Notes on Random Walks in the Cauchy Domain of Attraction

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Abstract. The goal of these notes is to fill some gaps in the literature about random walks in the Cauchy domain of attraction, which has been in many cases left aside because of its additional technical difficulties. We prove here several results in that case: a Fuk-Nagaev inequality and a local version of it ; a large deviation theorem ; two types of local large deviation theorems. We also derive two important applications of these results: a sharp estimate of the tail of the first ladder epochs, and renewal theorems – extending standard renewal theorems to the case of random walks. Most of our techniques carry through to the case of random walks in the domain of attraction of an α-stable law with $\alpha \in (0,2)$, so we also present results in that case, since some of them seem to be missing in the literature.

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1. Introduction

This paper initiated when, consulting some colleagues about random walks in the Cauchy domain of attraction, they all shared the same observation that this case was often left aside in the literature, and that many very natural results – to the best of our knowledge – were not proven. These notes therefore aim at filling as many gaps as possible, proving some new results as a by-product.

1.1. Setting and first notations.

Let $(X_i)_{i \geqslant 1}$ be a sequence of i.i.d., $\mathbb{Z}$-valued, random variables. We denote $S_n := \sum_{i=1}^n X_i$, and $M_n := \max_{1 \leqslant i \leqslant n}\{X_i\}$. We assume that $(S_n)_{n \geqslant 0}$ is in the domain of attraction of an α-stable distribution, with $\alpha \in (0,2)$. We will put emphasis on the case $\alpha = 1$, but we introduce notations in the general case.

More precisely, we assume that there is some $\alpha \in (0,2)$ and some slowly varying function $L(\cdot)$, such that as $x \to \infty$, $\mathbf{P}(|X_1| > x) \sim L(x)x^{-\alpha}$ and

$$\mathbf{P}(X_1 > x) \sim pL(x)\, x^{-\alpha}\,, \qquad \mathbf{P}(X_1 < -x) \sim qL(x)\, x^{-\alpha}, \tag{1.1}$$

with $p+q=1$. If $p=0$ (or $q=0$), then we interpret (1.1) as $o(L(x)x^{-\alpha})$.

We define a_n the scaling sequence, characterized up to asymptotic equivalence by the following relation

$$L(a_n)(a_n)^{-\alpha} \overset{n\to\infty}{\sim} \frac{1}{n} \quad \text{if } \alpha \in (0,2)\,. \tag{1.2}$$

We also define the recentering sequence, that we denote b_n. We set $\mu := \mathbf{E}[X_1]$ if X_1 is integrable (if X_1 is not integrable we abbreviate it as $|\mu| = +\infty$), and let

$$\begin{aligned} &b_n \equiv 0 \quad \text{if } \alpha \in (0,1)\,; \qquad\qquad b_n = n\mu \quad \text{if } \alpha > 1\,; \\ &b_n = n\mu(a_n) \quad \text{with } \mu(x) = \mathbf{E}\big[X_1 \mathbf{1}_{\{|X_1| \leqslant x\}}\big] \quad \text{if } \alpha = 1\,. \end{aligned} \tag{1.3}$$

We then have that $(S_n - b_n)/a_n$ converges in distribution to a non-trivial α-stable distribution, see e.g. [15, IX.8] (in particular p.315 (8.15) for the recentering). We denote Y a random variable with the limiting distribution, whose density is continuous and denoted by $g(\cdot)$. Under these assumptions, Gnedenko's local limit theorem gives (see e.g. [18]):

$$\sup_{x\in\mathbb{Z}^d} \Big| a_n \mathbf{P}\big(S_n = x\big) - g\Big(\frac{x-b_n}{a_n}\Big)\Big| \stackrel{n\to\infty}{\longrightarrow} 0\,. \tag{1.4}$$

This local limit result is sharp in the range when $|x - b_n|$ is of order a_n, but does not give much information when $|x-b_n|/a_n \to +\infty$: one aim of our paper is to provide large and local large deviations estimates, in particular in the case $\alpha = 1$ which was left aside in many cases such as [7] or [10, 13].

1.2. **Organization of the paper and outline of the results.** Let us now present a brief overview of the paper.

In Section 2, we present large and local large deviation theorems. First, our Theorem 2.1 gives a standard large deviation estimate which seemed to be missing in the case $\alpha = 1$ in full generality. We provide a Fuk-Nagaev inequality and a local version of it (Theorem 2.2), which is a cornerstone of our paper and in turn implies our local large deviation Theorem 2.3 that extends Caravenna and Doney's result [7, Thm. 1.1] to the case $\alpha = 1$. Furthermore, with an additional locality assumption on the distribution of X_1, we provide an improved local large deviation Theorem 2.4, extending that of Doney [13, Thm. A] to the case $\alpha \in [1, 2)$.

In Section 3, we give applications of the large and local large deviations to two distinct problems. First, we consider the first descending ladder epoch $T_- = \inf\{n; S_n < 0\}$, and give a sharp asymptotic of $\mathbf{P}(T_- > n)$ in the case $\alpha = 1$ with infinite mean (Theorem 3.2) which improves significantly Budd, Curien and Marzouk's result [6, Prop. 1]. We also treat the case $\alpha \in [1, 2)$ with finite mean (Theorem 3.3, as was done in [12] for $\alpha \neq 1$). Some subtleties arise in the case $\alpha = 1$ with $\mu = 0$, that we collect in Theorem 3.4. Second, we consider renewal theorems for transient random walks: in the case $\alpha = 1$, we give sufficient conditions for the random walk to be transient, and we give the asymptotics of the Green function $G(x) = \sum_{k=0}^{\infty} \mathbf{P}(S_k = x)$ as $x \to \infty$, see Theorem 3.5 in the "centered" case $p = q$ and Theorem 3.6 in the infinite mean case with $p \neq q$. We also give a result in the case $\alpha \in [1, 2)$ with finite non-zero mean (Theorem 3.7).

In Section 4, we consider several issues arising in the case $\alpha = 1$. In particular, we discuss the question of the transience/recurrence of the random walk in the case $|\mu| = +\infty$: this is actually quite subtle, since the random walk – even if it goes to $+\infty$ or $-\infty$ in probability in some cases – is shown not to drift to $+\infty$ or $-\infty$ (at least when $p, q \neq 0$, see Theorem 3.1). We give sufficient conditions for the random walk to be transient in the "centered" case $p = q$ (Proposition 4.1) or in the case $p \neq q$ (Proposition 4.2). Moreover, we provide useful estimates on slowly varying functions (more precisely de Haan functions) related to the case $\alpha = 1$ with infinite mean.

All the proofs are collected in Sections 5-6-7-8. In Section 5 we state and prove Fuk-Nagaev inequalities, which are the central tool for proving the large and local large deviation estimates, that are derived in Section 6. In Section 7, we focus on the ladder epochs theorems in the case $\alpha = 1$ with $|\mu| = \infty$ (and then adapt the proof to the case $\mu = 0$). In Section 8, we prove the renewal theorems, first in the case $\alpha = 1$ with infinite mean, and then in the finite mean case.

2. Large and local large deviations

Let us begin by stating a large deviation theorem which is standard when $\alpha \neq 1$ (see [20] and references therein, also [9]), but appears to be missing in the case $\alpha = 1$.

Theorem 2.1. *Assume that* (1.1) *holds with* $\alpha \in (0,2)$*, and define* a_n *as in* (1.2) *and* b_n *as in* (1.3)*. Then,*

$$\mathbf{P}(S_n - b_n > x) \sim npL(x)x^{-\alpha}, \qquad \text{as } x/a_n \to +\infty\,,$$
$$\mathbf{P}(S_n - b_n < -x) \sim nqL(x)x^{-\alpha}, \qquad \text{as } x/a_n \to +\infty\,.$$

(If $p = 0$ or $q = 0$, one interpret this as $o(nL(x)x^{-\alpha})$.)

This result asserts that the large deviation is realized by a so-called *one-jump* strategy (see [10] for a general setting). For the case $\alpha = 1$, [20] or [10] appears to be giving the correct behavior only when the step distribution is sufficiently centered, that is $\sup_n |b_n|/a_n < +\infty$ (see condition (27) in [10]), or when $x \geqslant \delta b_n$ for some $\delta > 0$. The contribution of Theorem 2.1 is therefore to extend the result in the case $\alpha = 1$ to the whole range $|x|/a_n \to +\infty$, without any restriction on b_n. In Section 5, we recall the central tool to prove this theorem, the so-called Fuk-Nagaev inequalities (we prove a new one in the case $\alpha = 1$), and we then prove Theorem 2.1 as a simple consequence of these inequalities.

2.1. **Local large deviations.** As far as a local version of Theorem 2.1 is concerned, results can be found in [10, §9] in the "centered" case $\sup_n |b_n|/a_n < +\infty$. Recently, Caravenna and Doney [7, Thm. 1.1] gave an improved local limit estimate in the case $\alpha \in (0,1) \cup (1,2)$ (and assuming $\mu = 0$ when $\alpha \in (1,2)$): given $\gamma > 0$, they prove that there is some constant $C_0 = C_0(\gamma) < \infty$ such that for all $x \geqslant 0$,

$$\mathbf{P}(S_n = x, M_n \leqslant \gamma x) \leqslant \frac{C_0}{a_n} \big(n\mathbf{P}(X_1 > x)\big)^{\lceil 1/\gamma \rceil}.$$

We extend their result to the case $\alpha = 1$ without any restriction on b_n, and we also prove a result in the case where $M_n \leqslant y$ with $y \ll x$ (which they do not considered).

It essentially corresponds to having a local version of some Fuk-Nagaev inequalities —we recall these inequalities for $\alpha \in (0,2)$ in Section 5. Here, we give a Fuk-Nagaev inequality in the case $\alpha = 1$ and a local version of it, which are new.

Theorem 2.2. *Assume that* (1.1) *holds with* $\alpha = 1$*. For every* $\varepsilon > 0$ *there exist constants* $c_1, c_2, c_3 > 0$ *such that for any* $0 \leqslant y \leqslant x$ *and* $x \geqslant a_n$*, we have*

$$\mathbf{P}(S_n - b_n \geqslant x; M_n \leqslant y) \leqslant \Big(c_1 nL(y)/x\Big)^{(1-\varepsilon)x/y} + e^{-c_2 \left(x/a_n\right)^{1/\varepsilon}} \tag{2.1}$$

$$c_3 a_n \times \mathbf{P}(S_n - \lfloor b_n \rfloor = x; M_n \leqslant y) \leqslant \Big(c_1 nL(y)/x\Big)^{(1-\varepsilon)x/2y} + e^{-c_2 \left(x/a_n\right)^{1/\varepsilon}}\,. \tag{2.2}$$

We stress that corresponding bounds hold in the case $\alpha \in (0,1) \cup (1,2)$, see Theorems 5.1-6.1, and that it can be improved in the case $\alpha = 1$ if we assume that X_1 has a symmetric distribution, or that X_1 is non-negative, see Theorem 5.2.

This has a simple consequence, which is an extension of the local limit theorem (1.4) to the case when $|x|/a_n \to +\infty$.

Theorem 2.3. *For any* $\alpha \in (0,2)$ *(the case* $\alpha \in (0,1) \cup (1,2)$ *is in* [7, Thm. 1.1.]*), there exists a constant* $C_0 > 0$ *such that for any* $x \in \mathbb{Z}$

$$\mathbf{P}(S_n - \lfloor b_n \rfloor = x) \leqslant \frac{C_0}{a_n}\, nL(|x|)(1+|x|)^{-\alpha}\,. \tag{2.3}$$

Moreover, if $p=0$, we obtain that as $x/a_n \to +\infty$

$$\mathbf{P}(S_n - \lfloor b_n \rfloor = x) = o\Big(\frac{1}{a_n} n L(x) x^{-\alpha}\Big)\,.$$

An analogous result holds in the case $q=0$, when $x/a_n \to -\infty$.

Let us underline that this statement is somehow optimal under the mere assumption (1.1): for any sequence $(\varepsilon_n)_{n\geqslant 0}$ with $\varepsilon_n \to 0$, one may find distributions verifying (1.1) and a sequence x_n such that $x_n/a_n \to +\infty$ with

$$\limsup_{n\to\infty} \frac{a_n \mathbf{P}(S_n = x_n)}{\varepsilon_n n L(x_n) x_n^{-\alpha}} = +\infty\,. \tag{2.4}$$

2.2. **Improved local large deviation.** We may improve Theorem 2.3 if we assume that the (left or right) tail of the distribution of X_1 verifies a more local condition than (1.1), as considered for example by Doney in [13] in the case $\alpha \in (0,1)$. The first natural condition – analogous to Eq. (1.9) in [13] – is that there exists a constant $C_1>0$ (resp. C_2) such that

$$\mathbf{P}(X_1 = x) \leqslant C_1 L(x)(1+x)^{-(1+\alpha)} \qquad \text{for all } x\in\mathbb{N}\,, \tag{2.5}$$

$$\mathbf{P}(X_1 = -x) \leqslant C_2 L(x)(1+x)^{-(1+\alpha)} \qquad \text{for all } x\in\mathbb{N}\,. \tag{2.6}$$

Another natural assumption – analogous to Eq. (1.3) in [13] – is that $\mathbf{P}(X_1 = x)$ (resp. $\mathbf{P}(X_1=-x)$) is regularly varying, that is that

$$\mathbf{P}(X_1 = x) \sim p\alpha L(x) x^{-(1+\alpha)} \qquad \text{as } x\to+\infty\,, \tag{2.7}$$

$$\mathbf{P}(X_1 = -x) \sim q\alpha L(x) x^{-(1+\alpha)} \qquad \text{as } x\to+\infty\,. \tag{2.8}$$

(If $p=0$ resp. $q=0$, we interpret this as $o(L(x)x^{-(1+\alpha)})$.)

Theorem 2.4. *Assume that (1.1) holds with $\alpha\in(0,2)$. If in addition we have (2.5), then there is a constant $C>0$ such that for any $x\geqslant a_n$,*

$$\mathbf{P}(S_n - \lfloor b_n \rfloor = x) \leqslant C n L(x) x^{-(1+\alpha)} \sim \frac{C}{x}\frac{L(x)}{L(a_n)}\left(\frac{x}{a_n}\right)^{-\alpha}\,. \tag{2.9}$$

If we have (2.7), then as $n\to\infty$, $x/a_n\to+\infty$

$$\mathbf{P}(S_n - \lfloor b_n \rfloor = x) \sim n p\alpha L(x) x^{-(1+\alpha)}\,. \tag{2.10}$$

The analogous conclusion to (2.9) (resp. (2.10)) holds for $\mathbf{P}(S_n - \lfloor b_n\rfloor = -x)$ if we assume (2.6) (resp. (2.8)).

3. Applications: ladder epochs and renewal theorems

In this section, we put more emphasis on the case $\alpha=1$ (although we will also state results in the case $\alpha\in(1,2)$). Slowly varying functions will be interpreted as functions of the integers as well as differentiable functions of positive real numbers.

In the case $\alpha=1$, we define

$$\begin{aligned} \ell(n) &:= \int_1^n \frac{L(u)}{u}\mathrm{d}u \overset{n\to\infty}{\sim} \sum_{k=1}^n \frac{L(k)}{k}, \quad \text{if } |\mu|=\infty\,,\\ \ell^\star(n) &:= \int_n^\infty \frac{L(u)}{u}\mathrm{d}u \overset{n\to\infty}{\sim} \sum_{k=n}^\infty \frac{L(k)}{k}, \quad \text{if } |\mu|<\infty\,. \end{aligned} \tag{3.1}$$

We have that both $\ell(\cdot)$ and $\ell^\star(\cdot)$ are slowly varying function (in fact, de Haan functions), and as $n \to \infty$, we have that $\ell(n) \to +\infty$ and $\ell^\star(n) \to 0$, with also $\ell(n)/L(n), \ell^\star(n)/L(n) \to +\infty$, see [3, Prop. 1.5.9.a.].

In the case $|\mu| = +\infty$, then because of (1.1) we have $\mu(t) \overset{t\to\infty}{\sim} (p-q)\ell(t)$ and $b_n \overset{n\to\infty}{\sim} (p-q)n\ell(a_n)$. Similarly, in the case $|\mu| < \infty$, we also get thanks to (1.1) that $\mu - \mu(t) \overset{t\to\infty}{\sim} (p-q)\ell^\star(t)$: we end up with $b_n \sim \mu n$ if $\mu \neq 0$, and $b_n \sim (q-p)n\ell^\star(a_n)$ if $\mu = 0$. We therefore see that $|b_n|/a_n \to +\infty$ because $\ell(n)/L(n), \ell^\star(n)/L(n) \to +\infty$ (recall (1.2)), except possibly when $p = q$ and $\mu = 0$ or $+\infty$. In the case $p = q$ and $|\mu| = +\infty$ (resp. $\mu = 0$), we get that $b_n = o(n\ell(a_n))$ (resp. $b_n = o(n\ell^\star(a_n))$, and the general study gets much more subtle since we do not have a priori the asymptotic behavior of b_n. We will sometimes consider the case where $b_n/a_n \to b \in \mathbb{R}$.

3.1. **Ladder epochs.** Denote $T_- = \inf\{n\,;\, S_n < S_0 = 0\}$ and $T_+ = \inf\{n\,;\, S_n > S_0 = 0\}$ the first descending and ascending ladder epochs. A very natural question is first to know whether T_-, T_+ are defective (if so, then S_n is said to *drift* to $+\infty$, resp. $-\infty$), and to obtain the asymptotics of the tail probabilities $\mathbf{P}(T_- > n), \mathbf{P}(T_+ > n)$.

A crucial tool for this study is our Theorem 2.1, which gives as an easy consequence the precise asymptotics (see Lemmas 7.1 and 7.3 below):

(i) in the case $\alpha = 1$ with infinite mean, we have $b_n/a_n \to +\infty$ if $p > q$ and $b_n/a_n \to -\infty$ if $p < q$, so that

$$\mathbf{P}(S_n < 0) \sim \frac{q}{p-q}\frac{L(|b_n|)}{\ell(|b_n|)} \text{ if } p > q, \quad \text{and} \quad \mathbf{P}(S_n > 0) \sim \frac{p}{q-p}\frac{L(|b_n|)}{\ell(|b_n|)} \text{ if } p < q\,; \tag{3.2}$$

(ii) in the case $\alpha = 1$ with $\mu = 0$, we have $b_n/a_n \to -\infty$ if $p > q$ and $b_n/a_n \to +\infty$ if $p < q$, so that

$$\mathbf{P}(S_n > 0) \sim \frac{p}{p-q}\frac{L(|b_n|)}{\ell^\star(|b_n|)} \text{ if } p > q, \quad \text{and} \quad \mathbf{P}(S_n < 0) \sim \frac{q}{q-p}\frac{L(|b_n|)}{\ell^\star(|b_n|)} \text{ if } p < q\,. \tag{3.3}$$

(One may naturally find asymptotics in the general case $\alpha \in (0,2)$.) From this we may use Theorem 2 in [15, XII.7] and Lemma 4.4 below, to get the following

Proposition 3.1. *Assume (1.1) holds with $\alpha = 1$, and assume that $\mu = 0$ or $|\mu| = +\infty$. Then if $q \neq 0$*

$$\sum_{n=1}^{+\infty} \frac{1}{n}\mathbf{P}(S_n < 0) = +\infty,$$

and the random walk does not drift to $+\infty$, in the sense that $T_- < \infty$ a.s. Analogously, if $p \neq 0$ then the random walk does not drift to $-\infty$, i.e. $T_+ < +\infty$ a.s.

Note that in the case of finite non-zero mean $\mu \neq 0$, the strong law of large numbers gives that $(S_n)_{n\geqslant 1}$ is transient, and drifts to $+\infty$ (resp. to $-\infty$) if $\mu > 0$ (resp. $\mu < 0$).

