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Abstract

Sufficient conditions are presented for regional stability and nonlinear $\mathcal{L}_{2m}$ gain analysis of linear systems subject to saturation, based on piecewise polynomial Lyapunov functions. The proposed conditions are formulated in terms of convex optimization problems and improve existing results both for the quadratic ($\mathcal{L}_2$ gain) and the polynomial ($\mathcal{L}_{2m}$ gain) cases.

1 Introduction and background

In this paper we use piecewise polynomial functions to study regional properties of linear saturating systems, proposing nontrivial extensions of the results in [1–4]. We consider a linear closed loop comprising saturation/deadzone nonlinearities:

$$\begin{aligned} \dot{x} &= Ax + B_q dz(y) + B_w w \\ y &= C_y x + D_{yq} dz(y) + D_{yw} w \\ z &= C_z x + D_{zq} dz(y) + D_{zw} w, \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^n$, $z \in \mathbb{R}^p$, $y \in \mathbb{R}^d$, $w \in \mathbb{R}^r$, all the matrices are real and have appropriate dimensions, and $dz(y) = y - sat(y)$ is a decentralized symmetric deadzone having limits $\overline{u} = [\overline{u}_1 \quad \dots \quad \overline{u}_d]$. In the following, |x| denotes the Euclidean norm of vector x and $\|w\|_{2m}$ denotes the $\mathcal{L}_{2m}$ norm of signal w. As it is well known in the saturated systems literature (see, e.g., [8,7]), a relevant problem when analyzing the properties of (1) is the following one.

Problem 1 *(Saturated stability/performance analysis) Given system (1), if the origin is locally exponentially stable (namely, matrix A is Hurwitz), determine:*
(i) an (inner) estimate of its region of attraction when $w = 0$;
(ii) an (outer) estimate of the maximum reachable set with $x(0) = 0$ and $\|w\|_{2m} \le \bar{\rho}$, for some $\bar{\rho} > 0$ and $m \in \mathbb{Z}_{\ge 1}$;
(iii) an upper bound $\gamma_{\bar{\rho}}$ of the worst case local $\mathcal{L}_{2m}$ gain, such that, for $x(0) = 0$, $\|z\|_{2m} \le \gamma_{\bar{\rho}}\|w\|_{2m}$, for all w satisfying $\|w\|_{2m} \le \bar{\rho}$, for some $\bar{\rho} > 0$ and $m \in \mathbb{Z}_{\ge 1}$.

In [1], piecewise quadratic Lyapunov functions (solving Problem 1.i) or Storage functions (solving Problem 1.ii, 1.iii) of the form

$$V(x, dz(u(x))) = \begin{bmatrix} x \\ dz(u(x)) \end{bmatrix}^T P \begin{bmatrix} x \\ dz(u(x)) \end{bmatrix} \tag{2}$$

were proposed to analyze stability and $\mathcal{L}_2$ gain performance of (1), based on the piecewise linear solution $x \mapsto u(x)$ of the implicit equation

$$u(x) - D_{yq} dz(u(x)) = C_y x, \tag{3}$$

arising from the algebraic loop of (1) when $w = 0$. Necessary and sufficient conditions for the well posedness of this nonlinear algebraic loop had been already given in [7, Claim 2]. We will assume that matrix D_{yq} satisfies these conditions so that function $x \mapsto u(x)$ is well defined and unique. The nice feature of the approach in [1] is that one does not need to compute explicitly function $x \mapsto u(x)$ because the Lyapunov construction directly follows from the implicit equation (3). In [3] a piecewise polynomial version of the function in (2) was introduced for the computation of $\mathcal{L}_{2m}$-gain bounds. An important aspect of the results in [1,3] is that the computation of *global* gain estimates was obtained with the solution of a *convex* Semidefinite Program (SDP). However one drawback of the corresponding local formulations described in [1,2] is that regional

