Mixed Deterministic and Random Optimal Control of Linear Stochastic Systems with Quadratic Costs

Ying Hu, Shanjian Tang

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August 22, 2017

Abstract

In this paper, we consider the mixed optimal control of a linear stochastic system with a quadratic cost functional, with two controllers—one can choose only deterministic time functions, called the deterministic controller, while the other can choose adapted random processes, called the random controller. The optimal control is shown to exist under suitable assumptions. The optimal control is characterized via a system of fully coupled forward-backward stochastic differential equations (FBSDEs) of mean-field type. We solve the FBSDEs via solutions of two (but decoupled) Riccati equations, and give the respective optimal feedback law for both deterministic and random controllers, using solutions of both Riccati equations. The optimal state satisfies a linear stochastic differential equation (SDE) of mean-field type. Both the singular and infinite time-horizonal cases are also addressed.

AMS subject classification. 93E20

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Keywords. Stochastic LQ, differential/algebraic Riccati equation, mixed deterministic and random control, singular LQ, infinite-horizon

1 Introduction and formulation of the problem

Let $T > 0$ be given and fixed. Denote by $\mathcal{S}^n$ the totality of $n \times n$ symmetric matrices, and by $\mathcal{S}^n_+$ its subset of all $n \times n$ nonnegative matrices. We mean by an $n \times n$ matrix $S \geq 0$ that $S \in \mathcal{S}^n_+$ and by a matrix $S > 0$ that S is positive definite. For a matrix-valued function $R : [0, T] \to \mathcal{S}^n$, we mean by $R \gg 0$ that $R(t)$ is uniformly positive, i.e. there is a positive real number α such that $R(t) \geq \alpha I$ for any $t \in [0, T]$.

In this paper, we consider the following linear control stochastic differential equation (SDE)

$$(1.1)\ \ dX_s = [A_s X_s + B^1_s u^1_s + B^2_s u^2_s]ds + \sum_{j=1}^{d} [C^j_s X_s + D^{1j}_s u^1_s + D^{2j}_s u^2_s]dW^j_s, \quad s > 0; \quad X_0 = x_0,$$

with the following quadratic cost functional

$$(1.2) \qquad J(u) \triangleq \frac{1}{2}\mathbb{E}\int_0^T \left[\langle Q_s X_s, X_s\rangle + \langle R^1_s u^1_s, u^1_s\rangle + \langle R^2_s u^2_s, u^2_s\rangle\right] ds + \frac{1}{2}\mathbb{E}[\langle G X_T, X_T\rangle].$$

Here, $(W_t)_{0\leq t\leq T} = (W^1_t, \cdots, W^d_t)_{0\leq t\leq T}$ is a d-dimensional Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Denote by $(\mathcal{F}_t)$ the augmented filtration generated by (W_t). A, B^1, B^2, C^j, D^{1j} and D^{2j} are all bounded Borel measurable functions from $[0, T]$ to $\mathbb{R}^{n\times n}, \mathbb{R}^{n\times l_1}, \mathbb{R}^{n\times l_2}, \mathbb{R}^{n\times n}, \mathbb{R}^{n\times l_1}$, and $\mathbb{R}^{n\times l_2}$, respectively. Q, R^1, and R^2 are nonnegative definite, and they are all essentially bounded measurable functions on $[0, T]$ with values in $\mathbb{S}^n, \mathbb{S}^{l_1}$, and $\mathbb{S}^{l_2}$, respectively. In the first four sections, R^1 and R^2 are further assumed to be positive definite. $G \in \mathbb{S}^n$ is positive semi-definite. For a The process $u \in L^2_{\mathcal{F}}(0,\, T;\, \mathbb{R}^l)$ is the control, and $X \in L^2_{\mathcal{F}}(\Omega;\, C(0,\, T;\, \mathbb{R}^n))$ is the corresponding state process with initial value $x_0 \in \mathbb{R}^n$.

We will use the following notation: $\mathbb{S}^l$: the set of symmetric $l\times l$ real matrices. $L^2_{\mathcal{G}}(\Omega;\, \mathbb{R}^l)$ the set of random variables $\xi : (\Omega, \mathcal{G}) \to (\mathbb{R}^l, \mathcal{B}(\mathbb{R}^l))$ with $\mathbb{E}\left[|\xi|^2\right] < +\infty$. $L^\infty_{\mathcal{G}}(\Omega;\, \mathbb{R}^l)$ is the set of essentially bounded random variables $\xi : (\Omega, \mathcal{G}) \to (\mathbb{R}^l, \mathcal{B}(\mathbb{R}^l))$. $L^2_{\mathcal{F}}(t,\, T;\, \mathbb{R}^l)$ is the set of $\{\mathcal{F}_s\}_{s\in[t,T]}$-adapted processes $f = \{f_s : t \leq s \leq T\}$ with $\mathbb{E}\left[\int_t^T |f_s|^2\, ds\right] < \infty$, and denoted by $L^2(t, T; \mathbb{R}^l)$ if the underlying filtration is the trivial one. $L^\infty_{\mathcal{F}}(t,\, T;\, \mathbb{R}^l)$:

the set of essentially bounded $\{\mathcal{F}_s\}_{s\in[t,T]}$-adapted processes. $L^2_{\mathcal{F}}(\Omega;\, C(t,\, T;\, \mathbb{R}^l))$: the set of continuous $\{\mathcal{F}_t\}_{s\in[t,T]}$-adapted processes $f=\{f_s: t\le s\le T\}$ with $\mathbb{E}\left[\sup_{s\in[t,T]}|f_s|^2\right]<\infty$. We will often use vectors and matrices in this paper, where all vectors are column vectors. For a matrix M, M' is its transpose, and $|M|=\sqrt{\sum_{i,j}m_{ij}^2}$ is the Frobenius norm. Define

