Sequential LMI Approach for Design of a BMI-Based Robust Observer State Feedback Controller with Nonlinear Uncertainties

Yan Wang[1*], Rajesh Rajamani[1], Ali Zemouche[2]

[1] *Department of Mechanical Engineering, University of Minnesota, Minneapolis, MN 55455, USA.*
[2] *University of Lorraine and EPI Inria DISCO, Laboratoire des Signaux et Systèmes, CNRS-CentraleSupélec, 91192 Gif-sur-Yvette, France*

SUMMARY

This paper aims at developing a robust observer based estimated state feedback control design method for an uncertain dynamical system that can be represented as a LTI system connected with an IQC-type nonlinear uncertainty. Traditionally in existing design methodologies, a convex semidefinite constraint is obtained at the cost of conservatism and unrealistic assumptions. On the other hand, the design of the robust observer state feedback controller in this paper is formulated as a feasibility problem of a bilinear matrix inequality (BMI) constraint. Unfortunately, the search for a feasible solution of a BMI constraint is a NP hard problem in general. The applicability of a linearization method, such as the variable change method or the congruence transformation, depends on the specific structure of the problem at hand and cannot be generalized. This paper transforms the feasibility analysis of the BMI constraint into an eigenvalue problem and applies the convex-concave based sequential LMI optimization method to search for a feasible solution. Furthermore, an augmentation of the sequential LMI algorithm to improve its numerical stability is presented. In the application part, a vehicle lateral control problem is presented to demonstrate the applicability of the proposed algorithm to a real-world estimated state feedback control design problem and the necessity of the augmentation for numerical stability.

KEY WORDS: Output Feedback; Robust Control; Convex Optimization; Control Nonlinearities; Automotive Control.

1. INTRODUCTION

State feedback control is one of the most commonly used control methods for both linear and nonlinear systems [1] [23] [28] [29]. Unfortunately, it is not possible or too expensive to measure all the state variables in many real-world applications. Thus, an observer is a necessary part in the implementation of the state feedback controller. However, model uncertainty is always the trouble maker for control engineers. It turns out that the well-known separation principle is not valid even for linear time invariant (LTI) systems in the presence of parametric uncertainty [24] [44]. Hence, the controller gain and observer gain need to be decided in a coupled way to guarantee the stability of the closed-loop system in the presence of model uncertainty. This paper aims at applying a state-of-the-art optimization algorithm to develop a robust observer state feedback controller for an uncertain dynamical system that can be represented in the linear fractional transformation (LFT) form in which a nominal LTI system is connected with an uncertain nonlinear operator satisfying the integral quadratic constraint (IQC).

*Correspondence to: Department of Mechanical Engineering, University of Minnesota, Minneapolis, MN 55455, USA.
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In the last two decades, semi-definite programming (SDP) has gained popularity in the control systems community [5] [13] [18] [7]. There are a number of successful applications of this numerical method in controller/observer design and system identification [25] [26] [27]. The essence of this state-of-the-art methodology is to represent the design of control systems as an optimization problem with a linear cost function together with a linear matrix inequality (LMI) as the constraint. The convex property of the LMI ensures that the optimization problem can be solved by the well-developed interior-point algorithm [30] [6].

However, the formulation of the design of control systems as a LMI based optimization problem is not always an easy task. Many constraints from practical considerations make the optimization problem a nonconvex one, such as the fixed-order controller design [19] [20] [14]. On the other hand, the bilinear matrix inequality (BMI) is a more general semidefinite constraint than a LMI that frequently appears in the study of control theory [7] [37] [18]. For example, the semidefinite condition for output feedback controller design contains the product of the Lyapunov matrix and controller gain parameters with a constant matrix in the middle. Unfortunately, this problem is traditionally treated as a NP hard problem [4]. Although various convexification methods, such as change variable and congruence transformation [18] [13] [12], are proposed in the literature to transform the BMIs into LMIs, the success of these methods depends on the specific structure of the semidefinite constraint at hand. In some literature, a convex inner approximation of the BMI can be obtained by trading off the conservatism for computational efficiency [12]. The same with the previous methods, the tricks that apply to a specific application cannot be generalized to others.

Besides those global convexification methods, there are also several results that apply the local optimization methods, such as sequential convex programming [17] [2], nonsmooth maximum eigenvalue minimization [3], augmented Lagrangian approach [41] [21] and Newton-like alternating projection method [31] to search for a local optimal solution. The successful applications to robust output feedback controller design show the effectiveness of those algorithms. All of these local methods start with a strictly feasible initial point such that all the subsequent iterations are constrained in the feasible set. However, searching for such a feasible initial point is not a trivial task for a problem with high-dimension decision variables.

In this paper, the design of the robust observer state feedback controller is formulated as the feasibility problem of a BMI constraint coming from the product of the Lyapunov matrix and the controller or observer gains. This paper transforms the feasibility analysis of the BMI constraint into an eigenvalue problem and applies the convex-concave based sequential LMI optimization method to search for a feasible solution. Furthermore, an augmentation of the sequential LMI algorithm to improve its numerical stability is presented.

The remainder of this paper is organized as follows. The model of the observer based estimated state feedback control system connected with an uncertainty block is presented in section 2. The design of the controller and observer is formulated as a BMI feasibility problem for the IQC-type uncertainty block in section 3. Then, a sequential LMI algorithm is developed to search for a feasible solution of the BMI constraint in section 4. The proposed algorithms are applied to a vehicle lateral control problem in section 5. Section 6 contains the final conclusions and recommendation for future work.

2. PRELIMINARIES

A brief review of the linear fractional transformation (LFT) representation for a LTI system with uncertainty together with a formulation of the observer based robust estimated state feedback control problem will be presented in this section.

2.1. Notations

The notations used in this paper are quite standard. $\mathcal{L}_2^n[0,\infty)$ is the Hilbert space of all square summable functions $f:[0,\infty)\rightarrow\mathbb{R}^n$. $\mathcal{L}_{2e}^n[0,\infty)$ is the extended space the element of which only needs to be square integrable on finite intervals. A mathematical operator means a function

$f: \mathcal{L}_{2e}^m[0,\infty) \to \mathcal{L}_{2e}^n[0,\infty)$. RL_∞ is the set of real rational transfer functions bounded on the imaginary axis. RH_∞ represents the set of real rational, proper and stable transfer functions. The set of $m \times n$ matrices with elements in RL_∞ or RH_∞ are denoted as $RL_\infty^{m\times n}$ or $RH_\infty^{m\times n}$. For the transfer matrix $T(s) \in RH_\infty^{m\times n}$, $\|T(s)\|_\infty$ denotes the $\mathcal{H}_\infty$-norm, the largest singular value of $T(j\omega), \forall\, \omega \in \mathbb{R} \cup \{\infty\}$. For a Hermitian matrix H, $H \prec 0, H \succ 0\,(H \preceq 0, H \succeq 0)$ means a negative and positive (semi) definite constraint for H.