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properties[1] could only be assessed via *non-convex* optimizations because the proposed conditions are given in terms of bilinear matrix inequality (BMI) constraints. Hence the proposed iterative formulations are only guaranteed to achieve local optima of the presented nonconvex optimizations. Another rather restrictive assumption on the piecewise quadratic/polynomial functions, $x \mapsto V(x, dz(u(x)))$ studied in [1–4] is that they were required to be positive definite with respect to both arguments, disregarding the relation between the deadzone function and the state variable x. For the quadratic case in (2), this assumption implies that the matrix P must satisfy the unnecessary constraint $P > 0$.

By introducing technical improvements related to the storage functions and the local sector inequalities, in this paper we enhance the results of [1–4]. Preliminary results in this direction were given in [2] where a nonconvex formulation for local generalizations of the global results in [1,4] was presented. The proposed improvements are: 1) Relax the constraint on the piecewise polynomial function to be positive on both its arguments; 2) Provide a convex SDP formulation in terms of Sum-of-squares (SOS) programmes allowing one to obtain bounds on the regional $\mathcal{L}_{2m}$-gains of linear saturating systems. These convex results also improve upon the nonlinear $\mathcal{L}_2$ gain estimates computed with the nonconvex conditions in [1], as shown by the example studies in Section 3.

Notation. To easily refer to the quantities in (2) and (3), we omit the explicit dependence on x by introducing the notation $q = dz(y(x)),\ \theta = dz(u(x))$. We also introduce $\dot{u} = du/dt$ and $\zeta(x, w) = d(dz(u))/dt = d\theta/dt$, so that the time derivatives of $u(x(t))$ and $\theta(t) = dz(u(x(t)))$, whenever they exist, satisfy

$$\dot{\theta}_i = \zeta_i(x, w) = \begin{cases} 0, & \text{if } \ |u_i| < \bar{u}_i \\ \dot{u}_i, & \text{if } \ |u_i| > \bar{u}_i. \end{cases} \tag{4}$$

Note that $\zeta_i(x, w)$ may not be defined where $u_i = \pm\bar{u}_i$, hence it is well defined almost everywhere (namely for all (x, w) except for a set of Lebesgue measure zero). We respectively denote by $\mathbb{P}[x]$ and by $\mathbb{P}^{d\times d}_{diag}[x]$ the set of positive polynomials functions and the set of diagonal polynomial matrix on variable $x \in \mathbb{R}^n$. We also denote by $\mathbb{D}^d_{>0} \subset \mathbb{R}^{d\times d}$ and by $\mathbb{D}^d_{\geq 0} \subset \mathbb{R}^{d\times d}$ the sets of real diagonal matrices with positive elements and with nonnegative elements, respectively.

We briefly recall the sector-like conditions of [3] where vector ξ is the generic argument of some diagonal multipliers, and $\overline{U} = \mathrm{diag}\,(\overline{u})$.

[1] Due to structural limitations of bounded stabilization, regional stability/performance properties are fundamental in the study of saturated systems (see, e.g., the local/generalized sector conditions in [6,5], and the related discussions).

Lemma 1 (Sector conditions) *Given any diagonal matrix* $\xi \mapsto \Pi(\xi)$ *such that* $\Pi(\xi) \geq 0$ *for all* ξ, *then*

(I) $\forall \psi \in \mathbb{R}^d,\ \phi \in \mathbb{R}^d, |\overline{U}^{-1}\phi|_\infty \leq 1$

$$dz(\psi)^T \Pi(\xi)(\psi - \phi - dz(\psi)) \geq 0,\ \forall \xi. \tag{5a}$$

(II) the following holds almost everywhere in x and w

$$\begin{cases} \theta^T \Pi(\xi)\,(\dot{u} - \zeta(x, w)) = 0 \\ \zeta(x, w)^T \Pi(\xi)\,(\dot{u} - \zeta(x, w)) = 0. \end{cases} \tag{5b}$$