This proposition tells that if $\alpha = 1$ with $|\mu| = +\infty$ or $\mu = 0$ and $q \neq 0$, then even if $b_n/a_n \to +\infty$ so that S_n goes to $+\infty$ in probability (more precisely S_n/b_n converges in probability to 1), the random walk does not drift to $+\infty$ —and $\liminf S_n = -\infty$ a.s. This is due to the fact that even if the random walk is in probability "close" to $b_n \to +\infty$, once in a while a large jump to the left occurs (of length of order b_n), making $S_n < 0$ (and in fact, of order $-b_n$). We will see below in Section 4.1 that the random walk may still be transient (in the sense that 0 may be visited only a finite number of times), but that determining transience/recurrence is a more complicated matter.

As far as the asymptotics of $\mathbf{P}(T_- > n)$ are concerned (the estimates for T_+ are symmetric by considering $(-S_n)_{n \geqslant 0}$), we refer to the seminal papers of Rogozin [21] and of Doney [11, 12], and to [5] or [22] for more recent results. However, the case $\alpha = 1$ (to our knowledge) does not appear to have been treated in the literature, apart from the recent work of Budd, Curien and Marzouk [6, Prop. 1] where a rough estimate on $\mathbf{P}(T_- > n)$ is given in the case where $L(n)$ is constant. We shall give a sharp asymptotic, in the case of a general slowly varying function.

We will give two levels of sharpness, according to whether we assume that $L(\cdot)$ in (1.1) is slowly varying in the Flajolet-Odlyzko sense (see conditions V1-V2 in [16]), that is verifies:

V1. there exists some $x_0 > 0$ and some $\phi \in (\pi/2, \pi)$ such that $L(z)$ is analytic in the region $\{z\,;\, \arg(z - x_0) \in [-\phi, \phi]\}$

V2. we have, for any $\theta \in [-\phi, \phi]$ and $x \geqslant x_0$

$$\Big|\frac{L(xe^{i\theta})}{L(x)} - 1\Big| \leqslant \varepsilon(x)\,, \qquad \Big|\frac{L(x\log x)}{L(x)} - 1\Big| \leqslant \varepsilon(x)\,, \qquad \text{for some } \varepsilon(x) \stackrel{x\to\infty}{\to} 0\,.$$

This is satisfied for example if $L(x)$ is equal to $(\log x)^a$ or $(\log\log x)^a$for some $a \in \mathbb{R}$, but we stress that V2 fails for instance if $L(x) = \exp((\log x)^b)$ for some $b \in (0,1)$.

3.1.1. *Case $\alpha = 1$, $|\mu| = +\infty$.*

Theorem 3.2. *Assume that (1.1) holds with $\alpha = 1$ and $|\mu| = +\infty$. Then, recalling the definition of $b_n = n\mu(a_n)$ in (1.3) and of $\ell(\cdot)$ in (3.1)*

(i) *If $p = q$ and if $b = \lim_{n\to\infty} b_n/a_n = \lim_{n\to\infty} \mathbb{E}[\frac{X_1}{1+(X_1/a_n)^2}]$ exists, then there exists a slowly varying function $\varphi(\cdot)$ such that*

$$\mathbf{P}(T_- > n) = \varphi(n)\, n^{-\rho} \qquad \text{with } \rho = \frac12 + \frac1\pi \arctan\Big(\frac{2b}{\pi}\Big)\,.$$

(ii) *If $p < q$ in (1.1), then $b_n \stackrel{n\to\infty}{\sim} -(q-p)n\ell(a_n) \to -\infty$, and*

$$\mathbf{P}(T_- > n) = \frac{L(|b_n|)}{n} \ell(|b_n|)^{\frac{p}{q-p} - 1 + o(1)}\,.$$

If additionally V1-V2 above holds, then we can make the $o(1)$ more precise: there exists a slowly varying function $\widetilde{L}(\cdot)$ such that $\ell(|b_n|)^{o(1)}$ can be replaced by $\widetilde{L}\big(\ell(|b_n|)\big)$.

(iii) *If $p > q$ in (1.1), then $b_n \stackrel{n\to\infty}{\sim} (p-q)n\ell(a_n) \to +\infty$, and*

$$\mathbf{P}(T_- > n) = \ell(b_n)^{-\frac{q}{p-q} + o(1)}\,.$$

If additionally V1-V2 above holds, then there exists a slowly varying function $\bar{L}(\cdot)$ such that $\ell(|b_n|)^{o(1)}$ can be replaced by $\bar{L}\big(\ell(|b_n|)\big)$.

We are also able to deal with the case $p = q = 1/2$ when $b_n/a_n \to +\infty$ but we did not state it here for conciseness, since it requires further notations: we refer to Section 7.5 for details, see in particular (7.22).

As an application of Theorem 3.2, we improve Proposition 1 of [6] in the case $p \neq q$: if $L(n)$ is constant equal to c (and obviously verifies V1-V2), then $\ell(n) \sim c\log n$, $a_n \sim cn$, and $b_n \sim (p-q)cn\log n$. Hence, we obtain that there exist some slowly varying functions $\bar{L}(\cdot), \widetilde{L}(\cdot)$ such that

$$\mathbf{P}(T_- > n) = \begin{cases} (\log n)^{-\frac{q}{p-q}}\, \bar{L}(\log n) & \text{if } p > q\,, \\ n^{-1}(\log n)^{\frac{p}{q-p} - 1}\, \widetilde{L}(\log n) & \text{if } p < q\,. \end{cases} \tag{3.4}$$

3.1.2. *Case* $|\mu| < +\infty$. Estimates for $\mathbf{P}(T_- > n)$ can be found in the case $\alpha \in (1,2)$ in [12] and in [4, 5], but the case $\alpha = 1$ also appears to have been left aside. We stress that the proofs in [4, 5] rely on an asymptotic estimate of $\mathbf{P}(S_n < 0)$ (or $\mathbf{P}(S_n > 0)$) as $n \to \infty$, that are given by Theorem 2.1: for instance if $\mu > 0$, we get that $\mathbf{P}(S_n < 0) \sim qnL(\mu n)(\mu n)^{-\alpha}$, using the fact that $b_n \sim n\mu$. We state the results for the sake of completeness, leaving aside the case $\alpha = 1$ $\mu = 0$ for the moment.

Theorem 3.3 (cf. Theorems 0-I of [12] and Theorem 1 of [4])**.** *Assume that* (1.1) *holds with* $\alpha \in [1,2)$ *and* $|\mu| < +\infty$.

(i) *If* $\mu > 0$, *then* T_- *is defective:* $p_\infty := \mathbf{P}(T_- = +\infty) = e^{-D_-}$ *with* $D_- := \sum_{k=1}^\infty k^{-1}\mathbf{P}(S_k < 0) < \infty$. *Moreover, if* $q \neq 0$ *we have*
$$\mathbf{P}(T_- = n) \overset{n\to\infty}{\sim} \frac{q\, e^{-D_-}}{\mu^\alpha} L(n) n^{-\alpha}\,.$$

(ii) *If* $\mu < 0$ *and* $p \neq 0$, *then we have* $D_+ := \sum_{k=1}^\infty \frac{1}{k}\mathbf{P}(S_k \geqslant 0) < \infty$ *and*
$$\mathbf{P}(T_- > n) \overset{n\to\infty}{\sim} \frac{p\, e^{D_+}}{|\mu|^\alpha} L(n) n^{-\alpha}\,.$$

(iii) *If* $\mu = 0$ *and* $\alpha \in (1,2)$, *then there exists a slowly varying function* $\varphi(\cdot)$ *such that*
$$\mathbf{P}(T_- > n) = \varphi(n)\, n^{-\rho}\,,$$
with $\rho := \mathbf{P}(Y > 0) = \frac12 + \frac{1}{\pi\alpha}\arctan\big((p-q)\tan(\pi\alpha/2)\big)$.

The proof of items (i)-(ii) easily translate from [4, Thm. 1] to the case $\alpha = 1$ with a finite non-zero mean, so we only sketch the proof of these items. For item (i), we simply use that in the case $\mu > 0$, as seen above, we have $n^{-1}\mathbf{P}(S_n < 0) \sim qL(n)(\mu n)^{-\alpha}$, which is regularly varying: then the result follows from Eq. (23) in [4] together with an application of [8, Thm. 1], using that $D_- := \sum_k k^{-1}\mathbf{P}(S_k > 0) < \infty$. Item (ii) is similar. For (iii), we have that $b_n \equiv 0$, and we can use that $\mathbf{P}(S_k > 0) = \mathbf{P}(S_k/a_k > 0)$ converges to $\mathbf{P}(Y > 0) =: \rho$ where Y is an α-stable law with skewness parameter $\beta = p - q$ (hence the formula for the positivity parameter ρ, see [25, Sec. 2.6]). This implies (see [21, 12]) that T_- is in the domain of attraction of a positive stable random variable with index ρ, and item (iii) follows.

3.1.3. *Case* $\alpha = 1$, $\mu = 0$. We left the case $\alpha = 1$, $\mu = 0$ outside of the statement of Theorem 3.3 since it is not a straightforward adaptation of [4, Thm. 1], in particular because $\sum_k k^{-1}\mathbf{P}(S_k > 0) = +\infty$ (see Proposition 3.1). We obtain results analogous to those of Theorem 3.2 (we leave aside the case $p = q$ for simplicity).

Theorem 3.4. *Assume that* (1.1) *holds with* $\alpha = 1$ *and assume that* $\mu = 0$. *Then (recall the definition* (3.1) *of* $\ell^\star(\cdot)$*),*

(i) *If* $p > q$ *we have* $b_n \sim -(p-q)n\ell^\star(a_n) \to -\infty$, *and*
$$\mathbf{P}(T_- > n) = \frac{L(|b_n|)}{n}\ell^\star(|b_n|)^{-\frac{p}{p-q}-1+o(1)}\,.$$

(ii) *If* $p < q$ *we have* $b_n \sim (q-p)n\ell^\star(a_n) \to +\infty$, *and*
$$\mathbf{P}(T_- > n) = \ell^\star(b_n)^{\frac{q}{q-p}+o(1)}\,.$$

If additionally assumption V1-V2 holds, then in each case there exists a slowly varying function $\widetilde{L}(\cdot)$ *such that such that* $\ell^\star(|b_n|)^{o(1)}$ *can be replaced by* $\widetilde{L}\big(\ell^\star(|b_n|)\big)$.

We prove this theorem in Section 7.6, where we discuss also the case $p = q$ —it is treated similarly to Theorem 3.2-(i), see in particular (7.24).

3.2. **Renewal theorems.** An interesting application of the local limit Theorems 2.3-2.4 is that we are able to obtain renewal theorems for transient random walks $(S_n)_{n\geqslant 1}$: we give the behavior, as $x \to \infty$, of the Green function $G(x) = \sum_n \mathbf{P}(S_n = x)$.

In the case of renewals, that is when the step variable X_1 is positive, $G(x)$ is interpreted as the renewal mass function $\mathbf{P}(x \in S)$, and has been studied in a variety of papers. The well-known renewal theorem gives that whenever $X_1 \geqslant 0$ and $\mu = \mathbf{E}[X_1] < +\infty$, then $\mathbf{P}(x \in S) \to 1/\mu$ as $x \to +\infty$. Assuming additionally that (1.1) holds with $\alpha \in (0,1]$ (and necessarily $p = 1, q = 0$ since $X_1 \geqslant 0$), then Garcia and Lamperti [17] showed the strong renewal theorem

$$\mathbf{P}(x \in S) \overset{x\to+\infty}{\sim} \frac{\alpha \sin(\pi\alpha)}{\pi} L(n)^{-1} n^{-(1-\alpha)} \qquad \text{if } \alpha \in (1/2, 1); \tag{3.5}$$

Erickson [14] also proved that,

$$\mathbf{P}(x \in S) \overset{x\to+\infty}{\sim} \frac{1}{\mu(x)} \qquad \text{if } \alpha = 1 \text{ with } |\mu| = +\infty. \tag{3.6}$$

Finally, Caravenna and Doney [7] gave very recently a necessary an sufficient condition for the above strong renewal theorem (3.5) to hold when $\alpha \in (0, 1/2]$.

When $(S_n)_{n\geqslant 1}$ is a (general) random walk rather than a renewal process, the Green function $G(x)$ has been considered in the case $\alpha \in (0,1)$ for example in [7, 24], but we are not aware of any references for the case $\alpha \geqslant 1$. We shall prove renewal theorems in the case $\alpha = 1$, under some specific assumptions that ensures the transience of the random walk (this excludes the case $\mu = 0$, where the random walk is known to be recurrent). We comment further in Section 4 the particularity of the case $\alpha = 1$, $|\mu| = +\infty$, where even the question of recurrence/transience of $(S_n)_{n\geqslant 1}$ is subtle.

The first renewal theorem we get is in the "centered" case $p = q = 1/2$.

Theorem 3.5. *Assume that (1.1) holds with $\alpha = 1$ and $p = q = 1/2$, and that $b := \lim_{n\to\infty} b_n/a_n$ exists, $b \in \mathbb{R}$. Assume also that $\sum_{n\geqslant 1} \frac{1}{nL(n)} < +\infty$. Then S_n is transient, and*

$$G(x) \overset{n\to\infty}{\sim} \frac{2}{\pi(1+(2b)^2)} \sum_{n>x} \frac{1}{nL(n)}\,,$$

which is vanishing as a slowly varying function as $x \to +\infty$.

In the case where (1.1) holds with $\alpha = 1$, $|\mu| = +\infty$ and $p \neq q$, we need the extra assumptions (2.5)-(2.6) to be able to derive a renewal theorem —otherwise it is not even clear if $(S_n)_{n\geqslant 0}$ is transient, see Proposition 4.2. Recall that in that case, we have $\mu(x) \sim (p-q)\ell(x)$ (and goes to $+\infty$ if $p > q$ and $-\infty$ if $p < q$).

Theorem 3.6. *Assume that (1.1) holds with $\alpha = 1$, $p \neq q$ and $|\mu| = +\infty$. Assume additionally that (2.5)-(2.6) hold. Then S_n is transient, and we have that $G(x) = O\big(1/|\mu(x)|\big)$ and also* $\liminf G(x)/\mu(x) \geqslant 1$ *if $p > q$.*

(i) *If $p > q$ and $\mathbf{P}(X_1 = -x) \overset{x\to\infty}{\sim} qL(x)x^{-2}$ (if $q = 0$, we interpret this as $o(L(x)x^{-2})$), then*

$$G(x) \overset{x\to+\infty}{\sim} \frac{1 + q/(p-q)}{\mu(x)}\,. \tag{3.7}$$

(ii) *If $p < q$ and $\mathbf{P}(X_1 = x) \overset{x\to\infty}{\sim} pL(x)x^{-2}$ (if $p = 0$, we interpret this as $o(L(x)x^{-2})$), then*

$$G(x) \overset{x\to+\infty}{\sim} \frac{p/(q-p)}{|\mu(x)|}\,. \tag{3.8}$$

We therefore recover Erickson's result (3.6) in the case of general random walks, at the expense of assumptions (2.5)-(2.6) (in the case of renewals we get $q = 0$).

Let us make a short comment on (3.7). When $p > q$ (so that $b_n \to +\infty$) the behavior in (3.7) comes from two types of contribution to $G(x) = \sum_{k=1}^{+\infty} \mathbf{P}(S_k = x)$: when the number of steps is of the order of k_x verifying $b_{k_x} = x$ (so that S_{k_x} is approximately x), and when the number of steps is much larger (S_k is much larger than x, but large jumps to the left still occur). For the first part, we prove that it is asymptotic to $1/\mu(x)$. For the second part, we are able to prove that it is $O(1/\mu(x))$ under (2.6), and we can get its asymptotic behavior if $\mathbf{P}(X_1 = -x) \overset{x\to\infty}{\sim} qL(x)x^{-2}$.

Finally, we give a renewal theorem also in the case of a finite non-zero mean.

Theorem 3.7. *We assume that (1.1) holds with $\alpha \in [1, 2)$ and $|\mu| < +\infty$.*

(i) *Case $\mu > 0$: in the case $\alpha = 1$, we assume additionally that (2.5)-(2.6) hold. Then we have that*

$$\lim_{x\to\infty} G(x) = \frac{1}{\mu}\,.$$

(ii) *Case $\mu < 0$: we assume additionally that $\mathbf{P}(X_1 = x) \sim p\alpha L(x)x^{-(1+\alpha)}$ as $x \to +\infty$. Then we have that, as $x \to +\infty$*

$$G(x) \sim \frac{p}{(\alpha-1)|\mu|^2}\, L(x)x^{1-\alpha} \qquad \text{if } \alpha > 1\,,$$
$$G(x) \sim \frac{p}{|\mu|^2}\ell^\star(x) \qquad \text{if } \alpha = 1\,,$$

with $\ell^\star(\cdot)$ defined in (3.1).

Item (ii) is reminiscent of Williamson's results [24] in the case $\alpha \in (0, 1)$. Our result is somehow different since it deals with a *drifting* random walk and may be seen as a generalization of [24, Thm. 1], where only the case of centered walks is considered.

We stress that one may use our techniques to prove that $G(x) \to \mu^{-1}$ as $x \to \infty$ also when S is in the domain of attraction of the Normal distribution, under a very weak assumption. We do not treat this case here since it will be proven (in a more general setting) in [2].

4. Further discussion and useful estimates in the case $\alpha = 1$

In this section, we focus on the case $\alpha = 1$, and we discuss the subtleties that might arise. One of the first difficulty is that the recentering term b_n is not *homogeneous*: the *recentered* walk $S_n - b_n$ is a sum of i.i.d. *recentered* random variables, but that recentering depends on n:

$$S_n - b_n = \sum_{i=1}^{n} (X_i - \mu(a_n))\,.$$

Hence, we are not able to simplify the problem by studying a random walk with *centered* increments, as it is customary when $\alpha > 1$.

Let us focus on the case when $|\mu| = +\infty$ for a moment, for the simplicity of exposition (analogous reasoning holds when $\mu = 0$). Recall the definition (3.1) of $\ell(\cdot)$, and let us discuss the behavior of the recentering constant b_n.

$*$ If $p > q$, then $b_n \sim (p-q)n\ell(a_n)$: we conclude that $b_n/a_n \to +\infty$, since $\ell(x)/L(x) \to +\infty$ and $a_n \sim nL(a_n)$. Hence S_n/b_n converges in probability to 1, and S_n goes in probability to $+\infty$, even though Proposition 3.1 tells that S_n does not drift to $+\infty$.

$*$ If $p = q$ then it is more tricky, and we can have all possible behaviors for b_n: $b_n = o(a_n)$ (in which case we may set $b_n \equiv 0$, as it is the case for a symmetric distribution) ; $0 < \limsup_{n\to+\infty} |b_n|/a_n < +\infty$; $\lim_{n\to\infty} b_n/a_n = +\infty$ (but still $b_n = o(n\ell(a_n))$) ; and it is not excluded that $\limsup_{n\to\infty} b_n/a_n = +\infty$ and $\liminf_{n\to\infty} b_n/a_n = +\infty$.

4.1. **About the transience/recurrence of S_n.** Recall Proposition 3.1: in the case $\alpha = 1$ with $\mu = 0$ or $\mu = +\infty$, the random walk is shown not to drift neither to $+\infty$ or $-\infty$. We recall that in the case of a finite mean, S_n is transient if $\mu \neq 0$ (by the strong law of large numbers), and recurrent if $\mu = 0$. The central – and subtle – question is therefore to know whether the random walk is transient or recurrent in the case $|\mu| = +\infty$. Let us now consider the Green function at 0, $\sum_n \mathbf{P}(S_n = 0)$, with the help of our local limit theorems.

First case. If $\sup_n |b_n|/a_n < +\infty$, then $\mathbf{P}(S_n = 0)$ is necessarily of order $1/a_n$: by the local limit theorem (1.4) $a_n \mathbf{P}(S_n = 0) = (1+o(1))g(-b_n/a_n)$, and is therefore bounded away from 0 and infinity. We conclude that the walk is transient if and only if $\sum (a_n)^{-1} < +\infty$, and Wei [23] gives another characterization.

