Monte-Carlo acceleration: importance sampling and hybrid dynamic systems

H. Chraibi[a], A. Dutfoy[a], *T. Galtier*[a,b], J. Garnier[c]

[a] *EDF R&D - MRI (departement de management des risques industriels)*
[b] *University Paris Diderot - LPMA (Laboratoire de probabilités et modèles aléatoires)*
[c] *Ecole polytechnique - CMAP (Centre de mathématiques appliqués)*

Abstract

The reliability of a complex industrial system can rarely be assessed analytically. As system failure is often a rare event, crude Monte-Carlo methods are prohibitively expensive from a computational point of view. In order to reduce computation times, variance reduction methods such as importance sampling can be used. We propose an adaptation of this method for a class of multi-component dynamical systems.

We address a system whose failure corresponds to a physical variable of the system (temperature, pressure, water level) entering a critical region. Such systems are common in hydraulic and nuclear industry. In these systems, the statuses of the components (on, off, or out-of-order) determine the dynamics of the physical variables, and is altered both by deterministic feedback mechanisms and random failures or repairs. In order to deal with this interplay between components status and physical variables we model trajectory using piecewise deterministic Markovian processes (PDMP).

We show how to adapt the importance sampling method to PDMP, by introducing a reference measure on the trajectory space, and we present a biasing strategy for importance sampling. A simulation study compares our importance sampling method to the crude Monte-Carlo method for a three-component-system.

Keywords: Monte-Carlo acceleration, importance sampling, hybrid dynamic system, piecewise deterministic Markovian process, cross-entropy, reliability, PyCATSHOO

1. Introduction

For both safety and quality issues, nuclear, hydraulic and other industries resort to probabilistic safety assessment to quantify the reliability of their systems. In recent years, dynamic reliability methods have been gaining interest as they avoid conservative static approximations of the systems and better capture the dynamics involved in the systems. When dealing with complex industrial systems, reliability analysis faces two main challenges: the first challenge is related to the modeling of such complex systems, the second one concerns the quantification of the reliability.

Email address: tgaltier@gmail.com (*T. Galtier*)

1.1. A model based on a PDMP

In many industrial systems, failure corresponds to a physical variable of the system (temperature, pressure, water level) entering a critical region. The physical variables usually can enter this region only if a sufficient number of the basic components of the system are damaged. In order to estimate the reliability we need an accurate model of the trajectories of the physical variables. In industrial systems, the physics of the system is often determined by ordinary differential equations which depend on the statuses of the components within the system (on, off or failed). It is also possible that the failure and repair rates of the components depend on the physical conditions, so the values of physical variables can impact the statuses of the component. In order to deal with this interplay between the physical variables and the statuses of components, we need to model their joint evolution. The vector gathering these two elements is called the state of the system. To address the challenge of modeling the trajectory of the state of the system, we fall in with the work of [15] and [7], as we model the evolution of the state of the system by a piecewise deterministic Markovian process (PDMP). PDMPs were introduced by M.H.A Davis in [5, 6], they are meant to represent a large class of Markovian processes that do not include diffusion, and as such they benefit from high modeling capacity. These processes can easily incorporate component aging, failure on demand, and delays before repairs.

1.2. Accelerate reliability assessment by using importance sampling

The second challenge is that the reliability of a complex industrial system can rarely be assessed analytically, so reliability analysis often relies on Monte-Carlo simulations techniques. EDF has recently developed the PyCATSHOO toolbox [3] [4], which allows the simulation and the modeling of dynamic hybrid systems. PyCATSHOO bases this modeling on PDMPs. Thanks to Monte Carlo simulation, PyCATSHOO evaluates the dependability criteria of the system among which is the reliability of the system. The objective of our work is to set up new algorithms to accelerate the reliability assessment with PyCATSHOO.

In the context of reliable systems, crude Monte-Carlo techniques perform poorly because the system failure is a rare event. Indeed, with the Monte-Carlo method when the probability of failure approaches zero, the number of simulations needed to get a reasonable precision on the relative error increases dramatically and so does the computational time. To reduce this computational burden, one option is to reduce the number of simulations needed by using a variance reduction method. Amongst variance reduction techniques, we may think at multilevel splitting techniques and at importance sampling techniques. In this article, we provide an adaptation of importance sampling to PDMP trajectories.

1.3. Prerequisite for importance sampling

Importance sampling consists in simulating from a more fragile system, while eliminating the induced bias by weighting the simulation outputs by a likelihood ratio. To define such a likelihood ratio for PDMP trajectories, it is necessary to have a measure dominating both the law of the trajectories of our system and the law of the weaker system used for simulations. PDMP are very degenerate processes, their laws involve hybrid random variables which have continuous and discrete parts. In this context, it is important to ensure we do

have a reference measure to define properly the likelihood ratio.

In our case the state of the system at time t is denoted by Z_t. It is given by both values of the physical variables, gathered in the vector X_t, and the statuses of all the components in the system, represented by a vector M_t, so $Z_t = (X_t, M_t)$. Throughout the paper we call X_t the position of the system, and M_t the mode of the system. $\mathbf{Z} = (Z_t)_{t\in[0,t_f)}$ represents a trajectory of our PDMP up to a final observation time t_f. We consider that the trajectories are all initiated in a state z_o. Let A be the critical region corresponding to system failure, then, we denote by $\mathcal{A}$ the set of the trajectories of $\mathbf{Z}$ that pass through A. We want to estimate the probability

$$p = \mathbb{P}\big(\mathbf{Z} \in \mathcal{A} | Z_0 = z_o\big) = \mathbb{P}_{z_o}\big(\mathbf{Z} \in \mathcal{A}\big).$$

Suppose ζ is a reference measure for $\mathbb{P}_{z_o}\big(\mathbf{Z} \in .\big)$, and f is the density of $\mathbf{Z}$ with respect to ζ, and g is the density of an importance process with respect to ζ. If ζ exists, and $\forall \mathbf{z} \in \mathcal{A},\ f(\mathbf{z}) \neq 0 \Rightarrow g(\mathbf{z}) \neq 0,$ then we can write:

$$\mathbb{P}_{z_o}\big(\mathbf{Z} \in \mathcal{A}\big) = \mathbb{E}_f\big[\mathbb{1}_{\mathcal{A}}(\mathbf{Z})\big] = \int_{\mathcal{A}} f(\mathbf{z})\, d\zeta(\mathbf{z}) = \int_{\mathcal{A}} \frac{f(\mathbf{z})}{g(\mathbf{z})} g(\mathbf{z})\, d\zeta(\mathbf{z}) = \mathbb{E}_g\left[\mathbb{1}_{\mathcal{A}}(\mathbf{Z})\frac{f(\mathbf{Z})}{g(\mathbf{Z})}\right] \tag{1}$$

Therefore if $\big(\mathbf{Z}'_1, \dots \mathbf{Z}'_{N_{sim}}\big)$ is a sample of independent trajectories simulated according to an importance process with density g, then $\mathbb{P}_{z_o}\big(\mathbf{Z} \in \mathcal{A}\big)$ can be estimated without bias by:

$$\hat{p}_{IS} = \frac{1}{N_{sim}} \sum_{i=1}^{N_{sim}} \mathbb{1}_{\mathcal{A}}(\mathbf{Z}'_i)\frac{f(\mathbf{Z}'_i)}{g(\mathbf{Z}'_i)} \qquad \text{with} \quad \mathbb{V}\mathrm{ar}(\hat{p}_{IS}) = \mathbb{E}_f\left[\mathbb{1}_{\mathcal{A}}\frac{f(\mathbf{Z})}{g(\mathbf{Z})}\right] - p^2 \tag{2}$$

When $\mathbb{E}_f\big[\mathbb{1}_{\mathcal{A}}(\mathbf{Z})\frac{f(\mathbf{Z})}{g(\mathbf{Z})}\big] < \infty$ and the conditions above are verified, we have a central limit theorem on $\hat{p}_{IS}$. The estimator $\hat{p}_{IS}$ is unbiased and $\sqrt{N}(\hat{p}_{IS} - p)$ converges in law to a centered Gaussian with variance $\mathbb{E}_f\big[\mathbb{1}_{\mathcal{A}}(\mathbf{Z})\frac{f(\mathbf{Z})}{g(\mathbf{Z})}\big] - p^2$. Theoretically the variance can be brought to zero if the importance density g is equal to $g^*(\mathbf{z}) = \frac{\mathbb{1}_{\mathcal{A}}(\mathbf{z})f(\mathbf{z})}{p}$, but this can be done only if we already know the value of p which we are trying to estimate. In practice, one tries to approach this optimal density choosing g as close as possible from g^* to reduce the variance.