(III) $\forall \psi_1, \psi_2 \in \mathbb{R}^d$,

$$(dz(\psi_1) - dz(\psi_2))^T\, \Pi(\xi)\,(sat(\psi_1) - sat(\psi_2)) \geq 0. \tag{5c}$$

Lemma 1.(I) states that dz belongs to the sector $[0, I]$. By setting $\Pi(\xi) = \Pi_0(\xi) \geq 0\ \forall \xi$, $\psi = u$ and $\phi = 0$ in (5a) along with (3) one gets

$$\Omega_0(\Pi_0(\xi)) := \theta^T \Pi_0(\xi)\,(C_y x + (D_{yq} - I_d)\theta) \geq 0, \tag{6a}$$

which holds for all ξ and for all x, q, w, θ satisfying (1). By setting $\Pi(\xi) = \Pi_i(\xi) > 0\ \forall\xi$ and $\phi = \rho^{-1}\Pi_i^{-1}(\xi)h_i(x, \theta)$ in (5a) $i = 1, 2$, with ρ a positive scalar, one obtains, respectively

$$\begin{aligned} \Omega_1(\Pi_1(\xi)) := \theta^T\,\Big(&\Pi_1(\xi)\,(C_y x + D_{yq}\theta) \\ &- \rho^{-\frac{1}{2m}} h_1(x, \theta) - \Pi_1(\xi)\theta\Big) \geq 0, \end{aligned} \tag{6b}$$

$$\begin{aligned} \Omega_2(\Pi_2(\xi)) := q^T\,\Big(&\Pi_2(\xi)\,(C_y x + D_{yq} q + D_{yw} w) \\ &- \rho^{-\frac{1}{2m}} h_2(x, \theta) - \Pi_2(\xi) q\Big) \geq 0, \end{aligned} \tag{6c}$$

which hold for all ξ and for all x, θ, q, w satisfying (1) and $|(\rho^{\frac{1}{2m}}\overline{u}_i^{-1}\Pi_i(\xi))^{-1}h_i(x, \theta)|_\infty \leq 1$, $i = 1, 2$. By the definition of u and by (1) and (4), we have $\dot{u} = C_y\dot{x} + D_{yq}\zeta(x, w)$. Then, relations (5b) provide the following equalities, which hold for all ξ and for all x, q, θ, $\zeta = d\theta/dt$, solutions to (1):

$$\Omega_3(\Pi_3(\xi)) := \theta^T \Pi_3(\xi)\,(C_y\dot{x} + (D_{yq} - I_d)\zeta) = 0; \tag{7a}$$

$$\Omega_4(\Pi_4(\xi)) := \zeta^T \Pi_4(\xi)\,(C_y\dot{x} + (D_{yq} - I_d)\zeta) = 0. \tag{7b}$$

Finally, by using (5c) with $\psi_1 = u$ and $\psi_2 = y$, for any $\Pi_5(\xi) \geq 0$, the following holds for all q, θ solutions to (1), and for all w and ξ:

$$\begin{aligned} \Omega_5(\Pi_5(\xi)) := (\theta - q)^T\, \Pi_5(\xi)\,(&(D_{yq} - I_d)\theta \\ &+ (I_d - D_{yq})q - D_{yw} w) \geq 0. \end{aligned} \tag{8}$$

2 Local stability and finite $\mathcal{L}_{2m}$ gain estimates with piecewise polynomial functions

We present a convex formulation for the estimation of the basin of attraction and the local $\mathcal{L}_{2m}$-gain of system (1). The regional analysis requires the following lemma that presents a sufficient condition for the inclusion of level sets of Lyapunov/storage functions

$$\mathcal{E}(V,\rho) := \{x \in \mathbb{R}^n \mid V(x, dz(u(x))) \le \rho\} \quad (9)$$

in the set where the local sector inequalities (6b)-(6c) hold, which according to Lemma 1.(I) are of the form

$$\mathcal{H} := \Big\{x \in \mathbb{R}^n \mid |\rho^{-\frac{1}{2m}} \overline{U}^{-1} \Pi^{-1}(x, dz(u)) h(x, dz(u))|_\infty \le 1\Big\}.$$