2.2. *LFT Representation of The Uncertain Plant*

In this paper, the uncertain system shown in Fig. 1 is studied. As can be seen, the uncertainty operator Δ is isolated from the nominal linear generalized plant, whose state space model is shown as [45] [40]

$$\begin{bmatrix} \dot{x} \\ \hline y \\ z \end{bmatrix} = \left[\begin{array}{c|cc} A & B_u & B_w \\ \hline C_y & D_{yu} & D_{yw} \\ C_z & D_{zu} & D_{zw} \end{array}\right] \begin{bmatrix} x \\ \hline u \\ w \end{bmatrix}, \quad w = \Delta(z) \tag{1}$$

where x, u, y denote the state vector, controller input and the measured output. w and z are the input and output of the uncertainty channel. Here, it is implicitly assumed that the interconnection of the generalized plant and the uncertainty block Δ is well-posed. The relationship of w and z is crucial for robustness analysis which depends on the structure of Δ [13] [15]. We will focus on those operators whose input and output satisfy the general integral quadratic constraint (IQC) in this paper.

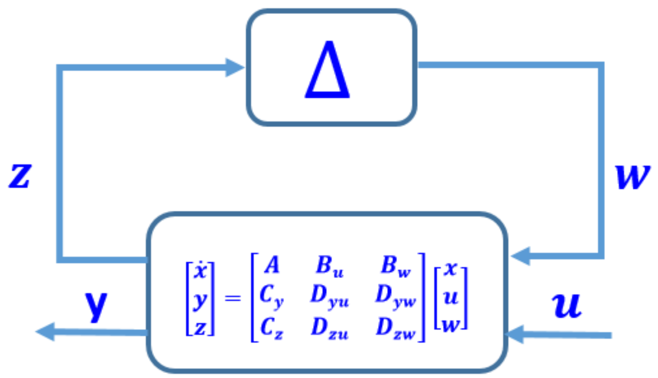

Figure 1. LFT representation of the open-loop system

2.3. *Nominal Observer Design*

To stabilize an uncertain system with an estimated state feedback controller, an observer is needed to estimate the state variables from the measurement signal y and control input u. Due to the lack of model information of the uncertain block Δ (besides its input-output relationship), the design of the observer is based on the model of the nominal linear system [24] [44]. The observer state equation is

$$\dot{\hat{x}} = A\hat{x} + B_u u + L(y - C_y\hat{x} - D_{yu}u) \tag{2}$$

The corresponding observer error state equation is

$$\dot{e} = (A - LC_y)e + (B_w - LD_{yw})w \tag{3}$$

where $e = x - \hat{x}$.

2.4. *Observer Based Estimated State Feedback Control*

With the observer based estimated state feedback control $u = -K\hat{x}$, the state space model of the closed-loop system isolated from uncertainty becomes

$$\left[\begin{array}{c} \dot{x} \\ \dot{e} \\ \hline z \end{array}\right] = \left[\begin{array}{cc|c} A - B_u K & B_u K & B_w \\ 0 & A - LC_y & B_w - LD_{yw} \\ \hline C_z - D_{zu}K & D_{zu}K & D_{zw} \end{array}\right] \left[\begin{array}{c} x \\ e \\ \hline w \end{array}\right] \tag{4}$$

The transfer matrix of the state space realization in (4) is denoted as $T(s)$ with $z(s) = T(s)w(s)$. The block diagram of the closed-loop system with the uncertainty block Δ can be seen in Fig. 2. The search for the controller gain K and observer gain L such that the closed-loop system is robustly asymptotically stable against the uncertainty Δ is the main task of this paper. To facilitate subsequent analysis, the closed-loop state space matrices A_c, B_c, C_c and D_c are defined as

$$\begin{aligned} A_c &= \begin{bmatrix} A - B_u K & B_u K \\ 0 & A - LC_y \end{bmatrix}, \quad B_c = \begin{bmatrix} B_w \\ B_w - LD_{yw} \end{bmatrix} \\ C_c &= \begin{bmatrix} C_z - D_{zu}K & D_{zu}K \end{bmatrix}, \quad D_c = D_{zw} \end{aligned} \tag{5}$$

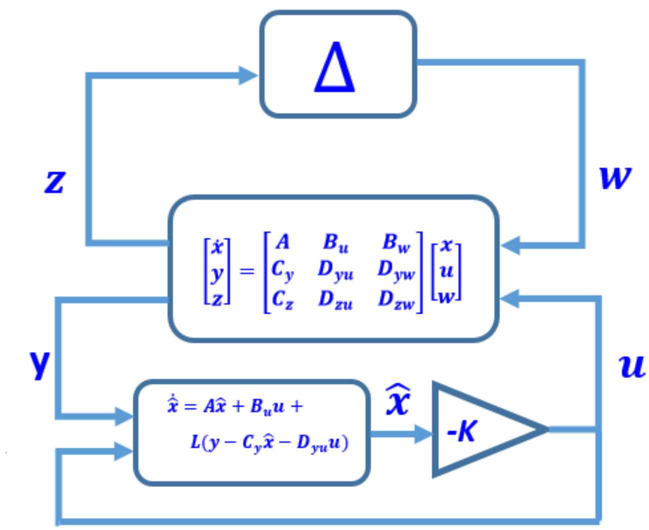

Figure 2. LFT representation of the closed-loop system with uncertainty

3. BMIS FOR ROBUST OBSERVER STATE FEEDBACK DESIGN

It will be shown in this section that the design of the controller gain K and the observer gain L such that the interconnection in Fig. 2 is robustly stable against the IQC-type uncertainty block Δ resorts to searching for a feasible solution of a BMI constraint. Unlike the existing results, such as [24] [44], the system under study is not limited to be a linear one in this paper because the integral quadratic constraint itself does not discriminate between the linear and nonlinear operator. It is also well known that many types of nonlinearities in control systems can be formulated as an IQC-type uncertainty [36]. Suppose $z \in \mathcal{L}_2^l$ and $w \in \mathcal{L}_2^m$, the IQC for the uncertainty operator Δ is defined as

$$\int_{-\infty}^{\infty} \begin{bmatrix} z(j\omega) \\ w(j\omega) \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} z(j\omega) \\ w(j\omega) \end{bmatrix} d\omega \geq 0 \tag{6}$$

where $z(j\omega)$ and $w(j\omega)$ represent the Fourier transforms of the signals z and w at frequency ω. $\Pi : j\mathbb{R} \to \mathbb{C}^{(l+m)\times(l+m)}$ is a Hermitian-value matrix function. It is also called the IQC multiplier in some literature.

Many important uncertainties in the feedback control loop can be modeled as a mathematical operator satisfying the IQC condition, such as uncertain linear time-invariant dynamics, delay, memoryless nonlinearity in a sector, slope-restricted monotonic odd nonlinearity and so on. The list of the IQCs for various types of uncertainties can be found in [36] [13] [8] [9] [10]. Once the IQC multiplier Π is obtained, the IQC Theorem can be applied to obtain the sufficient stability condition for the $T-\Delta$ interconnection in Fig. 2.

Theorem 1
(IQC Theorem [36]): Let $T(s) \in RH_\infty^{l\times m}$ and $\Delta : \mathcal{L}_2^l \to \mathcal{L}_2^m$ is a bounded causal operator. Assume that:

1) for every $\tau \in [0,1]$, the interconnection of $T(s)$ and $\tau\Delta$ is well-posed;
2) for every $\tau \in [0,1]$, the IQC defined in (6) is satisfied by $\tau\Delta$;
3) there exist some $\epsilon > 0$ such that

$$\begin{bmatrix} T(j\omega) \\ I \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} T(j\omega) \\ I \end{bmatrix} \leq -\epsilon I, \quad \forall \omega \in \mathbb{R} \tag{7}$$

Then, the $T-\Delta$ interconnection in Fig. 2 is stable.