$$B:=(B^1,B^2),\quad D:=(D^1,D^2),\quad R:=\mathrm{diag}(R^1,R^2),\quad u:=((u^1)',(u^2)')'; \tag{1.3}$$

and for a matrix K with suitable dimensions and $(t,x,u)\in[0,T]\times\mathbb{R}^n\times\mathbb{R}^l$,

$$(C_tx+D_tu)dW_t=\sum_{j=1}^d(C_t^jx+D_t^{1j}u^1+D_t^{2j}u^2)dW_t^j;\quad C_t'K:=\sum_{j=1}^d(C_t^j)'K^j;$$
$$D_tKD_t:=\sum_{j=1}^d(D_t^j)'KD_t^j,\quad C_t'KD_t:=\sum_{j=1}^d(C_t^j)'KD_t^j,\quad C_t'KC_t:=\sum_{j=1}^d(C_t^j)'KC_t^j.$$

If both u^1 and u^2 are adapted to the natural filtration of the underlying Brownian motion W (i. e., $u^i\in U^i_{\mathrm{ad}}=\mathscr{L}^2_{\mathscr{F}}(0,T;\mathbb{R}^{l_i})$ for $i=1,2$), it is well-known that the optimal control exists and can be synthesized into the following feedback of the state:

$$u_t=\left(R_t+D_t'K_tD_t\right)^{-1}\left(K_tB_t+C_tK_tD_t\right)'X_t,\quad t\in[0,T]. \tag{1.4}$$

Here K solves the following Riccati equation:

$$\begin{aligned}\frac{d}{ds}K_s &= A_s'K_s+K_sA_s+C_s'K_sC_s+Q_s\\ &\quad -(K_sB_s+C_s'K_sD_s)(R_s+D_s'K_sD_s)^{-1}(K_sB_s+C_s'K_sD_s)',\quad s\in[0,T];\\ K_T &= G.\end{aligned} \tag{1.5}$$

See Wonham [10], Haussmann [5], Bismut [2, 3], Peng [7], and Tang [8] for more details on the general Riccati equation arising from linear quadratic optimal stochastic control with both state- and control-dependent noises and deterministic coefficients.

In this paper, we consider the following situation: there are two controllers called the deterministic controller and the random controller: the former can impose a deterministic action u^1 only, i.e., $u^1\in U^1_{\mathrm{ad}}=L^2(0,T;\mathbb{R}^{l_1})$; and the latter can impose a random action u^2, more precisely $u^2\in U^2_{\mathrm{ad}}=L^2_{\mathcal{F}}(0,T;\mathbb{R}^{l_2})$. Firstly, we apply the conventional variational technique to characterize the optimal control via a system of fully coupled forward-backward stochastic differential equations (FBSDEs) of mean-field type. Then we give solution of the FBSDEs with two (but decoupled) Riccati equations, and derive the respective optimal feedback law for both deterministic and random controllers, using solutions of both Riccati equations. Existence and uniqueness is given to both Riccati equations. The optimal state is shown to satisfy a linear stochastic differential equation (SDE) of mean-field type. Both the singular and infinite time-horizonal cases are also addressed.

The rest of the paper is organized as follows. In Section 2, we give the necessary and sufficient condition of the mixed optimal Controls via a system of FBSDEs. In Section 3, we synthesize the mixed optimal control into linear closed forms of the optimal state. We derive two (but decoupled) Riccati equations, and study their solvability. We state our main result. In Section 4, we address some particular cases. In Section 5, we discuss singular linear quadratic control cases. Finally in Section 6, we discuss the infinite time-horizonal case.

2 Necessary and sufficient condition of mixed optimal Controls

Let u^* be a fixed control and X^* be the corresponding state process. For any $t \in [0, T)$, define the processes $(p(\cdot), (k^j(\cdot))_{j=1,\cdots,d}) \in L^2_{\mathcal{F}}(0, T; \mathbb{R}^n) \times (L^2_{\mathcal{F}}(0, T; \mathbb{R}^n))^d$ as the unique solution to

$$(2.1) \qquad \begin{cases} dp(s) &= -[A'_s p(s) + C'_s k(s) + Q_s X^*_s]\, ds + k'(s) dW_s, \quad s \in [0, T]; \\ p(T) &= G X^*_T. \end{cases}$$

The following necessary and sufficient condition can be proved in a straightforward way.

Theorem 2.1 *Let u^* be the optimal control, and X^* be the corresponding solution. Then there exists a pair of adjoint processes (p, k) satisfying the BSDE* (2.1)*. Moreover, the following optimality conditions hold true:*

$$(2.2) \qquad \mathbb{E}[(B^1_s)' p(s) + (D^1_s)' k(s) + R^1_s u^{1*}_s] = 0,$$

$$(2.3) \qquad (B^2_s)' p(s) + (D^2_s)' k(s) + R^2_s u^{2*}_s = 0;$$

and they are also sufficient for u^* to be optimal.