Lemma 2 *Given a function $V : \mathbb{R}^{n+d} \to \mathbb{R}$, a function $\Pi : \mathbb{R}^{n+d} \to \mathbb{D}^d_{>0}$, a function $h : \mathbb{R}^{n+d} \to \mathbb{R}^d$, if there exist diagonal matrix functions $\Pi_{aj} : \mathbb{R}^{n+d} \to \mathbb{D}^d_{\ge 0}$, $j = 1, 2$, and a function $\omega : \mathbb{R}^{n+d} \to \mathbb{R}^d$ satisfying the set of inequalities:*

$$\begin{aligned}
&\mathcal{I}(V, h, \Pi, \Pi_{a1}, \Pi_{a2}, \omega) := \Big\{ \\
&\Pi(x,\theta) - (|[\begin{smallmatrix} x \\ \theta \end{smallmatrix}]|)^{2m-2} I_m \ge 0, && (10a)\\
&(|[\begin{smallmatrix} x \\ \theta \end{smallmatrix}]|)^{2m-2} \omega_i(x,\theta) - \Omega_0(\Pi_{a1}(x,\theta)) - 2\nu h_i(x,\theta) + \nu^2 \ge 0, && (10b)\\
&V(x,\theta) - \Omega_0(\Pi_{a2}(x,\theta)) - 2\overline{u}_i^{-2} \nu^{2(m-1)} \omega_i(x,\theta) + \nu^{(2m)} \ge 0, && (10c)\\
&i = 1, \ldots, d, \forall x \in \mathbb{R}^n, \forall \theta \in \mathbb{R}^d, \forall \nu \in \mathbb{R}\Big\},
\end{aligned}$$

then $\mathcal{E}(V,\rho) \subseteq \mathcal{H}$ holds.

Proof. Define $\mathcal{H}_\theta := \Big\{(x,\theta) \mid \overline{u}_i^{-2} \Pi^{-2}_{(i,i)}(x,\theta) h_i^2(x,\theta) \le \rho^{\frac{1}{m}}$, $i = 1 \ldots d\}$, and notice that $\mathcal{H} = \mathcal{H}_\theta\Big|_{\theta = dz(u(x))}$. Also define $\mathcal{W}_\theta := \Big\{(x,\theta) \in \mathbb{R}^{n+d} \mid \overline{u}_i^{-2} \omega_i(x,\theta) \le \rho^{\frac{1}{m}}, i = 1 \ldots d\Big\}$. We respectively have $\mathcal{E}(V(x,\theta),\rho) \subseteq \mathcal{W}_\theta$ and $\mathcal{W}_\theta \subseteq \mathcal{H}_\theta$ if and only if for all x, θ

$$\begin{aligned}
&V(x,\theta) - \overline{u}_i^{-2m} (\omega_i(x,\theta))^m \ge 0, i = 1, \ldots, d, && (11a)\\
&\Pi^2_{(i,i)}(x,\theta)\omega_i(x,\theta) - h_i^2(x,\theta) \ge 0, i = 1, \ldots, d. && (11b)
\end{aligned}$$

We now show that, with $\theta = dz(u(x))$, (10) implies (11) in three successive steps **a)**, **b)**, and **c)** below.