Proposition 4.1. *If (1.1) holds with $\alpha = 1$, and $\sup_n |b_n|/a_n < +\infty$ (so in particular $p = q$) then*

$$(S_n)_{n\geqslant 0} \text{ is transient} \quad \iff \quad \sum_{n=1}^{+\infty} \frac{1}{a_n} < +\infty \quad \iff \quad \sum_{n=1}^{+\infty} \frac{1}{nL(n)} < +\infty\,.$$

Second Case. If $\lim_{n\to\infty} b_n/a_n \to +\infty$, we have that $S_n \to +\infty$ in probability, but we cannot conclude that S_n is transient, in particular because Proposition 3.1 tells that $\liminf S_n = -\infty$ a.s. We have $\mathbf{P}(S_n = 0) = \mathbf{P}(S_n - b_n = -b_n)$ and Theorem 2.3 gives that is is bounded by a constant times $(a_n)^{-1} n\mathbf{P}(X_1 > b_n) \sim p(b_n)^{-1} L(b_n)/L(a_n)$. Hence, a sufficient condition for the walk to be transient is that

$$\sum_{n\geqslant 1} (b_n^{-1}) L(b_n)/L(a_n) < +\infty.$$

However Theorem 2.3 does not provide a lower bound on $\mathbf{P}(S_n = 0)$, so the question of the recurrence/transience cannot be settled. Let us give a simple sufficient condition for the transience of the random walk.

Proposition 4.2. *Assume that (1.1) holds with $\alpha = 1$, and assume that $|\mu| = +\infty$. If $p > q$ (we then have $b_n/a_n \to +\infty$) and if additionally (2.6) holds, then S_n is transient.*

Note that the local assumption (2.6) we need is only on the left tail of X_1: since we already know that $S_n \to +\infty$ in probability when $p > q$, we simply need to control the (large) jumps to the left that might make the random walk visit 0.

Proof In the case $\alpha = 1$ with infinite mean and $p > q$, we use Theorem 2.4 to get that there is a constant C such that

$$\mathbf{P}(S_n = 0) = \mathbf{P}(S_n - \lfloor b_n \rfloor = -\lfloor b_n \rfloor) \leqslant C n b_n^{-2} L(b_n)\,.$$

Then, to show that $(S_n)_{n\geqslant 1}$ is transient, and since $b_n = n\mu(a_n) \sim (p-q)n\ell(a_n) \sim (p-q)n\ell(b_n)$ (see Lemma 4.3 below), it is therefore sufficient to show that

$$\sum_{n=1}^{+\infty} \frac{L(b_n)}{b_n \ell(b_n)} < +\infty \qquad \Longleftrightarrow \qquad \sum_{k=1}^{+\infty} \frac{L(k)}{k\ell(k)^2} < +\infty\,.$$

The equivalence simply comes from a comparison of the sums with corresponding integral (we may work with differentiable slowly varying functions see [3, Th. 1.8.2]), and a change of variable $k = b_n$, $dk = (p-q)\ell(b_n)dn$. Then Lemma 4.4-(i) below (more precisely (4.1)) shows the summability of the sum on the right-hand side, and concludes the proof. $\square$

Other cases. If for example $\limsup b_n/a_n = +\infty$ and $\liminf b_n/a_n < +\infty$, it is even less clear, and it seems hopeless to conclude anything without further assumptions.

4.2. **Useful estimates on $\ell(\cdot), \ell^\star(\cdot)$.** We now collect a few estimates on the slowly varying function $\ell(\cdot), \ell^\star(\cdot)$ defined in (3.1) (they are de Haan functions), which will be central in the proofs of Theorems 3.2 and 3.6.

Lemma 4.3. *Recall the definition (1.2) of a_n. Assume that $\ell(n) \to +\infty$ (otherwise the statement is trivial), we then have that*

$$\ell(a_n) \sim \ell\big(n\ell(a_n)\big) \qquad \text{as } n\to\infty\,.$$

Similarly, we have $\ell^\star(a_n) \overset{n\to\infty}{\sim} \ell^\star(n\ell^\star(a_n))$.

As a consequence, when $\alpha = 1$ with $|\mu| = +\infty$ (resp. $\mu = 0$) so that $b_n \sim (p-q)n\ell(a_n)$ (resp. $b_n \sim (q-p)n\ell^\star(a_n)$), if $p \neq q$, we get that $\ell(a_n) \sim \ell(b_n)$ (resp. $\ell^\star(a_n) \sim \ell^\star(b_n)$).

Proof The proof can be found in [1], but since it is very short, we include it here for the sake of completeness. We write

$$\frac{\ell(n\ell(a_n)) - \ell(a_n)}{\ell(a_n)} = \frac{1}{\ell(a_n)} \int_{a_n}^{n\ell(a_n)} \frac{L(u)}{u} \mathrm{d}u\,.$$

Then, we choose a sequence v_n that goes to infinity sufficiently slowly so that $L(u)/L(a_n) \leqslant 2$ for all $a_n \leqslant u \leqslant v_n a_n$ (with also $v_n a_n \leqslant n\ell(a_n)$). There is also a constant c such that for all $u \geqslant v_n a_n$ we have $u^{-1}L(u) \leqslant c(v_n a_n)^{-1} L(v_n a_n)$, so that we obtain

$$\begin{aligned}\frac{1}{\ell(a_n)} \int_{a_n}^{n\ell(a_n)} \frac{L(u)}{u}\mathrm{d}u &\leqslant 2\frac{L(a_n)}{\ell(a_n)} \int_{a_n}^{v_n a_n} \frac{\mathrm{d}u}{u} + cn\frac{L(v_n a_n)}{v_n a_n} \\ &\leqslant \frac{L(a_n)}{\ell(a_n)} \log \frac{\ell(a_n)}{L(a_n)} + 2cv_n^{-1/2} \overset{n\to\infty}{\to} 0\,.\end{aligned}$$

For the second inequality, we used that $v_n \leqslant n\ell(a_n)/a_n \sim \ell(a_n)/L(a_n)$ for the first term, and Potter's bound to get that $L(va_n) \leqslant 2v^{1/2}$ for all $v \geqslant 1$ (provided n is large) for the second term. The convergence to 0 comes from the fact that $\ell(a_n)/L(a_n) \to \infty$ and $v_n \to \infty$.

The statement also holds for $\ell^\star(\cdot)$ since $\ell^\star(u) - \ell^\star(v)$ is formally equal to $\ell(v) - \ell(u)$ (and $\ell^\star(x)/L(x) \to \infty$). $\square$

The next lemma is concerned with sums that appear naturally in the course of the proofs.

Lemma 4.4. *Consider the case $\alpha = 1$, and recall the definitions (3.1) of $\ell(\cdot)$ (in the case $|\mu| = +\infty$) and $\ell^\star(\cdot)$ (in the case $|\mu| < \infty$). Let f be a non-increasing function $\mathbb{R}_+ \to \mathbb{R}_+$.*

(i) *If $|\mu| = +\infty$, we have $\ell(n) \to +\infty$ and*
$$\text{if } \int_1^{+\infty} f(t)dt < +\infty, \quad \text{then} \quad \sum_{k=n}^{\infty} \frac{L(k)}{k} f\big(\ell(k)\big) \stackrel{n\to+\infty}{\sim} \int_{\ell(n)}^{+\infty} f(t)dt \to 0$$
$$\text{if } \int_1^{+\infty} f(t)dt = +\infty, \quad \text{then} \quad \sum_{k=1}^{n} \frac{L(k)}{k} f\big(\ell(k)\big) \stackrel{k\to+\infty}{\sim} \int_{1}^{\ell(n)} f(t)dt \to +\infty\,.$$
(ii) *If $|\mu| < \infty$, we have $\ell^\star(n) \to 0$ and*
$$\text{if } \int_0^{1} f(t)dt < +\infty, \quad \text{then} \quad \sum_{k=n}^{\infty} \frac{L(k)}{k} f\big(\ell^\star(k)\big) \stackrel{n\to+\infty}{\sim} \int_{0}^{\ell^\star(n)} f(t)dt \to 0$$
$$\text{if } \int_0^{1} f(t)dt = +\infty, \quad \text{then} \quad \sum_{k=1}^{n} \frac{L(k)}{k} f\big(\ell^\star(k)\big) \stackrel{n\to+\infty}{\sim} \int_{\ell^\star(n)}^{1} f(t)dt \to +\infty\,.$$
We therefore have that $\sum_{k\geqslant 1} k^{-1}L(k)f(\ell(k))$ (resp. $\sum_{k\geqslant 1} k^{-1}L(k)f(\ell^\star(k))$) is convergent if and only if $\int_1^{+\infty} f(t)dt$ (resp. $\int_0^1 f(t)dt$) is convergent.

As a consequence, in the case $\alpha = 1$ with infinite mean, we have
$$\sum_{k\geqslant 1} \frac{L(k)}{k\ell(k)^2} < +\infty \quad \text{and} \quad \sum_{k=n}^{+\infty} \frac{L(k)}{k\ell(k)^2} \stackrel{n\to\infty}{\sim} \frac{1}{\ell(n)}\,; \tag{4.1}$$
$$\sum_{k\geqslant 1} \frac{L(k)}{k\ell(k)} = +\infty \qquad \text{and} \quad \sum_{k=1}^{n} \frac{L(k)}{k\ell(k)} \stackrel{n\to\infty}{\sim} \log \ell(n)\,. \tag{4.2}$$

Proof For (i), the asymptotic equivalence comes from a simple comparison of the sum with the following integral (since $k^{-1}L(k)f(\ell(k))$ is asymptotically non-increasing this is straightforward), which is computed explicitly thanks to a change of variable $t = \ell(u)$, $dt = L(u)u^{-1}du$ (recall $\ell(u) \to +\infty$ as $u \to \infty$):
$$\int_n^{\infty} \frac{L(u)}{u} f\big(\ell(u)\big)du = \int_{\ell(n)}^{+\infty} f(t)dt\ ; \qquad \int_1^{n} \frac{L(u)}{u} f\big(\ell(u)\big)du = \int_1^{\ell(n)} f(t)dt.$$
For (ii), the result is proven in a similar manner, using the change of variable $t = \ell^\star(u)$, $dt = -L(u)u^{-1}du$ (recall $\ell^\star(u) \to 0$ as $u \to \infty$). □

5. Fuk-Nagaev's inequalities and local large deviations

From now on, we use $c, C, c', C', ...$ as generic (universal) constants, and will will keep the dependence on parameters when necessary, writing for example $c_\varepsilon, C_\varepsilon$ for constants depending on a parameter ε.

5.1. **Fuk-Nagaev inequalities.**
Our first result is an improved Fuk-Nagaev inequality in the case $\alpha = 1$. Let us first recall known Fuk-Nagaev inequalities —they are collected for example in [20], and are based on standard Cramér-type exponential moment calculations.

Theorem 5.1. *Assume that (1.1) holds with $\alpha \in (0, 2)$. There exist a constants c such that for any $y \leqslant x$,*

(i) *If $\alpha < 1$,*
$$\mathbf{P}\left(S_n \geqslant x; M_n \leqslant y\right) \leqslant \Big(1 + \frac{cx}{ny^{1-\alpha}L(y)}\Big)^{-x/y} \leqslant \Big(c^{-1}\,\frac{y}{x} nL(y)y^{-\alpha}\Big)^{x/y}\,.$$

(ii) *If* $\alpha > 1$,

$$\mathbf{P}\left(S_n - \mu n \geqslant x; M_n \leqslant y\right) \leqslant \Big(1 + \frac{cx}{ny^{1-\alpha}L(y)}\Big)^{-\frac{x}{y}} \leqslant \Big(c^{-1}\,\frac{y}{x} nL(y)y^{-\alpha}\Big)^{x/y}.$$

(iii) *If* $\alpha = 1$, *we have for any* $y \leqslant x$,

$$\mathbf{P}\left(S_n \geqslant x; M_n \leqslant y\right) \leqslant \Big(1 + \frac{cx}{nL(y)}\Big)^{-(x-n\mu(y))\frac{1}{y} - cnL(y)y^{-\alpha}}.$$

The case $\alpha < 1$ is given in [20, Thm 1.1] (take $1 \geqslant t > \alpha$ so that $A(t, Y) \leqslant cny^{t-\alpha}L(y)$). The case $\alpha \in [1, 2)$ is given in [20, Thm 1.2] (take $2 \geqslant t > \alpha$ so that $A(t, Y) \leqslant cny^{t-\alpha}L(y)$). All these results remain valid if we only assume an upper bound $\mathbf{P}(|X_1| > x) \leqslant c\,L(x)x^{-\alpha}$. Also, the case $\alpha = 2$ and more generally random walks in the Normal domain of attraction can also be dealt with, see Corollary 1.7 in [20].

For the case $\alpha = 1$, Theorem 5.1 gives some bound, but it is not optimal in general. However, if X_1 has a symmetric distribution, we have that $\mu(y) \equiv 0$ and $b_n \equiv 0$, so Theorem 5.1 yields immediately the inequality

$$\mathbf{P}\left(S_n - b_n \geqslant x; M_n \leqslant y\right) \leqslant \Big(1 + \frac{cx}{nL(y)}\Big)^{-x/y}. \tag{5.1}$$

The general case needs more work.

Theorem 5.2. *Assume that* (1.1) *holds with* $\alpha = 1$.

(i) *Assume in a first step that* X_1 *is non-negative, so* $y \mapsto \mu(y)$ *is non-decreasing. Then there exists a constant* $c > 0$, *and for every* $\varepsilon > 0$ *there is some* $C_\varepsilon > 0$, *such that for any* $x \geqslant C_\varepsilon a_n$ *and any* $y \leqslant x$

$$\begin{aligned} &\mathbf{P}\left(S_n - b_n \geqslant x; M_n \leqslant y\right) \leqslant \Big(cnL(y)/x\Big)^{(1-\varepsilon)x/y}, \\ &\mathbf{P}\left(S_n - b_n \leqslant -x\right) \leqslant \exp\Big(-\big(x/a_n\big)^{1/\varepsilon}\Big). \end{aligned} \tag{5.2}$$

(ii) *In the general case, for every* $\varepsilon > 0$ *there is some* $C_\varepsilon > 0$ *such that for any* $x \geqslant C_\varepsilon a_n$ *and any* $y \leqslant x$

$$\mathbf{P}\left(S_n - b_n \geqslant x; M_n \leqslant y\right) \leqslant \Big(cnL(y)/x\Big)^{(1-\varepsilon)x/y} + \exp\Big(-\big(x/a_n\big)^{1/\varepsilon}\Big). \tag{5.3}$$

Let us comment briefly this result. First, it gives the first part of Theorem 2.2, by possibly adjusting the constants to treat the case $x \geqslant a_n$ instead of $x \geqslant C_\varepsilon a_n$. Second, we need here that $x \geqslant C_\varepsilon a_n$, which was not the case for Theorem 5.1 —even if it has the same flavor. Note that in in (5.3), the large deviation may come from two different possibilities (see (5.11) for more details). Consider the positive part $X_1^+ := X_1 \mathbf{1}_{\{X_1>0\}}$ and the negative part $X_1^- := -X_1\mathbf{1}_{\{X_1<0\}}$ of X_1: then either the positive part makes a few jumps of length y (the number of such jumps is approximately $(1-\varepsilon)x/y$), giving the first term; either the negative part makes a large deviation to lower its value, giving rise to the second term (which is not affected by the truncation $M_n \leqslant y$).

5.2. **An easy consequence: Theorem 2.1.** We now prove Theorem 2.1, as a consequence of the above Fuk-Nagaev inequalities. We write it only for large deviations to the right (*i.e.* $x/a_n \to +\infty$), the other case being symmetric. For any fixed $\varepsilon > 0$, we write

$$\mathbf{P}(S_n - b_n > x) = \mathbf{P}(S_n - b_n > x; M_n > (1-\varepsilon)x) + \mathbf{P}(S_n - b_n > x; M_n \leqslant (1-\varepsilon)x). \tag{5.4}$$

First term. It gives the main contribution. A lower bound is, by exchangeability and independence of the X_i's

$$n\mathbf{P}\big(X_1 > (1+\varepsilon)x\big)\mathbf{P}\big(S_{n-1} - b_n > -(1-\varepsilon)x\big) \geqslant (1-\varepsilon)n\mathbf{P}\big(X_1 > (1+\varepsilon)x\big),$$

where we used that $(S_{n-1} - b_n)/a_n$ converges in distribution and $x/a_n \to +\infty$, the lower bound holding for n large enough. For an upper bound, we use simply a union bound to get

$$\mathbf{P}\big(M_n > (1-\varepsilon)x\big) \leqslant n\mathbf{P}\big(X_1 > (1-\varepsilon)x\big)\,.$$

By (1.1), the first term is therefore asymptotically bounded from below by $(p-\varepsilon)nL(x)x^{-\alpha}$ and from above by $(p+\varepsilon)nL(x)x^{-\alpha}$.

Second term. It remains to prove that, for any arbitrary $\varepsilon > 0$, the second term in (5.4) is $o(nL(x)x^{-\alpha})$ as $x/a_n \to +\infty$. We decompose again this probability into two part. The first part is

$$\mathbf{P}\Big(S_n - b_n > x; M_n \in (x/8, (1-\varepsilon)x]\Big) \leqslant n\mathbf{P}(X_1 \geqslant x/8)\mathbf{P}\big(S_{n-1} - b_n > \varepsilon x\big) = o(nL(x)x^{-\alpha})\,.$$

where the first inequality comes from the exchangeability and independence of the X_i's, and the second one comes from the convergence in distribution of $(S_{n-1} - b_n)/a_n$ together with $x/a_n \to +\infty$. The last part is controlled thanks to the above Fuk-Nagaev Theorems 5.1-5.2: we have that in any case (take $\varepsilon = 1/2$ in (5.3)), as $x/a_n \to +\infty$

$$\mathbf{P}\big(S_n - b_n > x; M_n \leqslant x/8\big) \leqslant \Big(cnL(x)x^{-\alpha}\Big)^4 + e^{-c(x/a_n)^2} = o\Big(\frac{L(x)}{L(a_n)}(x/a_n)^{-\alpha}\Big)\,,$$

where we used that $n \sim a_n^{-\alpha}L(a_n)$ (so that the last term is indeed $o(nL(x)x^{-\alpha})$).

In conclusion, since $\varepsilon > 0$ is arbitrary, we get Theorem 2.1. □

5.3. **Proof of Theorem 5.2.**

5.3.1. *The case of a non-negative X_1.* Here, we have that $\mu(x) := \mathbf{E}[X_1 \mathbf{1}_{\{|X_1|<x\}}]$ is non-decreasing.

Proof of the first part of (5.2). We start from Theorem 5.1, which states that for any $y \leqslant x'$

$$\mathbf{P}\left(S_n \geqslant x'; M_n \leqslant y\right) \leqslant \Big(1 + c\frac{x'}{nL(y)}\Big)^{-(x-n\mu(y))\frac{1}{y} - cst.nL(y)y^{-1}}\,.$$

Plugging $x' = b_n + x = n\mu(a_n) + x$ in this inequality, we get that for any $y \leqslant x$

$$\mathbf{P}\left(S_n - b_n \geqslant x; M_n \leqslant y\right) \leqslant \Big(1 + c\frac{n\mu(a_n) + x}{nL(y)}\Big)^{-\frac{x}{y} + n(\mu(y)-\mu(a_n))/y}\,. \tag{5.5}$$

Then it is just a matter of comparing $n(\mu(y) - \mu(a_n))$ to x (with $x \geqslant y$).

First, when $y \leqslant a_n$, then $\mu(y) - \mu(a_n) \leqslant 0$ since X_1 is non negative, so that

$$\mathbf{P}\left(S_n - b_n \geqslant x; M_n \leqslant y\right) \leqslant \Big(c\frac{x}{nL(y)}\Big)^{-\frac{x}{y}}\,.$$

When $y \geqslant a_n$, then we use the following claim (we prove it for $\ell(\cdot)$ defined in (3.1) whose definition also holds when $|\mu| < \infty$, but it obviously holds also for $\mu(\cdot)$ in the case of a non-negative X_1).