If $\Pi(j\omega) \in RL_\infty^{(l+m)\times(l+m)}$ (matrix of rational functions with no poles on the imaginary axis), it always can be factorized as the form in (8), though such a factorization is not unique [36] [38].

$$\Pi(j\omega) = \Psi(j\omega)^* M \Psi(j\omega) \tag{8}$$

where $M = M^T$, $\Psi \in RH_\infty$ (matrix of rational functions without poles in the closed right-half plane). Accordingly, the frequency domain inequality (FDI) in (7) becomes

$$\begin{bmatrix} T(j\omega) \\ I \end{bmatrix}^* \Psi(j\omega)^* M \Psi(j\omega) \begin{bmatrix} T(j\omega) \\ I \end{bmatrix} \leq -\epsilon I, \ \ \forall \omega \in \mathbb{R} \tag{9}$$

The state space realization of Ψ has the form shown in (10).

$$\dot{x}_\Psi = A_\Psi x_\Psi + \begin{bmatrix} B_{\Psi,z} & B_{\Psi,w} \end{bmatrix} \begin{bmatrix} z \\ w \end{bmatrix}, \quad y_\Psi = C_\Psi x_\Psi + \begin{bmatrix} D_{\Psi,z} & D_{\Psi,w} \end{bmatrix} \begin{bmatrix} z \\ w \end{bmatrix} \tag{10}$$

Furthermore, the corresponding state space model of the transfer function matrix $\Psi(s) \begin{bmatrix} T(s) \\ I \end{bmatrix}$ becomes [38]

$$\begin{aligned} \begin{bmatrix} \dot{x}_T \\ \dot{x}_\Psi \end{bmatrix} &= \underbrace{\begin{bmatrix} A_c & 0 \\ B_{\Psi,z} C_c & A_\Psi \end{bmatrix}}_{\bar{A}} \begin{bmatrix} x_T \\ x_\Psi \end{bmatrix} + \underbrace{\begin{bmatrix} B_c \\ B_{\Psi,w} + B_{\Psi,z} D_c \end{bmatrix}}_{\bar{B}} w \\ y_\Psi &= \underbrace{\begin{bmatrix} D_{\Psi,z} C_c & C_\Psi \end{bmatrix}}_{\bar{C}} \begin{bmatrix} x_T \\ x_\Psi \end{bmatrix} + \underbrace{(D_{\Psi,w} + D_{\Psi,z} D_c)}_{\bar{D}} w \end{aligned} \tag{11}$$

where $x_T = \begin{bmatrix} x^T & e^T \end{bmatrix}^T$ denotes the state of the transfer function matrix $T(s)$. The matrices A_c, B_c, C_c and D_c are defined in (5). The FDI in (9) is equivalent to the following semidefinite constraint according to the well-known Kalman-Yakubovich-Popov (KYP) Lemma [35] [16] [38].

$$\begin{gathered} P = P^T \\ \begin{bmatrix} \bar{A}^T P + P\bar{A} & P\bar{B} \\ \bar{B}^T P & 0 \end{bmatrix} + \begin{bmatrix} \bar{C}^T \\ \bar{D}^T \end{bmatrix} M \begin{bmatrix} \bar{C} & \bar{D} \end{bmatrix} \prec 0 \end{gathered} \tag{12}$$

Since the matrices $\bar{A}$, $\bar{B}$ shown in (11) are affine matrix functions of the gain parameters K and L, the semidefinite constraint in (12) is a BMI constraint for (K, L) and P.

Remark 1
A remarkable feature of the KYP Lemma is that the matrix decision variable P in the semidefinite constraint in (12) is not necessary to be positive definite.

4. A SEQUENTIAL LMI APPROACH

4.1. Eigenvalue Problem

The observer based estimated state feedback control design problem discussed above relies on a feasible solution of a BMI constraint. Unfortunately, no existing numerical algorithm can be directly applied to search for the feasible solution of such a nonconvex constraint. There exist algorithms to search for the local Karush-Kuhn-Tucker (KKT) point of a BMI optimization problem but only with an initial feasible solution [17], which is not assumed to be available in this paper. To make the local optimization algorithm applicable, a slack variable t is introduced such that the search for the feasible solution of the BMI constraint is equivalent to the following eigenvalue problem.

$$\begin{aligned} &\textbf{minimize} \quad t \\ &\textbf{subject to} \\ &B(X,Y) - tI \preceq 0 \end{aligned} \tag{13}$$

where $B(X,Y)$ denotes a bilinear matrix function shown below

$$B(X,Y) = X^T Y + Y^T X + F \tag{14}$$

with X, Y and F are affine matrix functions of the free variables $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $\beta = (\beta_1, \ldots, \beta_m)$ respectively.

$$X = X_0 + \sum_{i=1}^{n} \alpha_i X_i, \quad Y = Y_0 + \sum_{j=1}^{m} \beta_j Y_j, \quad F = F_0 + \sum_{i=1}^{n} \alpha_i F_{\alpha,i} + \sum_{j=1}^{m} \beta_j F_{\beta,j} \tag{15}$$

It is assumed that the slack variable t is included in either α or β in the remainder of the paper. Once a negative optimal solution of t in (13) appears, the feasible solution is naturally obtained. In principle, the application of the sequential LMI algorithm to the eigenvalue problem in (13) do not require the starting point to be a feasible one. However, it will be shown that this algorithm may fail due to poor numerical conditions before a negative value of t is obtained. An augmentation of the sequential LMI approach to improve the numerical stability will be presented in this section.

4.2. Convex matrix function

Before presenting the convex-concave decomposition of the bilinear matrix function, it is worth to review the definition of the convexity for a matrix function.

Definition 1
[39] [17] A matrix-valued mapping $G : R^n \to S^p$ is said to be positive semidefinite convex on a convex subset $E \subseteq R^n$ if for all $t \in [0,1]$ and $x, y \in E$, it satisfies the following semidefinite constraint.

$$G(tx + (1-t)y) \preceq tG(x) + (1-t)G(y) \tag{16}$$

An equivalent way to define the positive semidefinite convexity of a matrix-valued mapping is based on its scalarization as shown below.

Lemma 1
[17] A matrix-valued mapping $G : R^n \to S^p$ is said to be positive semidefinite convex on a convex subset $E \subseteq R^n$ if and only if for any $v \in R^p$ the function $v^T G(x) v$ is convex on E.

Although Definition 1 or Lemma 1 can be directly applied to the convexity analysis of any matrix-valued mapping, a more efficient way for the convexity verification of the quadratic matrix functions exists.