Proof. Using the convex perturbation, we obtain in a straightforward way the equivalent condition of the optimal control u^*:

$$\mathbb{E}\int_0^T \langle (B_s^i)'p(s) + (D_s^i)'k(s) + R_s^i u_s^{i*}, u_s^i\rangle ds \;=\; 0, \quad \forall u^i \in U_{\mathrm{ad}}^i; \quad i = 1, 2. \tag{2.4}$$

The sufficient condition can be proved in a standard way. □

3 Synthesis of the mixed optimal control

3.1 Ansatz

Define

$$\overline{X} := \mathbb{E}[X], \quad \widetilde{X} := X - \overline{X}; \quad \overline{u^2} := \mathbb{E}[u^2], \quad \widetilde{u^2} := u^2 - \overline{u^2}. \tag{3.1}$$

We expect a feedback of the following form

$$p_s = P_1(s)\widetilde{X}_s + P_2(s)\bar{X}_s. \tag{3.2}$$

Applying Ito's formula, we have

$$\begin{aligned} dp \;=\; & P_1'\widetilde{X}ds + P_1[A_s\widetilde{X} + B^2\widetilde{u^{2*}}]ds + P_1[C_sX_s + D_su_s^*]dW_s \\ & + P_2'\bar{X}_sds + P_2[A_s\bar{X}_s + B_s^1u_s^{1*} + B_s^2\overline{u_s^{2*}}]\,ds. \end{aligned} \tag{3.3}$$

Hence

$$k^j(s) = P_1(s)(C_s^jX_s + D_s^ju_s^*). \tag{3.4}$$

Define for $i = 1, 2$,

$$\Lambda_i(S) \;:=\; R^i + (D^i)'SD^i, \quad S \in \mathbb{S}^n; \tag{3.5}$$

$$\widehat{\Lambda}(S) \;:=\; \Lambda_1(S) - (D^1)'SD^2\Lambda_2^{-1}(S)(D^2)'SD^1, \quad S \in \mathbb{S}^n; \tag{3.6}$$

and

$$\Theta_i := (B^2)'P_i + (D^2)'P_1C. \tag{3.7}$$

Plugging equations (3.2) and (3.4) into the optimality conditions (2.2) and (2.3):

$$(B_s^1)'P_2(s)\bar{X}_s + (D_s^1)'P_1(s)(C_s\overline{X}_s + D_s^1 u_s^{1*} + D_s^2\overline{u_s^{2*}}) + R_s^1 u_s^{1*} = 0, \tag{3.8}$$

$$\begin{aligned} &(B_s^2)'[P_1(s)\widetilde{X}_s + P_2(s)\bar{X}_s] \\ &+(D_s^2)'P_1(s)[C_s(\widetilde{X}_s + \overline{X}_s) + D_s^1 u_s^{1*} + D_s^{2j} u_s^{2*}] + R_s^2 u_s^{2*} = 0; \end{aligned} \tag{3.9}$$

From the last equality, we have

$$u^{2*} = -\Lambda_2^{-1}(P_1)[\Theta_1\widetilde{X} + \Theta_2\overline{X} + (D^2)'P_1 D^1 u^{1*}] \tag{3.10}$$

and consequently

$$\overline{u^{2*}} = -\Lambda_2^{-1}(P_1)[\Theta_2\overline{X} + D_2' P_1 D_1 u^{1*}]. \tag{3.11}$$

In view of (3.8), we have

$$\begin{aligned} &(B_s^1)'P_2(s)\bar{X}_s + (D_s^1)'P_1(s)C_s\overline{X}_s \\ &+(D_s^1)'P_1(s)D_s^2\overline{u_s^{2*}} + \Lambda_1(P_1(s))u_s^{1*} = 0 \end{aligned} \tag{3.12}$$

and therefore,

$$\begin{aligned} &\Lambda_1(P_1(s))u_s^{1*} + (B_s^1)'P_2(s)\bar{X}_s + (D_s^1)'P_1(s)C_s\overline{X}_s \\ &-(D_s^1)'P_1(s)D_s^2\Lambda_2^{-1}(P_1)[\Theta_2(s)\overline{X}_s + (D_s^2)'P_1(s)D_s^1 u_s^{1*}] = 0 \end{aligned} \tag{3.13}$$

or equivalently

$$\begin{aligned} &[\Lambda^1(P_1) - (D^1)'P_1 D^2\Lambda_2^{-1}(P_1)(D^2)'P_1 D^1]u^{1*} \\ &= -[(B^1)'P_2 + (D^1)'P_1 C - (D^1)'P_1 D^2\Lambda_2^{-1}(P_1)\Theta_2]\overline{X}_s. \end{aligned} \tag{3.14}$$

We have

$$u^1 = M^1\overline{X}, \quad u^2 = M^2\widetilde{X} + M^3\overline{X} \tag{3.15}$$

where

$$\begin{aligned} M^1 &:= -[\Lambda_1(P_1) - (D^1)'P_1 D^2\Lambda_2^{-1}(P_1)(D^2)'P_1 D^1]^{-1} \\ &\quad\times[(B^1)'P_2 + (D^1)'P_1 C - (D^1)'P_1 D^2\Lambda_2^{-1}(P_1)\Theta_2], \end{aligned} \tag{3.16}$$

$$M^2 := -\Lambda_2^{-1}(P_1)\Theta_1, \tag{3.17}$$

$$M^3 := -\Lambda_2^{-1}(P_1)[\Theta_2 + (D^2)'P_1 D^1 M^1]. \tag{3.18}$$

In view of (3.3) and (2.1), we have

$$
\begin{aligned}
dp &= P_1'\widetilde{X}ds + P_1[A_s\widetilde{X} + B^2M^2\widetilde{X}]ds + k_s'dW_s \\
&\quad +P_2'\bar{X}_sds + P_2[A_s\bar{X}_s + B_s^1M^1\overline{X}_s + B_s^2M_s^3\overline{X}_s]\,ds \qquad (3.19)\\
&= -\Big\{A_s'(P_1(s)\widetilde{X}_s + P_2(s)\bar{X}_s) + (Q_s + C_s'P_1(s)C_s)(\overline{X}_s + \widetilde{X}_s) \qquad (3.20)\\
&\quad +C_s'P_1(s)[D_s^1M_s^1\overline{X}_s + D_s^2(M^2\widetilde{X} + M_s^3\overline{X}_s)]\Big\}ds \\
&\quad +k_s'dW_s.
\end{aligned}
$$