Claim 5.3. *For every $\delta > 0$, there is a constant c_δ s.t. for every $u \geqslant v \geqslant 1$ we have*

$$\frac{1}{L(v)}\big(\ell(u) - \ell(v)\big) \leqslant c_\delta (u/v)^\delta \,. \tag{5.6}$$

Moreover, considering two sequences $(u_n), (v_n) \to +\infty$, if there is a constant $c > 0$ such that $u_n \leqslant v_n \leqslant c u_n$ for all n then we have

$$\frac{1}{L(u_n)}\big(\ell(v_n) - \ell(u_n)\big) \overset{n\to\infty}{\sim} \log\big(v_n/u_n\big) \tag{5.7}$$

Proof of the Claim. We write

$$\frac{\ell(u) - \ell(v)}{L(v)} = \int_u^v \frac{L(t)t^{-1}}{L(v)} \mathrm{d}t \leqslant c_\delta \Big(\frac{u}{v}\Big)^{\delta/2} \int_u^v \frac{\mathrm{d}t}{t}$$

where we used Potter's bound to get that there is a constant c_δ such that uniformly for $t \geqslant v$ we have $L(t)/L(v) \leqslant c_\delta (t/v)^{\delta/2}$. Then, the last integral is equal to $\log(u/v) \leqslant c_\delta(u/v)^{\delta/2}$ so the first part of the claim is proven. For the second part, this is standard and comes from the same computation, together with the fact that $L(t)/L(v_n) \to 1$ uniformly for $t \in [u_n, v_n] \subset [u_n, cu_n]$:

$$\frac{\ell(v_n) - \ell(u_n)}{L(u_n)} = \int_{u_n}^{v_n} \frac{L(t)t^{-1}}{L(v_n)} \mathrm{d}t = (1 + o(1)) \int_{u_n}^{v_n} \frac{\mathrm{d}t}{t} \overset{n\to\infty}{\sim} \log(v_n/u_n) \,.$$

$\square$

From Claim 5.3 (take $\delta = 1/2$) we get that, for any $y \geqslant a_n$, and since $x \geqslant y$

$$n(\mu(y) - \mu(a_n)) \leqslant c_\delta n L(a_n)(y/a_n)^{1/2} \leqslant c\frac{nL(a_n)}{a_n}(x/a_n)^{-1/2} \times x.$$

Therefore, by choosing $C_\varepsilon > 0$ large enough, we get that $n(\mu(y) - \mu(a_n)) \leqslant \varepsilon x$ for $x \geqslant C_\varepsilon a_n$, $a_n \leqslant y \leqslant x$. Plugged in (5.5), we get

$$\mathbf{P}\left(S_n - b_n \geqslant x; M_n \leqslant y\right) \leqslant \Big(c\frac{n\mu(a_n) + x}{nL(y)}\Big)^{-(1-\varepsilon)\frac{x}{y}} \leqslant \Big(c\frac{nL(y)}{x}\Big)^{-(1-\varepsilon)\frac{x}{y}} .$$

Hence, the first part of (5.2) is proven.

Proof of the second part of (5.2). We write, for any $t > 0$ and any $a_n \leqslant x \leqslant b_n$

$$\mathbf{P}\left(S_n - b_n \leqslant -x\right) \leqslant e^{-t(x+b_n)}\mathbf{E}[e^{-tX_1}]^n \,.$$

We also use that, because $X_1 \geqslant 0$ and thanks to (1.1), we have that there is a constant $c > 0$ such that for any $t \leqslant 1$

$$\mathbf{E}[e^{-tX_1}] - 1 + t\mu(1/t) \leqslant ctL(1/t) \,. \tag{5.8}$$

Indeed, one simply writes that the absolute value of the left hand side is

$$\begin{aligned}
&\Big| \sum_{n=1}^{1/t} \Big(1 - e^{-tn} - tn\Big)\mathbf{P}(X_1 = n) + \sum_{n > 1/t} (1 - e^{-tn})\mathbf{P}(X_1 = n)\Big| \\
&\qquad\qquad \leqslant \sum_{n=1}^{1/t} t^2 n^2 \mathbf{P}(X_1 = n) + \sum_{n>1/t} (1 + e^{-1})\mathbf{P}(X_1 = n) \\
&\qquad\qquad \leqslant t^2 \mathbf{E}\big[(X_1)^2 \mathbf{1}_{\{X_1 \leqslant 1/t\}}\big] + (1 + e^{-1})\mathbf{P}(X_1 > 1/t) \,.
\end{aligned}$$

Then, one easily get that thanks to (1.1), both terms are $O(tL(1/t))$.

Thanks to (5.8), and using that $1+x \leqslant e^x$, we get that for any $t \leqslant 1$

$$\begin{aligned}\mathbf{P}\left(S_n - b_n \leqslant -x\right) &\leqslant \exp\Big(-t(x+b_n) - nt\mu(1/t) + cntL(1/t)\Big)\\ &= \exp\Big(-tx + nt\Big[\mu(a_n) - \mu(1/t) + cL(1/t)\Big]\Big)\,. \end{aligned} \tag{5.9}$$

Then, we fix $\varepsilon > 0$ and choose $t := (a_n)^{-1} \times (x/a_n)^{(1-\varepsilon)/\varepsilon}$, so that $1/t < a_n$: thanks to Claim 5.3 we get that there is a constant c_ε such that $\mu(a_n) - \mu(1/t) \leqslant c_\varepsilon (ta_n)^\varepsilon L(a_n)$, and Potter's bound also gives that $L(1/t) \leqslant c_\varepsilon (ta_n)^\varepsilon L(a_n)$. We therefore get that the r.h.s. of (5.9) is bounded by

$$\exp\Big(-tx + (1+c_\varepsilon)nt(ta_n)^\varepsilon L(a_n)\Big) \leqslant \exp\Big(-\Big(\frac{x}{a_n}\Big)^{1/\varepsilon} + c'_\varepsilon\Big(\frac{x}{a_n}\Big)^{(1-\varepsilon^2)/\varepsilon}\Big) \tag{5.10}$$

where we used the definition of t, together with the fact that $nL(a_n) \sim a_n$ for the second term in the exponential. Hence, there exists some $C_\varepsilon > 0$ such that provided that $x/a_n \geqslant C_\varepsilon$ we have

$$\mathbf{P}\left(S_n - b_n \leqslant -x\right) \leqslant \exp\Big(-\frac{1}{2}(x/a_n)^{1/\varepsilon}\Big)\,,$$

which ends the proof of the second part of (5.2), the factor $1/2$ being irrelevant.

5.3.2. *General case: proof of* (5.3). When X_1 can be negative as well as positive, we separate the X_i's into a positive and a negative part:

$$X_i^+ := X_i \mathbf{1}_{\{X_i > 0\}}\,, \quad X_i^- := X_i \mathbf{1}_{\{X_i < 0\}}\,,$$

so that both X_i^+ and X_i^- are non-negative. Naturally, we also define $S_n^+ := \sum_{i=1}^n X_i^+$ and $S_n^- := \sum_{i=1}^n X_i^-$ so that $S_n = S_n^+ - S_n^-$; and also $b_n^+ := n\mathbf{E}[X_1^- \mathbf{1}_{\{X_1^- \leqslant a_n\}}]$ and $b_n^- := n\mathbf{E}[X_1^- \mathbf{1}_{\{X_1^- \leqslant a_n\}}]$, so that b_n^+ (resp. b_n^-) is a centering sequence for S_n^+ (resp. S_n^-), and $b_n = b_n^+ - b_n^-$. Then, the probability we are after can be bounded by

$$\begin{aligned}\mathbf{P}\Big(S_n - b_n \geqslant x\,;\, M_n \leqslant y\Big) \leqslant \mathbf{P}\Big(S_n^+ - b_n^+ \geqslant (1-\varepsilon)x\,;\, \max_{1\leqslant i\leqslant n} X_i^+ \leqslant y\Big)\\ +\mathbf{P}\Big(S_n^- - b_n^- \leqslant -\varepsilon x\Big)\,. \end{aligned} \tag{5.11}$$

Then, we may use (5.2) for both terms, and we obtain (5.3) (changing possibly the value of ε and of the constant C_ε).

6. Local large deviations

6.1. **Local versions of Fuk-Nagaev inequalities.** We prove Theorem 2.2-(2.2), and along the way the following local version of Theorem 5.1.

Theorem 6.1. *Assume that* (1.1) *holds with* $\alpha \in (0,1) \cup (1,2)$. *There exist constants* $c, C > 0$ *such that for any* $y \leqslant x$

$$Ca_n \times \mathbf{P}\left(S_n - b_n = x; M_n \leqslant y\right) \leqslant \Big(1 + \frac{cx}{ny^{1-\alpha}L(y)}\Big)^{-x/2y} \leqslant \Big(c^{-1}\frac{y}{x}nL(y)y^{-\alpha}\Big)^{x/2y}\,.$$

Note that this result is similar to [7, Thm. 1.1], but here we have an estimate even when $y \ll x$ which is not the case in [7]. We might also be able to improve the exponent $x/2y$ to $\lceil x/y \rceil$ as in [7, Thm. 1.1] (at least when y is a constant times x), but we do not pursue this level of optimality here.

Proof We will focus on the proof of Theorem 2.2-(2.2), *i.e.* in the case $\alpha = 1$, but the proof is identical for proving Theorem 6.1. Let us denote $\widehat{S}_n := S_n - \lfloor b_n \rfloor$ the "recentered" walk. We decompose $\mathbf{P}(\widehat{S}_n = x)$ according to whether $S_{\lfloor n/2 \rfloor} - \frac12 \lfloor b_n \rfloor \geqslant x/2$ or not, so that we obtain

$$\mathbf{P}\big(\widehat{S}_n = x; M_n \leqslant y\big) \leqslant \mathbf{P}\Big(\widehat{S}_n = x; S_{\lfloor n/2 \rfloor} - \tfrac12 \lfloor b_n \rfloor \geqslant x/2\,;\, M_{\lfloor n/2 \rfloor} \leqslant y\Big) \\ + \mathbf{P}\Big(\widehat{S}_n = x; S_n - S_{\lfloor n/2 \rfloor} - \tfrac12 \lfloor b_n \rfloor \geqslant x/2\,;\, \max_{\lfloor n/2 \rfloor < i \leqslant n} X_i \leqslant y\Big)\,.$$

The two terms are treated similarly, so we only focus on the first one. We have

$$\begin{aligned} &\mathbf{P}\Big(\widehat{S}_n = x\,;\, S_{\lfloor n/2 \rfloor} - \tfrac12 \lfloor b_n \rfloor \geqslant x/2; M_{\lfloor n/2 \rfloor} \leqslant y\Big) \\ &\qquad = \sum_{z \geqslant \frac12 \lfloor b_n \rfloor + x/2} \mathbf{P}\big(S_{\lfloor n/2 \rfloor} = z; M_{\lfloor n/2 \rfloor} \leqslant y\big) \mathbf{P}\big(S_n - S_{\lfloor n/2 \rfloor} = \lfloor b_n \rfloor + x - z\big) \\ &\qquad \leqslant \frac{C}{a_n} \sum_{z \geqslant \frac12 \lfloor b_n \rfloor + x/2} \mathbf{P}\big(S_{\lfloor n/2 \rfloor} = z; M_{\lfloor n/2 \rfloor} \leqslant y\big) \\ &\qquad = \frac{C}{a_n} \mathbf{P}\Big(S_{\lfloor n/2 \rfloor} - \tfrac12 \lfloor b_n \rfloor \geqslant x/2; M_{\lfloor n/2 \rfloor} \leqslant y\Big)\,, \end{aligned} \tag{6.1}$$

where we used Gnedenko's local limit theorem (1.4) to get that there is a constant $C > 0$ such that for any $k \geqslant 1$ and $y \in \mathbb{Z}$, we have $\mathbf{P}(S_k = y) \leqslant C/a_k$.

Then, we want to use Fuk-Nagaev inequalities (*i.e.* Theorems 5.1-5.2) to estimate the last probability: for that, we need to control $\frac12 \lfloor b_n \rfloor - b_{\lfloor n/2 \rfloor}$. When $\alpha \in (0,1)$ we have that $b_n \equiv 0$ so this quantity is equal to 0, and when $\alpha \in (1,2)$ we have $b_k = k\mu$ in which case we get $\frac12 \lfloor b_n \rfloor - \lfloor b_{n/2} \rfloor \geqslant -|\mu|$. When $\alpha = 1$, this is more delicate but not too hard:

$$\begin{aligned} \tfrac12 n\mu(a_n) - \lfloor n/2 \rfloor \mu(a_{\lfloor n/2 \rfloor}) &\geqslant \frac{n}{2}\Big[\mu(a_n) - \mu(a_{\lfloor n/2 \rfloor})\Big] - |\mu(a_{\lfloor n/2 \rfloor})| \\ &\geqslant -c_0\, nL(a_n) - |\mu(a_{\lfloor n/2 \rfloor})| \geqslant -2c_0 a_n\,. \end{aligned}$$

where for the second inequality we used the Claim 5.3 (separating the positive and negative parts of X_1, using also that $a_n/a_{\lfloor n/2 \rfloor}$ is bounded from above by a constant), and in the last inequality we used the definition of a_n (and the fact that $|\mu(a_{\lfloor n/2 \rfloor})| \ll a_n$). In all cases, and provided that $x \geqslant C_\varepsilon a_n$ with some constant C_ε large enough, we get that $\frac12 \lfloor b_n \rfloor - b_{\lfloor n/2 \rfloor} \geqslant -\varepsilon x/2$, so that

$$\mathbf{P}\Big(S_{\lfloor n/2 \rfloor} - \tfrac12 \lfloor b_n \rfloor \geqslant x/2; M_{\lfloor n/2 \rfloor} \leqslant y\Big) \leqslant \mathbf{P}\Big(S_{\lfloor n/2 \rfloor} - b_{\lfloor n/2 \rfloor} \geqslant (1-\varepsilon)x/2; M_{\lfloor n/2 \rfloor} \leqslant y\Big)\,.$$

Then, an application of Theorems 5.1-5.2, plugged into (6.1), gives Theorem 2.3. (Note we do not need to take $x \geqslant C_\varepsilon a_n$ if $\alpha \in (0,1) \cup (1,2)$.) ☐

Also, one may obtain the following local analogous of (5.1) and (5.2): if $\alpha = 1$ in (1.1) then for every $\varepsilon > 0$ there are constants c_1, c_2 such that for any $x \geqslant c_2 a_n$,

- if X_1 has a symmetric distribution, we have

$$C a_n \times \mathbf{P}(S_n = x; M_n \leqslant y) \leqslant \Big(cnL(y)/x\Big)^{(1-\varepsilon)x/2y}\,; \tag{6.2}$$

- if $X_1 \geqslant 0$, we have

$$C a_n \times \mathbf{P}(S_n - \lfloor b_n \rfloor = x; M_n \leqslant y) \leqslant \Big(cnL(y)/x\Big)^{(1-\varepsilon)x/2y}\,. \tag{6.3}$$

6.2. **Improved local large deviations: proof of Theorem 2.4.** We only consider the large deviation to the right, i.e. $x \geqslant a_n$, since the other case is symmetric. We give the proof of (2.9) and (2.10) together, the latter using the same estimates. We fix $\varepsilon > 0$ (we take $\varepsilon = 1/8$ when we prove (2.9), and we will choose ε arbitrarily small when we prove (2.10)), and we write (recall $\widehat{S}_n = S_n - \lfloor b_n \rfloor$)

$$\begin{aligned} \mathbf{P}\big(\widehat{S}_n = x\big) = \mathbf{P}\big(\widehat{S}_n = x, M_n \geqslant (1-\varepsilon)x\big) \\ + \mathbf{P}\big(\widehat{S}_n = x, M_n \in (\varepsilon x, (1-\varepsilon)x)\big) + \mathbf{P}\big(\widehat{S}_n = x, M_n \leqslant \varepsilon x\big)\,. \end{aligned} \tag{6.4}$$

The first term in (6.4) is also the main one: by exchangeability of the X_i's, we get that

$$\begin{aligned} \mathbf{P}\big(\widehat{S}_n = x, M_n \geqslant (1-\varepsilon)x\big) &= \sum_{y \geqslant (1-\varepsilon)x} \mathbf{P}(\widehat{S}_n = x, M_n = y) \\ &= \sum_{y \geqslant (1-\varepsilon)x} n\mathbf{P}(X_1 = y)\mathbf{P}(S_{n-1} - \lfloor b_n \rfloor = x - y, M_{n-1} \leqslant y) \\ &\leqslant n \sup_{y \geqslant (1-\varepsilon)x} \mathbf{P}(X_1 = y) \sum_{y \geqslant (1-\varepsilon)x} \mathbf{P}(S_{n-1} - \lfloor b_n \rfloor = x - y) \\ &\leqslant n \sup_{y \geqslant (1-\varepsilon)x} \mathbf{P}(X_1 = y)\,. \end{aligned} \tag{6.5}$$

Then, if we assume (2.5), we get that

$$\mathbf{P}\big(\widehat{S}_n = x, M_n \geqslant (1-\varepsilon)x\big) \leqslant C n L(x) x^{-(1+\alpha)}\,, \tag{6.6}$$

and if we assume (2.7), we may replace the constant C by $(p + 3\varepsilon)$, provided that x is large enough.

For the second term in (6.4), we have:

$$\begin{aligned} \mathbf{P}\big(\widehat{S}_n = x, M_n \in (\varepsilon x, (1-\varepsilon)x)\big) &= \sum_{y=\varepsilon x}^{(1-\varepsilon)x} \mathbf{P}(\widehat{S}_n = x, M_n = y) \\ &\leqslant \sum_{y=\varepsilon x}^{(1-\varepsilon)x} n\mathbf{P}(X_1 = y)\mathbf{P}(S_{n-1} - \lfloor b_n \rfloor = x - y) \\ &\leqslant C_\varepsilon\, n L(x) x^{-(1+\alpha)}\, \mathbf{P}(S_{n-1} - \lfloor b_n \rfloor \geqslant \varepsilon x)\,, \end{aligned} \tag{6.7}$$

where we used (2.5) in the last inequality. Moreover, since $(S_n - b_n)/a_n$ converges in distribution, we get that $\mathbf{P}(S_{n-1} - \lfloor b_n \rfloor \geqslant \varepsilon x) \to 0$ if $x/a_n \to +\infty$, so the second term is $o(nL(x)x^{-(1+\alpha)})$.