Lemma 2
Let X and Y be affine matrix functions of the free variables $\alpha_1, \ldots, \alpha_n$ and $\beta_1, \ldots, \beta_m$ respectively.

$$X = X_0 + \sum_{i=1}^{n} \alpha_i X_i, \qquad Y = Y_0 + \sum_{j=1}^{m} \beta_j Y_j \tag{17}$$

Then, the quadratic matrix function $G : R^{n+m} \to S^p$ shown in (18) is positive semidefinite convex on a convex subset $E \subseteq R^{n+m}$ if the matrix Q is positive semidefinite.

$$G(x) = \begin{bmatrix} X^T & Y^T \end{bmatrix} Q \begin{bmatrix} X \\ Y \end{bmatrix} \tag{18}$$

where $x = [\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_m]^T$.

Proof
The matrix $G(x)$ can be scalarized by any real vector $v \in R^p$ as

$$v^T G(x) v = \begin{bmatrix} \alpha^T & \beta^T \end{bmatrix} \Gamma^T Q \Gamma \begin{bmatrix} \alpha \\ \beta \end{bmatrix} + 2v^T \begin{bmatrix} X_0^T & Y_0^T \end{bmatrix} Q \Gamma \begin{bmatrix} \alpha \\ \beta \end{bmatrix} + v^T \begin{bmatrix} X_0^T & Y_0^T \end{bmatrix} Q \begin{bmatrix} X_0 \\ Y_0 \end{bmatrix} v \tag{19}$$

where $\alpha = [\alpha_1, \ldots, \alpha_n]^T$, $\beta = [\beta_1, \ldots, \beta_m]^T$. The matrix Γ is

$$\Gamma = \begin{bmatrix} X_1 v, \ldots, X_n v & 0 \\ 0 & Y_1 v, \ldots, Y_m v \end{bmatrix} \tag{20}$$

where 0 denotes the zero matrix with a compatible dimension. The positive semidefiniteness of Q implies the same property of $\Gamma^T Q \Gamma$ for any real vector $v \in R^p$. Hence, the quadratic polynomial of α and β in (19) is a convex function [6]. According to Lemma 1, it can be concluded that the quadratic matrix function $G(x)$ in (18) is positive semidefinite convex on a convex subset $E \subseteq R^{n+m}$. $\square$

Remark 2
Different from the scalar quadratic function that the positive semidefiniteness of the matrix Q is just the sufficient condition for the positive semidefinite convexity of the matrix function $G(x)$ not necessary. The reason lies in the fact that the positive semidefiniteness of $\Gamma^T Q \Gamma$ does not guarantee the same property of Q if the matrix Γ is rank deficient.

4.3. Convex-Concave decomposition

A convex-concave decomposition (or DC decomposition) of the bilinear matrix function $B(X, Y)$ in (14) is a necessary step to apply the sequential LMI approach. Besides the decomposition in [17], another form of the DC decomposition will be presented in what follows.

$$B(X, Y) = \begin{bmatrix} X^T & Y^T \end{bmatrix} \underbrace{\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}}_{Q} \begin{bmatrix} X \\ Y \end{bmatrix} + F \tag{21}$$

The matrix Q is an indefinite matrix, the eigenvalues of which distribute at -1 and 1 equally. Furthermore, the indefinite matrix Q can always be decomposed as the difference of two positive semidefinite matrices [22].

$$Q = Q_1 - Q_2, \qquad Q_1, Q_2 \succeq 0 \tag{22}$$

Such a convex-concave decomposition is not unique. In general, the search for Q_1 and Q_2 resorts to the feasibility problem of a LMI with element-wise linear equality constraints. Hence, the interior-point algorithm can be used. However, a more simpler way to obtain such two positive semidefinite matrices Q_1 and Q_2 is based on the eigenvalue decomposition of Q.

$$\begin{aligned}\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} &= V \begin{bmatrix} -I & 0 \\ 0 & I \end{bmatrix} V^T \\ &= \underbrace{V \begin{bmatrix} 0 \\ I \end{bmatrix} \begin{bmatrix} 0 & I \end{bmatrix} V^T}_{Q_1} - \underbrace{V \begin{bmatrix} I \\ 0 \end{bmatrix} \begin{bmatrix} I & 0 \end{bmatrix} V^T}_{Q_2}\end{aligned} \tag{23}$$

where the matrix V is composed to the eigenvectors of Q. It is quite obvious that Q_1 and Q_2 in (23) satisfy the positive semidefinite constraint in (22). Then, the bilinear matrix $B(X,Y)$ can be represented as the following convex-concave form

$$B(X,Y) = \begin{bmatrix} X^T & Y^T \end{bmatrix} Q_1 \begin{bmatrix} X \\ Y \end{bmatrix} + F - \begin{bmatrix} X^T & Y^T \end{bmatrix} Q_2 \begin{bmatrix} X \\ Y \end{bmatrix} \tag{24}$$

In the following algorithm analysis, the two convex matrix functions in (24) are denoted as $G_1(X,Y)$ and $G_2(X,Y)$ respectively.

$$G_1(X,Y) = \begin{bmatrix} X^T & Y^T \end{bmatrix} Q_1 \begin{bmatrix} X \\ Y \end{bmatrix} + F, \quad G_2(X,Y) = \begin{bmatrix} X^T & Y^T \end{bmatrix} Q_2 \begin{bmatrix} X \\ Y \end{bmatrix} \tag{25}$$

4.4. Linearization of a BMI

Next, it will be shown that a convex constraint for the decision variables can be obtained through the linearization of just the concave matrix function around the feasible solution of the relaxed BMI constraint. Hence, this method provides the possibility to search for a better solution with a smaller value of t by applying an efficient convex optimization algorithm. At first, let's review the derivative of a matrix function.

Definition 2
[17] Suppose $G(x)$ is a matrix-valued mapping from R^n to $R^{p\times p}$, The derivative DG at x^* is defined as

$$DG(x^*)h = \sum_{i=1}^{n} h_i \frac{\partial G}{\partial x_i}(x^*), \quad \forall\, h \in R^n \tag{26}$$

Suppose X and Y are affine matrix functions of the decision variables $\alpha_1, \dots, \alpha_n$ and $\beta_1, \dots, \beta_m$ defined in (15). Then, the convex matrix function $G_2(X,Y)$ in (25) becomes

$$G_2(X,Y) = \begin{bmatrix} X_0 + \sum\limits_{i=1}^{n} \alpha_i X_i \\ Y_0 + \sum\limits_{j=1}^{m} \beta_j Y_j \end{bmatrix}^T Q_2 \begin{bmatrix} X_0 + \sum\limits_{i=1}^{n} \alpha_i X_i \\ Y_0 + \sum\limits_{j=1}^{m} \beta_j Y_j \end{bmatrix} \tag{27}$$