a) We have $\Pi_{(i,i)}(x,\theta)\omega_i(x,\theta) \ge |[\begin{smallmatrix} x \\ \theta \end{smallmatrix}]|^{2(m-1)} \omega_i(x,\theta)$ if (10a) holds. Then, for (11b) it is enough to prove, for each $i = 1, \ldots, d$,

$$(|[\begin{smallmatrix} x \\ \theta \end{smallmatrix}]|)^{2m-2} \omega_i(x,\theta) - h_i^2(x,\theta) \ge 0, \forall\, x, \theta. \quad (12)$$

b) To enforce (12), recall from (6b) that $\Omega_0(\Pi_{a1}) \ge 0$, therefore (10b) implies $(|[\begin{smallmatrix} x \\ \theta \end{smallmatrix}]|)^{2m-2} \omega_i(x,\theta) - 2h_i(x,\theta)\nu + \nu^2 \ge 0$, $\forall (x,\theta,\nu) \in \mathbb{R}^{n+d+1}$. This implies (12) since it corresponds to the particular choice $\nu = h_i(x,\theta)$ in the above inequality.

c) Similarly, for (11a), we recall from (10a) that $\Omega_0(\Pi_{a2}) \ge 0$ therefore (10c) implies $V(x,\theta) - 2\overline{u}_i^{-2} \times \omega_i(x,\theta)\nu^{2(m-1)} + \nu^{2m} \ge 0$ $i = 1, \ldots, d$, $\forall (x,\theta,\nu) \in \mathbb{R}^{n+d+1}$. This implies (11a) that corresponds to the particular choice $\nu = \left(\overline{u}_i^{-2}\omega_i(x,\theta)\right)^{\frac{1}{2}}$ in the above inequality. Thus the set inclusion $\mathcal{E}(V,\rho) \subseteq \mathcal{H}$ holds under (10) and the lemma is proven. ⋄

An important feature of Lemma 2 is that the inequalities (10) are affine in V, h, Π, Π_{aj}, and ω. We use those inequalities to study stability and performance of system (1) with a piecewise polynomial Lyapunov/storage function in the affine formulation below.

Theorem 1 *Consider system (1) with a well-posed algebraic loop* (3) *and consider the set of inequalities*

$$\begin{aligned}
&V(x,\theta) - \Omega_0(\Pi_0(\xi)) - \epsilon|x|^{k_1} \ge 0, && (13a)\\
&-\dot{V}(x,q,\theta,\zeta,w) - \sum_{i=1}^{5}\Omega_i(\Pi_i(\xi)) - \Psi - \epsilon|x|^{k_2} \ge 0, && (13b)\\
&\mathcal{I}(V, h_1, \Pi_1, \Pi_6, \Pi_7, \omega_1),\ \mathcal{I}(V, h_2, \Pi_2, \Pi_8, \Pi_9, \omega_2), && (13c)
\end{aligned}$$

where Ω_i, $i = 0, \ldots, 5$, are given in (6)-(8) and $\dot{V}(x,q,\theta,\zeta,w)$ is a shorthand notation for

$$\left\langle \nabla V(x,\theta), \begin{bmatrix} Ax + B_q q + B_w w \\ \zeta \end{bmatrix} \right\rangle.$$

If there exist a polynomial function $V(x,\theta) \in \mathbb{P}[(x,\theta)]$, polynomial matrices $\Pi_i(\xi) \in \mathbb{P}^{d\times d}_{diag}[\xi]$, $i \in \{0,1,2,5,6,7,8,9\}$, two reals $k_1, k_2 \ge 2$ and a scalar $\epsilon > 0$ such that (13) *holds with*

(1) $\Psi = 0$ and $w = 0$ (hence $\theta = q$ giving $\Omega_5(\Pi_5(\xi)) = 0$ and, $h_1 = h_2$, giving $\Omega_1 + \Omega_2 = 2\Omega_1$);
(2) $\Psi = -(w^T w)^m$;
(3) $\Psi = \gamma_d (z^T z)^m - (w^T w)^m$ with $\gamma_d \in \mathbb{R}_{>0}$;

for all x, θ, q, ζ, w and ξ, then the following properties, respectively, hold:

(1) the origin of system (1) *is locally asymptotically stable and an estimate of its basin of attraction is given by $\{x | V(x, dz(u(x))) \le \rho\}$;*
(2) for each $\rho > 0$, $x(0) = 0$ and $\|w\|_{2m} \le \rho^{\frac{1}{2m}}$, it holds $x(t) \in \{x | V(x, dz(u(x))) \le \rho\}$, for all $t \ge 0$ with $u(x)$ the unique solution to (3);
(3) $x(0) = 0$ implies $\|z\|_{2m} \le \left(\frac{1}{\gamma_d}\right)^{\frac{1}{2m}} \|w\|_{2m}$ $\forall w \in \{w \mid \|w\|_{2m} \le \rho^{\frac{1}{2m}}\}$.

Proof. Let $V(x,\theta), \Pi_i(\xi), \in \mathbb{P}^{d\times d}_{diag}, i \in \{0,1,2,5,6,7,8,9\}$, $\Pi_1(\xi), \Pi_2(\xi) > 0\ \forall\xi$ and k_1, k_2 satisfy (13). Define the piecewise polynomial candidate Lyapunov function $W(x) = V(x, dz(u(x)))$. Function W is continuous and piecewise smooth, and it is non differentiable in a set of measure zero. Based on the reasoning in [9, page 99], it is sufficient to guarantee a uniform decrease, or dissipativity condition on W almost everywhere in $\mathbb{R}^n$. From (13a) we have $W(x) \geq \epsilon|x|^{k_1}$ which proves positive definiteness and radial unboundedness of W. From (13c) we have, from Lemma 2 that $\mathcal{E}(V,\rho) \subseteq \mathcal{H}$ holds with $\Pi = \Pi_i$ and $h = h_i$, $i = 1,2$, which, from Lemma 1, implies that $\Omega_i(\Pi_i(\xi)) \geq 0$, $i = 1,2$ for all $x \in \{x|V(x, dz(u(x))) \leq \rho\}$. Moreover, since the non-negativity of $\Omega_i(\Pi_i(\xi))$, $i = 1,\ldots,5$, established in (6)-(8), holds along any solution of (1), constraint (13b) implies the following, respectively, in the three cases.

1) With $w = 0$, $q = \theta$ and $\Psi = 0$ we have $\dot{W}(x) \leq -\epsilon|x|^{k_2}$ for almost all x. Global asymptotic stability of the origin then follows from standard Lyapunov derivations.
2) With $\Psi = -(w^T w)^m$, set $\dot{W}(x,w) := \langle \nabla_x W(x), Ax + B_q dz(y(x,w)) + B_w w\rangle$. Then $\dot{W}(x,w) - (w^T w)^m \leq -\epsilon|x|^{k_2}$ for almost all x, w, which implies $\dot{W}(x,w) \leq (w^T w)^m$ for almost all x, w. Considering $x(0) = 0$ and integrating both sides of the previous inequality along any solution x to (1) under an input w with $\mathcal{L}_{2m}$ norm bounded by $\rho^{\frac{1}{2m}}$, we obtain $W(x(t)) \leq \int_0^t (w(\tau)^T w(\tau))^m d\tau = (\|w\|_{2m})^{2m} \leq \rho$ for all $t \geq 0$.
3) With $\Psi = \gamma_d(z^T z)^m - (w^T w)^m$, we have $\dot{W}(x,w) + \gamma_d(z^T z)^m - (w^T w)^m \leq -\epsilon|x|^{k_2}$ for almost all x, w which implies $\dot{W}(x,w) + \gamma_d(z^T z)^m \leq (w^T w)^m$ for almost all x, w. Considering $x(0) = 0$ and $w \in \mathcal{L}_{2m}$, integrating both sides from 0 to t and taking the limit as $t \to \infty$, we have that $\|z\|_{2m} \leq \left(\frac{1}{\gamma_d}\right)^{\frac{1}{2m}} \|w\|_{2m}$. ◇

The key difference of the above result with respect to [2, Theorem 3.2] is that the inequalities (13) are affine on the problem variables whereas the inequalities describing the inclusion condition and the regional sector inequalities used in [2, Theorem 3.2] are not affine. This affine dependence on the variables allows for the convex optimization formulation described in the next section. Note that the choice of k_1 and k_2 in (13) has to be compatible with m in items 2 and 3 and the degree of the polynomial V. For instance, in Example 2 in the next section it is sufficient to choose $k_1 = k_2 = 2$ for $m = 1$.