We expect the following system for (P_1, P_2):

$$
\begin{aligned}
&P_1' + P_1A + A'P_1 + C'P_1C + Q \\
&-(P_1B^2 + C'P_1D^2)\Lambda_2^{-1}(P_1)(P_1B^2 + C'P_1D^2)' = 0, \\
&P_1(T) = G
\end{aligned}
\tag{3.21}
$$

and

$$
\begin{aligned}
P_2' + P_2A + A'P_2 + C'P_1C + Q + C'P_1D^1M^1 + C'P_1D^2M^3 \\
+P_2B^1M^1 + P_2B^2M^3 = 0.
\end{aligned}
\tag{3.22}
$$

The last equation can be rewritten into the following one:

$$
P_2' + P_2\widetilde{A}(P_1) + \widetilde{A}'(P_1)P_2 + \widetilde{Q}(P_1) - P_2\mathcal{N}(P_1)P_2 = 0, \quad P_2(T) = G \tag{3.23}
$$

where for $S \in \mathbb{S}_+^n$,

$$
\begin{aligned}
U(S) &:= S - SD^2\Lambda_2^{-1}(S)\left(D^2\right)'S; \\
\widetilde{Q}(S) &:= Q + C'U(S)C - C'U(S)D^1\widehat{\Lambda}^{-1}(S)(D^1)'U(S)C, \\
\widetilde{A}(S) &:= A - B^2\Lambda_2^{-1}(S)\left(D^2\right)'SC \\
&\quad - \left[B^1 - B^2\Lambda_2^{-1}(S)\left(D^2\right)'SD^1\right]\widehat{\Lambda}^{-1}(S)\left(D^1\right)'U(S)C, \\
\mathcal{N}(S) &:= B^2\Lambda_2^{-1}(S)\left(B^2\right)' \\
&\quad + \left[B^1 - B^2\Lambda_2^{-1}(S)\left(D^2\right)'SD^1\right]\widehat{\Lambda}^{-1}(S)\left[B^1 - B^2\Lambda_2^{-1}(S)\left(D^2\right)'SD^1\right]'.
\end{aligned}
$$

We have the following representation for M^1 and M^2:

$$
\begin{aligned}
M^1 &= -\widehat{\Lambda}^{-1}(P_1)\left[(B^1)'P_1 + (D^1)'U(P_1)C - (D^1)'P_1D^1\Lambda_2^{-1}(P_1)(B^2)'P_2\right], \\
M^3 &= -\Lambda_2^{-1}(P_1)\Big\{(B^2)'P_2 + (D^2)'P_1C \\
&\quad -(D^2)'P_1D^1\widehat{\Lambda}^{-1}(P_1)\left[(B^1)'P_1 + (D^1)'U(P_1)C - (D^1)'P_1D^1\Lambda_2^{-1}(P_1)(B^2)'P_2\right]\Big\}.
\end{aligned}
\tag{3.24}
$$

Lemma 3.1 *For $S \in \mathbb{S}^n_+$, we have $\widetilde{Q}(S) \geq 0$.*

Proof. First, we show that $U(S) \geq 0$. In fact, we have (setting $\widehat{D^2} := S^{1/2} D^2$)

$$
\begin{aligned}
U(S) &= S - S^{1/2}\widehat{D^2}\left[R^2 + \left(\widehat{D^2}\right)'\widehat{D^2}\right]^{-1}\left(\widehat{D^2}\right)' S^{1/2} && (3.25)\\
&\geq S - S^{1/2} I S^{1/2} = 0. && (3.26)
\end{aligned}
$$

Here we have used the following well-known matrix inequality:

$$
D(R + D'FD)^{-1}D' \leq F^{-1} \tag{3.27}
$$

for $D \in \mathbb{R}^{n\times m}$, and positive matrices $F \in \mathbb{S}^n$ and $R \in \mathbb{S}^m$.

Using again the inequality (3.27), we have (setting $\widehat{D^1} := [U(S)]^{1/2} D^1$)

$$
\begin{aligned}
\widetilde{Q}(S) &= Q + C'U(S)C \\
&\quad - C'U(S)D^1\left[R^1 + (D^1)'SD^1 - (D^1)'SD^2\Lambda_2^{-1}(S)(D^2)'SD^1\right]^{-1}(D^1)'U(S)C \\
&= Q + C'U(S)C - C'U(S)D^1\left[R^1 + (D^1)'U(S)D^1\right]^{-1}(D^1)'U(S)C \\
&= Q + C'U(S)C - C'[U(S)]^{1/2}\widehat{D^1}\left[R^1 + \left(\widehat{D^1}\right)'\widehat{D^1}\right]^{-1}\left(\widehat{D^1}\right)'[U(S)]^{1/2}C \\
&\geq Q + C'U(S)C - C'[U(S)]^{1/2} I [U(S)]^{1/2} C \geq 0. && (3.28)
\end{aligned}
$$