According to the definition in (26), the derivative of the matrix function $G_2(X,Y)$ at the available feasible solution $\alpha^{(k)} = (\alpha_1^{(k)}, \dots, \alpha_n^{(k)})$ and $\beta^{(k)} = (\beta_1^{(k)}, \dots, \beta_m^{(k)})$ has the following form

$$\begin{aligned}&DG_2(X^{(k)}, Y^{(k)})h^{(k)} = \\ &\begin{bmatrix} \sum\limits_{i=1}^{n} \left(\alpha_i - \alpha_i^{(k)}\right) X_i \\ \sum\limits_{j=1}^{m} \left(\beta_j - \beta_j^{(k)}\right) Y_j \end{bmatrix}^T Q_2 \begin{bmatrix} X_0 + \sum\limits_{i=1}^{n} \alpha_i^{(k)} X_i \\ Y_0 + \sum\limits_{j=1}^{m} \beta_j^{(k)} Y_j \end{bmatrix} + \begin{bmatrix} X_0 + \sum\limits_{i=1}^{n} \alpha_i^{(k)} X_i \\ Y_0 + \sum\limits_{j=1}^{m} \beta_j^{(k)} Y_j \end{bmatrix}^T Q_2 \begin{bmatrix} \sum\limits_{i=1}^{n} \left(\alpha_i - \alpha_i^{(k)}\right) X_i \\ \sum\limits_{j=1}^{m} \left(\beta_j - \beta_j^{(k)}\right) Y_j \end{bmatrix} \\ &= \begin{bmatrix} X \\ Y \end{bmatrix}^T Q_2 \begin{bmatrix} X^{(k)} \\ Y^{(k)} \end{bmatrix} + \begin{bmatrix} X^{(k)} \\ Y^{(k)} \end{bmatrix}^T Q_2 \begin{bmatrix} X \\ Y \end{bmatrix} - 2 \begin{bmatrix} X^{(k)} \\ Y^{(k)} \end{bmatrix}^T Q_2 \begin{bmatrix} X^{(k)} \\ Y^{(k)} \end{bmatrix}\end{aligned} \tag{28}$$

where the elements of the vector $h^{(k)}$ are composed of $\alpha_i - \alpha_i^{(k)}$, $\beta_j - \beta_j^{(k)}$ with $i = 1, \dots, n, j = 1, \dots, m$.

Lemma 3
[17] A matrix function $G(\cdot)$ is positive convex on a convex subset $E \subseteq R^n$ if and only if the following semidefinite constraint is satisfied for all $x, y \in E$

$$G(y) - G(x) \succeq DG(x)(y - x) \tag{29}$$

Applying (29), the lower bound of the convex matrix function $G_2(X, Y)$ can be obtained as

$$G_2(X, Y) \succeq G_2(X^{(k)}, Y^{(k)}) + DG_2(X^{(k)}, Y^{(k)})h^{(k)} \tag{30}$$

In the subsequent analysis, the linearization of the matrix function $G_2(X, Y)$ at $(X^{(k)}, Y^{(k)})$ will be denoted as $LG_2(X^{(k)}, Y^{(k)})$:

$$LG_2(X^{(k)}, Y^{(k)}) = G_2(X^{(k)}, Y^{(k)}) + DG_2(X^{(k)}, Y^{(k)})h^{(k)} \tag{31}$$

Consequently, an upper bound of the bilinear matrix function $B(X, Y) - tI$ is

$$B(X, Y) - tI \preceq G_1(X, Y) - LG_2(X^{(k)}, Y^{(k)}) - tI \tag{32}$$

Instead of considering the nonconvex BMIs constraint $B(X, Y) - tI \preceq 0$, the following convex quadratic matrix inequality (QMI) constraint will be studied.

$$G_1(X, Y) - LG_2(X^{(k)}, Y^{(k)}) - tI \preceq 0 \tag{33}$$

An improved solution with a lower value for t can be obtained by solving the following LMI optimization problem.

$$\begin{aligned} &\textbf{minimize} \quad t + \tfrac{\rho_\alpha}{2}\|\alpha - \alpha^{(k)}\|_2^2 + \tfrac{\rho_\beta}{2}\|\beta - \beta^{(k)}\|_2^2 \\ &\textbf{subject to} \\ &\begin{bmatrix} -I & \begin{bmatrix} 0 & I \end{bmatrix} V^T \begin{bmatrix} X \\ Y \end{bmatrix} \\ * & F - LG_2(X^{(k)}, Y^{(k)}) - tI \end{bmatrix} \preceq 0 \end{aligned} \tag{34}$$

where $*$ can be deduced from symmetry. $\rho_\alpha, \rho_\beta > 0$ are the regularization parameters and 0, I are the zero and identity matrices with compatible dimensions. The LMI constraint in (34) is equivalent to the convex QMI constraint in (33) according to Schur Complement [13] [7].

Remark 3
Since the decision variable t as a cost function in (13) is just a linear one, then the two regularization parameters ρ_α, ρ_β in (34) have to be strictly positive to enforce the strong convexity, which sufficiently guarantees the optimal value of t in (34) satisfying $t - t^{(k)} < 0$ as proved in [17].

4.5. Augmented Sequential LMI algorithm

Although the convex LMI optimization problem in (34) can be applied iteratively to obtain a sequence of the slack variable $t^{(0)}, t^{(1)}, t^{(2)}, \dots$ that decreases monotonically, the numerical condition of the LMI constraint may continuously deteriorate. For example, the decision variable β is composed of the Lyapunov matrix P as $\beta = \text{vec}(P)$ in the output feedback control design application. The condition number of P may become unexpected large as the sequential LMI algorithm proceeds. Unfortunately, such a bad scaled Lyapunov matrix P returned by the solver in the current iteration will become a given matrix parameter in the subsequent iterations, which will result in a failure of the interior point algorithm before the convergence to a local KKT point or the appearance of a negative value of t [5]. To improve the numerical stability, a constraint on the condition number of the Lyapunov matrix is necessary, which results in an augmentation of the optimization problem in (34) is shown as

$$\begin{array}{ll}\textbf{minimize} & t + \frac{\rho_\gamma}{2}\|\gamma_c\|_2^2 + \frac{\rho_\alpha}{2}\|\alpha - \alpha^{(k)}\|_2^2 + \frac{\rho_\beta}{2}\|\beta - \beta^{(k)}\|_2^2 \\ \textbf{subject to} & \\ & \mu I \preceq P \preceq \gamma_c \mu I \\ & \begin{bmatrix} -I & \begin{bmatrix} 0 & I \end{bmatrix} V^T \begin{bmatrix} X \\ Y \end{bmatrix} \\ * & F - LG_2(X^{(k)}, Y^{(k)}) - tI \end{bmatrix} \preceq 0 \end{array} \tag{35}$$

where $\rho_\gamma, \mu > 0$ are given penalty parameters and lower bound of the eigenvalues of the Lyapunov matrix P respectively. The additional decision variable γ_c denotes the upper bound of the condition number of the Lyapunov matrix.

Another difference with [17] is the selection of the starting point of the sequential LMI algorithm. Although the convergence condition requires the BMI constraint to be strictly feasible at the initial solution, which is not easy in general, finding a set of decision variables, such that all the eigenvalues of the bilinear matrix function in (13) are strictly less than a large enough value of t, is not a hard task. An LMI optimization method with a prior selection of one of the two sets of the decision variables to search for an initial solution will be presented below. Suppose $\beta = \text{vec}(P)$ and a set of numerical values $\beta_1^{(0)}, \ldots, \beta_m^{(0)}$ are preselected as the candidate for the feasible solution in the BMI constraint in (13). Then, the feasible solution $\alpha_1^{(0)}, \ldots, \alpha_n^{(0)}$ of the other set of decision variables $\alpha_1, \ldots, \alpha_n$ together with t can be found by solving the following LMI optimization problem.

$$\begin{array}{ll}\textbf{minimize} & t \\ \textbf{subject to} & \\ & B(X, Y^{(0)}) - tI \preceq \epsilon I \end{array} \tag{36}$$

where $Y^{(0)} = Y_0 + \sum_{j=1}^m \beta_j^{(0)} Y_j$ and $\epsilon > 0$ is a given parameter that guarantees the strict feasibility of the initial solution in the BMI constraint in (13). The solution of (36) will be denoted as $X^{(0)} = X_0 + \sum_{i=1}^n \alpha_i^{(0)} X_i$ and $t^{(0)}$. Since the initial condition number $\gamma_c^{(0)}$ is naturally predetermined together with $\beta^{(0)}$, it is unnecessary to add a penalty term in the cost function and the LMI constraint for γ_c as in (35).