Thus the use importance sampling on PDMP trajectories requires the following three conditions:

(C1) We have a measure ζ on the trajectory space, and the trajectory $\mathbf{Z}$ of the system state has density f with respect to ζ

(C2) We are able to simulate trajectories according to an importance process $\mathbf{Z}'$ which has density g with respect to ζ on $\mathcal{A}$ such that $\mathbb{E}_f\big[\mathbb{1}_{\mathcal{A}}(\mathbf{Z})\frac{f(\mathbf{Z})}{g(\mathbf{Z})}\big] < \infty$.

(C3) $\forall\, \mathbf{z} \in \mathcal{A},\ \ f(\mathbf{z}) \neq 0 \Rightarrow g(\mathbf{z}) \neq 0$

Many authors have used importance sampling on particular cases of PDMP sometimes without noting it, see [12] [10] [11] [13]. Most of the models considered in the literature rarely used all the possibilities offered by PDMPs. Authors tend to consider only constant failure rates and constant repair rates, not taking into account component aging and dependency of failure rate on physical conditions. Or sometimes, they avoid considering automatic control mechanisms which activate and deactivate components depending on the values of physical variables. In [13], and in the more recent work [14], importance sampling is used on PDMP while taking into account automatic control mechanisms but the reference measure is not clearly identified. In part 3, we show how to find a reference measure ζ defining the likelihood ratio and we try to highlight the possible kinds of importance processes associated with ζ.

1.4. Optimization of the variance reduction

Finding the optimal importance process is equivalent to solving the following minimization problem:

$$g^* = \underset{g}{\operatorname{argmin}}\ \mathbb{E}_f\big[\mathbb{1}_{\mathcal{A}}(\mathbf{Z})\frac{f(\mathbf{Z})}{g(\mathbf{Z})}\big] - p^2$$

Minimizing a quantity on a density space being difficult, we usually consider a parametric family of importance densities $\{g_\alpha\}$ and look for a parameter α which yields an estimator with the smallest possible variance.
One generally tries to approach the optimal importance density by specifying a certain kind of parametric family whose form is determined by a large deviation analysis. But such large deviation behavior can be difficult to describe with degenerate processes like PDMP.
Therefore we focus on other methods which rather try to minimize an approximation of the distance between the importance density g and the optimal one g^*. For instance, if the approximated distance happens to be $D(g, g^*) = \mathbb{E}_f\big[\frac{g^*(\mathbf{Z})}{g(\mathbf{Z})}\big]$ it is equivalent to minimize the variance of the estimator, and if we consider the Kullback-Leibler divergence in place of a distance so that $D(g, g^*) = \mathbb{E}_{g^*}\left[\log\left(\frac{g^*(\mathbf{Z})}{g(\mathbf{Z})}\right)\right]$, we would be using the Cross-Entropy method [1]. These two options have been compared on a set of standard cases in [2]. They yielded similar results, though results obtained with the Cross-Entropy seemed slightly more stable than with the other option. In [16], the Cross-Entropy method was applied on a model equivalent to a PDMP without boundaries and showed good efficiency. Therefore we choose this method to select the parameters of the importance process in our paper. Of course, the efficiency of this procedure strongly depends on the choice of the parametric family of importance densities. In this article, we give an example of parametric family for systems consisting in parallel identical components in redundant back up.

The rest of the paper is organized as follows: Section 2 introduces our model of multi-component system based on a Piecewise deterministic Markovian process. In section 3, we introduce a reference measure on the space of the PDMP trajectories which allows us to properly define the likelihood ratio involved in the importance sampling weighting. Finally, in section 4 we present a possible biasing strategy, and we compare our adaptation of the importance sampling technique with the Monte-Carlo on a three-component system.

2. A model for multi-component system based on PDMP

2.1. Possible state of the system

We consider a system with N_c components and d physical variables. Remember we call position the vector $X \in \mathbb{R}^d$ which represents physical variables of the system, and we call mode the vector $M = (M^1, M^2, ..., M^{N_c})$ gathering the statuses of the N_c components. The state of the system Z includes the position and the mode: $Z = (X, M)$. For ease of the presentation, we consider the status of a component can be alternatively On, or Off, or out-of-order (noted F), so that the set of modes is $\mathbb{M} = \{On, Off, F\}^{N_c}$, but as long as $\mathbb{M}$ stays countable, it is possible to consider more options for the statuses of the components: For instance, one could consider different regimes of activity instead of the simple status On, or different types of failure instead of the status F. Note that we can also deal with continuous degradations, like the size of a breach in a pipe for instance: the presence of the degradation can be included in the mode and its size in the position.
The components in the system can be programmed to activate or deactivate when the position takes specific values. To take into account these automatic control mechanisms, within a mode m the physical variables are restricted to an open and connected set $\Omega_m \subset \mathbb{R}^d$. With $E_m = \{(x, m), x \in \Omega_m\}$, the state space is then:

$$E = \bigcup_{m \in \mathbb{M}} E_m = \bigcup_{m \in \mathbb{M}} \Big\{(x, m), x \in \Omega_m\Big\} \tag{3}$$

2.2. Flow functions

In a given mode m, i.e. a given combination of statuses of components, the evolution of the position is determined by an ordinary differential equation. We note ϕ_x^m the solution of that equation initiated in x. If we consider a position state Z_t at time t, there exists a time $T > 0$ such that $\forall s \in [0, T)$, $X_{t+s} = \phi_{X_t}^{M_t}(s)$ and $M_{t+s} = M_t$. For an initial state $z \in E$, we can introduce the flow function Φ_z with values in E. Regarding the evolution of the trajectory after a state $Z_t = (X_t, M_t)$, the next states are locally given by Φ_{Z_t}:

$$\begin{aligned} &\exists T > 0,\ \forall s \in [0, T), \\ &Z_{t+s} = \Phi_{Z_t}(s) = \big(\phi_{X_t}^{M_t}(s), M_t\big) = \big(X_{t+s}, M_t\big) \end{aligned} \tag{4}$$

2.3. Jumps

The trajectory of the state can also evolve by jumping. This typically happens because of control mechanisms, failures, repairs, or natural discontinuities in the physical variables. When such a jump is triggered, the current state moves to another one by changing its mode and/or its position.
We note $\overline{E}$ the closure of E, and $\mathcal{B}(E)$ the Borelian σ-algebra on E. If a jump occurs at time T, then the destination of the jump is determined according to a transition Kernel $K_{Z_T^-}$ where $Z_T^- \in \overline{E}$ is the departure state of the jump. If $Z_T^+ \in E$ is the arrival state, and $\forall z^- \in \overline{E}$, ν_{z^-} is a σ-finite measure on E, then the Kernel is defined by:

$$\forall B \in \mathcal{B}(E), \quad \mathbb{P}\left(Z_T^+ \in B | Z_T^- = z^-\right) = \int_B K_{z^-}(z)\, d\nu_{z^-}(z)\ . \tag{5}$$

The Kernel density must verify $K_z(z) = 0$ so we can not jump on the departure state.

2.4. Jump times

Jumps at boundaries

For $m \in \mathbb{M}$, let $\partial\Omega_m$ be the boundary of Ω_m. The boundary of the set E_m is the set $\partial E_m = \{(x,m), x \in \partial\Omega_m\}$. For $z = (x,m) \in E$, we define $t_z^* = \inf\{s > 0, \Phi_z(s) \in \partial E_m\}$ the time until the flow hits the boundary. We take the convention $t_z^* = +\infty$ if $\{s > 0, \Phi_z(s) \notin E_m\} = \emptyset$. Assume that the system starts in state $z = (x,m)$. When the flow leads the position out of its restricted set Ω_m, i.e the state touches ∂E_m, an automatic jump is triggered (see the scheme in 1).