3 SOS relaxations and Numerical Results

We compute estimates of the $\mathcal{L}_{2m}$ gain solving the inequalities in item 3 of Theorem 1. For fixed values of ρ the SOS relaxation of the inequalities (13) (analogous to the relaxations in [2, Prop. 3.2]) yield polynomial constraints that are affine on the decision variables, which are convex constraints. We solve the associated SDP (semi-definite program) by maximizing the value of γ_d, which implies the minimization of the upper-bound on the gain for bounded values of the disturbance. When compared to the constraints of [2, Prop. 3.2], the inequalities (13) introduce variables, Π_i, $i \in \{0,6,7,8,9\}$, thus increasing the number of variables of the underlying SDPs to be solved. The algorithm proposed to build the gain curve in [2] requires a line search to compute the induced gain for each fixed value of ρ. Conversely, the approach proposed here simplifies the computation for a fixed ρ, thanks to the inclusion conditions of Lemma 2, requiring the solution of a single SDP (a comparison with the results of [2] is reported in Example 3 below).

The following example provides estimates of the global $\mathcal{L}_{2m}$ gain, which are obtained by considering the SOS relaxation of inequalities (13a)-(13b) and by imposing $h_1 = h_2 = 0$.

Example 1 Consider the system from [3, Example 1], for which no $\mathcal{L}_2$-gain could be computed when imposing $P = \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^T & P_{22} \end{bmatrix} > 0$ in the quadratic function $V(x,\theta) = \begin{bmatrix} x^T & \theta^T \end{bmatrix} P \begin{bmatrix} x \\ \theta \end{bmatrix}$. The proposed inequality (13a) relaxes the conditions imposed on the matrix P. The bound on the global $\mathcal{L}_2$-gain turns out to be $\gamma_2 = 2.71$ and the matrix P is defined by

$$P_{11} = \begin{bmatrix} 2.1406 & -0.2324 & -0.4720 \\ -0.2324 & 0.9320 & -0.5652 \\ -0.4720 & -0.5652 & 1.2056 \end{bmatrix}; P_{12} = \begin{bmatrix} -0.1112 & -0.0588 \\ 0.6144 & 0.2306 \\ -0.6267 & -0.4603 \end{bmatrix};$$

$$P_{22} = \begin{bmatrix} -0.0064 & 0.0235 \\ 0.0235 & -0.1412 \end{bmatrix}.$$

Note that the P_{22} block is negative definite. We also compute a bound on the global $\mathcal{L}_4$-gain, $\gamma_4 = 2.54$, which significantly improves the bound $\gamma_4 = 105$ computed in [3].

Example 2 Consider the linear saturating system with an algebraic loop studied in [1, Example 2], where the proposed piecewise quadratic function structure was compared to previous quadratic and non-quadratic LF structures from the literature. However, in [1] instead of (13a), the LF is required to satisfy $V(x,\theta) - \epsilon|\begin{bmatrix} x \\ \theta \end{bmatrix}|^2 \geq 0$. The example showed performance curves for $D_{yq} \in \left\{D_{yq1} = \begin{bmatrix} -2 & -1 \\ -2 & -4 \end{bmatrix}, D_{yq2} = \begin{bmatrix} -2 & -1.3 \\ -2.3 & -4 \end{bmatrix}, D_{yq3} = \begin{bmatrix} -2 & -1 \\ -2 & -4 \end{bmatrix}\right\}$. For all these cases, the solutions of [1] outperformed the bounds obtained from previous results in the literature. Table 1 reports a comparison between the proposed conditions for the global $\mathcal{L}_2$ performance and the results in [1]. Figure 1 depicts the nonlinear $\mathcal{L}_2$ gain curve as a function of the norms of the disturbances for the case $D_{yq} = D_{yq1}$. Similar curves were obtained for D_{yq2} and D_{yq3}.

