\textit{L}^\infty\text{-Stability of IMEX-BDF2 Finite Volume Scheme for Convection-Diffusion Equation}

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Abstract In this paper, we propose a finite volume scheme for solving a two-dimensional convection-diffusion equation on general meshes. This work is based on a implicit-explicit (IMEX) second order method and it is issued from the seminal paper [2]. In the framework of MUSCL methods, we will prove that the local maximum property is guaranteed under an explicit Courant-Friedrichs-Levy condition and the classical hypothesis for the triangulation of the domain.

Key words: Convection-diffusion equation, finite volume scheme, IMEX-BDF2 scheme, \( L^\infty \)-stability

MSC (2010): 65M99, 76M12, 76E17

1 Introduction

Convection-diffusion processes appear in many areas of science, e.g. fluid dynamics or heat and mass transfer. In the study of the evolution of a mixture, the system of PDEs derives from the compressible Navier-Stokes equations. The mixture of two viscous fluids is described by the density \( \rho \geq 0 \), the mass velocity field \( \mathbf{v} \) (which is not solenoidal) and the pressure \( p \). Following Kazhikhov and Smagulov [11], we set

\[
\mathbf{u} = \mathbf{v} + \lambda \nabla \ln(\rho),
\]

(1)
for some mass diffusion coefficient $\lambda > 0$. This Fick’s law describes the diffusive fluxes of one fluid into the other. Clearly, the volume velocity field $u$ satisfies $\text{div} u = 0$ and we obtain the non-standard constraint $\text{div} v = -\text{div}(\lambda \nabla \ln(p))$, which is relied on the definition of the pressure $p$. Using (1), the mass conservation equation becomes

$$\frac{\partial \rho}{\partial t} + \text{div}(\rho u) = \lambda \Delta \rho. \tag{2}$$

The momentum equation can also be rewritten in order to obtain the Kazhikhov-Smagulov model [11]. This model was firstly studied in [14, 1] (see also references therein). The mathematical analysis in a three-dimensional domain of Kazhikhov-Smagulov type models was carried out in recent works [5, 8], where the authors study the Kazhikhov-Smagulov models with a specific Korteweg stress tensor. The numerical study of a Kazhikhov-Smagulov model for the two-dimensional case can be found in [4], where the authors propose an hybrid finite volume-finite element method combined with the backward Euler method in time. In order to generalize the analysis given in [4] to second-order methods in time and space, the first goal is to recover the $L^\infty$-stability of the finite volume method used for the convection-diffusion equation. This is the purpose of this paper.

2 Description of the numerical scheme

This section is devoted to the design of a numerical scheme to approximate (2), using the vertex-based MUSCL finite volume methods introduced in [13] and used in [2] for a second-order accuracy in space, and an implicit-explicit (IMEX) linear multistep methods [10] for a second-order in time.

Mesh definitions and notations. Let $\Omega$ be an open bounded polygonal subset on $\mathbb{R}^2$, with sufficiently regular boundary $\partial \Omega$, and $[0,T]$ the time interval, for $T > 0$. The discretization of (2) will be carried out on an unstructured triangular mesh. We denote by $\mathcal{T}_h$ a partition of $\Omega$ composed of conforming and isotropic triangles $T_k$, $k \in [1,K]$, with $K \in \mathbb{N}^*$. The $\mathcal{T}_h$ is called the primal mesh. We suppose the following hypotheses:

(H1) Let $\{ \mathcal{T}_h \}_{h > 0}$ be a regular family of triangulations of $\Omega$.

(H2) The triangulation $\mathcal{T}_h$ is of weakly acute type (no triangle with an angle greater than $\pi/2$).

For each element $T \in \mathcal{T}_h$, we denote $B_T$ the barycenter of the triangle, $|T|$ the area of $T$, and $M_i, M_{j1}, M_{j2}$ the three vertices of $T$. We also denote respectively $M_{ij}$ and $M_{ij2}$ the middles of $[M_iM_{j1}]$ and $[M_iM_{j2}]$.