For the last term in (6.4), we decompose it into two parts,

$$\mathbf{P}\big(\widehat{S}_n = x, M_n \leqslant \varepsilon x\big) \leqslant \mathbf{P}\big(\widehat{S}_n = x, M_n \leqslant c a_n\big) + \mathbf{P}\big(\widehat{S}_n = x, M_n \in (c a_n, \varepsilon x)\big)$$

The first part is controlled thanks to the local Fuk-Nagaev inequalities Theorems 2.2-6.1: using that $x \geqslant a_n$ and that $nL(a_n)a_n^{-\alpha} \to 1$, we get that

$$\mathbf{P}(\widehat{S}_n = x, M_n \leqslant c a_n) \leqslant \frac{C}{a_n}\Big(c' \frac{x}{a_n}\Big)^{-c' x / a_n} \leqslant \frac{c}{x} \times e^{-c'' x / a_n}, \tag{6.8}$$

which is negligible compared to (2.9) (*i.e.* (6.6)) as $x/a_n \to \infty$. For the second part, we have

$$
\begin{aligned}
\mathbf{P}\big(\widehat{S}_n = x, M_n \in [ca_n, \varepsilon x)\big) &= \sum_{j=\log_2(1/\varepsilon)}^{\log_2(cx/a_n)} \mathbf{P}\Big(\widehat{S}_n = x, M_n \in [2^{-(j+1)}, 2^{-j})x\Big) \\
&\leqslant \sum_{j=\log_2(1/\varepsilon)}^{\log_2(cx/a_n)} \sum_{y\in[2^{-(j+1)},2^{-j})x} n\mathbf{P}\big(X_1 = y\big)\mathbf{P}\big(S_{n-1} - \lfloor b_n \rfloor = x-y, M_n = y\big) \\
&\leqslant C \sum_{j=\log_2(1/\varepsilon)}^{\log_2(cx/a_n)} nL(2^{-j}x)(2^{-j}x)^{-(1+\alpha)}\mathbf{P}\Big(S_{n-1} - \lfloor b_n \rfloor \geqslant x/2, M_n \leqslant 2^{-j}x\Big), \qquad (6.9)
\end{aligned}
$$

where we used (2.5) to bound $\mathbf{P}(X_1 = y)$ uniformly for $y \in [2^{-(j+1)}, 2^{-j})x$. Then, we use Fuk-Nagaev's inequalities Theorem 5.1-Theorem 5.2 —leave aside the case $\alpha = 1$ for the moment— to get that (replacing $S_{n-1} - \lfloor b_n \rfloor$ by $S_n - b_n$ for simplicity)

$$
\mathbf{P}\Big(S_n - b_n \geqslant x/2, M_n \leqslant 2^{-j}x\Big) \leqslant \Big(1 + \frac{c2^j}{nL(2^{-j}x)(2^{-j}x)^{-\alpha}}\Big)^{-2^{j-2}} \leqslant \Big(1 + c2^j\Big)^{-2^{j-2}},
$$

where we used that $2^{-j}x \geqslant a_n$ for the range considered, so $n\mathbf{P}(X_1 > 2^{-j}x) \leqslant n\mathbf{P}(X_1 > a_n)$ and is bounded from above by a universal constant. Plugged in (6.9), and using Potter's bound to get that $L(2^{-j}x) \leqslant c2^j L(x)$ for all $j \geqslant 1$, we therefore get that

$$
\begin{aligned}
\mathbf{P}\Big(\widehat{S}_n = x, M_n \in [ca_n, \varepsilon x)\Big) &\leqslant CnL(x)x^{-(1+\alpha)} \sum_{j=\log_2(1/\varepsilon)}^{\log_2(x/a_n)} 2^{(2+\alpha)j}\Big(1 + c2^j\Big)^{-2^{j-2}} \\
&\leqslant c_\varepsilon nL(x)x^{-(1+\alpha)}, \qquad (6.10)
\end{aligned}
$$

where the constant c_ε can be made arbitrarily small by choosing ε small. In the case $\alpha = 1$, Theorem 2.2 gives an additional e^{-x/a_n} in bounding $\mathbf{P}\big(S_n - b_n \geqslant x/2, M_n \leqslant 2^{-j}x\big)$ for any $j \leqslant \log_2(cx/a_n)$. Hence in (6.10) we obtain an additionnal

$$
CnL(x)x^{-(1+\alpha)} \sum_{j=1}^{\log_2(cx/a_n)} 2^{(2+\alpha)j}e^{-x/a_n} \leqslant CnL(x)x^{-(1+\alpha)} \times \Big(\frac{x}{a_n}\Big)^{3+\alpha} e^{-x/a_n},
$$

which is $o(nL(x)x^{-(1+\alpha)})$ as $x/a_n \to \infty$.

In conclusion, combining (6.6)-(6.7)-(6.8)-(6.10), we proved that fixing $\varepsilon = 1/8$ we get (2.9). Assuming additionally (2.6), in view of the remark made after (6.6) we obtained that for any $\eta > 0$we can find $\varepsilon > 0$ (sufficiently small) such that, if n and x/a_n are large enough,

$$
\mathbf{P}\big(S_n - b_n = x\big) \leqslant (p+\eta)nL(x)x^{-(1+\alpha)}\,.
$$

This proves the upper bound part in Theorem 2.4.

To get the lower bound in (2.10), assume that $p > 0$ (otherwise there is nothing to prove), and write

$$\begin{aligned}\mathbf{P}\big(\widehat{S}_n = x\big) &\geqslant \mathbf{P}\Big(\exists\, i \text{ s.t. } X_i \in \big((1-\varepsilon)x, (1+\varepsilon)x\big)\,;\, \forall j \neq i\ X_j \leqslant x/2\,;\, \widehat{S}_n = x\Big)\\ &= \sum_{y=(1-\varepsilon)x}^{(1+\varepsilon)x} n\mathbf{P}(X_1 = y)\mathbf{P}\Big(S_{n-1} - \lfloor b_n \rfloor = x - y; M_{n-1} \leqslant x/2\Big)\\ &\geqslant (1-3\varepsilon)npL(x)x^{-(1+\alpha)}\mathbf{P}\Big(S_{n-1} - b_n \in [-\varepsilon x, \varepsilon x]; M_{n-1} \leqslant x/2\Big),\end{aligned}$$

where we used that $\mathbf{P}(X_1 = y) \geqslant (1-3\varepsilon)pL(x)x^{-(1+\alpha)}$ uniformly for $y \in \big((1-\varepsilon)x, (1+\varepsilon)x\big)$, because of (2.6). Then, the last probability converges to 1 as $n \to \infty$ because $(S_{n-1} - \lfloor b_n \rfloor)/a_n$ and M_{n-1}/a_n both converge in distribution, and $x/a_n \to \infty$. Hence we have that for any $\eta > 0$, we can find $\varepsilon > 0$ (sufficiently small) such that if n and x/a_n are large enough,

$$\mathbf{P}\big(S_n - \lfloor b_n \rfloor = x\big) \geqslant (1-\eta)npL(x)x^{-(1+\alpha)}\,,$$

which concludes the proof. □

7. Ladder epochs: proof of Theorems 3.2-3.4

In order to prove Theorem 3.2, a crucial identity follows from the Wiener-Hopf factorization (see e.g. Theorem 4 in [15, XII.7]): set $p_k := \mathbf{P}(T_- > k)$ for every $k \geqslant 0$, then for any $s \in [0,1)$

$$p(s) := \sum_{k=0}^{\infty} p_k s^k = \exp\Big(\sum_{m=1}^{+\infty} \frac{s^m}{m}\mathbf{P}(S_m \geqslant 0)\Big)\,. \tag{7.1}$$

Also, Theorem 2 in [15, XII.7] characterizes the defectiveness of T_-, T_+ in terms of convergence of the series $\sum_{k\geqslant 1} k^{-1}\mathbf{P}(S_k > 0)$.

We present the proof in the case $\alpha = 1$ with infinite mean, *i.e.* Theorem 3.2 (it captures all the ideas needed), and then we adapt the proof to the case $\mu = 0$ (*i.e.* Theorem 3.4) in Section 7.6.

7.1. Preliminaries.

We first give the following lemma, which is the core of our proofs.

Lemma 7.1. *Assume that (1.1) holds, with $\alpha = 1$ and $|\mu| = +\infty$. Recall the definition (3.1) of $\ell(\cdot)$ and (1.3) of b_n. Then, if $p > q$, $b_n \sim (p-q)n\ell(a_n)$ and*

$$\mathbf{P}(S_n < 0) \overset{n\to\infty}{\sim} \frac{q}{p-q}\frac{L(b_n)}{\ell(b_n)}.$$

Moreover we have that

$$\sum_{k=1}^{n} \frac{1}{k}\mathbf{P}(S_k < 0) \overset{n\to\infty}{\sim} \frac{q}{p-q}\log \ell(b_n)\,.$$

If $q = 0$, we interpret this as $o\big(\log \ell(b_n)\big)$. The case $p < q$ is symmetric.
If $p = q = 1/2$, then $b_n = o(n\ell(a_n))$ but if $b_n/a_n \to +\infty$ we have

$$\mathbf{P}(S_n < 0) \overset{n\to\infty}{\sim} \frac{nL(b_n)}{2b_n}\,, \qquad \sum_{k=1}^{+\infty} k^{-1}\mathbf{P}(S_k < 0) = \infty\,.$$

Proof First, by Theorem 2.1 and since $b_n/a_n \to +\infty$ (because $p > q$), we get that

$$\mathbf{P}(S_n < 0) = \mathbf{P}(S_n - b_n < -b_n) \overset{n\to\infty}{\sim} nqL(b_n)b_n^{-1} \overset{n\to\infty}{\sim} \frac{q}{p-q}\,\frac{L(b_n)}{\ell(b_n)}, \tag{7.2}$$

where we used that $b_n = n\mu(a_n)$ with $\mu(a_n) \overset{n\to\infty}{\sim} (p-q)\ell(a_n) \overset{n\to\infty}{\sim} (p-q)\ell(b_n)$ (see Lemma 4.3). Note that the first asymptotic equivalence remains true as soon as $b_n/a_n \to \infty$.

Then, it remains to estimate the sum $\sum_{k=1}^n k^{-1}\mathbf{P}(S_k < 0)$ or, because of (7.2), of $q\sum_{k=1}^n \frac{L(b_k)}{b_k}$. A comparison with an integral and a change of variable $t = b_k$ (so $dt \sim (p-q)\ell(b_k)dk$, since we may assume that we work with differentiable function, see [3, Thm. 1..2]) gives that

$$\sum_{k=1}^n \frac{L(b_k)}{b_k} \overset{n\to\infty}{\sim} \frac{1}{p-q}\sum_{t=1}^{b_n}\frac{L(t)}{t\ell(t)} \overset{n\to\infty}{\sim} \frac{1}{p-q}\log \ell(b_n)\,,$$

where for the last identity we used Lemma 4.4-(i) —or more directly (4.2).

In the case where $p = q = 1/2$ (so $b_k = o(k\ell(a_k))$) we use that according to (7.2) and provided that $b_n/a_n \to +\infty$, there is a constant $c > 0$ such that

$$k^{-1}\mathbf{P}(S_k < 0) \geqslant cL(b_k)/b_k \geqslant c\frac{L(k\ell(a_k))}{k\ell(a_k)}\,.$$

And we proved just above that $\sum_{k=1}^{+\infty}\frac{L(k\ell(a_k))}{k\ell(a_k)} = +\infty$. □

A simple consequence of Lemma 7.1 is Proposition 3.1 (in the case $|\mu| = +\infty$, the case $\mu = 0$ being treated in Lemma 7.3 below), thanks to [15, XII.7 Thm. 2]. Indeed, we get that $\sum_{k\geqslant 1} k^{-1}\mathbf{P}(S_k < 0) = +\infty$ as soon as $q \neq 0$: if $p > q$ or $p = q$ with $b_n/a_n \to +\infty$, this is directly Lemma 7.1; if $\sup_n |b_n|/a_n < +\infty$, then this is just a consequence of the convergence in distribution of $(S_n - b_n)/a_n$ to get that $\mathbf{P}(S_k < 0) = \mathbf{P}\big((S_k - b_k)/a_k < -b_k/a_k\big)$ is uniformly bounded away from 0, so that $\sum_{k\geqslant 1} k^{-1}\mathbf{P}(S_k < 0) = +\infty$; the general case when $p = q = 1/2$ can be dealt with similarly, by observing that there is a constant c such that $\mathbf{P}(S_k < 0) \geqslant c\frac{L(k\ell(a_k))}{k\ell(a_k)}$.

7.2. **The case** $\lim_{n\to\infty} b_n/a_n = b$**.** We first prove the case (i) in Theorem 3.2, which is standard, cf. Rogozin [21]. The sequence $\mathbf{P}(S_k > 0) = \mathbf{P}((S_k - b_k)/a_k > -b_k/a_k)$ converges to $\mathbf{P}(Y > -b)$, where Y is the limit in distribution of $(S_n - b_n)/a_n$, that is a symmetric Cauchy(1/2) distribution ($p = q = 1/2$), and $b = \lim_{n\to\infty} b_n/a_n$. Note that we could also characterize b_n/a_n by $\mathbf{E}\big[\frac{X_1}{1+(X_1/a_n)^2}\big]$, see [3, Thm. 8.3.1]. We therefore get that

$$\lim_{n\to+\infty}\frac{1}{n}\sum_{k=1}^n \mathbf{P}(S_k > 0) = \mathbf{P}(Y > -b) =: \rho \in (0,1),$$

with $\rho = \frac{1}{2} + \frac{1}{\pi}\arctan(2b/\pi)$. This is Spitzer's condition: [3, Thm. 8.9.12] implies that T_- is in the domain of attraction of a positive stable random variable with index ρ, and (i) follows.

7.3. **The case** $p < q$**.** We will first prove the weak result with the $o(1)$ in the exponent, and then turn to the precise statement under assumptions V1-V2.

General Case. Denote

$$f(s)=\sum_{m=1}^{+\infty}\frac{s^m}{m}\mathbf{P}(S_m\geqslant 0)\,,\qquad f'(s)=\sum_{m=0}^{+\infty}s^m\mathbf{P}(S_{m+1}\geqslant 0)\,. \tag{7.3}$$

We are able to obtain the behavior of $f(s)$ and $f'(s)$ as $s\uparrow 1$. Since $p<q$, Lemma 7.1 gives that

$$\sum_{k=1}^{n}k^{-1}\mathbf{P}(S_k\geqslant 0)\overset{n\to\infty}{\sim}\frac{p}{q-p}\log\ell(|b_n|)\quad\text{ and }\quad\sum_{k=1}^{n}\mathbf{P}(S_k\geqslant 0)\overset{n\to\infty}{\sim}\frac{p}{q-p}\frac{nL(|b_n|)}{\ell(|b_n|)}\,, \tag{7.4}$$

(because $\mathbf{P}(S_k\geqslant 0)\sim\frac{p}{q-p}L(|b_k|)/\ell(|b_k|)$). Therefore, Corollary 1.7.3 in [3] gives that

$$f(s)\sim\frac{p}{q-p}\log\ell(|b_{1/(1-s)}|)\,,\qquad f'(s)\sim\frac{p}{q-p}\frac{1}{1-s}\frac{L(|b_{1/(1-s)}|)}{\ell(|b_{1/(1-s)}|)}\qquad\text{as } s\uparrow 1\,. \tag{7.5}$$

Then, (7.1) gives that $p(s)=e^{f(s)}$ for any $s\in[0,1)$ so that $p'(s)=f'(s)e^{f(s)}$: the estimates (7.5) allows us to derive that

$$p'(s)=\frac{L(|b_{1/(1-s)}|)}{1-s}\Big(\ell(|b_{1/(1-s)}|)\Big)^{\frac{p}{q-p}-1+o(1)}\quad\text{ as } s\uparrow 1\,, \tag{7.6}$$

where the term $p/(q-p)$ has been absorbed in $\ell(|b_{1/(1-s)}|)^{o(1)}$ From this we would like to conclude that $\sum_{k=1}^{n}kp_k=nL(|b_n|)\big(\ell(|b_n|)\big)^{\frac{p}{q-p}-1+o(1)}$, but we cannot directly apply Corollary 1.7.3 in [3] since we do not have a proper asymptotic equivalence. We therefore prove it directly.

Upper bound. First, take $s=1-1/n$ in (7.6), so that we get, as $n\to\infty$

$$\sum_{k=0}^{+\infty}kp_k\Big(1-\frac{1}{n}\Big)^{k-1}=nL(|b_n|)\big(\ell(|b_n|)\big)^{\frac{p}{q-p}-1+o(1)}\,. \tag{7.7}$$

Then, using that p_k is non-increasing, we can write that

$$\sum_{k=0}^{+\infty}kp_k\Big(1-\frac{1}{n}\Big)^{k-1}\geqslant\sum_{k=0}^{n}kp_n\Big(1-\frac{1}{n}\Big)^{n}\geqslant c\,n^2p_n,$$

and we therefore get the upper bound

$$p_n\leqslant c\frac{1}{n}L(b_n)\big(\ell(b_n)\big)^{\frac{p}{q-p}-1+o(1)}=\frac{L(|b_n|)}{n}\big(\ell(|b_n|)\big)^{\frac{p}{q-p}-1+o(1)}\,. \tag{7.8}$$

Lower bound. The lower bound is a bit trickier. Let $\varepsilon>0$, and define $t_n:=\ell(|b_n|)^{\varepsilon}\to\infty$ as $n\to\infty$. Setting $s=1-1/(nt_n)$ in (7.6), we get that for n sufficiently large

$$\begin{aligned}\sum_{k=0}^{+\infty}kp_k\Big(1-\frac{1}{nt_n}\Big)^{k-1}&=nt_nL(|b_{n/t_n}|)\big(\ell(|b_{n/t_n}|)\big)^{\frac{p}{q-p}-1+o(1)}\\&\geqslant nt_n^{1/2}L(|b_n|)\big(\ell(|b_n|)\big)^{\frac{p}{q-p}-1+o(1)}.\end{aligned} \tag{7.9}$$

For the second inequality, we used that $|b_{n/t_n}|\geqslant t_n^{-2}|b_n|$ for n large enough (since b_k is regularly varying with index -1), and than Potter's bound to get that $L(t_n^{-2}|b_n|)\leqslant t_n^{-1/4}L(|b_n|)$ and $\ell(t_n^{-2}|b_n|)^{\frac{p}{q-p}-1+o(1)}\leqslant t_n^{-1/4}$ for n large enough.

Now we may write, since p_k is non-increasing,

$$\sum_{k=0}^{+\infty} kp_k\Big(1-\frac{1}{nt_n}\Big)^{k-1} \leqslant \sum_{k=0}^{n} kp_k + p_n \sum_{k=n+1}^{+\infty} k\Big(1-\frac{1}{nt_n}\Big)^{k-1} \tag{7.10}$$

For the first sum, we get thanks to (7.8) that

$$\sum_{k=0}^{n} kp_k \leqslant c\, nL(|b_n|)\big(\ell(|b_n|)\big)^{\frac{p}{q-p}-1+o(1)} = o\Big(nt_n^{1/2}L(|b_n|)\big(\ell(|b_n|)\big)^{\frac{p}{q-p}-1+o(1)}\Big),$$

where we used the definition of t_n, which is such that $\ell(|b_n|)^{o(1)} = o(t_n^{1/2})$. Now we have $\sum_{k\geqslant 1} ks^{k-1} = (1-s)^{-2}$, so that the second term in (7.10) is bounded by $p_n(nt_n)^2$. Hence, plugging (7.10) (and the subsequent estimates) in (7.9), we obtain that

$$nt_n^{1/2}L(|b_n|)\big(\ell(|b_n|)\big)^{\frac{p}{q-p}-1+o(1)} \leqslant o\Big(nt_n^{1/2}L(b_n)\big(\ell(|b_n|)\big)^{\frac{p}{q-p}-1+o(1)}\Big) + p_n(nt_n)^2,$$

so that we conclude that

$$p_n \geqslant \frac{c}{nt_n^{3/2}}L(|b_n|)\big(\ell(|b_n|)\big)^{\frac{p}{q-p}-1+o(1)}\,.$$

Recalling that $t_n = \ell(|b_n|)^{\varepsilon}$, and since $\varepsilon > 0$ is arbitrary, we get that

$$p_n \geqslant \frac{L(|b_n|)}{n}\big(\ell(|b_n|)\big)^{\frac{p}{q-p}-1+o(1)}\,. \tag{7.11}$$

$\square$

Under assumption V1-V2. Let us first introduce some notations. We construct $\widetilde{b}_t$ an analytic function such that its derivative is given by $\ell(\widetilde{b}_t)$. Define

$$H(x) = \Big(\int_1^t \frac{\mathrm{d}x}{\ell(x)}\Big)^{-1}, \quad \text{and} \quad \widetilde{b}_t = H^{-1}(1/t)\,. \tag{7.12}$$

Then, it is easy to verify that $H'(x) = -H(x)/\ell(x)$, so that $\partial_t \widetilde{b}_t = \ell(\widetilde{b}_t)$ (using also that $H(\widetilde{b}_t) = 1/t$). Notice that we also have easily that $H(x) \stackrel{x\to\infty}{\sim} \ell(x)/x$, so that $\widetilde{b}_t \stackrel{t\to\infty}{\sim} t\ell(\widetilde{b}_t)$ and thanks to Lemma 4.3 we get that $\widetilde{b}_t \stackrel{t\to\infty}{\sim} t\ell(a_t)$. We therefore get that $b_n \sim -(q-p)\widetilde{b}_n$.