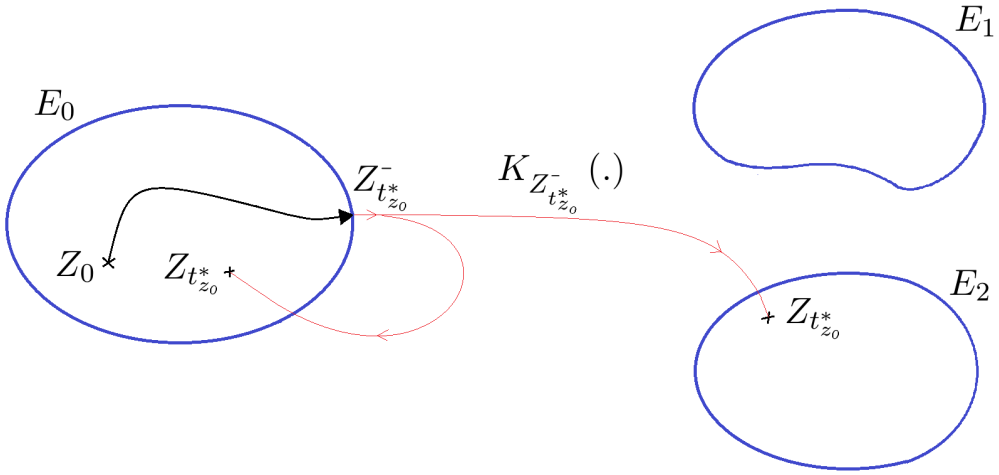


Figure 1: A jump at boundary.

Boundaries can be used to model automatic control mechanisms, or any automatic change in the status of a component. For instance in a dam, if the water level X reaches a given threshold x_{max} the evacuation valve automatically opens to avoid overflow. If $M = C,\ O,\ F$ represent respectively the modes where the valve is closed, or opened, or failed, this control system could be modeled by setting $\Omega_C = (0, x_{max})$ and $K_{(x_{max},C)}(\{(x_{max}, O)\}) = 1$.
Boundaries also allow to include failure on demand, by incorporating a probability of jumping to a failed mode at the boundary. In our example this would be modeled by setting $K_{(x_{max},C)}(\{(x_{max}, O)\}) = 1 - \gamma$ and $K_{(x_{max},C)}(\{(x_{max}, F)\}) = \gamma$, where $\gamma \in [0,1]$ is the probability of failure on demand on this boundary.

Spontaneous jumps

The trajectory can also jump to another state when a random failure or a repair occurs (see Figure 2). The distribution of the random time at which it happens is usually modeled through a state-related intensity function $\lambda : E \to \mathbb{R}_+$. For $z \in E$, $\lambda(z)$ represents the instantaneous risk of having a failure or a repair at state z. If $Z_t = z$ and T is the duration until the next jump, $\forall s < T$ we have $Z_{t+s} = \Phi_z(s)$. To simplify the notations in the later, we introduce the time-related intensity λ_z such that $\lambda_z(s) = \lambda(\Phi_z(s))$ and $\Lambda_z(s) =$

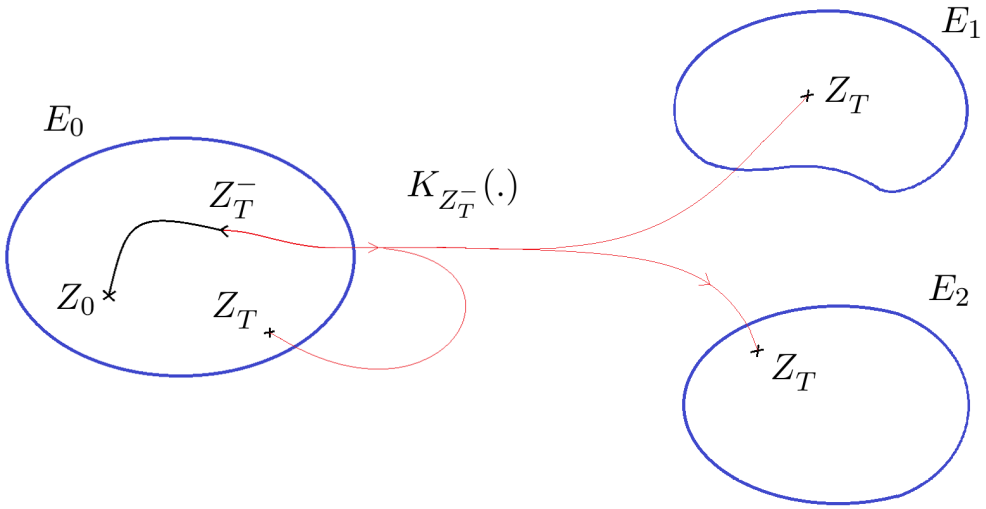


Figure 2: A spontaneous jump.

$\int_0^s \lambda\big(\Phi_z(u)\big)du$. If $\mathbb{P}_z(.)$ is the probability of an event knowing $Z_t = z$, we have:

$$\mathbb{P}_z(T \leq s) = \begin{cases} 1 - \exp\left[-\Lambda_z(s)\right] & \text{if } s < t_z^*\,, \\ 1 & \text{if } s \geq t_z^*\,. \end{cases} \tag{6}$$

The law of T has a continuous and a discrete part (see Figure 3). As T has no density with

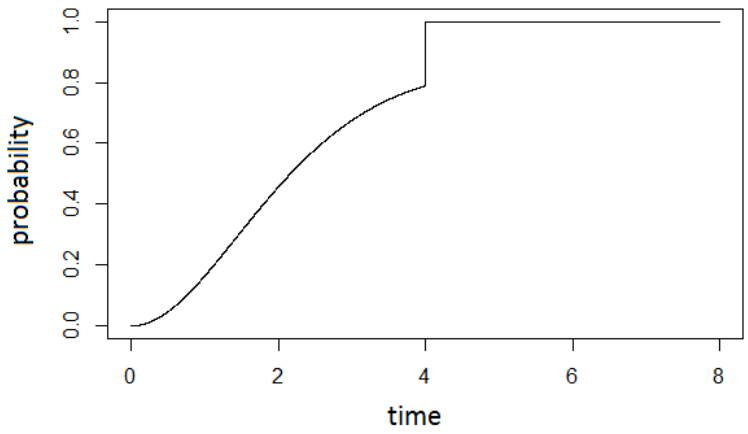


Figure 3: An example of the cdf of T,
where $x \in \mathbb{R}^+$, $z = (x, m)$, $\Phi_z(t) = (x + t, m)$, $\lambda(z) = \frac{x(5-x)}{12}$, and $t_z^* = 4$

respect to the Lebesgue measure, we introduce the following reference measure

$$\mu_z(.) = leb(. \cap (0, t_z^*)) + \delta_{t_z^*}(.)\ , \tag{7}$$

where $leb(.)$ corresponds to the Lebesgue measure. The measure $\mu_z(.)$ will be useful to define the dominant measure ζ in section 3. It also allows to reformulate the law of T under an

integral form:

$$\mathbb{P}_z(T \leq t) = \int_{(0,t]} \Big(\lambda_z(u)\Big)^{\mathbb{1}_{u<t_z^*}} \exp\Big[-\Lambda_z(u)\Big] d\mu_z(u) \ . \tag{8}$$