Let us construct the dual mesh $\mathcal{C}_h = \{ \mathcal{C}_i, i \in [1,I] \}$, which defines a second partition of $\Omega$, ($I \in \mathbb{N}^*$ is the number of vertices of $\mathcal{T}_h$). The dual finite volume $\mathcal{C}_i$ associated with each vertex $M_i$, $i \in [1,I]$, is a closed polygon obtained in the following way: we join the barycenter $B_T$ of every triangle $T \in \mathcal{T}_h$ which share the vertex $M_i$ with
Here, we describe the finite volume scheme for solving (2). In order to obtain the density reconstruction on the interfaces $\Gamma_{i}^{(T)}$, we use the MUSCL technique with a multislope gradient reconstruction. Concerning the time discretization, we adapt the implicit-explicit (IMEX) linear 2-step methods using extrapolated BDF2 scheme for the convective term combined with implicit BDF2 scheme for the diffusive term. The velocity field $u(t, x) \in \mathbb{R}^2$ is a given function verifying the divergence free condition. For the space discretization, the usual vertex-based finite volume scheme on control volume $\mathcal{C}_i$, for all $i \in [1, I]$, reads:

$$\sum_{j=1}^{2} |\Gamma_{ij}^{(T)}| n_{ij}^{(T)} = -|T| \nabla \psi_i,$$

where $\psi_i$ is the $P_1$ basis function associated to the vertex $M_i$ of $T$. For every $\mathcal{C}_i \in \mathcal{E}_h$, the boundary of $\mathcal{C}_i$ is

$$\partial \mathcal{C}_i = \bigcup_{T, M_i \in T} \left( \Gamma_{ij_1}^{(T)} \cup \Gamma_{ij_2}^{(T)} \right).$$

**IMEX-BDF2 finite volume scheme.** Here, we describe the finite volume scheme for solving (2). In order to obtain the density reconstruction on the interfaces $\Gamma_{i}^{(T)}$, we use the MUSCL technique with a multislope gradient reconstruction. Concerning the time discretization, we adapt the implicit-explicit (IMEX) linear 2-step methods using extrapolated BDF2 scheme for the convective term combined with implicit BDF2 scheme for the diffusive term. The velocity field $u(t, x) \in \mathbb{R}^2$ is a given function verifying the divergence free condition. For the space discretization, the usual vertex-based finite volume scheme on control volume $\mathcal{C}_i$, for all $i \in [1, I]$, reads:

![Fig. 1 Dual mesh-Vertex based control volume $\mathcal{C}_i$ around the node $M_i$.](image)
\[
\frac{d}{dt} \int_{\mathcal{G}_i} \rho(t, x) dx + \int_{\partial \mathcal{G}_i} \rho(t, x) \mathbf{u}(t, x) \cdot \mathbf{n} d\sigma = \lambda \int_{\partial \mathcal{G}_i} \nabla \rho(t, x) \cdot \mathbf{n} d\sigma. \tag{5}
\]

We denote by \( \Delta t \) the time step and \( t^n = n \Delta t, \ n \geq 0 \), but variable time steps can also be used. Then, the approximate solution \( \rho^n_i, \ i \in [1, l] \), at time \( t^n \), verifies

\[
\rho^n_i \approx \frac{1}{|\mathcal{G}_i|} \int_{\mathcal{G}_i} \rho(t^n, x) dx.
\]

In particular, the numerical approximation of the density is a piecewise constant function in space on the control volume \( \mathcal{G}_i \). For the time discretization, we consider the implicit BDF2 scheme and an extrapolated BDF2 scheme following [10]. With this choice, we obtain a second-order accuracy in time. Then, the equation (5) is rewritten as follows, for each \( i \in [1, l] \) and \( n \geq 1 \),

\[
\rho^{n+1}_i - \frac{2\lambda}{3} \int_{|\mathcal{G}_i|} \nabla \rho(t^{n+1}, x) \cdot \mathbf{n} d\sigma = \frac{4}{3} \rho^n_i - \frac{4}{3} \left( \int_{\partial \mathcal{G}_i} \rho(t^n, x) \mathbf{u}(t^n, x) \cdot \mathbf{n} d\sigma \right) - \frac{1}{3} \rho^{n-1}_i + \frac{2}{3} \left( \int_{|\mathcal{G}_i|} \rho(t^{n-1}, x) \mathbf{u}(t^{n-1}, x) \cdot \mathbf{n} d\sigma \right).
\tag{6}
\]

In order to approximate \( \nabla \rho(t^{n+1}, x) \) in (6), we consider a \( P_1 \)-finite element approach for the density such that

\[
\rho^{n+1}_{|T} \approx \sum_{M_j \in T} \psi_j \rho^{n+1}_j, \quad \text{for all } T \in \mathcal{T}_h,
\]

with \( \{ \psi_j \}_{j \in [1, l]} \) the canonical basis of the usual \( P_1 \) finite element space. Using (3) and (4), we find \( \rho^{n+1}_i, \ i \in [1, l], \ n \geq 1 \), verifying the following second-order IMEX-BDF2 finite volume scheme:

\[
\rho^{n+1}_i + \frac{2\lambda}{3} \frac{\Delta t}{|\mathcal{G}_i|} \sum_{T, M_j \in T} |T| \sum_{M_j \in T} \nabla \psi_i \cdot \nabla \rho^{n+1}_j = \frac{4}{3} \rho^n_i - \frac{4}{3} \frac{\Delta t}{|\mathcal{G}_i|} \sum_{T, M_j \in T} 2|\Gamma_{ij}^{(T)}| \sum_{k=1}^l G_i^{n} \left( \rho^n_{ij}, \rho^n_{jk} \right) \tag{7}
\]

\[
- \frac{1}{3} \rho^{n-1}_i + \frac{2}{3} \frac{\Delta t}{|\mathcal{G}_i|} \sum_{T, M_j \in T} 2|\Gamma_{ij}^{(T)}| \sum_{k=1}^l G_i^{n-1} \left( \rho^{n-1}_{ij}, \rho^{n-1}_{jk} \right).
\]

Here we denote by \( G_{ij} \left( \rho_1, \rho_2 \right) \) a numerical flux that satisfies the consistency, conservativity and monotonicity properties. In particular, for any constant function \( \rho_1 \), we have

\[
\sum_{k \in T(i)} |\Gamma_{ik}^{(T)}| G_{ik} \left( \rho_1, \rho_1 \right) = 0. \tag{8}
\]

In [2], \( G_{ij} \) is the upstream flux, but many other numerical fluxes can be considered, as for instance Lax-Friedrichs or Engquist-Osher fluxes. We underline that for multi-
physics coupled models, a particular attention must be paid in the approximation of the continuous velocity associated to any point of ∂Ω_h (see [3]). In (7), ρ_{ij} and ρ_{ji} denote the density reconstructions on the segments \( \Gamma_{ij}^{(T)} \), for \( l = 1, 2 \). In order to reach a second-order accuracy in space, we use the MUSCL technique [13] with a multislope gradient reconstruction. Introducing \( M = \{ M_i, M_j \} \cap (M_i A_{ij}^{(T)}) \) and \( \tilde{N} \in [N_i, N_j] \cap (M_i A_{ij}^{(T)}) \), we define

\[
\begin{align*}
p_{ij}^{\text{up}} &= \frac{\rho_i - \rho_{\tilde{h}}}{||M_{i\tilde{N}}||} \quad \text{and} \quad p_{ij}^{\text{down}} = \frac{\rho_{\tilde{M}} - \rho_i}{||M_{i\tilde{M}}||}.
\end{align*}
\]

Then, \( \rho_{ij} \) is the density evaluated at node \( A_{ij}^{(T)} \), defined as:

\[
\rho_{ij} = \rho_i + p_{ij} ||M_{A_{ij}^{(T)}}||, \quad \text{with} \quad p_{ij} = p_{ij}^{\text{up}} \text{Lim} \left( \frac{p_{ij}^{\text{down}}}{p_{ij}^{\text{up}}} \right),
\]

where \( \text{Lim} \) is a so-called ”τ-limiter” (for details see [2]). In particular, they have the following result:

**Lemma 1.** There exists some coefficients \( \omega_{ijk} \geq 0 \), \( k \in \mathcal{V}(i) \), such that

\[
\rho_{ij} - \rho_i = \sum_{k \in \mathcal{V}(i)} \omega_{ijk} (\rho_i - \rho_k)
\]

holds, and furthermore, they verify \( \sum_{k \in \mathcal{V}(i)} \omega_{ijk} \leq \frac{12}{\tau} C_{\beta_h} \), where the constant \( C_{\beta_h} \) characterizes the mesh regularity (but it is more general than the classical Ciarlet ratio) and \( \tau > 0 \) is used in the definition of the τ-limiter.

### 3 \( L^\infty \)-stability of the numerical scheme

The IMEX-BDF2 finite volume scheme (7) is rewritten as linear system:

\[
A^n \rho^{n+1} = F^n,
\]

where the matrix \( A \) and the right hand side \( F^n \) are defined as follows:

\[
A_{i,j} = 1 + 2 \lambda \Delta t \sum_{T, M, \in \mathcal{E}} ||\nabla \psi_i||^2, \quad A_{i,j} = 2 \lambda \Delta t \sum_{T, M, \in \mathcal{E}} \nabla \psi_i \cdot \nabla \psi_j, \quad \forall i, j \in [1, I],
\]

\[
F^n_i = \frac{4}{3} \rho_i^n - \frac{4}{3} \frac{\Delta t}{\|\mathcal{E}_i\|} \sum_{T, M, \in \mathcal{E}} \sum_{l=1}^2 ||\Gamma_{ij}^{(T)}|| G^n_{ij} (\rho^n_i, \rho^n_{ji}),
\]

\[
- \frac{1}{3} \rho_i^{n-1} + \frac{2}{3} \frac{\Delta t}{\|\mathcal{E}_i\|} \sum_{T, M, \in \mathcal{E}} \sum_{l=1}^2 ||\Gamma_{ij}^{(T)}|| G^{n-1}_{ij} (\rho^{n-1}_i, \rho^{n-1}_{ji}), \quad \forall i \in [1, I].
\]