The proof is complete. □

3.2 Existence and uniqueness of optimal control

Theorem 3.2 *Assume that $R^1 \gg 0$ and $R^2 \gg 0$. Riccati equations* (3.21) *and* (3.23) *have unique nonnegative solutions P_1 and P_2. The optimal control is unique and has the following feedback form:*

$$
u^{1*} = M^1\overline{X}, \quad u^{2*} = M^2\widetilde{X}^* + M^3\overline{X}^* = M^2X^* + (M^3 - M^2)\overline{X}^*. \tag{3.29}
$$

Define $\overline{X}^*_t := \mathbb{E}[X_t]$ and $\widetilde{X}^*_t := X^*_t - \overline{X}^*_t$ The optimal feedback system is given by

$$
\begin{aligned}
X_t &= x + \int_0^t [(A + B^2M^2)X_s + (B^1M^1 - B^2M^2 + B^2M^3)\overline{X}_s]\, ds \\
&\quad + \int_0^t [(C + D^2M^2)X_s + (D^1M^1 - D^2M^2 + D^2M^3)\overline{X}_s]\, dW_s, \quad t \geq 0. && (3.30)
\end{aligned}
$$

It is a mean-field stochastic differential equation. The expected optimal state $\overline{X}_t^*$ is governed by the following ordinary differential equation:

$$\overline{X}_t = x + \int_0^t (A + B^1M^1 + B^2M^3)\overline{X}_s\,ds, \quad t \geq 0; \tag{3.31}$$

and $\widetilde{X}_t^*$ is governed by the following stochastic differential equation:

$$\begin{aligned} \widetilde{X}_t &= \int_0^t (A + B^2M^2)\widetilde{X}_s\,ds \\ &\quad + \int_0^t [(C + D^2M^2)\widetilde{X}_s + (C + D^1M^1 + D^2M^3)\overline{X}_s]\,dW_s, \quad t \geq 0. \end{aligned} \tag{3.32}$$

The optimal value is given by

$$J(u^*) = \langle P_2(0)X(0), X(0)\rangle. \tag{3.33}$$

Proof. Define

$$u^{1*} := M^1\bar{X}^*, \quad u^{2*} := M^2\widetilde{X^*} + M^3\bar{X}^* \tag{3.34}$$

and

$$p^* = P_1(s)\widetilde{X^*} + P_2\bar{X}^*, \tag{3.35}$$

$$k^* = P_1[CX^* + Du^*]. \tag{3.36}$$

We can check that (X^*, u^*, p^*, k^*) is the solution to FBSDE, satisfying the optimality condition. Hence, u^* is optimal.

If (X, u, p, k) is alternative solution to FBSDE, satisfying the optimality condition, then setting:

$$\delta p = p - (P_1\widetilde{X} + P_2\bar{X}), \quad \delta k = k - P_1(CX + Du).$$

Substituting

$$p = \delta p + P_1\widetilde{X} + P_2\bar{X}, \quad k = \delta k + P_1(CX + Du)$$

into (2.2) and (2.3), we have

$$\mathbb{E}\left\{(B^1)'(\delta p + P_1\widetilde{X} + P_2\bar{X}) + (D^1)'[\delta k + P_1(CX + Du)] + R^1u^1\right\} = 0, \tag{3.37}$$

$$(B^2)'(\delta p + P_1\widetilde{X} + P_2\bar{X}) + (D^2)'(\delta k + P_1(CX + Du)) + R^2u_s^2 = 0. \tag{3.38}$$

From the last equation, we have

$$\overline{u^2} = -\Lambda_2^{-1}(P_1)[(B^2)'\overline{\delta p} + (D^2)'\overline{\delta k} + \Theta_2\overline{X} + D_2'P_1D_1u^{1*}]. \tag{3.39}$$

In view of (3.37) and (3.38), we have from the last equation,

$$u^1 = L^1\overline{\delta p} + L^2\overline{\delta k} + M^1\overline{X} \tag{3.40}$$

and

$$u^2 = L^3\delta p + L^4\delta k + L^5\overline{\delta p} + L^6\overline{\delta k} + M^2\widetilde{X} + M^3\overline{X}$$

where

$$\begin{aligned}
L^1 &:= -\widehat{\Lambda}^{-1}(P_1)[(B^1)' - (D^1)'P_1D^2\Lambda_2^{-1}(P_1)(B^2)'],\\
L^2 &:= -\widehat{\Lambda}^{-1}(P_1)[(D^1)' - (D^1)'P_1D^2\Lambda_2^{-1}(P_1)(D^2)'],\\
L^3 &:= -\Lambda_2^{-1}(P_1)(B^2)',\\
L^4 &:= -\Lambda_2^{-1}(P_1)(D^2)',\\
L^5 &:= -\Lambda_2^{-1}(P_1)(D^2)'P_1D^1L^1,\\
L^6 &:= -\Lambda_2^{-1}(P_1)(D^2)'P_1D^1L^2.
\end{aligned}$$

Define the new function f as follows:

$$\begin{aligned}
& f(s,p,k,P,K)\\
= \; & [A_s' + P_1(s)B_s^2L_s^3 + C_s'P_1(s)D_s^2L_s^3]p + [C_s' + P_1(s)B_s^2L^4 + C'P_1(s)D_s^2L_s^4]k\\
& +[C_s'P_1(s)D_s^1L_s^1 + P_2(s)B_s^1L_s^1 + P_2(s)B_s^2L_s^3 - P_1(s)B_s^2L_s^3 + P_2(s)B_s^2L_s^5 + C_s'P_1(s)D_s^2L_s^5]P\\
& +[C_s'P_1(s)D_s^1L_s^2 + P_2(s)B_s^1L_s^2 + P_2(s)B_s^2L_s^4 - P_1(s)B_s^2L_s^4 + P_2(s)B_s^2L_s^6 + C_s'P_1(s)D_s^2L_s^6]K.
\end{aligned}$$