Destination of a jump

Note that equations (6) or (8) give the time of the next jump, but do not tell whether it is a failure, or a repair, or an automatic control mechanism. To specify the nature of the jump, we use the transition Kernel $K_{Z_t^-}$. Consider each transition from a departure mode $M^- = m$ to an arrival mode M^+ is indexed by a number in $\{1, ..., J_m\}$ where J_m is the number of the possible transitions. Let T^j be the time of occurrence of the transition indexed by j if the physical variables had followed the flow $\phi_x^{m^-}$ indefinitely. Let $\lambda^j : E \to \mathbb{R}_+$ be its associated state-related intensity function, such that $\mathbb{P}_z(T^j \leq t) = 1 - \exp\Big[\int_0^t \lambda^j\big(\Phi_z(u)\big)\Big] du$. The function λ^j is either a failure rate or a repair rate depending on the nature of the transition j. Knowing the initial state $z = (x, m)$, and therefore, knowing the indefinite trajectory of the positions given by ϕ_x^m, we make the assumption that the times T^j are independent. This assumption is true if the position gathers all the variables affecting failures or repairs when the system is in mode m. As $T = \min\{T^1, T^2, \ldots, T^{J_M}, t_z^*\}$, this conditional independence implies that:

$$\forall z = (x, m) \in E, \quad \lambda(z) = \sum_{j=1}^{J_m} \lambda^j(z) \ , \tag{9}$$

and then we have:

$$\forall z^+ = \big(x^+, m^+\big) \in E, \quad K_z\big(z^+\big) = \frac{\lambda^{j(m,m^+)}(z)}{\sum_{j=1}^{J_m} \lambda^j(z)} \, q_z\big(x^+\big) \tag{10}$$

where $j(m, m^+)$ is the index of the transition from m to m^+ and $q_z(x^+)$ is the density of a transition Kernel for positions. Typically if the physical variables are all continuous then $q_z(x^+) = \mathbb{1}_{x=x^+}$ and the reference measure of the transition Kernel is defined by $\forall B \in \mathcal{B}(E), \quad \nu_z(B) = \sum_{w \in \mathbb{M}\setminus\{m\}} \delta_{(x,w)}(B)$.

2.5. Generate a trajectory

To generate a realization of the PDMP, one can follow these steps:

- Start at a state $Z_0 = z$
- Generate T the time of the next jump using (8)
- Follow the flow Φ until T using (4)
- Generate $Z_T = z_T$ the arrival state of the jump knowing the departure state is $Z_T^- = \Phi_z(T)$ using (5) and (10)
- Repeat starting with z_T until you get a trajectory of size t_f

2.6. Example

As an example of system, we consider a room heated by three identical heaters. X_t represents the temperature of the room at time t. x_e is the exterior temperature. β_1 is the rate of the heat transition with the exterior. β_2 is the heating power of each heater. The differential equation giving the evolution of the position (i.e the temperature of the room) take this form:

$$\frac{d\,X_t}{dt} = \beta_1(x_e - X_t) + \beta_2 \mathbb{1}_{M_t^1 \text{ or } M_t^2 \text{ or } M_t^3 = On} \cdot$$

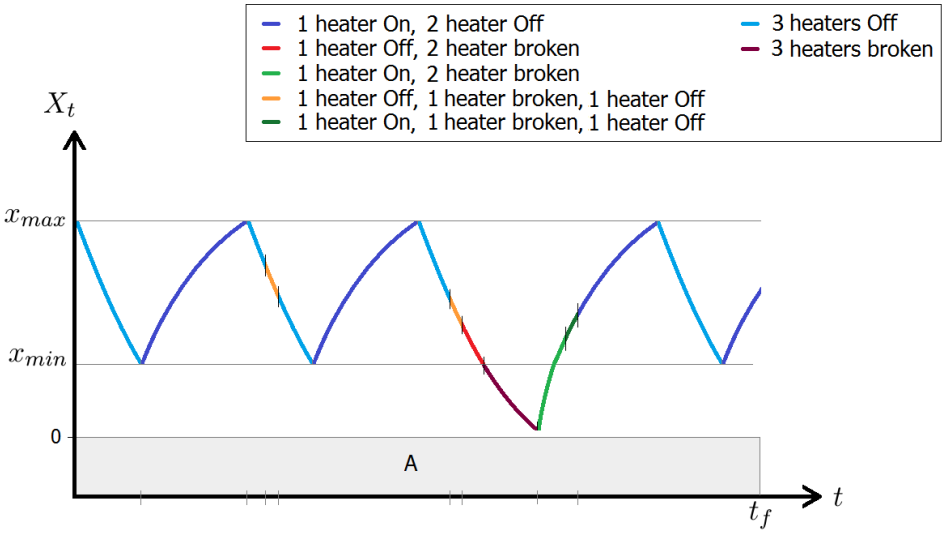


Figure 4: Scheme of a trajectory of the heated-room system
(mode is represented with colors)

The heaters are programmed to maintain the temperature within an interval (x_{min}, x_{max}) where $x_e < 0 < x_{min}$. Heaters can be on, off, or out-of-order, so $\mathbb{M} = \{On, Off, F\}^3$. We consider that the three heaters are in passive redundancy in the sense that: when $X \leq x_{min}$ the second heater activates only if the first one is failed, and the third one activates only if the two other heaters are failed. When a repair of a heater occurs, if $X \leq x_{min}$ and all other heaters are failed the heater status is set to On, else the heater status is set to Off. To handle the programming of the heaters, we set $\Omega_m = (-\infty, x_{max})$ when all the heaters are failed $m = (F, F, F)$ or when at least one is activated, otherwise we set $\Omega_m = (x_{min}, x_{max})$.
Due to the continuity of the temperature, the reference measure for the Kernel is $\forall B \in \mathcal{B}(E)$, $\nu_{(x,m)}(B) = \sum_{m^+ \in \mathbb{M}\setminus\{m\}} \delta_{(x,m^+)}(B)$. On the top boundary in x_{max}, heaters turn off with probability 1. On the bottom boundary in x_{min}, when a heater is supposed to turn on, there is a probability $\gamma = 0.01$ that the heater will fail on demand: For instance, if $z^- = \big(x_{min}, (Off, F, Off)\big)$, we have $K_{z^-}\big(x_{min}, (On, F, Off)\big) = 1 - \gamma$, $K_{z^-}\big(x_{min}, (F, F, On)\big) = \gamma(1-\gamma)$, and $K_{z^-}\big(x_{min}, (F, F, F)\big) = \gamma^2$.
Let j be a transition from m to m^+. For the spontaneous jumps that happen outside boundaries, if the transition j corresponds to the failure of a heater, then: $\lambda^j(x, m) = 0.003\frac{x}{x_{max}}$

and, if the transition corresponds to a repair, then $\lambda^j(x,m) = 0.2$ when $M^j = F$. An scheme of a possible trajectory of the state of this system is presented in figure 4. Here the system failure occurs when the temperature of the room falls below zero, so $A = \{(x,m) \in E, x < 0\}$.

3. A reference measure for trajectories

We have seen in section 2.3 that when the position is restricted to a bounded set in some modes, the time to the next jump can be a hybrid random variable. We have to be cautious when considering the density of a trajectory of a PDMP for several reasons: first the reference measure for such hybrid random variable is a mixture of Dirac and Lebesgue measures, secondly these hybrid jumps are involved multiple times and in a nested way in the law of the trajectory of PDMP. Indeed, with these mixtures of Dirac and Lebesgue measures involved, the existence of a sigma-finite reference measure on the trajectory space is not obvious, yet it is mandatory to properly define the density of a trajectory. The existence of a reference measure is therefore crucial, because it preconditions the existence of the likelihood ratio needed to apply the importance sampling method.

We begin this section by introducing a few notations: For a trajectory $\mathbf{Z}$ on the observation interval $[0, t_f)$, we note N the number of jumps before t_f, and S_k the time of the k-th jump with the convention $S_0 = 0$, and $\forall k < N$, $T_k = S_{k+1} - S_k$ is the duration between two jumps and $T_N = t_f - S_n$ is the remaining duration between the last jump and t_f. One can easily verify the sequence of the $(Z_{S_k}, S_{k+1} - S_k)$ is a Markov chain: it is called the embedded Markov chain of the PDMP.