The augmented sequential LMI algorithm for the feasibility analysis of the BMI constraint is summarized in Algorithm 1.

Algorithm 1: Sequential LMI Algorithm for the BMI Constraint $B(X, Y) \preceq 0$

Step 1: Select a set of numerical values $\beta_1^{(0)}, \ldots, \beta_m^{(0)}$ and solve the LMI optimization problem in (36) to obtain $\alpha_1^{(0)}, \ldots, \alpha_n^{(0)}$ and $t^{(0)}$

Step 2: If the optimized solution $t^{(0)} < 0$, stop the algorithm and return $\alpha_1^{(0)}, \ldots, \alpha_n^{(0)}$ and $\beta_1^{(0)}, \ldots, \beta_m^{(0)}$ as the solution. Otherwise, jump to Step 3

Step 3: Linearize the concave matrix function $G_2(X, Y)$ around the available feasible solution $\alpha_1^{(k)}, \ldots, \alpha_n^{(k)}$ and $\beta_1^{(k)}, \ldots, \beta_m^{(k)}$

Step 4: Solve the LMI optimization problem in (35) to obtain the new solution $\alpha_1^{(k+1)}, \ldots, \alpha_n^{(k+1)}$ and $\beta_1^{(k+1)}, \ldots, \beta_m^{(k+1)}$ together with $t^{(k+1)}$

Step 5: If $t^{(k+1)} < 0$, stop the iteration and return $\alpha_1^{(k+1)}, \ldots, \alpha_n^{(k+1)}$ and $\beta_1^{(k+1)}, \ldots, \beta_m^{(k+1)}$ as the solution. Otherwise, go back to Step 3 until the stop criterion is reached.

4.6. Convergence Analysis

Although the Algorithm 1 will generate a monotonic decreasing sequence $\{t^{(k)}\}$, it is not enough to guarantee the convergence to the KKT point. In general, the application of the convex-concave approach for the nonlinear semidefinite programming problem requires to formulate the optimization problem in the following form

$$\begin{array}{c}\textbf{minimize} \quad f(x) \\ \textbf{subject to} \\ Q_1(x) - Q_2(x) \preceq 0\end{array} \tag{37}$$

where all the decision variables are collected in $x \in \mathbb{R}^n$ and $f(x)$ is a convex cost function. $Q_1(x), Q_2(x)$ are both positive semidefinite convex matrix functions. The convex subproblem that corresponds to (35) for the eigenvalue problem in the sequential algorithm is

$$\begin{array}{c}\textbf{minimize} \quad f(x) + \frac{\rho_k}{2}\|W_k(x - x^{(k)})\|_2^2 \\ \textbf{subject to} \\ Q_1(x) - DQ_2(x^{(k)})(x - x^{(k)}) \preceq 0\end{array} \tag{38}$$

where $\rho_k \geq 0$ and W_k is a given projection matrix. The semidefinite constraint in (38) is strictly feasible if the available solution $x^{(k)}$ makes the semidefinite constraint in (37) strictly feasible, which guarantees the existence of the KKT point of the convex subproblem [6] [17]. Applying the interior-point algorithm, both the primal variable $x^{(k+1)}$ and the dual variable $\Lambda^{(k+1)}$ satisfying the KKT condition of (38) can be obtained. The convergence of the sequential convex optimization algorithm for the nonlinear semidefinite programming problem in (37) together with the prerequisite assumptions is rigorously stated in Theorem 2.

Theorem 2
([17]): Let $\mathbb{D}$ denotes the feasible set of the nonlinear semidefinite constraint in (37) that is bounded in $\mathbb{R}^n$. Suppose that $f(x)$ in (37) is bounded from below on $\mathbb{D}$ under the following three assumptions.

1) The relative interior of $\mathbb{D}$ is nonempty;
2) The matrix function $Q_1(x)$ in (37) is Schur positive definite convex. In addition, $f(x)$ is convex quadratic on $\mathbb{R}^n$ with a convexity parameter $\rho_f \geq 0$;
3) The convex subproblem in (38) is solvable and satisfies the strong second order sufficient condition.

Let $\{x^{(k)}, \Lambda^{(k)}\}$ be a sequence of the primal and dual variables generated by Algorithm 1 in [17]. If either $f(x)$ is strongly convex or $\rho_k \equiv \rho > 0$ and $W_k \equiv W$ is full-row-rank for all $k \geq 0$ then every accumulation point $\{x^*, \Lambda^*\}$ of $\{x^{(k)}, \Lambda^{(k)}\}$ is a KKT point of (37). Moreover, if the set of the KKT points of (37) is finite, then the sequence $\{x^{(k)}, \Lambda^{(k)}\}$ converges to a KKT point of (37).

The eigenvalue problem defined in (13) is a special case of the nonlinear semidefinite programming problem in (37). The following corollary proves the convergence of the convex-concave based Algorithm 1 to a KKT point.

Corollary 1
For the BMI constrained eigenvalue problem in (13), the sequence of the primal variables $\{\alpha^{(k)}, \beta^{(k)}\}$ together with the dual variable $\Lambda^{(k)}$ generated by the Algorithm 1 converges to a KKT point of the following augmented eigenvalue problem.

$$\begin{array}{c}\textbf{minimize} \quad t + \frac{\rho_\gamma}{2}\|\gamma_c\|_2^2 \\ \textbf{subject to} \\ \mu I \preceq P \preceq \gamma_c \mu I \\ B(X, Y) - tI \preceq 0\end{array} \tag{39}$$

where $\mu > 0$ is a given parameter.

Proof
First, it is obvious that the convex subproblem in (35) comes from the linearization of the concave part of the BMI constraint in the augmented eigenvalue problem in (39). Thus, the relationship between (39) and (35) exactly matches that between (37) and (38) for the general nonlinear semidefinite programming problem. Due to the strictly positiveness of ϵ, the initial solution obtained from the LMI problem in (36) makes the BMI constraint in the augmented eigenvalue problem in (39) strictly feasible, which implies the satisfaction of the first assumption in Theorem 2. Next, the matrix function $Q_1(x)$ in (37) is Schur positive definite convex as proved in (23) and the cost function $t + \frac{\rho_\gamma}{2}\|\gamma_c\|_2^2$ is also a convex quadratic function in spite of the zero convexity parameter. For the third assumption, the convex subproblem in (35) is just a standard LMI optimization problem, in which the strict feasibility is guaranteed by the strict feasible initial solution. Hence, this convex subproblem is solvable and satisfies the strong second order sufficient condition. At last, the regularization parameters ρ_α, ρ_β are strictly positive and the projection matrix W is an identity matrix in all the iterations of the convex subproblem.