Then $(\delta p, \delta k)$ satisfies the following linear homogeneous BSDE of mean-field type:

$$d\delta p \;=\; -f(s,\delta p_s,\delta k_s,\overline{\delta p}_s,\overline{\delta k}_s)\,ds + \delta k dW, \quad \delta p(T)=0. \tag{3.41}$$

In view of Buckdahn, Li and Peng [4, Therem 3.1], it admits a unique solution $(\delta p, \delta k) = (0,0)$. Therefore, $X = X^*$ and $u = u*$.

The formula (3.33) is derived from computation of $\langle p_T, X_T^*\rangle$ with the Itô's formula. $\square$

4 Particular cases

4.1 The classical optimal stochastic LQ case: $B^1 = 0$ and $D^1 = 0$.

In this case, let P_1 is the unique nonnegative solution to Riccati equation (3.21). Then, P_1 is also the solution of Riccati equation (3.23), and the optimal control reduces to the conventional feedback form.

4.2 The deterministic control of linear stochastic system with quadratic cost: $B^2 = 0$ and $D^2 = 0$.

In this case, $B = B^1$ and $D = D^1$, and Riccati equation (3.21) takes the following form (we write $R = R^1$ for simplifying exposition):

$$P_1' + P_1 A + A' P_1 + C' P_1 C + Q = 0, \quad P_1(T) = G,$$

which is a linear Liapunov equation. Riccati equation (3.23) takes the following form:

$$P_2' + P_2 \widetilde{A} + \widetilde{A}' P_2 + \widetilde{Q} - P_2 B' (R + D' P_1 D)^{-1} B P_2 = 0, \quad P_2(T) = G$$

with

$$\widetilde{A} := A - B'(R + D' P_1 D)^{-1} D' P_1 C$$

and

$$\widetilde{Q} := Q + C' P_1 C - C' P_1 D (R + D' P_1 D)^{-1} D' P_1 C.$$

The optimal control takes the following feedback form:

$$u = -(R + D' P_1 D)^{-1} (B P_2 + D' P_1 C) \bar{X}.$$

5 Some solvable singular cases

In this section, we study the possibility of $R^1 = 0$ or $R^2 = 0$. We have

Theorem 5.1 *Assume that* $R^1 \gg 0$ *and*

$$R^2 \geq 0, \quad (D^2)' D^2 \gg 0, \quad G > 0. \tag{5.1}$$

Then Riccati equations (3.21) *and* (3.23) *have unique nonnegative solutions* $P_1 \gg 0$ *and* P_2*, respectively. The optimal control is unique and has the following feedback form:*

$$u^{1*} = M^1\overline{X}, \quad u^{2*} = M^2\widetilde{X}^* + M^3\overline{X}^* = M^2X^* + (M^3 - M^2)\overline{X}^*. \tag{5.2}$$

The optimal feedback system and the optimal value take identical forms to those of Theorem 3.2.

Proof. In view of the conditions (5.1), the existence and uniqueness of solution $P_1 \gg 0$ to Riccati equations (3.21) can be found in Kohlmann and Tang [6, Theorem 3.13, page 1140], and those of solution $P_2 \geq 0$ to Riccati equations (3.21) comes from the fact that $\widehat{\Lambda}(P_1) \gg 0$ as a consequence of the condition that $R^1 \gg 0$.

Other assertions can be proved in an identical manner as Theorem 3.2. □

Theorem 5.2 *Assume that* $R^2 \gg 0$ *and*

$$R^1 \geq 0, \quad (D^1)'D^1 \gg 0, \qquad G > 0. \tag{5.3}$$

Then Riccati equations (3.21) *and* (3.23) *have unique nonnegative solutions* $P_1 \gg 0$ *and* P_2*, respectively. The optimal control is unique and has the following feedback form:*

$$u^{1*} = M^1\overline{X}, \quad u^{2*} = M^2\widetilde{X}^* + M^3\overline{X}^* = M^2X^* + (M^3 - M^2)\overline{X}^*. \tag{5.4}$$

The optimal feedback system and the optimal value take identical forms to those of Theorem 3.2. .

Proof. The existence and uniqueness of solution P_1 to Riccati equations (3.21) are well-known. In view of the condition $G > 0$, we have $P_1 \gg 0$. We now prove those of solution $P_2 \geq 0$ to Riccati equations (3.21).

In view of the well-known matrix inverse formula:

$$\left(A + BD^{-1}C\right)^{-1} = A^{-1} - A^{-1}B\left(D + CA^{-1}B\right)^{-1}CA^{-1} \tag{5.5}$$

for $B \in \mathbb{R}^{n\times m}, C \in \mathbb{R}^{m\times n}$ and invertible matrices $A \in \mathbb{R}^{n\times n}, D \in \mathbb{R}^{m\times m}$ such that $A + BD^{-1}C$ and $D + CA^{-1}B$ are invertible, we have the following identity:

$$\begin{aligned} \widehat{\Lambda}(P_1) &= R^1 + \left(D^1\right)'\left\{P_1 - P_1D^2\left[R^2 + \left(D^2\right)'P_1D^2\right]^{-1}(D^2)'P_1\right\}D^1 \\ &= R^1 + \left(D^1\right)'\left[P_1^{-1} + D^2\left(R^2\right)^{-1}\left(D^2\right)'\right]^{-1}D^1. \end{aligned} \tag{5.6}$$

Noting the condition $(D^1)'D^1 \gg 0$, we have $\widehat{\Lambda}(P_1) \gg 0$.