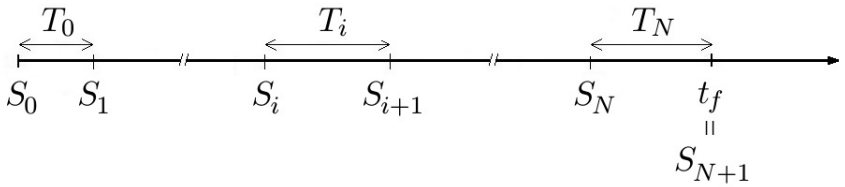


Figure 5: notations

3.1. The law of the trajectories

The main idea in building the law of the trajectory $\mathbf{Z}$ is to summarize the trajectory by the truncated embedded Markov chain of the process: the vector $\big(Z_{S_0}, T_0, Z_{S_1}, T_1, \ldots, Z_{S_N}, T_N\big)$. As the trajectory is piecewise deterministic, we only need to keep the states of the arrivals of the jumps and the durations between the jumps to describe the trajectory. If we have the vector $\big(Z_{S_k}, T_k\big)_{k \leq N}$ and we know the flow function Φ, we have enough information to reconstruct the trajectory using (4). Noting Θ the map that changes $\mathbf{Z}$ into $\big(Z_{S_k}, T_k\big)_{k \leq N}$, the law of $\mathbf{Z}$ can be defined as the image law of $\big(Z_{S_k}, T_k\big)_{k \leq N}$ through Θ.

We can get the law of $\big(Z_{S_k}, T_k\big)_{k\leq N}$, by using the dependencies between its coordinates. Thanks to (8) and (5) we can get the density of T_k knowing Z_{S_k} with respect to $\mu_{Z_{S_k}}$, and the density of $Z_{S_{k+1}}$ knowing $\big(Z_{S_k}, T_k\big)$ with respect to $\nu_{Z^-_{S_{k+1}}}$, where $Z^-_{S_{k+1}} = \Phi_{Z_{S_k}}(T_k)$:

$$f_{T_k|Z_{S_k}=z}(u) = \Big(\lambda_z(u)\Big)^{\mathbb{1}_{u<t^*_z}} \exp\Big[-\Lambda_z(u)\Big] \ , \tag{11}$$

$$f_{Z_{S_{k+1}}|Z_{S_k},T_k}(z) = K_{Z^-_{S_{k+1}}}(z) \ . \tag{12}$$

Using the Markov structure of the sequence $\big(Z_{S_k}, T_k\big)_{k\leq N}$, the law of $\big(Z_{S_k}, T_k\big)_{k\leq N}$ can be expressed as an integral of the product of the conditional densities given by (11) and (12).

We define the σ-algebra $\mathcal{S}$ on the set of the possible values of $\big(Z_{S_k}, T_k\big)_{k\leq N}$ as the σ-algebra generated by the sets in $\bigcup_{n\in\mathbb{N}^*} \mathcal{B}\Big(\Big\{\big(z_{s_k}, t_k\big)_{k\leq n} \in (E\times\mathbb{R}^*_+)^n, \sum_{i=0}^{n} t_i = t_f\Big\}\Big)$, where $\mathcal{B}(.)$ indicates the Borelians of a set. We get that for $B \in \mathcal{S}$:

$$\begin{aligned}\mathbb{P}_{z_o}\Big(\mathbf{Z}\in\Theta^{-1}(B)\Big) = \int_B \prod_{k=0}^{n} \Big(\lambda_{z_k}(t_k)\Big)^{\mathbb{1}_{t_k<t^*_{z_k}}} \exp\Big[-\Lambda_{z_k}(t_k)\Big] \prod_{k=1}^{n} K_{z_k^-}(z_k) \\ \times d\delta_{t^*_n}(t_n)\ d\nu_{z^-_n}(z_n)\ d\mu_{t^*_{z_{n-1}}}(t_{n-1})\ ...\ d\nu_{z^-_1}(z_1)\ d\mu_{t^*_{z_o}}(t_0)\ ,\end{aligned} \tag{13}$$

where $z^-_j = \Phi_{z_{j-1}}(t_{j-1})$, and $t^*_n = t_f - \sum_{i=0}^{n-1} t_i$. Note that with our construction, this is a probability law on the space of the trajectories that satisfy (4), not on the set of all the trajectories with values in E.

3.2. The dominant measure and the density

We define the measure ζ so that

$$\zeta(\Theta^{-1}(B)) = \int\limits_{(z_k,t_k)_{k\leq n}\in B} d\delta_{t^*_n}(t_n)\ d\nu_{z^-_n}(z_n)\ d\mu_{t^*_{z_{n-1}}}(t_{n-1})\ ...\ d\nu_{z^-_1}(z_1)\ d\mu_{t^*_{z_o}}(t_0) \tag{14}$$

If $\exists C>0, \forall z\in\overline{E},\ \nu_z(E)<C$ and $t_f<\infty$, then ζ is a σ-finite measure (see the Appendix B for the proof), and by Radon-Nikodym theorem, the density of a trajectory $\mathbf{z} = \Theta\big((z_0,t_0),\ ...\ ,(z_n,t_n)\big)$ is

$$f(\mathbf{z}) = \prod_{k=0}^{n} \Big(\lambda_{z_k}(t_k)\Big)^{\mathbb{1}_{t_k<t^*_{z_k}}} \exp\Big[-\Lambda_{z_k}(t_k)\Big] \prod_{k=1}^{n} K_{z_k^-}(z_k)\ , \tag{15}$$

Note that it is always possible to choose the measures ν_{z^-} so they are all bounded by the same constant. Indeed the transition Kernel is itself bounded by 1, as it is a probability measure. So, to get a measure ζ that is σ-finite, we can simply take the measures ν equal to the transition Kernel :

$$\forall B\in\mathcal{B}(E),\ \nu_{z^-}(B) = \mathbb{P}\big(Z^+\in B|Z^-=z^-\big) \quad \Rightarrow \quad \zeta \text{ is } \sigma\text{-finite}$$

which shows that the densities can be properly defined when the observation time t_f is finite.

3.3. Possible importance processes

The elements relative to the importance process are noted with a $'$.

3.3.1. Conditions and some possible importance process

Recall that a possible importance process is a process whose law is absolutely continuous with respect to ζ (condition C2) and which has a density g with respect to ζ that satisfies $\forall\, \mathbf{z} \in \mathcal{A},\ \ f(\mathbf{z}) \neq 0 \Rightarrow g(\mathbf{z}) \neq 0$ (condition C3).
Condition C2 implies that the importance process must generate trajectories in E which satisfy equation (4), so the importance process has to follow the same flows piecewisely. Therefore the bias must focus on the timing and nature of changes in flow, i.e. jumps.
To generate an importance process, we keep generating trajectories by successively generating the arrival state of a jump (Z'_{S_k}) and the time until the next jump (T'_k). As their is no requirement for the importance process to be Markovian, we consider that the law of a point of the trajectory Z'_t only depends on the past values of states. In other words, the law of Z'_{S_k} can depend on $\left(Z'_{S_i}, T'_i\right)_{i<k}$ and the law of T'_k can depend on $\left(Z'_{S_i}, T'_i\right)_{i<k}$ and Z'_{S_k}. Note that in order to ensure condition C2, the law of T'_k still has to be dominated by $\mu_{Z'_{S_k}}$, and the law of $Z'_{S_{k+1}}$ still has to be dominated by $\nu_{Z'^{-}_{S_k}}$. This means that the boundaries of the Ω_m's remain unchanged. For a jump time S_k, we note $\underline{Z}'_{S_k} = \left((Z'_{S_i}, T'_i)_{i\leq k}, Z'_{S_k}\right)$, and we denote $\lambda'_{\underline{z}_k}(.)$ the intensity function associated to T'_k when $\underline{Z}'_{S_k} = \underline{z}_k$, we have that:

$$\forall t \in (0, t^*_{z_0}], \quad \mathbb{P}(T'_k \leq t | \underline{Z}'_{S_k} = \underline{z}_k) = \int_{(0,t]} \left(\lambda'_{\underline{z}_k}(u)\right)^{\mathbb{1}_{u<t^*_{\underline{z}_k}}} \exp\Big[-\Lambda'_{\underline{z}_k}(u)\Big] d\mu_{z_k}(u) \qquad (16)$$

Noting $\underline{Z}'_{S_k}{}^{-} = \left((Z'_{S_i}, T'_i)_{i<k\text{-}1}\right)$ and $K'_{\underline{z}^-}$ the importance Kernel when $\underline{Z}'_{S_k}{}^{-} = \underline{z}_k{}^{-}$, we have that:

$$\forall B \in \mathcal{B}(E), \quad \mathbb{P}(Z'_{S_k} \in B | \underline{Z}'_{S_k}{}^{-} = \underline{z}_k{}^{-}) = \int_B K'_{\underline{z}^-_k}(z) d\nu_{z^-_k}(z) \qquad (17)$$

Another possibility is that the intensity function of T'_k does not have to be of the form $\lambda' \circ \phi_{z_k}$ where λ' is a positive function on E. This means that at the time t, the intensity function can depend on the arrival state of the last jump and on previous pairs (Z'_{S_i}, T'_i), and not necessarily only on the state Z'_t as it is the case in a PDMP.
Note that this way of generating the trajectory jump by jump knowing the past states is not necessarily the only possible one. We considered it essentially because in terms of implementation, it is similar to the way PDMPs are generated in PyCATSHOO, and therefore it should be convenient to incorporate it in the PyCATSHOO toolbox.