Example 3 In [2], a coordinate-wise approach has been used to compute bounds on the nonlinear $\mathcal{L}_4$-gain for

Table 1
Global bounds from [1] and from this paper.

Lyapunov function	D_{yq1}	D_{yq2}	D_{yq3}
PW-quadratic in [1]	15.13	17.19	25.86
PW-quadratic satisfying (13a)	12.04	12.39	17.79

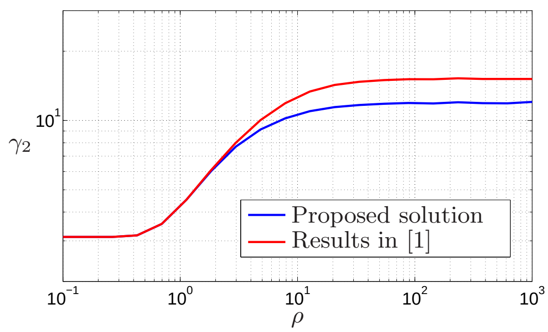

Fig. 1. Nonlinear $\mathcal{L}_2$ gain estimates with D_{yq1} obtained by solving inequalities (13) (blue line) and from [1] (red line).

the linear system described in [2, Example 5.2] by solving suitable BMI conditions. For the the same system, we have solved the LMI conditions associated to the SOS relaxations of the inequalities in (13) to compute bounds on the $\mathcal{L}_4$-gain curve. The points on this curve were obtained by selecting different values of the scalar ρ that appear in terms Ω_1 and Ω_2, which establishes a bound on the disturbance and also establish bounds of the disturbance as $\|w\|_4 \leq \rho^{\frac{1}{4}}$. Note that, from item 2 of Theorem 1, ρ defines upper-bounds for the $\mathcal{L}_{2m}$ bound of the disturbance. The gain curves are depicted in Figure 2, highlighting the improvement of the obtained results with respect to the results reported in [2].

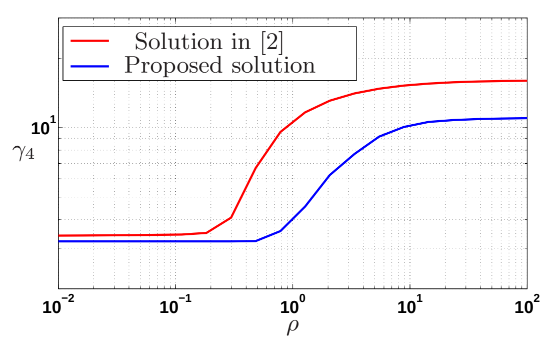

Fig. 2. Upper bounds of the nonlinear $\mathcal{L}_4$-gain for the system studied in [2, Example 5.2] obtained by solving inequalities (13) (blue line) and obtained with the coordinate-wise search to solve the BMIs in [2] (red line).

4 Conclusion

We presented convex conditions for regional stability and nonlinear $\mathcal{L}_{2m}$ gain analysis of linear systems subject to saturation. In addition to providing polynomial tools for stability/performance analysis, our results extend and improve the existing results also for the quadratic case ($\mathcal{L}_2$ gain). Including the treatment of parametric uncertainties is a straightforward extension of the results presented in this paper, by following the approach for the global case presented in [3]. We are currently investigating the use of piecewise quadratic functions for the nonlinear state feedback design with algebraic loops.

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