Then, we define

$$g(s) := \log \ell\Big(\widetilde{b}_{\frac{1}{1-s}}\Big)\,.$$

Using that $\partial_t \widetilde{b}_t = \ell(\widetilde{b}_t)$, we get that

$$g'(s) = \frac{1}{(1-s)^2}\frac{L\Big(\widetilde{b}_{\frac{1}{1-s}}\Big)}{\widetilde{b}_{\frac{1}{1-s}}} \stackrel{s\uparrow 1}{\sim} \frac{1}{1-s}\frac{L\Big(\widetilde{b}_{\frac{1}{1-s}}\Big)}{\ell\Big(\widetilde{b}_{\frac{1}{1-s}}\Big)}\,.$$

Since $L(\cdot)$ satisfies V1-V2, so does $L\big(\widetilde{b}_{\frac{1}{1-s}}\big)/\ell\big(\widetilde{b}_{\frac{1}{1-s}}\big)$, and we may apply Theorem 5 in [16] to get that $g'(s) = \sum_{n=0}^{\infty} a_n s^n$ with $a_n \stackrel{n\to\infty}{\sim} L(\widetilde{b}_n)/\ell(\widetilde{b}_n)$. We therefore end up with

$$g(s) := \log \ell\Big(\widetilde{b}_{\frac{1}{1-s}}\Big) = \sum_{n=1}^{\infty} g_n s^n \qquad \text{with } g_n \stackrel{n\to\infty}{\sim} \frac{L(\widetilde{b}_n)}{n\ell(\widetilde{b}_n)} \stackrel{n\to\infty}{\sim} \frac{q-p}{p}\frac{\mathbf{P}(S_n \geqslant 0)}{n}\,. \tag{7.13}$$

In view of (7.3), we get that

$$f(s)=\frac{p}{q-p}g(s)+\sum_{n=1}^{\infty}v_n s^n \quad \text{with } v_n=o\Big(\frac{L(\widetilde{b}_n)}{n\ell(\widetilde{b}_n)}\Big), \tag{7.14}$$

so that

$$p'(s)=f'(s)\ell\Big(\widetilde{b}_{\frac{1}{1-s}}\Big)^{\frac{p}{q-p}}\psi\big(\tfrac{1}{1-s}\big) \quad \text{with } \psi\big(\tfrac{1}{1-s}\big):=\exp\Big(\sum_{n=1}^{\infty}v_n s^n\Big)\,. \tag{7.15}$$

We show below the following lemma.

Lemma 7.2. *There exists some slowly varying function $\widetilde{L}(\cdot)$ such that*

$$\psi\Big(\frac{1}{1-s}\Big)=\widetilde{L}\Big(\ell\Big(\widetilde{b}_{\frac{1}{1-s}}\Big)\Big)\,.$$

With this lemma in hand, we get that $p'(s)$ is regularly varying with index -1,

$$p'(s)\sim\frac{L(\widetilde{b}_{\frac{1}{1-s}})}{1-s}\ell\Big(\widetilde{b}_{\frac{1}{1-s}}\Big)^{\frac{p}{q-p}-1}\widetilde{L}\Big(\ell\Big(\widetilde{b}_{\frac{1}{1-s}}\Big)\Big) \qquad \text{as } s\uparrow 1\,,$$

so that by Corollary 1.7.3 in [3] we get that

$$\sum_{k=1}^{n}kp_k \overset{n\to\infty}{\sim} nL\big(\widetilde{b}_n\big)\ell\big(\widetilde{b}_n\big)^{\frac{p}{q-p}-1}\widetilde{L}\big(\ell\big(\widetilde{b}_n\big)\big)\,.$$

The result follows by using the monotonicity of p_n (and the fact that $|b_n|\sim(q-p)\widetilde{b}_n$).

Proof of Lemma 7.2 Set $Q(t)=\ell(\widetilde{b}_t)$ for simplicity, which is an increasing function. We want to show that $\widetilde{L}(t):=\psi\big(Q^{-1}(t)\big)$ is slowly varying as $t\to\infty$ ($t=(1-s)^{-1}$), *i.e.* for any $b>0$,

$$\frac{\psi\big(Q^{-1}(bt)\big)}{\psi\big(Q^{-1}(t)\big)}=\exp\left(\sum_{n=1}^{\infty}v_n\Big[\Big(1-\frac{1}{Q^{-1}(bt)}\Big)^n-\Big(1-\frac{1}{Q^{-1}(t)}\Big)^n\Big]\right)\to 1 \quad \text{as } t\to\infty\,.$$

Since $v_n=o\big(L(\widetilde{b}_n)/\widetilde{b}_n\big)$, we write $v_n=\varepsilon_n L(\widetilde{b}_n)/\widetilde{b}_n$. In order to show that the sum in the exponential goes to 0 as $t\to\infty$, we split it into three parts.

Part 1. For $n\leqslant Q^{-1}(t)$ we use that

$$\Big(1-\frac{1}{Q^{-1}(bt)}\Big)^n-\Big(1-\frac{1}{Q^{-1}(t)}\Big)^n\leqslant 1-\Big(1-\frac{n}{Q^{-1}(t)}\Big)=\frac{n}{Q^{-1}(t)}.$$

Hence the sum up to $n=Q^{-1}(t)$ is bounded by $\frac{1}{Q^{-1}(t)}\sum_{n=1}^{Q^{-1}(t)}nv_n$, and since $nv_n\to 0$ we get that this first part goes to 0. Indeed, we have that $nv_n\sim\varepsilon_n L(\widetilde{b}_n)/\ell(\widetilde{b}_n)$, and $L(x)/\ell(x)\to 0$ as $x\to 0$.

Part 2. For $Q^{-1}(t)<n\leqslant Q^{-1}(2bt)$, we simply bound the sum by

$$\sum_{n=Q^{-1}(t)}^{Q^{-1}(2bt)}v_n\leqslant\sup_{n\geqslant Q^{-1}(t)}\varepsilon_n\times\int_{Q^{-1}(t)}^{Q^{-1}(2bt)}\frac{L(\widetilde{b}_u)}{\widetilde{b}_u}\mathrm{d}u\,. \tag{7.16}$$

Recalling that $Q(t)=\ell(\widetilde{b}_t)$ we get that $Q'(t)=\frac{L(\widetilde{b}_t)}{\widetilde{b}_t}\ell(\widetilde{b}_t)$, so that the integral is exactly $\log Q(s)\big|_{Q^{-1}(t)}^{Q^{-1}(2bt)}=\log 2b$. Since $\varepsilon_n\to 0$, this second part also goes to 0 as $t\to\infty$.

Part 3. For $n > Q^{-1}(2bt)$, we bound the sum by

$$\sum_{k=1}^{\infty} \sum_{n=Q^{-1}(2^k bt)}^{Q^{-1}(2^{k+1}bt)} \varepsilon_n \frac{L(\widetilde{b}_n)}{\widetilde{b}_n} \Big(1 - \frac{1}{Q^{-1}(bt)}\Big)^{Q^{-1}(2^k bt)}$$
$$\leqslant \sup_{n \geqslant Q^{-1}(t)} \varepsilon_n \times \sum_{k=1}^{\infty} e^{-Q^{-1}(2^k bt)/Q^{-1}(bt)} \int_{Q^{-1}(2^k bt)}^{Q^{-1}(2^{k+1}bt)} \frac{L(\widetilde{b}_u)}{\widetilde{b}_u} \mathrm{d}u \,.$$

As above, the integral is equal to $\log 2$. Moreover, since $Q(\cdot)$ is slowly varying we get that for t large enough, for any $k \geqslant 1$

$$Q\big(2^k Q^{-1}(bt)\big) \leqslant 2^k Q\big(Q^{-1}(bt)\big) = Q\big(Q^{-1}(2^k bt)\big) \,,$$

giving that $2^k Q^{-1}(bt) \leqslant Q^{-1}(2^k bt)$. Hence, for t large enough the third part is bounded by

$$\sup_{n \geqslant Q^{-1}(t)} \varepsilon_n \times \log 2 \sum_{k \geqslant 1} e^{-2^k} \to 0 \quad \text{as } t \to 0 \,.$$

□

7.4. **The case $p > q$.** This case is similar. We only prove the general case (with the $o(1)$ in the exponent), the improvement under assumption V1-V2 being identical to what is done above.

Using the same definition of $f(s)$, and writing $\mathbf{P}(S_m \geqslant 0) = 1 - \mathbf{P}(S_m < 0)$, we get that

$$f(s) = \sum_{m=1}^{\infty} \frac{s^m}{m} \mathbf{P}(S_m \geqslant 0) = \log\Big(\frac{1}{1-s}\Big) - h(s) \quad \text{with } h(s) = \sum_{m=1}^{\infty} \frac{s^m}{m} \mathbf{P}(S_m < 0) \,. \tag{7.17}$$

Then, Lemma 7.1 gives that $\sum_{k=1}^{n} k^{-1} \mathbf{P}(S_k < 0) \overset{n\to\infty}{\sim} \frac{q}{p-q} \log \ell(b_n)$, and Corollary 1.7.3 in [3] gives that $h(s) \sim \frac{q}{p-q} \log \ell(b_{1/(1-s)})$ as $s \uparrow 1$.

Hence, we conclude thanks to (7.1) that

$$p(s) = \frac{1}{1-s} e^{-h(s)} = \frac{1}{1-s} \Big(\ell(b_{1/(1-s)})\Big)^{-q/(p-q)+o(1)} \qquad \text{as } s \uparrow 1 \,,$$

and we deduce from this the behavior of p_n in the same way as above.
Upper bound. Taking $s = 1 - 1/n$, and using that p_k is non-increasing, we get that

$$n\Big(\ell(b_n)\Big)^{q/(p-q)+o(1)} = \sum_{k=0}^{\infty} p_k \Big(1 - \frac{1}{n}\Big)^k \geqslant \sum_{k=0}^{n} p_n \Big(1 - \frac{1}{n}\Big)^k \geqslant c n p_n,$$

and therefore

$$p_n \leqslant c \Big(\ell(b_n)\Big)^{-q/(p-q)+o(1)} = \Big(\ell(b_n)\Big)^{-q/(p-q)+o(1)} \,. \tag{7.18}$$

Lower bound. As above, we fix $\varepsilon > 0$ and define $t_n = \ell(b_n)^{\varepsilon}$. Taking $s = 1 - 1/(nt_n)$, we get as in (7.9) that for n large enough

$$\sum_{k=0}^{\infty} p_k \Big(1 - \frac{1}{nt_n}\Big)^k = nt_n \Big(\ell(b_{nt_n})\Big)^{-q/(p-q)+o(1)} \geqslant nt_n^{1/2} \Big(\ell(b_n)\Big)^{q/(p-q)+o(1)} \,. \tag{7.19}$$

On the other hand, as in (7.10), since p_k is non-increasing we have

$$\sum_{k=0}^{\infty} p_k \Big(1-\frac{1}{nt_n}\Big)^k \leqslant \sum_{k=0}^{n} p_k + p_n \sum_{k\,\geqslant\, n+1} \Big(1-\frac{1}{nt_n}\Big)^k$$
$$= o\Big(n t_n^{1/2} \Big(\ell(b_n)\Big)^{-q/(p-q)+o(1)}\Big) + n t_n p_n \,,$$

where we used (7.18) for the first term, together with the fact that $\ell(b_n)^{o(1)} = o(t_n^{1/2})$, and a standard computation for the second term. Combining this with (7.19) we get that

$$p_n \geqslant t_n^{-1/2} \Big(\ell(b_n)\Big)^{-q/(p-q)+o(1)},$$

and since $t_n = \ell(b_n)^{\varepsilon}$ with $\varepsilon > 0$ arbitrary, we get that $p_n \geqslant \ell(b_n)^{-q/(p-q)+o(1)}$.

7.5. **Further remarks on the case $p = q$.** In the case $p = q = \frac{1}{2}$, then $b_n = o(n\ell(a_n))$. If $\lim_{n\to\infty} b_n/a_n = b$, then Theorem 3.2 gives the correct asymptotic for $\mathbf{P}(T_- > n)$. In the case $\lim_{n\to\infty} b_n/a_n = +\infty$ (the case where the limit is $-\infty$ is symmetric), then we still have as in (7.2) that

$$\mathbf{P}(S_k < 0) \overset{n\to\infty}{\sim} \frac{1}{2} k L(b_k)(b_k)^{-1}. \tag{7.20}$$

Hence, since b_n is regularly varying with exponent -1, we get that

$$r(n) := \sum_{k=1}^{n} \frac{1}{k} \mathbf{P}(S_k < 0) \overset{n\to\infty}{\sim} \sum_{k=1}^{n} \frac{L(b_n)}{2b_n} \tag{7.21}$$

is slowly varying. Additionally, we get that $r(n) = o(\log n)$ (since $\mathbf{P}(S_n < 0) \to 0$), and also $r(n) \gg \log(\ell(a_n))$ in view of Lemma 7.1, since $b_n = o(n\ell(a_n))$. We therefore have that that

$$\ell(a_n)^{1/o(1)} \leqslant e^{r(n)} \leqslant n^{o(1)} \,.$$

Then, the same scheme of proof as above gives the behavior of $p(s)$ and $p'(s)$: we get that

$$\mathbf{P}(T_- > n) = e^{(1+o(1))r(n)} \quad \text{and} \quad \mathbf{P}(T_+ > n) = \frac{L(b_n)}{b_n} e^{(1+o(1))r(n)} \,. \tag{7.22}$$

Details are straightforward and left to the reader.

7.6. **Case $\alpha = 1$, $\mu = 0$: proof of Theorem 3.4.** First of all, we state the analogous of Lemma 7.1 in the case $\alpha = 1$, $\mu = 0$.

Lemma 7.3. *Assume that (1.1) holds, with $\alpha = 1$ and $\mu = 0$. Recall the definition (3.1) of $\ell^\star(\cdot)$ and (1.3) of b_n. Then, if $p > q$, $b_n \sim -(p-q)n\ell^\star(a_n) \to -\infty$ and*

$$\mathbf{P}(S_n > 0) \overset{n\to\infty}{\sim} \frac{p}{p-q} \frac{L(|b_n|)}{\ell^\star(|b_n|)}.$$

Moreover we have that

$$\sum_{k=1}^{n} \frac{1}{k} \mathbf{P}(S_k > 0) \overset{n\to\infty}{\sim} -\frac{p}{p-q} \log \ell^\star(|b_n|) \,.$$

The case $p < q$ is symmetric.

We skip the proof here since it is identical to that of Lemma 7.1, using Lemma 4.4-(ii) in place of Lemma 4.4-(i). We now turn to the proof of Theorem 3.3, which is very similar to that of Theorem 3.2, the key identity being the Wiener-Hopf factorization (7.1).

(i) The case $p > q$. Denoting $f(s)$ as in (7.3), Lemma 7.3 and Corollary 1.7.3 in [3] give analogously to (7.5)

$$f(s) \sim \frac{-p}{p-q} \log \ell^\star(|b_{1/(1-s)}|)\,, \quad f'(s) \sim \frac{p}{p-q} \frac{1}{1-s} \frac{L(|b_{1/(1-s)}|)}{\ell^\star(|b_{1/(1-s)}|)} \qquad \text{as } s \uparrow 1\,. \tag{7.23}$$

Therefore,

$$p'(s) = f'(s) e^{f(s)} = \frac{L(|b_{1/(1-s)}|)}{1-s} \Big(\ell^\star\big(|b_{1/(1-s)}|\big) \Big)^{-\frac{p}{p-q} - 1 + o(1)} \quad \text{as } s \uparrow 1\,,$$

which can be turned into $p_n = n^{-1} L(|b_n|) (\ell^\star(b_n))^{-1-\frac{p}{p-q}+o(1)}$ as done in Section 7.3. Again, as in Section 7.3, the term $\ell^\star(|b_n|)^{o(1)}$ can be replaced by $\widetilde{L}(\ell^\star(|b_n|))$ for some slowly varying function $\widetilde{L}$, under the assumption V1-V2.

(ii) The case $p < q$. Here also, the method of Section 7.4 is easily adapted. Denote $h(s)$ as in (7.17). Then Lemma 7.3 gives that $\sum_{k=1}^n k^{-1} \mathbf{P}(S_k < 0) \overset{n\to\infty}{\sim} -\frac{q}{q-p} \log \ell^\star(b_n)$ so that together with Corollary 1.7.3 in [3] it gives that $h(s) \sim -\frac{q}{q-p} \log \ell^\star(b_{1/(1-s)})$ as $s \uparrow 1$. Therefore, we get

$$p(s) = \frac{e^{-h(s)}}{1-s} = \frac{1}{1-s} \Big(\ell^\star(b_{1/(1-s)}) \Big)^{q/(q-p)+o(1)}\,,$$

which can be turned into $p_n = (\ell^\star(b_n))^{q/(q-p)+o(1)}$ as done in Section 7.4. Here again, $\ell^\star(|b_n|)^{o(1)}$ can be replaced by $\bar{L}(\ell^\star(|b_n|))$ for some slowly varying function $\bar{L}$, under the assumption V1-V2.

(iii) The case $p = q$. In the case $\lim_{n\to\infty} b_n/a_n = b \in \mathbb{R}$, the proof is identical to that of Section 7.2: we get that

$$\lim_{n\to+\infty} \frac{1}{n} \sum_{k=1}^n \mathbf{P}(S_k > 0) = \mathbf{P}(Y > -b) =: \rho \in (0,1),$$

with $\rho = \frac{1}{2} + \frac{1}{\pi} \arctan(2b/\pi)$. This is Spitzer's condition, and [3, Thm. 8.9.12] implies that T_- is in the domain of attraction of a positive stable random variable with index ρ, so that there exists a slowly varying function $\varphi(\cdot)$ such that

$$\mathbf{P}(T_- > n) \overset{n\to\infty}{\sim} \varphi(n) n^{-\rho}. \tag{7.24}$$

The case $\lim_{n\to\infty} b_n/a_n = +\infty$ can be treated similarly to Section 7.5: we get the same conclusion as in (7.22).

8. Renewal theorems: proof of Theorems 3.5-3.6-3.7

8.1. The case $\alpha = 1$ with infinite mean.

8.1.1. *The case* $\lim_{n\to+\infty} b_n/a_n = b \in \mathbb{R}$. Let us set k_x an integer such that $a_{k_x} \sim x$. We fix $\varepsilon > 0$ such that $\sup_n |b_n|/a_n \leqslant 1/\sqrt{\varepsilon}$, and we write

$$G(x) := \Big(\sum_{k \leqslant \frac{1}{\varepsilon} k_x} + \sum_{k > \frac{1}{\varepsilon} k_x} \Big) \mathbf{P}(S_k = x). \tag{8.1}$$

The first term is estimated thanks to Theorem 2.3, which gives that there is a constant C such that for any $x \geqslant 1$ and any k, $\mathbf{P}(S_k = x) \leqslant C(a_k)^{-1} k L(x) x^{-1}$, so that

$$\sum_{k \leqslant \frac{1}{\varepsilon} k_x} \mathbf{P}(S_k = x) \leqslant C' \Big(\frac{k_x}{\varepsilon}\Big)^2 (a_{k_x/\varepsilon})^{-1} L(x) x^{-1} \leqslant \frac{C''}{\varepsilon} L(x)^{-1}. \tag{8.2}$$

where we used that $a_{k_x/\varepsilon} \sim \varepsilon^{-1} a_{k_x}$ and that $k_x \sim a_{k_x} L(a_{k_x})^{-1} \sim x L(x)^{-1}$.