Concerning Condition C3, things are more delicate, indeed with complex systems, the set $\mathcal{A}$ can be very hard to manipulate. So we will only provide a sufficient condition to satisfy C3. To verify condition C3 a sufficient condition is to impose for any $z_k \in E$ and $z^- \in \overline{E}$:

$$\lambda(\Phi_{z_k}(t)) > 0 \ \Rightarrow \lambda'_{\underline{z}_k}(t) > 0$$
$$K'_{z^-}(z^+) > 0 \ \Rightarrow K'_{\underline{z}^-}(z^+) > 0$$

3.3.2. Extension trick

In order to devise a more efficient bias, one can use an extension trick as mentioned in random weight importance sampling in [8] and in [9]. The idea is to consider we are no longer working only on trajectories $\mathbf{Z}$, but on a pair $(\mathbf{Z}, Y)$ where Y is a random vector, and $(\mathbf{Z}, Y)$ admits $f(\mathbf{z})$ as marginal. If h is the density of the pair $(\mathbf{Z}, Y)$ and h' the density of $(\mathbf{Z}', Y')$ the estimator of $\mathbb{P}(\mathbf{Z} \in \mathcal{A})$ becomes $\hat{p}_{ISext} = \frac{1}{N_{sim}} \sum_{i=1}^{N_{sim}} \mathbb{1}_{\mathcal{A}}(\mathbf{Z}'_i) \frac{h(\mathbf{Z}'_i, Y'_i)}{h'(\mathbf{Z}'_i, Y'_i)}$. The advantage is that the law of the trajectories in the importance process can now depend on the added random variables, which makes the construction of an efficient importance process easier. Although the variance tends to increase when the dimension of the simulation output increases, this is generally compensated by the gain of variance offered by a more efficient importance process. If one dispose of the analytical expression of $\mathbb{E}_{f_{Y|\mathbf{Z}}}\big[\frac{h(\mathbf{Z}', Y')}{h'(\mathbf{Z}', Y')}\big]$, one can use the Rao-Blackwellized estimator: $\hat{p}_{ISRB} = \frac{1}{N_{sim}} \sum_{i=1}^{N_{sim}} \mathbb{1}_{\mathcal{A}}(\mathbf{Z}'_i) \mathbb{E}_{f_{Y|\mathbf{Z}=Z'_i}}\big[\frac{h(\mathbf{Z}'_i, Y')}{h'(\mathbf{Z}'_i, Y')}\big]$. The Rao-Blackwellization compensates the increase of the variance due to the increase of the dimension of the simulation outputs.

3.3.3. Optimal bias

In the importance process, generate the trajectories jump by jump by using (16) and (17) is not necessarily restrictive. Indeed, it still theoretically allows to simulate according to a process that cancels the variance of the estimator.
If we set:

$$f'_{T'_k|\underline{Z}'_{S_k}=\underline{z}_k}(u) = \frac{\mathbb{E}\big[\mathbb{1}_{\mathcal{A}}(\mathbf{z})|T_k = u, \underline{Z}_{S_k} = \underline{z}_k\big]}{\mathbb{E}\big[\mathbb{1}_{\mathcal{A}}(\mathbf{z})|\underline{Z}_{S_k} = \underline{z}_k\big]} \Big(\lambda_{z_k}(u)\Big)^{\mathbb{1}_{u<t^*_{z_k}}} \exp\Big[-\Lambda_{z_k}(u)\Big] \tag{18}$$

with $f'_{T'_k|\underline{Z}'_{S_k}=\underline{z}_k}(u) = 0$ when $\mathbb{E}\big[\mathbb{1}_{\mathcal{A}}(\mathbf{z})|\underline{Z}_{S_k} = \underline{z}_k\big] = 0$, and

$$K'_{\underline{z}_k{}^-}(z) = \frac{\mathbb{E}\big[\mathbb{1}_{\mathcal{A}}(\mathbf{z})|\underline{Z}_{S_k} = \underline{z}_k\big]}{\mathbb{E}\big[\mathbb{1}_{\mathcal{A}}(\mathbf{z})|\underline{Z}^-_{S_k} = \underline{z}^-_k\big]} K_{z_k^-}(z_k) \tag{19}$$

then we have that $\forall \mathbf{z} \in \mathcal{A}$, such that $\mathbf{z} = \Theta\big((z_0, t_0), \ldots, (z_n, t_n)\big)$:

$$\begin{aligned} f'(\mathbf{z}) &= \frac{\prod_{k=0}^{n} \Big(\lambda_{z_k}(t_k)\Big)^{\mathbb{1}_{t_k<t^*_{z_k}}} \exp\Big[-\Lambda_{z_k}(t_k)\Big] \prod_{k=1}^{n} K_{z_k^-}(z_k)}{\mathbb{E}_{z_0}\big[\mathbb{1}_{\mathcal{A}}(\mathbf{z})\big]} \mathbb{1}_{\mathcal{A}}(\mathbf{z}) \\ f'(\mathbf{z}) &= \frac{\mathbb{1}_{\mathcal{A}}(\mathbf{z}) f(\mathbf{z})}{\mathbb{E}_{z_0}\big[\mathbb{1}_{\mathcal{A}}(\mathbf{z})\big]} = g^*(\mathbf{z}) \end{aligned} \tag{20}$$

where $g^*(\mathbf{z})$ is the density of the optimal process. So equations (18) and (19) give some optimal densities for the jumps, and as such we will note these densities $g^*_{T'_k|\underline{Z}'_{S_k}=\underline{z}_k}$ and $K^*_{\underline{z}_k{}^-}$.

If there exists a constant $K > 1$ such that the importance densities for the jumps verify

$$f'_{T'_k|\underline{Z}'_{S_k}=\underline{z}_k}(u) = g^*_{T'_k|\underline{Z}'_{S_k}=\underline{z}_k}(u)\,\varepsilon_1(u) \qquad \text{with } \frac{1}{K} \leq \varepsilon_1(u) \leq K$$
$$K'_{\underline{z}_k{}^-}(z) = K^*_{\underline{z}_k{}^-}(z)\,\varepsilon_2(z) \qquad \text{with } \frac{1}{K} \leq \varepsilon_2(z) \leq K$$

then, providing the probability of having a trajectory with N jumps while being in $\mathcal{A}$ fades out quickly enough as N tends to infinity, we get the following inequalities :

$$p\mathbb{E}\big[\mathbb{1}_A(\mathbf{Z})K^{-2N}\big] - p^2 \;\leq\; \mathbb{V}\mathrm{ar}(\hat{p}_{IS}) \;\leq\; p\mathbb{E}\big[\mathbb{1}_A(\mathbf{Z})K^{2N}\big] - p^2$$