In summary, all the assumptions in Theorem 2 are satisfied for the BMI constrained eigenvalue problem defined in (39). Hence, the convergence of the Algorithm 1 to a KKT point is guaranteed. □

Although the additional penalty of the condition number in (39) leads to a less optimal value of t compared with the original eigenvalue problem in (13), this trade-off between performance and numerical stability is necessary for the success of the sequential LMI algorithm as shown in the next section. Finally, it is also worth to mention that the above sequential LMI algorithm will only converge to a local optimum of $t + \frac{\rho_\gamma}{2}\|\gamma_c\|_2^2$. Therefore, it cannot be concluded that the given BMIs constraint is infeasible in case that the algorithm fails to converge to a negative value.

5. VEHICLE LATERAL CONTROL APPLICATION

5.1. *Vehicle Dynamical Model and Lateral Control*

The above BMI based robust observer state feedback control design method will be applied to the development of a vehicle steering control system. While steering control for normal road and tire conditions has been extensively studied in literature [34]. A major challenge in this field is to design a rigorous control system that is robust to highly nonlinear and uncertain tire-road friction forces and the existing observer state feedback design methodology presented in [44] cannot generate such a robust control solution. Hence, it is a perfect example to demonstrate the superiority of the robust design method presented in this paper.

In the application, a general nonlinear tire model shown in (40) is adopted [34] [33] [42].

$$F_\alpha(\alpha) = \begin{cases} \mu F_z(3\theta\alpha - 3\theta^2\alpha^2\mathrm{sgn}(\alpha) + \theta^3\alpha^3), & \text{if } |\alpha| \le \frac{1}{\theta} \\ \mu F_z \mathrm{sgn}(\alpha) & \text{if } |\alpha| > \frac{1}{\theta} \end{cases} \tag{40}$$

where the slip angle α denotes either α_f or α_r and $F_\alpha(\alpha)$ is the corresponding tire force. The parameter θ is defined as $\theta = \frac{4a_c b_c k}{3\mu F_z}$, where μ and F_z represent the tire-road friction coefficient and normal load of the wheel. a_c is half-length of contact patch, b_c is half-width of the contact patch, and k is isotropic stiffness of tire elements per unit area of the belt surface. $1/\theta$ represents the value of slip at which a saturation of lateral tire force is reached. A typical curve of the nonlinear lateral tire-road friction model in (40) with $\mu = 0.85$, $F_z = 6000N$, $\theta = 4.5$ is shown in Fig. 3 below.

To simplify the controller design, the nonlinear part in the tire model in (40) is treated as an additive uncertainty to the linear part.

$$F_{\alpha,f} = \kappa_f \alpha_f + \Delta(\alpha_f), \qquad F_{\alpha,r} = \kappa_r \alpha_r + \Delta(\alpha_r) \tag{41}$$

where κ_f and κ_r are the coefficients such that the nonlinear function $\Delta(\alpha)$ with $\alpha = \alpha_f$ or α_r is a slope-restricted nonlinearity. The state space matrices in the linear generalized plant model in (1)

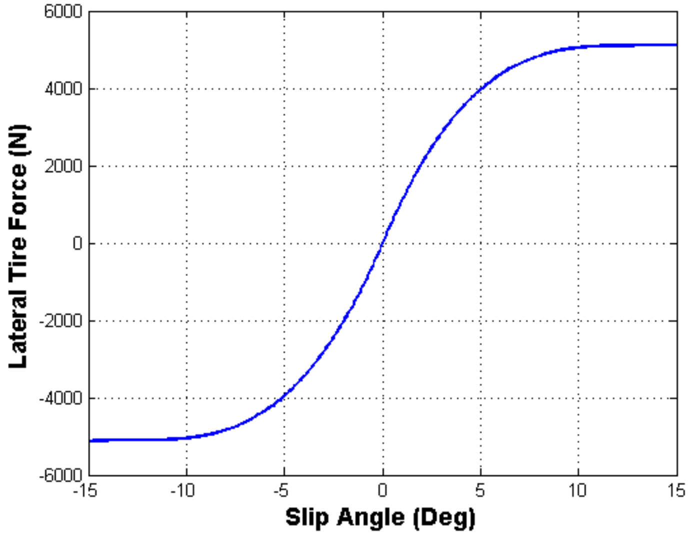

Figure 3. A typical curve of the nonlinear tire model in Eq. (40)

are shown below

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -\frac{\kappa_f+\kappa_r}{MV_x} & \frac{\kappa_f+\kappa_r}{M} & \frac{-\kappa_f l_f+\kappa_r l_r}{MV_x} \\ 0 & 0 & 0 & 1 \\ 0 & -\frac{\kappa_f l_f-\kappa_r l_r}{I_z V_x} & \frac{\kappa_f l_f-\kappa_r l_r}{I_z} & -\frac{\kappa_f l_f^2+\kappa_r l_r^2}{I_z V_x} \end{bmatrix}, \quad \begin{bmatrix} B_u & B_w \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ \frac{\kappa_f}{M} & \frac{1}{M} & \frac{1}{M} \\ 0 & 0 & 0 \\ \frac{\kappa_f l_f}{I_z} & \frac{l_f}{I_z} & -\frac{l_r}{I_z} \end{bmatrix} \tag{42}$$

where the state vector $x \in R^{4\times 1}$. The control and uncertainty input consist of the steering angle δ and $w = \begin{bmatrix} w_f & w_r \end{bmatrix}^T$ that denote the nonlinear parts in the front and rear tire model. The matrices corresponding to the measured output y and the output of the uncertain channel $z = \begin{bmatrix} \alpha_f & \alpha_r \end{bmatrix}^T$ are

$$C = \begin{bmatrix} 1 & 0 & d_s & 0 \\ 0 & -\frac{1}{V_x} & 1 & -\frac{l_f}{V_x} \\ 0 & -\frac{1}{V_x} & 1 & \frac{l_r}{V_x} \end{bmatrix}, \quad \begin{bmatrix} D_{yu} & D_{yw} \\ D_{zu} & D_{zw} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{43}$$

The nonlinear uncertainty block can be represented as $\begin{bmatrix} w_f & w_r \end{bmatrix}^T = \begin{bmatrix} \Delta(\alpha_f) & \Delta(\alpha_r) \end{bmatrix}^T$. Since the nonlinear function $\Delta(\alpha)$ is a slope-restricted nonlinearity with $\partial\Delta/\partial\alpha \in [0, \beta]$, the IQC condition can be constructed from the well-known Zames-Falb multiplier [43] [32].

$$\Pi(j\omega) = \begin{bmatrix} 0 & \beta M^*(j\omega) \\ \beta M(j\omega) & -M^*(j\omega) - M(j\omega) \end{bmatrix} \tag{44}$$

where $M(j\omega)$ is a diagonal complex matrix

$$M(j\omega) = \begin{bmatrix} 1 - h(j\omega) & 0 \\ 0 & 1 - h(j\omega) \end{bmatrix} \tag{45}$$