The second term is in fact the main one. Thanks to the local limit theorem (1.4), for any $\eta > 0$ there is some $\varepsilon > 0$ small and some k_0 such that for any $k \geqslant k_0$,

$$g\Big((x - b_k)/a_k\Big) - \eta \leqslant a_k \mathbf{P}(S_k = x) \leqslant g\Big((x - b_k)/a_k\Big) + \eta\,.$$

Then for $k > \varepsilon^{-1} k_x$, and because a_k is regularly varying with index -1, we have that $a_k \geqslant \frac{1}{2}\varepsilon^{-1} a_{k_x} \geqslant x/(4\varepsilon)$ for x sufficiently large. We therefore get that if $k \geqslant \varepsilon^{-1} k_x$ with x large enough, then $|x/a_k| \leqslant 4\varepsilon$ and also $|b_k/a_k - b| \leqslant \varepsilon$, so that by continuity of g, we get that provided that ε is small and x is large enough

$$g(-b) - 2\eta \leqslant a_k \mathbf{P}(S_k = x) \leqslant g(-b) + 2\eta \qquad \text{for all } k \geqslant k_x/\varepsilon.$$

We stress that since we are in the symmetric case, with $p = q = 1/2$, and by our definition (1.2) of a_n, $g(\cdot)$ is the density of a symmetric Cauchy$(1/2)$ distribution, so that $g(-b) = \frac{2}{\pi(1+(2b)^2)}$.

Hence, for any $\eta' > 0$ and provided that ε is small enough and x large enough, the second sum in (8.1) is

$$\frac{2(1-\eta')}{\pi(1+(2b)^2)} \sum_{k > \frac{1}{\varepsilon} k_x} \frac{1}{a_k} \leqslant \sum_{k > \frac{1}{\varepsilon} k_x} \mathbf{P}(S_k = x) \leqslant \frac{2(1+\eta')}{\pi(1+(2b)^2)} \sum_{k > \frac{1}{\varepsilon} k_x} \frac{1}{a_k},$$

and we estimate the last sum. We use a comparison with the following (convergent) integral

$$\int_{\varepsilon^{-1} k_x}^{\infty} \frac{dt}{a_t} \sim \int_{a_{\varepsilon^{-1} k_x}}^{\infty} \frac{du}{u L(u)} \sim \int_{\varepsilon^{-1} x}^{+\infty} \frac{du}{u L(u)} \qquad \text{as } x \to \infty,$$

where we used a change of variable $u = a_t$ so that $t \sim u/L(u)$ (see (1.2)) and $dt = L(u)^{-1} du$, and then used that $a_{\varepsilon^{-1} k_x} \sim \varepsilon^{-1} a_{k_x} \sim \varepsilon^{-1} x$. Since $v \mapsto \int_v^\infty \frac{du}{u L(u)}$ is a slowly varying function (vanishing as $v \to \infty$), we get that for $\varepsilon > 0$ small enough and x large enough (how large depends on ε)

$$\frac{2(1-2\eta')}{\pi(1+(2b)^2)} \sum_{n > x} \frac{1}{n L(n)} \leqslant \sum_{k > \frac{1}{\varepsilon} k_x} \mathbf{P}(S_k = x) \leqslant \frac{2(1+2\eta')}{\pi(1+(2b)^2)} \sum_{n > x} \frac{1}{n L(n)}. \tag{8.3}$$

In conclusion, combining (8.2) and (8.3), and since $L(x)^{-1} = o\big(\sum_{n > x} \frac{1}{n L(n)}\big)$ and η' is arbitrary, we get that

$$G(x) \overset{x \to \infty}{\sim} \frac{2}{\pi(1+(2b)^2)} \sum_{n > x} \frac{1}{n L(n)}\,.$$

8.1.2. *The case $p > q$.* Let us define k_x to be a solution of $b_{k_x} = k_x \mu(a_{k_x}) = x$ (k_x may not be an integer, but we may replace it by its integer part). Then we identify the range of k's for which we may apply the local limit theorem (1.4) to $\mathbf{P}(S_k = x)$: they are the k's such that $x - b_k$ is of order a_k, and we find that they are in the range $k = k_x + \Theta\big(a_{k_x}/\mu(a_{k_x})\big)$. Let us mention the results of [1, 19] where this heuristic is confirmed: if N_x the number of renewals before reaching x, it is shown that $(a_{k_x}/\mu(a_{k_x}))^{-1}(N_x - k_x)$ converges in distribution.

Let us stress right away that $\mu(a_{k_x}) \sim \mu(b_{k_x}) = \mu(x)$. Indeed, since $p > q$ we have that $\mu(x) \sim (p-q)\ell(x)$, and Lemma 4.3 gives that $\ell(a_n) \sim \ell(b_n)$.

We fix $\varepsilon > 0$ and decompose $G(x)$ into five sums

$$\begin{aligned} G(x) &= \Big(\sum_{k < \frac12 k_x} + \sum_{k = \frac12 k_x}^{k_x - \frac{1}{\varepsilon} a_{k_x}/\mu(x)} + \sum_{k = k_x - \frac{1}{\varepsilon} a_{k_x}/\mu(x)}^{k_x + \frac{1}{\varepsilon} a_{k_x}/\mu(x)} + \sum_{k_x + \frac{1}{\varepsilon} a_{k_x}/\mu(x)}^{2k_x} + \sum_{k > 2k_x} \Big) \mathbf{P}(S_k = x) \\ &=: \quad I \quad + \quad II \quad + \quad III \quad + \quad IV \quad + \quad V\,. \end{aligned} \tag{8.4}$$

The main contributions are the sums III and V, so we start by estimating those two terms

Term III. By the local limit theorem (1.4), we get that as $x \to +\infty$ (so $k_x \to +\infty$)

$$III = (1 + o(1)) \sum_{k = k_x - \frac{1}{\varepsilon} a_{k_x}/\mu(x)}^{k_x + \frac{1}{\varepsilon} a_{k_x}/\mu(x)} \frac{1}{a_k} g\Big(\frac{x - k\mu(a_k)}{a_k} \Big)\,.$$

Then, we use the fact that a_{k_x} is negligible compared to $b_{k_x} = k_x \mu(a_{k_x}) \sim k_x \mu(x)$: we get that uniformly for the k's in the range considered, we have $k = (1 + o(1))k_x$ so that $a_k = (1 + o(1))a_{k_x}$. Setting $j = k - k_x$, we also have that for the range of k considered (using that $a_k \sim a_{k_x}$ and $\mu(a_{k_x}) \sim \mu(x)$), since $x = k_x \mu(a_{k_x})$

$$\frac{x - k\mu(a_k)}{a_k} = (1 + o(1)) \frac{j\mu(x)}{a_{k_x}} + \frac{k_x\big(\mu(a_{k_x}) - \mu(a_k)\big)}{(1 + o(1))a_{k_x}} = (1 + o(1)) \frac{j\mu(x)}{a_{k_x}} + o(1). \tag{8.5}$$

For the second identity, we used Claim 5.3 to get that $|\mu(a_{k_x}) - \mu(a_k)| = o(L(a_{k_x})) = o(a_{k_x}/k_x)$ for the range of k considered. In the end, and since g is continuous, we get that

$$III = (1 + o(1)) \sum_{j = -\frac{1}{\varepsilon} a_{k_x}/\mu(x)}^{\frac{1}{\varepsilon} a_{k_x}/\mu(x)} \frac{1}{a_{k_x}} g\Big(j \times \frac{\mu(x)}{a_{k_x}} \Big) = \frac{1 + o(1)}{\mu(x)} \int_{-1/\varepsilon}^{1/\varepsilon} g(u) du\,, \tag{8.6}$$

where we used a Riemann sum approximation in the last identity. Then, since $\int_{-\infty}^{+\infty} g(u)du = 1$, we get that for any $\eta > 0$ we can choose $\varepsilon > 0$ such that for all sufficiently large x (how large depend on ε)

$$\frac{1 - \eta}{\mu(x)} \leqslant III \leqslant \frac{1 + \eta}{\mu(x)}\,. \tag{8.7}$$

Term V. For the last term in (8.4), we use (2.6) to get from Theorem 2.4 that for $k \geqslant 2k_x$

$$\mathbf{P}(S_k = x) = \mathbf{P}(S_k - b_k = x - b_k) \leqslant C k L(b_k) b_k^{-(1+\alpha)},$$

where we used that $b_k \geqslant b_{2k_x} \geqslant \frac{3}{2} b_{k_x} = \frac{3}{2}x$ provided that x is large enough, so that $|x - b_k| \geqslant \frac{1}{2} b_k \gg a_k$. Then we get that (we have $\alpha = 1$)

$$V \leqslant C \int_{k_x}^{\infty} \frac{uL(b_u)}{b_u^2} \mathrm{d}u \leqslant C' \int_x^{\infty} \frac{L(t)}{t\ell(t)^2} \mathrm{d}t = \frac{C'}{\ell(x)}\,, \tag{8.8}$$

where we used a change of variable $t = b_u \sim (p-q)u\ell(b_u)$ (using also $dt \sim (p-q)\ell(b_u)du$ and $b_{k_x} = x$), and then Lemma 4.4.

Moreover, if one has that $\mathbf{P}(X_1 = -x) \sim qL(x)x^{-2}$, then we write

$$V = \sum_{k=2k_x}^{\varepsilon^{-1}k_x} \mathbf{P}(S_k = x) + \sum_{k > \varepsilon^{-1}k_x} \mathbf{P}(S_k = x)\,.$$

As above, the first term is comparable to $\int_{2k_x}^{\varepsilon^{-1}k_x} \frac{uL(b_u)}{b_u^2} \mathrm{d}u \leqslant C \int_x^{\varepsilon^{-1}x} \frac{L(t)}{t\ell(t)^2} \mathrm{d}t$, which is $o(1/\ell(x))$ because of Lemma 4.4 and since $\ell(x)$ is slowly varying (so $\ell(\varepsilon^{-1}x) \sim \ell(x)$). For the second term, we use Theorem 2.4 which gives that for any $\eta > 0$, and provided that ε is fixed small enough and that x is large enough, we have for all $k \geqslant \varepsilon^{-1}k_x$ (so $x = b_{k_x} \leqslant \frac{1}{2}\varepsilon^{-1} b_k$)

$$\mathbf{P}(S_k = x) = \mathbf{P}(S_k - b_k = x - b_k) \begin{cases} \leqslant (q+\eta)kL(b_k)b_k^{-2}\,, \\ \geqslant (q-\eta)kL(b_k)b_k^{-2}\,. \end{cases}$$

Moreover, by a change of variable $t = b_u$ ($dt \sim (p-q)\ell(b_u)du$) we get

$$\int_{\varepsilon^{-1}k_x}^{\infty} \frac{uL(b_u)}{b_u^2} \mathrm{d}u \sim \int_{\varepsilon^{-1}x}^{\infty} \frac{1}{(p-q)^2} \frac{L(t)}{t\ell(t)^2} \mathrm{d}t = \frac{1}{(p-q)^2} \frac{1}{\ell(\varepsilon^{-1}x)} \sim \frac{1}{(p-q)\mu(x)}, \tag{8.9}$$

where we used that $b_{\varepsilon^{-1}k_x} \sim \varepsilon^{-1} b_{k_x} = \varepsilon^{-1}x$ and that $\ell(\cdot)$ is slowly varying, with $\mu(x) \sim (p-q)\ell(x)$. In the end, and since η is arbitrary, we get that as $x \to \infty$,

$$V = (1 + o(1)) \frac{q}{p-q} \frac{1}{\mu(x)}\,. \tag{8.10}$$

To conclude the proof of the statement, we need to show that the terms I, II and IV are negligible compared to $1/\mu(x)$.

Term I. Thanks to (2.5) and Theorem 2.4, we obtain that there is a constant $C > 0$ such that for any $k \leqslant \frac{1}{2}k_x$ (so that $x \geqslant \frac{2}{3} b_k \gg a_k$ provided that x is large engouh) we have $\mathbf{P}(S_k = x) \leqslant CkL(x)x^{-2}$. Then the first term in (8.4) is bounded by a constant times

$$\sum_{k=1}^{\frac{1}{2}k_x} kL(x)x^{-2} \leqslant \frac{1}{8} k_x^2 L(x) x^{-2} \leqslant \frac{c}{\ell(x)} \frac{L(x)}{\ell(x)}, \tag{8.11}$$

where we used that $k_x \sim \frac{1}{p-q} x\ell(x)^{-1}$ as $x \to \infty$. (indeed we have $x = b_{k_x} \sim (p-q)k_x\ell(b_{k_x})$). Now, because $L(x)/\ell(x) \to 0$, we get that $I = o(1/\ell(x))$.

Term II. We set $j = k_x - k$. Then, the range of k considered corresponds to $j \in \big[\frac{1}{\varepsilon} a_{k_x}/\mu(x), \frac{1}{2}k_x\big]$, and for that range we have similarly to (8.5)

$$\frac{x - b_k}{a_k} = \frac{k_x \mu(a_{k_x}) - (k_x - j)\mu(a_{k_x - j})}{a_k} \geqslant \frac{k_x}{a_k} \big|\mu(a_{k_x}) - \mu(a_{k_x-j})\big| + j\frac{j\mu(x)}{2a_k}\,. \tag{8.12}$$

We used that $\mu(a_{k_x-j}) \sim \mu(a_{k_x}) \sim \mu(x)$ (since $j \leqslant k_x/2$ with $\mu(\cdot)$ slowly varying). Then, we may use Claim 5.3 to get that

$$|\mu(a_{k_x}) - \mu(a_{k_x-j})| \leqslant cL(a_{k_x}) \log\big(a_{k_x}/a_{k_x-j}\big) \leqslant c' L(a_{k_x}) j/k_x\,,$$

where in the second inequality we used that there is a constant $c > 0$ such that uniformly for $j \in [k/2, k]$, $a_k/a_{k-j} \leqslant 1 + cj/k$ (using that a_k is regularly varying with index 1). Plugging this in (8.12), and since $L(a_{k_x}) = o(\mu(a_{k_x})) = o(\mu(x))$, we get that

$$x - b_k \geqslant \frac{1}{4} j\mu(x), \qquad \text{for } j := k_x - k \in \big[\varepsilon^{-1}\frac{a_{k_x}}{\mu(x)}, \frac{1}{2}k_x\big]\,. \tag{8.13}$$

Then, since for the range considered we have $j\mu(x) \geqslant \varepsilon^{-1}a_{k_x}$ we have that $x - b_k \geqslant \frac{1}{4}j\mu(x) \geqslant a_k$ ($k \geqslant k_x/2$), so that we may apply Theorem 2.4. We get that for the range of k considered and with $j = k_x - k$,

$$\begin{aligned}\mathbf{P}(S_k = x) = \mathbf{P}(S_k - b_k = x - b_k) &\leqslant CkL(x - b_k)(x - b_k)^{-2}\\ &\leqslant Ck_x L(j\mu(x))(j\mu(x))^{-2}\,.\end{aligned} \tag{8.14}$$

Hence we get that

$$\begin{aligned}II = \sum_{k=\frac{1}{2}k_x}^{k_x - \varepsilon^{-1}a_{k_x}/\mu(x)} \mathbf{P}(S_k = x) &\leqslant Ck_x \int_{\varepsilon^{-1}a_{k_x}/\mu(x)}^{\frac{1}{2}k_x} \frac{L(j\mu(x))}{(j\mu(x))^2}\mathrm{d}j\\ &\leqslant Ck_x \frac{1}{\mu(x)} \int_{\varepsilon^{-1}a_{k_x}}^{\infty} \frac{L(t)}{t^2}\mathrm{d}t \leqslant C\frac{k_x}{\mu(x)}\frac{L(\varepsilon^{-1}a_{k_x})}{\varepsilon^{-1}a_{k_x}}\,,\end{aligned}$$

where in the second inequality we made a change of variable $t = j\mu(x)$. Now, since $L(\cdot)$ is slowly varying, and because of the definition (1.2) of a_n, we get that $k_x L(\varepsilon^{-1}a_{k_x})/a_{k_x} \to 1$ as $x \to \infty$. Hence, there exists a constant $C > 0$ (independent of ε) such that for x sufficiently large (how large depends on ε)

$$II \leqslant \frac{C\varepsilon}{\mu(x)}\,. \tag{8.15}$$

Term IV. It is treated similarly to the term II. Setting $j = k - k_x$, one gets exactly as in (8.13) that provided that x is large enough,

$$x - b_k \leqslant -\frac{1}{4}j\mu(x)\,, \qquad \text{for } j := k - k_x \in \big[\varepsilon^{-1}\frac{a_{k_x}}{\mu(x)}, 2k_x\big], \tag{8.16}$$

Then we can apply Theorem 2.4 (we have that $|x - b_k| \geqslant a_k$ for the range considered) to get analogously to (8.14) that there is a constant C such that for any $j := k - k_x$ in the range considered,

$$\mathbf{P}(S_k = x) = \mathbf{P}(S_k - b_k = x - b_k) \leqslant Ck_x L(j\mu(x))\big(j\mu(x)\big)^2\,.$$

Therefore, we can bound the term IV

$$IV = \sum_{k=k_x+\varepsilon^{-1}a_{k_x}/\mu(x)}^{2k_x} \mathbf{P}(S_k = x) \leqslant Ck_x \sum_{j=\varepsilon^{-1}a_{k_x}/\mu(x)}^{2k_x} \frac{L(j\mu(x))}{\big(j\mu(x)\big)^2}\,,$$

so that as for the term II, we get that there is a constant C (independent of ε) such that for x sufficiently large

$$IV \leqslant \frac{C\varepsilon}{\mu(x)}\,. \tag{8.17}$$

Conclusion. Assuming (2.5)-(2.6), we get from the estimates (8.7)-(8.8) and (8.11)-(8.15)-(8.17) that there is a constant C such that

$$G(x) \leqslant \frac{C}{\mu(x)}\,. \tag{8.18}$$

If we additionally assume that $\mathbf{P}(X_1 = -x) \sim qL(x)x^{-1}$, we can use (8.10) instead of (8.8). According to (8.7)-(8.15)-(8.17) and to (8.10), we find that for every $\eta > 0$, we can choose $\varepsilon > 0$ such that for x sufficiently large (how large depends on ε) we get

$$\frac{1-\eta}{\mu(x)} + (1+o(1))\frac{q}{p-q}\frac{1}{\mu(x)} \leqslant G(x) \leqslant \frac{1+3\eta}{\mu(x)} + (1+o(1))\frac{q}{p-q}\frac{1}{\mu(x)}. \tag{8.19}$$

Since $\eta > 0$ is arbitrary, we get (3.7).

8.1.3. *The case $p < q$.* Here, we have that $b_k = k\mu(a_k) \to -\infty$ as a regularly varying function, since $\mu(k) \sim (p-q)\ell(k)$. We define k_x to verify $b_{k_x} = -x$. Let us fix $\varepsilon > 0$, and split the sum in G into two parts this time.

$$G(x) = \Big(\sum_{k=1}^{\varepsilon^{-1}k_x} + \sum_{k>\varepsilon^{-1}k_x}\Big)\mathbf{P}(S_k = x) =: I + II. \tag{8.20}$$

Term I. Since $x - b_k \geqslant |b_k|$ with $|b_k| \geqslant a_k$, we may use (2.5) to get from Theorem 2.4 that

$$\mathbf{P}(S_k = x) = \mathbf{P}(S_k - b_k = x - b_k) \leqslant Ck\frac{L(x-b_k)}{(x-b_k)^2}\,. \tag{8.21}$$

Then, since $x - b_k \geqslant x$ (except possibly for finitely many k's for which b_k is positive), we get that $\mathbf{P}(S_k = x) \leqslant CkL(x)x^{-2}$ for all $k \leqslant \varepsilon^{-1}k_x$ (and k larger than a constant). Hence, we get that the term I in (8.20) is bounded by

$$I \leqslant C\big(k_x/\varepsilon\big)^2 L(x)x^{-2} \leqslant C'\varepsilon^{-2}\frac{L(x)}{\ell(x)^2}\,, \tag{8.22}$$

where we used for the second inequality that $k_x \sim \frac{1}{q-p}x\ell(x)^{-1}$ (indeed $x = -b_{k_x} \sim (p-q)k_x\ell(|b_{k_x}|)$). Then, since $L(x)/\ell(x) \to 0$ as $x \to \infty$, we get that $I = o(1/\ell(x))$.