Although these bounds on the variance get very loose when K is far from 1, this inequality shows that the variance tends to zero when the constant K tends to 1. Therefore when specifying the bias through the times between jumps and the arrivals of jumps, one should try to specify densities as close as possible from equations (18) and (19).
This is of course difficult as we do not know the value of the conditional expectations involved in (18) and (19), but despite that, these two equations can still give us information on how to specify importance densities for the jumps. For instance, as $\mathbb{E}\big[\mathbb{1}_{\mathcal{A}}(\mathbf{z})|\underline{Z}_{S_k} = \underline{z}_k\big] = \mathbb{E}\big[\mathbb{1}_{\mathcal{A}}(\mathbf{z})|T_k = u,\, \underline{Z}_{S_k} = \underline{z}_k\big] = 1$ when $\tau_A < S_k$, the optimal densities remain unchanged after τ_A, so there is no need to apply a bias after τ_A. It is also possible to derive the optimal intensity associated to the time of a spontaneous jump as we will show in the following.
To ease the presentation, we slightly modify the state space by adding a boundary at the frontier of A and we add a coordinate on the mode which indicates if the trajectory has already visited A. So the state becomes $Z = \big(X, (M, M_A)\big)$ where $M_A = 0$ if A has not been visited, and 1 if it has. This way, for any time t we have $Z_t = (X_t, (M_t, \mathbb{1}_{\tau_A \leq t})$. For instance, with heated-room system the set of modes therefore becomes $\mathbb{M} = \{On, Off, F\}^3 \times \{0, 1\}$. The Kernel $K_{Z^-}(Z)$ is unchanged when $M_A^- = M_A^+$, and is null when $M_A^- \neq M_A^+$, except at the frontier of A where $K_{(0,(F,F,F,0))}\big(0, (F, F, F, 1)\big) = 1$. Now, noting

$$U^*(z,s) = \mathbb{E}\big[\mathbb{1}_{\mathcal{A}}(\mathbf{Z})|Z_s = z\big] \qquad \text{and} \qquad U^{\text{-}}(z^{\text{-}},s) = \int_E U^*(z^+,s)K_{z^{\text{-}}}(z^+)d\nu_{z^{\text{-}}}(z^+) \tag{21}$$

equations (18) and (19) can be rewritten as follows:

$$g^*_{T'_k|Z'_{S_k},\,S_k=z,s}(u) = \frac{U^{\text{-}}\big(\Phi_z(u), s+u\big)}{U^*\big(z,s\big)}\Big(\lambda_z(u)\Big)^{\mathbb{1}_{u<t^*_z}} \exp\Big[-\Lambda_z(u)\Big]\,, \tag{22}$$

$$K^*_{z^{\text{-}},s}(z) = \frac{U^*\big(z,s\big)K_{z^{\text{-}}}(z)}{U^{\text{-}}\big(z^{\text{-}},s\big)}\,. \tag{23}$$

Using (22) we can get the expression of the optimal intensity of the time of the k-th jump knowing $(Z'_{S_k},\, S_k) = (z, s)$, we have:

$$\lambda^*_{z,s}(u) = \frac{U^{\text{-}}\big(\Phi_z(u), s+u\big)\Big(\lambda_z(u)\Big)^{\mathbb{1}_{u<t^*_z}} \exp\Big[-\Lambda_z(u)\Big]}{\int_{(u,t^*_z]} U^{\text{-}}\big(\Phi_z(v), s+v\big)\Big(\lambda_z(v)\Big)^{\mathbb{1}_{v<t^*_z}} \exp\Big[-\Lambda_z(v)\Big]d\mu_z(v)}\,, \tag{24}$$

which can be rewritten as:

$$\lambda^*_{z,s}(u) = \frac{U^{\text{-}}\big(\Phi_z(u), s+u\big)}{U^*\big(\Phi_z(u), s+u\big)}\lambda_z(u) \ . \tag{25}$$

The detailed proof of this result is in Appendix B. As it plays an important role in the expression of the optimal process, we looked for more information about the function U^*. We found that U^* verifies these two properties:
U^* is Kernel invariant on boundaries:

$$\forall z \in E, \qquad U^{\text{-}}\big(\Phi_z(t^*_z), s+t^*_z\big) = \lim_{t \nearrow t^*_z} U^*\big(\Phi_z(t), s+t\big) \ . \tag{26}$$

And if $u \to U^{\text{-}}\big(\Phi_z(u), s+u\big)$ and $u \to \lambda_z(u)$ are continuous almost everywhere on $[0, t^*_z)$, then almost everywhere U^* is derivable along the flow, with:

$$\frac{\partial U^*\big(\Phi_z(v), s+v\big)}{\partial v} = U^*\big(\Phi_z(v), s+v\big)\lambda_z(v) - U^{\text{-}}\big(\Phi_z(v), s+v\big)\lambda_z(v) \tag{27}$$

The proof of these two properties of U^* is in Appendix B.

3.3.4. Parametric bias

To find an importance process that gives a good variance reduction, we usually restrict the search within a parametric family of importance densities, in order to use optimization routines. In the case of PDMP, we propose to use the cross-entropy method presented in [1] to select the parameters of the importance density as it was done in [16]. However, to our knowledge, there is no guaranty that the minimization routine used in the cross entropy method converges to a global optimum. Therefore, to avoid falling in a local optimum, one should run several times the cross entropy method with different initial values for the vector of parameters. Note that the parametrization must be chosen carefully: indeed the family of the importance densities must contain densities that are close to the zero-variance density $g(\mathbf{z}) = \frac{\mathbb{1}_{\mathcal{A}}(\mathbf{z})f(\mathbf{z})}{p}$ to obtain a good variance reduction, otherwise we could even obtain a variance increase. Also the parametric family should contain the original density f. When this is the case, the density associated to the best parameters is at least better than f, therefore it ensures that the variance decreases. Ideally we want to specify parametric densities that increase the likelihoods of *all* the elements in $\mathcal{A}$. In practice, we increase the probability of $\mathcal{A}$, while trying to simulate the elements of $\mathcal{A}$ in proportion to their natural probabilities.

4. Simulation study on a test case

In this section we present how we built an importance process for the heated room system presented in section 2.6.

4.1. A biasing strategy

In order to specify the importance densities close to the optimal ones, we propose to consider a parametric approximation of $\mathbb{E}\big[\mathbb{1}_{\mathcal{A}}(\mathbf{Z})|Z_s = z\big]$, and then combine it with equations (25) and (22) to get the form of the importance Kernel and of the importance intensities. Let $U_\alpha(z,s)$ be our approximation of $U^*(z,s)$. The corresponding importance intensities and Kernel are:

$$\lambda'_{z,s}(u) = \frac{U_\alpha^{\text{-}}\big(\Phi_z(u), s+u\big)}{U_\alpha\big(\Phi_z(u), s+u\big)}\lambda_z(u)\ , \tag{28}$$

$$K'_{z^{\text{-}},s}(z^+) = \frac{U_\alpha\big(z^+, s\big)K_{z^{\text{-}}}(z^+)}{U_\alpha^{\text{-}}\big(z^{\text{-}}, s\big)}\ , \tag{29}$$

where $U_\alpha^{\text{-}}\big(z^{\text{-}}, s\big) = \int_E U_\alpha(w,s)K_{z^{\text{-}}}(w)d\nu_{z^{\text{-}}}(w)$

In the heated-room system, the three heaters are identical and are in parallel redundancy, so we expected the probability $\mathbb{E}\big[\mathbb{1}_{\mathcal{A}}(\mathbf{z})|Z_s = z\big]$ to increase with the number of failed heaters in state z. Therefore, noting $b(z)$ the number of failed heaters in state z, we start by setting

$$U_\alpha(z,s) = H_\alpha\big(b(z)\big)\, Q(x,s) \tag{30}$$

where Q is some function on position and time, and H_α a function on integers. We set $H_\alpha(0) = 1$, so H_α has to be an increasing function, if we want $U_\alpha(z,s)$ to increase with $b(z)$. If T denotes the time to the next jump after being in state z, then

$$U^*(z,s) = \mathbb{E}\big[U^*(Z_T, s+T)\big|Z_s = z\big]$$

As the repair rates are larger than the failure rates by one order of magnitude, when there is at least one failed heater, the probability of arriving in a more degraded state Z_T is much lower than the probability of having repair. (This last remark can be applied to any reliable industrial systems. See for instance [4].) Ideally we would like U_α to mimic the property of U^* so would like too have

$$U_\alpha(z,s) = \mathbb{E}\big[U_\alpha(Z_T, s+T)\big|Z_s = z\big] \tag{31}$$

which can be reformulated as :

$$H_\alpha\big(b(z)\big) = \sum_{m^+\in\mathbb{M}} H_\alpha\big(b(x,m^+)\big)\int_{(0,t_z^*]} K_{\Phi_z(u)}\big((\phi_x^m(u), m^+)\big)Q(\phi_x^m(u), s+u)\exp\big[-\Lambda_z(u)\big]d\mu_z(u) \tag{32}$$