Other assertions can be proved in an identical manner as Theorem 3.2. □

6 The infinite time-horizontal case

In this section, we consider the time-invariant situation of all the coefficients A, B, C, D, Q and R in the linear control stochastic differential equation (SDE)

$$dX_s = [AX_s + B^1u_s^1 + B^2u_s^2]ds + [C_sX_s + D^1u_s^1 + D^2u_s^2]dW_s, \quad t > 0; \quad X_0 = x_0, \tag{6.1}$$

and the quadratic cost functional

$$J(u) \triangleq \frac{1}{2}\mathbb{E}\int_0^\infty \left[\langle QX_s, X_s\rangle + \langle R^1u_s^1, u_s^1\rangle + \langle R^2u_s^2, u_s^2\rangle\right] ds. \tag{6.2}$$

The admissible class of controls for the deterministic controller u^1 is $L^2(0,\infty;\mathbb{R}^{l_1})$ and for the random controller u^2 is $\mathcal{L}^2_{\mathcal{F}}(0,\infty;\mathbb{R}^{l_2})$. For simplicity of subsequent exposition, we assume that $Q > 0$.

Assumption 6.1 *There is $K \in \mathbb{R}^{l_2\times n}$ such that the unique solution X to the following linear matrix stochastic differential equation*

$$dX_s = (A + B^2K)X_s\, ds + (C + D^2K)X_s dW_s, \quad t > 0; \quad X_0 = I, \tag{6.3}$$

lies in $\mathcal{L}^2_{\mathcal{F}}(0,\infty;\mathbb{R}^{n\times n})$. That is, our linear control system (6.1) *is stabilizable using only control u^2.*

We have

Lemma 6.2 *Assume that $Q > 0$ and Assumption 6.1 is satisfied. Then, Algebraic Riccati equations*

$$\begin{aligned} &P_1A + A'P_1 + C'P_1C + Q \\ &-(P_1B^2 + C'P_1D^2)\Lambda_2^{-1}(P_1)(P_1B^2 + C'P_1D^2)' = 0 \end{aligned}$$

and

$$P_2\widetilde{A}(P_1) + \widetilde{A}'(P_1)P_2 + \widetilde{Q}(P_1) - P_2\mathcal{N}(P_1)P_2 = 0 \tag{6.4}$$

have positive solutions P_1 and P_2. Here for $S \in \mathbb{S}^n_+$,

$$\begin{aligned}
U(S) &:= S - SD^2\Lambda_2^{-1}(S)\left(D^2\right)' S;\\
\widetilde{Q}(S) &:= Q + C'U(S)C - C'U(S)D^1\widehat{\Lambda}^{-1}(S)(D^1)'U(S)C,\\
\widetilde{A}(S) &:= A - B^2\Lambda_2^{-1}(S)\left(D^2\right)' SC\\
&\quad - \left[B^1 - B^2\Lambda_2^{-1}(S)\left(D^2\right)' SD^1\right]\widehat{\Lambda}^{-1}(S)\left(D^1\right)' U(S)C,\\
\mathcal{N}(S) &:= B^2\Lambda_2^{-1}(S)\left(B^2\right)'\\
&\quad + \left[B^1 - B^2\Lambda_2^{-1}(S)\left(D^2\right)' SD^1\right]\widehat{\Lambda}^{-1}(S)\left[B^1 - B^2\Lambda_2^{-1}(S)\left(D^2\right)' SD^1\right]'.
\end{aligned}$$

Proof. Existence and uniqueness of positive solution P_1 to Algebraic Riccati equation (6.4) is well-known, and is referred to Wu and Zhou [9, Theorem 7.1, page 573]. Now we prove the existence of positive solution to Algebraic Riccati equation (6.4). We use approximation method by considering finite time-horizontal Riccati equations.

For any $T > 0$, let P_1^T and P_2^T be unique solutions to Riccati equations (3.21) and (3.23), with $G = 0$. It is well-known that P_1^T converges to the constant matrix P_1 as $T \to \infty$. We now show the convergence of P_2^T. Firstly, $P_2^T(t)$ is nondecreasing in T for any $t \geq 0$ due to the following representation formula: for $(t, x) \in [0, T] \times \mathbb{R}^n$,

$$\langle P_2^T(t)x, x\rangle = \inf_{\substack{u^1 \in L^2(t,T;\mathbb{R}^{l_1})\\ u^2 \in \mathcal{L}^2_{\mathcal{F}}(t,T;\mathbb{R}^{l_2})}} \frac{1}{2}\mathbb{E}^{t,x}\int_t^T \left[\langle QX_s, X_s\rangle + \langle R^1u_s^1, u_s^1\rangle + \langle R^2u_s^2, u_s^2\rangle\right] ds, \tag{6.5}$$

whose proof is identical to that of the formula (3.33). From Assumption 6.1, it is straightforward to show that there is $C_t > 0$ such that $|P_2^T(t)| \leq C_t$. Then $P_2^T(t)$ converges to $P_2(t)$ as $T \to \infty$. Furthermore, since all the coefficients are time-invariant and $(P_1^T(T), P_2^T(T)) = 0$ for any $T > 0$, we have

$$\left(P_1^{T+s}(t+s), P_2^{T+s}(t+s)\right) = \left(P_1^T(t), P_2^T(t)\right). \tag{6.6}$$

Taking the limit $T \to \infty$ yields that $P_2(t+s) = P_2(t)$. Therefore, P_2 is a constant matrix.