Term II. Again, (8.21) is valid. Here, we use that $x - b_k \geqslant |b_k|$ to get that $\mathbf{P}(S_k = x) \leqslant CkL(|b_k|)|b_k|^2$. Then, we obtain that

$$II \leqslant C\sum_{k \geqslant \varepsilon^{-1}k_x} k\frac{L(|b_k|)}{|b_k|^2} \leqslant \frac{C}{\ell(x)} \tag{8.23}$$

where we used the same calculation as in (8.9), with $|b_k| \sim (q-p)k\ell(|b_k|)$.

Hence we proved that there is a constant $C > 0$ such that $G(x) \leqslant C/|\mu(x)|$ (since $|\mu(x)| \sim -(q-p)\ell(x)$). Combined with (8.18) in the case $p > q$, we get that $G(x) = O(1/|\mu(x)|)$ in any case.

Let us now obtain the right asymptotic equivalence, assuming (2.7). We may use Theorem 2.4 to obtain that for any $\eta > 0$, we can choose $\varepsilon > 0$ so that if x is large enough we get for any $k \geqslant \varepsilon^{-1}k_x$ we have $|b_k| \leqslant x - b_k \leqslant (1+2\varepsilon)|b_k|$ and

$$\mathbf{P}(S_k = x) = \mathbf{P}(S_k = b_k = x - b_k)\begin{cases} \leqslant (p+\eta)kL(|b_k|)|b_k|^{-2}\,, \\ \geqslant (p-\eta)kL(|b_k|)|b_k|^{-2}\,. \end{cases}$$

Then the same calculation as in (8.9) gives that

$$\sum_{k>\varepsilon^{-1}k_x} \frac{kL(|b_k|)}{|b_k|^2} \sim \frac{1}{(q-p)|\mu(x)|}, \tag{8.24}$$

using that $|b_k| \sim (q-p)k\ell(|b_k|)$ and $|\mu(x)| \sim (q-p)\ell(x)$.

Then, since η is arbitrary, we get that as $x \to \infty$

$$V = \sum_{k > \varepsilon^{-1} k_x} \mathbf{P}(S_k = x) \overset{x\to\infty}{\sim} \frac{p}{q-p} \frac{1}{|\mu(x)|} \tag{8.25}$$

8.2. **The finite mean case.** In the case $\mu = 0$, then the walk is recurrent, so we have $G(x) = +\infty$ for all $x \in \mathbb{Z}$. We therefore consider only the cases $\mu > 0$ and $\mu < 0$.

8.2.1. *Case* $\mu > 0$. We follow exactly the same scheme as for the case $\alpha = 1$ with infinite mean. Here, we set k_x so that $b_{k_x} = x$ (hence $k_x \sim x/\mu$), and we decompose $G(x)$ as in (8.4): we fix $\varepsilon > 0$ and write

$$\begin{aligned} G(x) &= \Big(\sum_{k=1}^{k_x/2} + \sum_{k=k_x/2}^{k_x - \varepsilon^{-1} a_{k_x}} + \sum_{k=k_x-\varepsilon^{-1}a_{k_x}}^{k_x+\varepsilon^{-1}a_{k_x}} + \sum_{k=k_x+\varepsilon^{-1}a_{k_x}}^{2k_x} + \sum_{k>2k_x} \Big) \mathbf{P}(S_k = x) \\ &=: \quad I \quad + \quad II \quad + \quad III \quad + \quad IV \quad + \quad V\,. \end{aligned}$$

Let us now consider all the terms, the main term being the third one.

Term III. We proceed as for the term III in the infinite mean case, but it is easier here. The local limit theorem gives that we have, as $x \to \infty$

$$III = (1+o(1)) \sum_{k=k_x-\varepsilon^{-1}a_{k_x}}^{k_x+\varepsilon^{-1}a_{k_x}} \frac{1}{a_k} g\Big(\frac{x-b_k}{a_{k_x}}\Big) = (1+o(1)) \sum_{j=-\varepsilon^{-1}a_{k_x}}^{\varepsilon^{-1}a_{k_x}} \frac{1}{a_{k_x}} g\Big(\frac{j\mu}{a_{k_x}}\Big)$$

where we set $j = k - k_x$, together with the fact that $a_k \sim a_{k_x}$ uniformly for the range of k considered and that g is continuous. Then, a Riemann sum approximation gives that

$$III = (1+o(1)) \frac{1}{\mu} \int_{-\varepsilon^{-1}/\mu}^{\varepsilon^{-1}/\mu} g(u) du\,. \tag{8.26}$$

Since $\int_{-\infty}^{+\infty} g(u)du = 1$, we get as for (8.7) that for any $\eta > 0$ we may choose $\varepsilon > 0$ small enough, and then x large enough (how large depend on ε) such that

$$\frac{1-\eta}{\mu} \leqslant III \leqslant \frac{1+\eta}{\mu}. \tag{8.27}$$

Term I. For the first term, we get thanks to Theorem 2.3 that uniformly for the range of k considered (which implies that $x - b_k \geqslant x/4$)

$$\mathbf{P}(S_k = x) \leqslant \frac{C}{a_k} k L(x-\mu k)(x-\mu k)^{-\alpha} \leqslant C \frac{k}{a_k} L(x) x^{-\alpha}\,. \tag{8.28}$$

Then, since k/a_k is regularly varying with exponent $1 - 1/\alpha \geqslant 0$, we get

$$I \leqslant C L(x) x^{-\alpha} \sum_{k=1}^{k_x/2} k/a_k \leqslant C' L(x) x^{-\alpha} \times x^2/a_x,$$

where we used that $k_x \leqslant x$. If $\alpha > 1$, the right hand side is regularly varying with exponent $2 - \alpha - 1/\alpha < 0$, so that we obtain $I = o(1)$.

If $\alpha = 1$, then since we have $a_x \sim x L(a_x)$, we obtain the upper bound $CL(x)/L(a_x)$ and we cannot conclude: we need to improve (8.28) by using (2.5). Assuming (2.5), we get that there is a constant $C > 0$ such that for the range of k considered ($x - \mu k \geqslant x/4 \geqslant a_k$),

$$\mathbf{P}(S_k = x) \leqslant C k L(x-b_k)(x-b_k)^{-2} \leqslant C' k L(x) x^{-2}\,.$$

Then we can bound

$$I \leqslant C' L(x) x^{-2} \sum_{k=1}^{k_x} k \leqslant C'' L(x)\,.$$

Now, since $|\mu| < \infty$ we have that $L(n) \to 0$ ($L(n) = o\big(\sum_{k>n} L(k)k^{-1}\big)$), so we get that $I = o(1)$.

Term II. We may apply Theorem 2.3 to get that, setting $j = k_x - k$ so that $x - b_k \geqslant \frac{1}{2}\mu j$,

$$\mathbf{P}(S_k = x) = \mathbf{P}(S_k - b_k = x - b_k) \leqslant C \frac{k}{a_k} L(j) j^{-\alpha}\,. \tag{8.29}$$

We therefore get, since $k/a_k \leqslant 2k_x/a_{k_x}$ in the range considered, that

$$II \leqslant C \frac{k_x}{a_{k_x}} \sum_{j \,\geqslant\, \varepsilon^{-1} a_{k_x}} L(j) j^{-\alpha}\,. \tag{8.30}$$

If $\alpha > 1$, then we obtain

$$II \leqslant C \frac{k_x}{a_{k_x}} L(\varepsilon^{-1} a_{k_x}) (\varepsilon^{-1} a_{k_x})^{1-\alpha} \leqslant C' \varepsilon^{\alpha - 1}\,, \tag{8.31}$$

where we used the definition (1.2) of a_n to get that $L(\varepsilon^{-1} a_{k_x})(a_{k_x})^{-\alpha} \sim k_x^{-1}$.

If $\alpha = 1$, then $\sum_{j \,\geqslant\, \varepsilon^{-1} a_{k_x}} j^{-1} L(j) = \ell^\star(\varepsilon^{-1} a_{k_x}) \gg L(a_{k_x})$, and we cannot conclude. Assuming (2.5) and using Theorem 2.4 (we have $x - b_k \geqslant \frac{1}{2}\mu j \geqslant a_k$ for the range considered), we may improve (8.29) to

$$\mathbf{P}(S_k = x) = \mathbf{P}(S_k - b_k = x - b_k) \leqslant C k L(j) j^{-2} \quad \text{with } j = k_x - k\,. \tag{8.32}$$

We therefore get that

$$II \leqslant C k_x \sum_{j \,\geqslant\, \varepsilon^{-1} a_{k_x}} L(j) j^{-2} \leqslant C k_x L(\varepsilon^{-1} a_{k_x}) (\varepsilon^{-1} a_{k_x})^{-1}\,. \tag{8.33}$$

Using that $L(\varepsilon^{-1} a_{k_x})(a_{k_x})^{-1} \sim k_x^{-1}$ we get that $II \leqslant C\varepsilon$.

Term IV. This is similar to the term II. As for (8.29) we get that setting $j = k - k_x$ so that for the range considered we have for the range considered $|x - b_k| \geqslant \frac{1}{2}\mu j$,

$$\mathbf{P}(S_k = x) = \mathbf{P}\big(S_k - b_k = x - b_k\big) \leqslant C \frac{k}{a_k} L(j) j^{-\alpha} \quad \text{with } j = k - k_x\,.$$

Hence, for $\alpha > 1$ we get as for (8.30)-(8.31) that $IV \leqslant C' \varepsilon^{\alpha-1}$.

In the case $\alpha = 1$, one need to assume additionally that (2.6) holds: using Theorem 2.4 (we have $|x - b_k| \geqslant \frac{1}{2}\mu j \geqslant a_k$ for the range considered), we get that

$$\mathbf{P}(S_k = x) = \mathbf{P}\big(S_k - b_k = x - b_k\big) \leqslant C k L(j) j^{-2} \quad \text{with } j = k - k_x\,.$$

As in (8.33) we then get that $II \leqslant C\varepsilon$.

Term V. Here, using that for $k \geqslant 2k_x$ we have $|x - b_k| \geqslant \frac{1}{2}k$, to get that

$$\mathbf{P}(S_k = x) = \mathbf{P}\big(S_k - b_k = x - b_k\big) \leqslant C \times \begin{cases} \frac{k}{a_k} L(k) k^{-\alpha} & \text{in the general case,} \\ k L(k) k^{-2} & \text{if } \alpha = 1 \text{ and (2.6) holds.} \end{cases}$$

Hence, if $\alpha > 1$, we get that

$$IV \leqslant C \sum_{k \,\geqslant\, 2k_x} L(k) k^{1-\alpha}/a_k \to 0, \qquad \text{as } k_x \to \infty,$$

since $k^{1-\alpha}/a_k$ is regularly varying with index $1 - \alpha - 1/\alpha < -1$.

If $\alpha = 1$, we get that

$$IV \leqslant C \sum_{k \geqslant 2k_x} L(k)k^{-1} = C\ell^\star(k_x) \to 0, \qquad \text{as } k_x \to \infty.$$

Conclusion. Combining all the estimates (and assuming that (2.5)-(2.6) holds in the case $\alpha = 1$), we get that for any fixed η, we may choose $\varepsilon > 0$ sufficiently small so that for x large enough (how large depends on ε) so that

$$(1-\eta)\frac{1}{\mu} \leqslant G(x) \leqslant (1+3\eta)\frac{1}{\mu}\,,$$

8.2.2. *Case $\mu < 0$.* Recall that we assume that $\mathbf{P}(X_1 = x) \sim p\alpha L(x) x^{-(1+\alpha)}$ as $x \to \infty$ —that is (2.5). We fix $\varepsilon > 0$, and split $G(x)$ into three parts,

$$G(x) = \Big(\sum_{k=1}^{\varepsilon x} + \sum_{k=\varepsilon x}^{\varepsilon^{-1}x} + \sum_{k \geqslant \varepsilon^{-1} x} \Big) \mathbf{P}(S_k = x) =: I + II + III\,. \tag{8.34}$$

The main term is the second one in the case $\alpha \in (1,2)$ and the third one in the case $\alpha = 1$.

Term I. We use that for the range considered $x - b_k \geqslant x \geqslant a_k$ to get that from Theorem 2.4

$$\mathbf{P}(S_k = x) = \mathbf{P}(S_k - b_k = x - b_k) \leqslant CkL(x)x^{-(1+\alpha)}\,,$$

so that

$$I \leqslant C \sum_{k=1}^{\varepsilon x} kL(x)x^{-(1+\alpha)} \leqslant C\varepsilon^2 L(x) x^{1-\alpha}\,. \tag{8.35}$$

Term II. We get thanks to Theorem 2.4 that uniformly for $\varepsilon x \leqslant k \leqslant \varepsilon^{-1}x$, and since $b_k \sim \mu k$

$$\mathbf{P}(S_k = x) = \mathbf{P}(S_k - b_k = x + b_k) \sim p\alpha k L(x)(x + |\mu| k)^{-(1+\alpha)} \quad \text{as } x \to \infty\,,$$

where we also used that $L(x + |\mu|k) \sim L(x)$ uniformly for $\varepsilon x \leqslant k \leqslant \varepsilon^{-1}x$. Then, we get that

$$\begin{aligned} II &= (1+o(1)) p\alpha L(x) x^{1-\alpha} \times \frac{1}{x} \sum_{k=\varepsilon x}^{\varepsilon^{-1}} \frac{k}{x} \Big(1 + |\mu| \frac{k}{x}\Big)^{-(1+\alpha)} \\ &= (1+o(1)) p\alpha L(x) x^{1-\alpha} \int_{\varepsilon}^{\varepsilon^{-1}} \frac{u\, du}{(1+|\mu|u)^{(1+\alpha)}}\,, \end{aligned} \tag{8.36}$$

where we used a Riemann sum approximation for the second identity.

Term III. For the last term, we use that $x - b_k \geqslant ck \gg a_k$, so that Theorem 2.4 gives

$$\mathbf{P}(S_k = x) = \mathbf{P}(S_k - b_k = x - b_k) \leqslant CkL(k)k^{-(1+\alpha)}\,.$$

Therefore, if $\alpha > 1$ we get that

$$III \leqslant C \sum_{k \geqslant \varepsilon^{-1}x} L(k)k^{-\alpha} \leqslant C\varepsilon^{\alpha-1} L(x) x^{1-\alpha}\,. \tag{8.37}$$

In the case $\alpha = 1$, we obtain thanks to Theorem 2.4 that

$$\mathbf{P}(S_k = x) \sim pkL(x + |\mu|k)(x + |\mu|k)^{-2}\,.$$

And since $k \geqslant \varepsilon^{-1}x$, we get that for any $\eta > 0$ we can choose $\varepsilon > 0$ small so that for x sufficiently large,

$$II \leqslant (p+\eta)|\mu|^{-2} \sum_{k \geqslant \varepsilon^{-1}x} L(k)k^{-1} = (p+\eta)|\mu|^{-2}\ell^\star(\varepsilon^{-1}x)\,,$$
$$II \geqslant (p-\eta)|\mu|^{-2} \sum_{k \geqslant \varepsilon^{-1}x} L(k)k^{-1}(p-\eta)|\mu|^{-2}\ell^\star(\varepsilon^{-1}x)\,.$$

Conclusion. In the case $\alpha > 1$, the integral in the term II is convergent. Since $\varepsilon > 0$ is arbitrary, we obtain that the terms I and III are negligible compared to the term II. We therefore get that

$$G(x) \overset{x\to\infty}{\sim} \frac{p}{(\alpha-1)|\mu|^2} L(x)x^{1-\alpha}\,,$$

where we computed $\int_0^\infty u(1+|\mu|u)^{-(1+\alpha)}du = \big(|\mu|^2\alpha(\alpha-1)\big)^{-1}$ by integration by part.

In the case $\alpha = 1$, then the term III is dominant, since $\ell^\star(t)$ is a slowly varying with $\ell^\star(t)/L(t) \to +\infty$. We therefore get that, since η is arbitrary

$$G(x) \overset{x\to\infty}{\sim} \frac{p}{|\mu|^2}\ell^\star(t)\,.$$

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References

[1] K. K. Anderson and K. B. Athreya, *A note on conjugate* Π*-variation and a weak limit theorem for the number of renewals*, Stat. Probab. Letters **6**, pp. 151–154, 1988.

[2] Q. Berger, *Local large deviation and renewal theorems for multivariate renewals*, work in progress.

[3] N. H. Bingham, C. M. Goldie and J. L. Teugels, *Regular Variations*, Cambridge University Press, Cambridge, 1987.

[4] A. A. Borovkov, *On the asymptotics of distributions of first-passage times. I.*, Math. Notes, **75** no. 1, pp. 24–39, 2004.

[5] A. A. Borovkov, *On the asymptotics of distributions of first-passage times. II.*, Math. Notes, **75** no. 3, pp. 322–330, 2004.

[6] T. Budd, N. Curien and C. Marzouk, *Infinite random planar maps related to Cauchy processes*, preprint, arXiv:1704.05297

[7] F. Caravenna and R. Doney, *Local Large Deviations and the Strong Renewal Theorem*, preprint, arXiv:1612.07635v1

[8] J. Chover and P. Ney, and S. Wainger, *Function of probability measures*, J. Analyse. Math., **26**, pp. 255–302, 1973.

[9] D. B. H. Cline and T. Hsing, *Large deviation probabilities for sums of random variables with heavy or subexponential tails*, Technical report, Texas A& M Univ, 1994.

[10] D. Denisov, A. B. Dieker and V. Shneer, *Large deviations for random walks under subexponentiality: the big-jump domain*, Ann. Probab., **36** (5), pp. 1946-199, 2008.

[11] R. A. Doney, *On the exact asymptotic behavior of the distribution of ladder epochs*, Stoch. Proc. Appl. **12** pp. 203–214, 1982.

[12] R. A. Doney, *On the asymptotic behaviour of first passage times for transient random walks*, Probab. Theory Relat. Fields, **18**, pp. 239–246, 1989.

[13] R. A. Doney, *One-sided local large deviations and renewal theorems in the case of infinite mean*, Probab. Th. Rel. Fields **107**, pp. 451–465, 1997.

[14] K. B. Erickson, *Strong renewal theorems with infinite mean*, Transaction of the Americ. Math. Soc., **151**, 1970.
[15] W. Feller, *An introduction to probability theory and its applications. Vol. II*, Second edition. John Wiley & Sons, Inc., New York-London-Sydney, 1971.
[16] P. Flajolet and A. Odlyzko *Singularity analysis of generating functions*, SIAM J. Discrete Math. **3**, 2 (1990), 216–240
[17] A. Garsia and J. Lamperti, *A discrete renewal theorem with infinite mean*, Comm. Math. Helv. **37**, pp. 221–234, 1963.
[18] B. V. Gnedenko and A. N. Kolmogorov, *Limit Theorems for Sums of Independent Random Variables*, Addison-Wesley, Cambridge, 1954.
[19] L. de Haan and S.I. Resnick, *Conjugate* Π*-variation and process inversion*, Ann. Probab. **7**, pp. 1028–1035, 1979.
[20] A. V. Nagaev, *Large deviations of sums of independent random variables*, The Annals of Probability, Vol. 7, No. 5, 745–789, 1979.
[21] B. A. Rogozin, *On the distribution of the first ladder moment and height and fluctuations of a random walk*, Theory Probab. Appl., **16** pp. 575-595, 1971.
Trans. Amer. Math. Soc., 82:323–339, 1956.
[22] V. A. Vatutin and V. Wachtel, *Local probabilities for random walks conditioned to stay positive*, Probab. Theory Relat. Fields **143**, pp. 177–217, 2009.
[23] R. Wei, *On the long-range directed polymer model*, J. Stat. Phys, **165** (2), 320–350, 2016.
[24] J. A. Williamson, *Random Walks and Riesz Kernels*, Pacific J. Math., **25** (2), 1968.
[25] V. M. Zolotarev, *One-dimensional stable distributions*, Amer. Math. Soc., Providence RI, 1986.

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