Under the hypotheses (H1) and (H2) on the mesh \( \mathcal{H}_h \), the matrix \( A \) is an M-matrix.
Remark 1. The hypothesis (H2), necessary to establish error estimates, is classical for the vertex-centered finite volume scheme [12] or the combined finite volume-finite element scheme [9]. Obviously, the M-matrix property still holds for Delaunay triangulations (see [6], Sect. 3.4).

Now, we prove the following result:

**Proposition 1.** If for any \( n \geq 1 \), we have \( \rho^{n-1} \geq 0 \) and \( \rho^n \geq 0 \), then the right hand side of linear system (9) satisfy \( F^n \geq 0 \) under the CFL condition:

\[
\Delta t \leq \min_{1 \leq i \leq l} \left( \frac{7\tau}{12} C_{\delta_i} + 2 \right) \| u \|_{i,\infty} \sum_{T,M_i \in T} \left( \| R^{(T)}_{ij} \| + | R^{(T)}_{ij} | \right) \tag{10}
\]

with \( \| u \|_{i,\infty} = \max_{T,M_i \in T} \| u_T \|_{L^2(\mathbb{R}^2)} \) where \( u_T \) is the cell average velocity.

**Proof.** Let \( i \in [1,l] \) and \( n \geq 1 \). Thanks to (8), the i-th row of (9) is given by:

\[
\begin{align*}
\left( A \rho^{n+1} \right)_i &= \frac{4}{3} \rho_i^n - \frac{4}{3} \sum_{T,M_i \in T} \sum_{l=1}^2 \left( G^{(T)}_{ij} \left( \rho^n_{ij}, \rho^n_{lj} \right) - G^{(T)}_{ij} \left( \rho^n_{ij}, \rho^n_{lj} \right) \right) \\
&\quad - \frac{1}{3} \rho_i^{n-1} + \frac{2}{3} \Delta t \sum_{T,M_i \in T} \sum_{l=1}^2 \left( G^{(T)}_{ij} \left( \rho^n_{ij}, \rho^n_{lj} \right) - G^{(T)}_{ij} \left( \rho^n_{ij}, \rho^n_{lj} \right) \right) \tag{11}
\end{align*}
\]

Let us introduce some definitions and notations dropping the time indices, such that

\[
\Delta \rho_{ij} = \rho_{ij} - \rho_i, \quad \Delta \rho_{ij} = \rho_{ij} - \rho_i, \quad \text{for } l = 1, 2.
\]

Thanks to Lemma 1, there exists for \( l = 1, 2 \), some coefficients \( \omega_{ijk} \geq 0, k \in \mathcal{V}(i) \), such that

\[
\Delta \rho_{ij} = \sum_{k \in \mathcal{V}(i)} \omega_{ijk} \left( \rho_i - \rho_k \right), \quad \text{with } \sum_{k \in \mathcal{V}(i)} \omega_{ijk} \leq \frac{7\tau}{12} C_{\delta_i}.
\]

Also, there exists for \( l = 1, 2 \), some coefficients \( \bar{\omega}_{ijk} \geq 0, k \in \mathcal{V}(i) \), such that

\[
\bar{\Delta} \rho_{ij} = \sum_{k \in \mathcal{V}(i)} \bar{\omega}_{ijk} \left( \rho_k - \rho_i \right), \quad \text{with } \sum_{k \in \mathcal{V}(i)} \bar{\omega}_{ijk} \leq 2.
\]

Next, for \( 0 < \delta_{ij} < 1, l = 1, 2 \), we consider the following quantities:

\[
\begin{align*}
E_{ij} &= \frac{| R^{(T)}_{ij} |}{| G_i |} \frac{\partial G_{ij}}{\partial \rho_1} \left( \rho_i + \delta_{ij} \Delta \rho_{ij}, \rho_i + \delta_{ij} \bar{\Delta} \rho_{ij} \right), \quad l = 1, 2, \\
F_{ij} &= -\frac{| R^{(T)}_{ij} |}{| G_i |} \frac{\partial G_{ij}}{\partial \rho_2} \left( \rho_i + \delta_{ij} \Delta \rho_{ij}, \rho_i + \delta_{ij} \bar{\Delta} \rho_{ij} \right), \quad l = 1, 2.
\end{align*}
\]