Taking the limit $T \to \infty$ in the integral form of Riccati equation (3.23), we show that P_2 solves Algebraic Riccati equation (6.4).

Finally, in view of $Q > 0$, we have $P_2^1(0) > 0$. Hence $P_2 \geq P_2^1(0) > 0$. □

Theorem 6.3 *Let Assumption 6.1 be satisfied. Assume that $Q > 0$ and either of the following three sets of conditions holds true:*

(i) $R^1 > 0$ *and* $R^2 > 0$*;*

(ii) $R^1 > 0, R^2 \geq 0, (D^2)'D^2 > 0,$ *and* $G > 0$*; and*

(iii) $R^1 \geq 0, (D^1)'D^1 > 0, R^2 > 0,$ *and* $G > 0$.

Then the optimal control is unique and has the following feedback form:

$$u^{1*} = M^1\overline{X}, \quad u^{2*} = M^2\widetilde{X}^* + M^3\overline{X}^* = M^2X^* + (M^3 - M^2)\overline{X}^*. \tag{6.7}$$

Define $\overline{X}_t^* := \mathbb{E}[X_t]$ *and* $\widetilde{X}_t^* := X_t^* - \overline{X}_t^*$ *The optimal feedback system is given by*

$$\begin{aligned} X_t &= x + \int_0^t [(A + B^2M^2)X_s + (B^1M^1 - B^2M^2 + B^2M^3)\overline{X}_s]\, ds \\ &\quad + \int_0^t [(C + D^2M^2)X_s + (D^1M^1 - D^2M^2 + D^2M^3)\overline{X}_s]\, dW_s, \quad t \geq 0. \end{aligned} \tag{6.8}$$

It is a mean-field stochastic differential equation. The expected optimal state $\overline{X}_t^*$ *is governed by the following ordinary differential equation:*

$$\overline{X}_t = x + \int_0^t (A + B^1M^1 + B^2M^3)\overline{X}_s\, ds, \quad t \geq 0; \tag{6.9}$$

and $\widetilde{X}_t^*$ *is governed by the following stochastic differential equation:*

$$\begin{aligned} \widetilde{X}_t &= \int_0^t (A + B^2M^2)\widetilde{X}_s\, ds \\ &\quad + \int_0^t [(C + D^2M^2)\widetilde{X}_s + (C + D^1M^1 + D^2M^3)\overline{X}_s]\, dW_s, \quad t \geq 0. \end{aligned} \tag{6.10}$$

The optimal value is given by

$$J(u^*) = \langle P_2X(0), X(0)\rangle. \tag{6.11}$$

Proof. The uniqueness of the optimal control is an immediate consequence of the strict convexity of the cost functional in both control variables u^1 and u^2. We now show that u^* is optimal.

For any admissible pair (u^1, u^2), from Theorem 3.2, we have

$$J^T(u) \geq \langle P_2^T(0)x, x\rangle.$$

Therefore, letting $T \to \infty$, we have $J(u) \geq \langle P_2(0)x, x\rangle$.

For $0 \leq s \leq T < \infty$, let $(u^{*,T}, X^{*,T})$ be the optimal pair corresponding to the time-horizon $T > 0$, and the associated adjoint process is denoted by p^T. Using Itô's formula to compute the inner product $\langle p^T, X^{*,n}\rangle$, noting that $p_s^T = P_1^T(s)\widetilde{X}_s^{*,T} + P_2^T(s)\overline{X}_s^{*,T}$, we have

$$\mathbb{E}\left[\langle P_1^T(s)\widetilde{X}_s^{*,T} + P_2^T(s)\overline{X}_s^{*,T}, X_s^{*,T}\rangle\right] + J^s(u^{*,T}) = \langle P_2^T(0)x, x\rangle. \tag{6.12}$$

From stability of solutions of stochastic differential equations, we have for any $s > 0$,

$$\lim_{T\to\infty} \mathbb{E} \max_{0\le t\le s} |X_t^{*,T} - X_t^*|^2 = 0, \quad \lim_{T\to\infty} \mathbb{E} \int_0^s |u_t^{*,T} - u_t^*|^2 = 0.$$

Passing to the limit $T \to \infty$ in (6.12), we have for any $s \ge 0$

$$\mathbb{E}\left[\langle P_1 \widetilde{X}_s^* + P_2 \overline{X}_s^*, X_s^*\rangle\right] + J^s(u^*) = \langle P_2 x, x\rangle. \tag{6.13}$$

Since

$$\mathbb{E}\left[\langle P_1 \widetilde{X}_s^* + P_2 \overline{X}_s^*, X_s^*\rangle\right] = \mathbb{E}\left[\langle P_1 \widetilde{X}_s^*, \widetilde{X}_s^*\rangle\right] + \mathbb{E}\left[\langle P_2 \overline{X}_s^*, \overline{X}_s^*\rangle\right] \ge 0,$$

we have $J^s(u^*) \le \langle P_2 x, x\rangle$, and thus X^* is stable and u^* is admissible .

Passing to the limit $s \to \infty$ in (6.13), we have

$$J^s(u^*) = \langle P_2 x, x\rangle. \tag{6.14}$$

Finally, the last formula implies the uniqueness of the positive solution to Algebraic Riccati equation (6.4). □

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