities; $\lambda_{12} = 1$, $\lambda_{13} = 1.2$, $\lambda_{23} = 1.7$, $\lambda_{24} = 0.5$ and $\lambda_{34} = 2$ and iii) the critical traffic densities for free-flow condition; $\sigma_{12} = 50$, $\sigma_{13} = 50$, $\sigma_{23} = 50$, $\sigma_{24} = 30$ and $\sigma_{34} = 100$. For a given constant input flow demand $d_1^* = 100$, equilibrium points were found by solving the constrained optimization problem described in Subsection 3.1 using the optimization toolbox in Matlab *fgoalattain*. Nevertheless, one is not able to guarantee a global minimizer but a local one because the cost function is not convex. Using a Pareto front of the multi-objective optimization problem J, one can obtain the following minimizer decision variables for J: $w_1^* = 0.95$, $w_2^* = 0.95$, $w_3^* = 0.95$, $w_4^* = 0.95$ and $u_{12}^* = 0.36$, $u_{23}^* = 0.54$. The steady-states are $z_1^* = 19$, $z_2^* = 1.98$, $z_3^* = 13.12$, $z_4^* = 5.6$ and $q_{12}^* = 35$, $q_{13}^* = 60$, $q_{23}^* = 18.25$, $q_{24}^* = 15$ and $q_{34}^* = 74.3$. With this values, we obtain that $J_1 = 84.8$ and that $J_2 = 2.19$. We aim at minimizing the asymptotic gain in open-loop subject to (24). This can be formulated as an optimization problem involving a bilinear matrix inequality (BMI), which can be solved by a line search algorithm (on μ) to get successive LMIs to be solved using semi-definite programing. At each iteration, hypothesis of Theorem 1 are satisfied. Once the optimization problem is solved, Theorem 1 holds with optimal matrices

$$
P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 20.96 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1.40 \end{pmatrix}, Q = \begin{pmatrix} 4.25 & 0 & 0 & 0 & 0 \\ 0 & 3.41 & 0 & 0 & 0 \\ 0 & 0 & 7.023 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}
$$ along

with optimal values $\mu = 0.11$, $\nu = 0.055$ and $\gamma = 3.59$. Then, a bound of the asymptotic gain obtained is $\frac{\gamma}{2\mu}e^{2\mu} = 40.48$. Figure 5 shows the total output function $e_4(t)$ (right). It can be observed that it remains close to the optimal steady-state value given by $J_1 = e_4^* = 84.8$. However, as expected, no convergence is guaranteed to
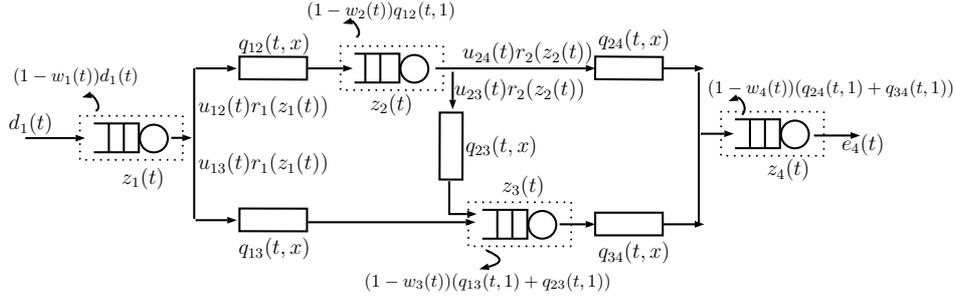
Fig. 4. Network of compartments made up of 4 buffers and 5 transmission lines.
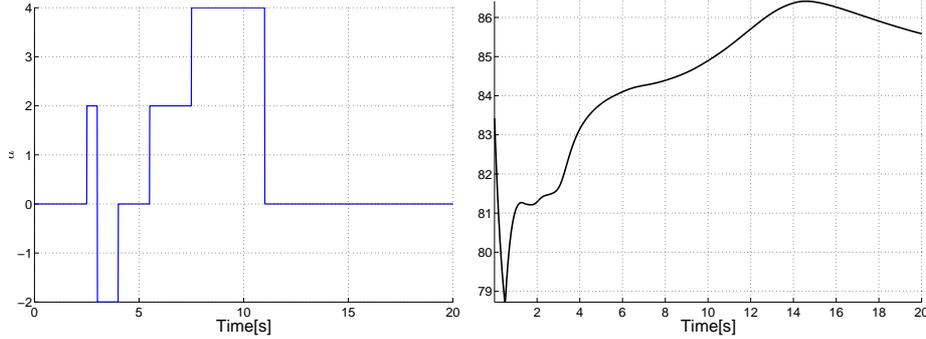

Fig. 5. Input flow demand $\tilde{d}(t)$ (left) and total output flow $e_4(t)$ of the network (right).

such a optimal value due to the influence of the input disturbance (unless $d_1 \equiv 0$) which has a profile shown in Figure 5 on the left.

5. CONCLUSION

Fluid-flow modeling for a network along with stability analysis has been studied. The motivating application is a communication network. We studied a free-flow operating point for which we proved Input-to-state stability properties. We deduced a bound of the asymptotic gain which measures the impact of the input flow demands into the network. It could be fruitful to study feedback control setting, using the two control actions described in this work. We would expect to reduce the asymptotic gain and to improve the performance of the network.

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