Of course, by monotonicity of the numerical flux, we have \( E_{ij} \geq 0 \) and \( F_{ij} \geq 0 \). Hence, using the mean value theorem, the numerical scheme (11) is rewritten as
follows:

$$\left( A \rho^{n+1} \right)_i = \frac{4}{3} \rho^n_i - \frac{4}{3} \Delta t \sum_{T,M \in T} \sum_{k \in \mathcal{Y}(i)} \left( \sum_{l=1}^{2} \left( \omega_{ijkl} E_{ijkl}^n (\rho^n_i - \rho^n_k) - \tilde{\omega}_{ijkl} F_{ijkl}^n (\rho^n_k - \rho^n_l) \right) \right) - \frac{1}{3} \rho^{n-1}_i + \frac{2}{3} \Delta t \sum_{T,M \in T} \sum_{k \in \mathcal{Y}(i)} \left( \sum_{l=1}^{2} \left( \omega_{ijkl} E_{ijkl}^{n-1} (\rho^{n-1}_i - \rho^{n-1}_k) - \tilde{\omega}_{ijkl} F_{ijkl}^{n-1} (\rho^{n-1}_k - \rho^{n-1}_l) \right) \right).$$

(12)

Finally, we obtain the following equations for each $i \in [1,I]$ and for all $n \geq 1$:

$$\left( A \rho^{n+1} \right)_i = a_{ii} \rho^n_i + b_{ii} \rho^{n-1}_i + \sum_{k \in \mathcal{Y}(i)} (a_{ik} \rho^n_k + b_{ik} \rho^{n-1}_k),$$

(13)

where $a_{ii}$, $b_{ii}$, $a_{ik}$ and $b_{ik}$ are easily determined from (12). Clearly, we have

$$a_{ii} + b_{ii} + \sum_{k \in \mathcal{Y}(i)} (a_{ik} + b_{ik}) = 1.$$  

(14)

Moreover, by choosing the time step $\Delta t$ such that for all $i \in [1,I]$,

$$\Delta t \leq \left( \sum_{T,M \in T} \sum_{k \in \mathcal{Y}(i)} \left( \sum_{l=1}^{2} \left( \omega_{ijkl} E_{ijkl}^n + \tilde{\omega}_{ijkl} F_{ijkl}^n \right) - \frac{2}{3} \left( \omega_{ijkl} E_{ijkl}^{n-1} + \tilde{\omega}_{ijkl} F_{ijkl}^{n-1} \right) \right) \right)^{-1},$$

(15)

we have

$$0 \leq a_{ii} + b_{ii} \leq 1 \quad \text{and} \quad 0 \leq a_{ik} + b_{ik} \leq 1.$$  

(16)

Hence, (13), (14) and (16) allow us to conclude that for each $i \in [1,I]$,

$$\left( A \rho^{n+1} \right)_i$$

is written as convex combination of $\rho^n_i$, $\rho^{n-1}_i$, $\rho^n_k$ and $\rho^{n-1}_k$, $k \in \mathcal{Y}(i)$. □

Finally, as a consequence of Proposition 1, and recalling that an M-matrix is invertible with positive inverse, we obtain:

**Theorem 1.** Let the velocity field $u$ divergence free and the initial density $\rho_0$ such that $\rho_0(x) \geq 0$. Then, under the CFL condition (10) and the hypotheses (H1) and (H2) on the mesh, the linear system (9) is invertible, and

$$\rho^{n+1} \geq 0, \quad \forall \ n \geq 1.$$  

(17)

**Numerical results.** Here we consider structured meshes on $\Omega = [-1,1]^2$, a stationary rotating velocity field $u = (x_2, -x_1)$ and a small diffusion coefficient $\lambda = 10^{-6}$. Setting $r = \sqrt{(x_1 + 0.5)^2 + x_2^2}$, the discontinuous initial condition is $\rho_0 = 1000$ if $r \leq 0.25$ and $\rho_0 = 1$ if $r > 0.25$. The computations are performed for different values of $h \geq 0.004$, until $T = 0.3$. In Fig. 2 we show the evolution of the density contours (left) and the solution profiles for some horizontal sections (right). We can remark that the maximum principle is well verified using the IMEX-BDF2 scheme, unlike other classical order two schemes, such as Crank-Nicolson Adams-Bashforth or Crank-Nicolson Runge-Kutta. Some other numerical results can be found in [7].
Fig. 2 The evolution of the density contours (left) and the solution profiles (right) for $\lambda = 10^{-6}$.

References