The rational function $h(s)$ is required to satisfy the $\mathcal{L}_1$ constraint $\|h\|_1 \leq 1$ [43] [32] [10]. One such example is $H(s) = \frac{b}{s+a}$ with $b < a$ [11]. The IQC multiplier $\Pi(j\omega)$ in (44) can be factorized as

$$\Pi(j\omega) = \begin{bmatrix} \Gamma^*(j\omega) & I \end{bmatrix} \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \begin{bmatrix} \Gamma(j\omega) \\ I \end{bmatrix} \tag{46}$$

with $\Gamma(j\omega)$ defined as

$$\Gamma(j\omega) = \begin{bmatrix} 0 & 0 \\ \beta M(j\omega) & -M(j\omega) \end{bmatrix} \tag{47}$$

By matching the state space matrices in (42) together with the factorization of the IQC multiplier in (46) to those in (5) (11) for the closed-loop system, the BMIs constraint in (12) can be obtained for this application. The Algorithm 1 is applied to search for the feasible solution of this BMIs constraint. Those regularization parameters used in the algorithm are shown below.

$$\rho_\gamma = 10^{-5}, \quad \mu = 10^{-4}, \quad \rho_\alpha = \rho_\beta = 2.5 \times 10^{-2}$$

The output of the sequential LMI algorithm can be seen in Fig. 4, which shows that the slack variable t starts from an initial value around 1.4 and approaches a value below 0 after 45 iterations. The simulation result of the vehicle control system can be seen in Fig. 5 which shows the asymptotic convergence of the vehicle states and the observer error in the presence of the IQC-type uncertain tire forces.

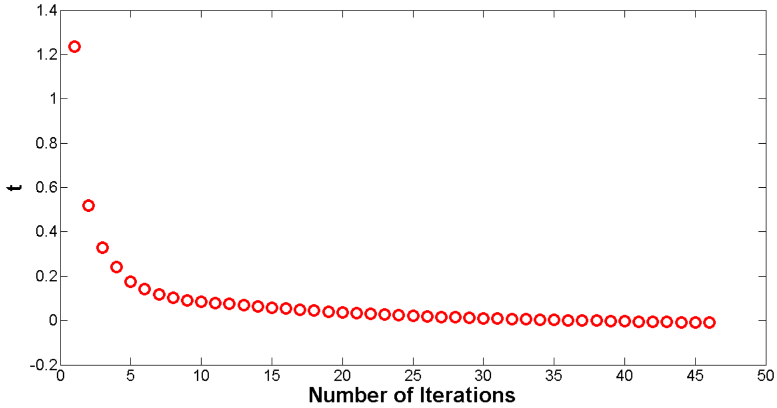

Figure 4. Sequence of t produced by the Sequential LMI Algorithm 1

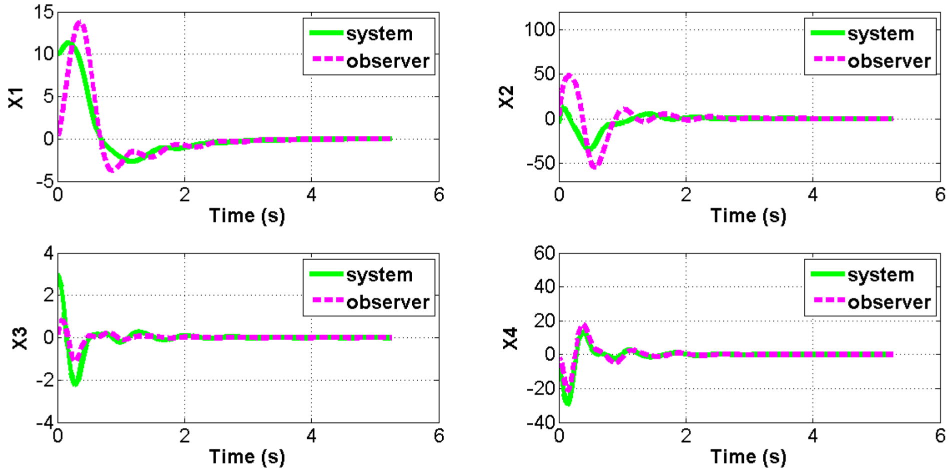

Figure 5. The simulation result of the vehicle lateral control system with uncertain tire forces

5.2. Condition Number Regularization

Finally, it is also worth to review the sequence of the condition number of the Lyapunov matrix generated by the sequential LMI algorithm. The result with the penalty term of γ_c can be seen in Fig. 6. It shows that the condition number is constrained below 9×10^4, which can be well handled by the SDP solver, before the appearance of a negative value of t. On the other hand, the result without the penalty term of γ_c is shown in Tab. I. As can be seen that the condition number grows rapidly as the iteration proceeds. After 4 iterations, the SDP solver fails to return a reasonable solution due to the extremely large condition number.

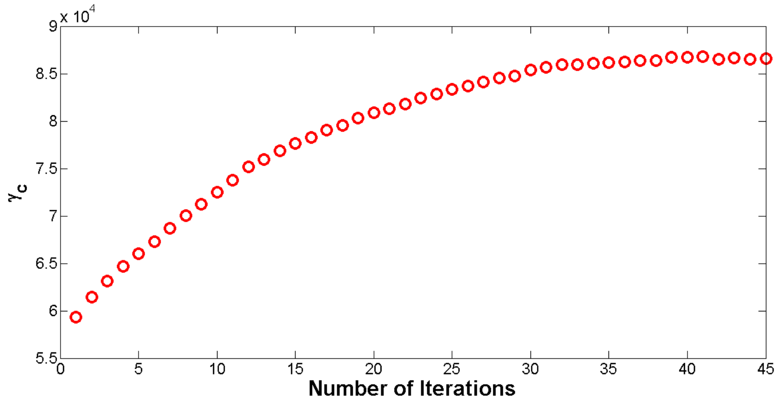

Figure 6. Sequence of γ_c with a penalty term in the cost function

Iteration	t	γ_c
1	1.2468	3.5142×10^5
2	0.4106	8.4916×10^7
3	0.2264	5.2843×10^{11}
4	0.1391	1.0364×10^{14}

Table I. Sequence of γ_c without a penalty term in the cost function

6. CONCLUSION AND FUTURE WORK

This paper studies the robust observer state feedback control design for the interconnection of the linear nominal system and the uncertainty block Δ. The searching for the controller gain and observer gain parameters against different structure of the uncertainty Δ is formulated as a feasibility problem of BMIs constraints, which is traditionally treated as a NP hard problem. To solve the dilemma, an augmented sequential LMI algorithm, which includes a penalty of the condition number of the Lyapunov matrix, is proposed to search for such a feasible solution. Finally, the proposed design methodology and the sequential LMI algorithm is applied to a vehicle lateral control problem. The simulation examples demonstrate the applicability of the proposed algorithm and necessity of the regularization of the condition number.

Future research of this convex-concave based sequential LMI algorithm will focus on a better way to improve the numerical stability and some further study of the application of this algorithm to other control systems design methodologies, such as fix-order controller design and robust model predictive control design.

ACKNOWLEDGEMENT

This work was supported by funding from the National Science Foundation under Grant CMMI 1562006 and was also supported in part by EPI Inria DISCO, L2S, CentraleSupélec, during the accademic position (Délégation Inria) of A. Zemouche from september 2016 to August 2017.

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