As repair is much more likely than failure, if $j(m,m^+)$ indexes a repair $K_{\Phi_z(u)}\big((\phi_x^m(u), m^+)\big)$ is larger than if it had indexes a failure. So, (32) implies that, when $b(z) > 1$, the value of $H_\alpha(b(z))$ is closer from $H_\alpha(b(z)-1)$ than from $H_\alpha(b(z)+1)$. As H_α was supposed

increasing, it must be convex. So we propose that $H_\alpha(b(z)) = \exp\left[\alpha_1 b(z)^2\right]$, with $\alpha_1 > 0$. The associated failure rates are then:

$$\lambda'^{j}_{z,s}(u) = \lambda^j_z(u) \exp\left[\alpha_1\big(2b(z)+1\big)\right] , \tag{33}$$

and the associated repair rates are :

$$\lambda'^{j}_{z,s}(u) = \lambda^j_z(u) \exp\left[-\alpha_1\big(2b(z)-1\big)\right] , \tag{34}$$

with

$$\lambda'_{z,s}(u) = \sum_j \lambda'^{j}_{z,s}(u) \qquad \text{and} \qquad K'_{\Phi_{z(u)^-},s}(z^+) = \frac{\lambda'^{j(m,m^+)}_{z,s}(u)}{\lambda'_{z,s}(u)} q_z(x^+) \ . \tag{35}$$

Remark that plugging U_α into the equations (18) and (19) imposes some kind of symmetry in the biasing of failure and repair rates. It is especially visible in equations (33) and (34): On one hand the failure rate associated to the transition from a state z^- to z^+ is multiplied by a factor $\exp\left[\alpha_1\big(2b(z^-)+1\big)\right]$, and on the other hand the repair rate corresponding to the reversed transition (from state z^+ to state z^-) is divided by the same factor. The equations (18) and (19) not only imply that the failures should be enhanced and the repairs inhibited, but it also states that the magnitude of the bias should be equivalent for a transition and its reversal.

The square in H_α's formula was introduced to strengthen the failure rates when the number of broken heaters gets larger. The idea was to shorten the duration where several heaters are simultaneously failed in the simulated trajectories. Indeed, as repair is exponential and faster than failure, the shorter the durations with a failed heater are the more likely is the trajectory. Increase the failure rates with the number of broken heaters is a mean to simulate more trajectories in $\mathcal{A}$ while maintaining the natural proportion between the likelihoods of the trajectories, which should decrease the variance.

As the failure on demand was likely to play an important role in the system failure, we choose to separate it from spontaneous failure in our parametrisation setting $U_\alpha((x_{min}, m), s) = \exp[-\alpha_2 b(z)^2] H_\alpha(x_{min}, s)$. This allows to better fit U_α to U^*. Under this assumption, the equation (29) implies that when $z^- = (x_{min}, m)$, the importance Kernel has this form:

$$K'_{z^-}(z^+) = \frac{K_{z^-}(z^+) \exp\left[-\alpha_2\, b(z^+)^2\right]}{\int_E K_{z^-}(z) \exp\left[-\alpha_2\, b(z)^2\right] d\nu_{z^-}(z)} \tag{36}$$

4.2. Results

For the Monte-Carlo method simulations have been carried out using the Python library PyCATSHOO currently developed by EDF R&D. As the Cross-Entropy method was not implemented yet in PyCATSHOO, we used a specific Python code for the Cross-Entropy and the importance sampling methods. Note the Cross-Entropy method will be integrated to the PyCATSHOO tool in 2017. The system parameters used in the simulation were the following: $x_{min} = 0.5$, $x_{max} = 5.5$, $x_e = -1$, $\beta_1 = 0.1$, $\beta_2 = 5$, $t_f = 100$. Trajectories were

all initiated in the state $z_0 = \big(7.5,\ (Off, Off, Off)\big)$. The probability of having a system failure before t_f was estimated to $p = 1.29 \times 10^{-5}$ with an intensive Monte-Carlo estimation based on 10^8 runs.
For the cross-entropy method, the approximations of the Kullback-Leiber divergence between g^* and g_α were realized with a sample of 10^5 simulations. For the first step of the method, the approximation was obtained by simulating from a biased density with parameters $(0.5, 0.5)$. Note that it is important to ensure that all the types of failing trajectories are represented in the sample used in the approximation of the divergence in the first step. Otherwise the cross-entropy method may yield a vector of parameters that favors only the types of trajectories present in the sample, which generally leads to underestimate both the probability p and the variance of the estimator.
The values of the parameters selected by the cross-entropy method were $\alpha_1 \simeq 0.915$ and $\alpha_2 \simeq 1.197$. A comparison between Monte-Carlo and the associated importance sampling estimate is presented in table 1, where we display the number of simulations used for each method, the estimates of the probability, the associated empirical variances and confidence intervals. To compare the two methods we look at the ratio between the relative variance of the Monte-Carlo estimator $p(1-p)$ and the estimated relative variance of the importance sampling estimator $N_{IS} \times \hat{\sigma}^2_{IS}$. This ratio indicates that our importance sampling strategy reduced the variance by approximately a factor 25 500. For 10^6 simulations the results shows the Monte-Carlo estimator has not converged yet, whereas the importance sampling estimate is very accurate.

	N_{sim}	$\hat{p}$	$\hat{\sigma}^2$	IC$\times 10^5$
IS	10^6	1.288×10^{-5}	5.05×10^{-16}	$[1.283, 1.292]$
MC	10^6	0.4×10^{-5}	4.00×10^{-12}	$[0.01, 0.79]$
	10^7	1.3×10^{-5}	1.28×10^{-12}	$[1.07, 1.51]$

Table 1: Comparison between Monte-Carlo and importance sampling estimations

5. Discussion

Our work shows that importance sampling is applicable on any PDMPs with boundaries. We have given an expression of the intensities and Kernel of the optimally biased distribution, and we have seen that it depends on the function $U^*(z,s) = \mathbb{E}\big[\mathbb{1}_{\mathcal{A}}(\mathbf{Z})|Z_s = z\big]$. Although we do not have a closed form of the function U^*, these expressions are important for two reasons: 1) They prove the existence of an optimal bias, which ensures that the importance sampling technique can be very efficient on PDMPs. 2) They can guide the specification of the biasing strategy. This expression shows that the optimal bias presents a symmetry between a transition and its reversal. By replacing U^* by an approximation in the optimal expressions of the transition rates and kernels, we preserve this specific structure. The presented method

therefore helps designing a bias which has the same behavior as the optimal one, which yielded an efficient biasing strategy for our case study.

This biasing strategy can be applied on any system, but the parametric shape of the approximation of U^* may have to be adapted from case to case. The parametric shape presented in this article is suited to any system with similar components in terms of failure rates and repair rates and containing one minimal group. (A minimal group being a group of components that need to fail so that the system can fail.) For a system with a different configuration, we expect the shape of the function U^* will differ, and the method may require a different parametric approximation for the function U^*.

When choosing the importance process there is a risk of over-biasing the importance process. Over-biasing happens when one type of failing trajectories is over represented in the importance distribution comparatively to other types of failing trajectories. This phenomenon can result in underestimating the probability of the system failure and in underestimating the variance. To avoid it, we must satisfy two points: 1) We must design a parametric importance density that can increase the likelihoods of each type of failing trajectories separately. 2) We need to initiate the Cross-Entropy method with a sample of trajectories that contains all types of failing trajectories. It is therefore preferable to apply this method only on systems of reasonable complexity, for which it is possible to determine the different types of failing trajectories.

6. Conclusion

We have presented a model for multi-component systems based on PDMPs. In order to speed up reliability assessment on such systems, we have adapted the importance sampling method to trajectories of PDMP. We have given a dominant measure for PDMP trajectories, allowing to properly define the likelihood ratio needed to apply the importance sampling method on such processes. The possible kinds of importance processes were discussed, and the theoretical optimal biasing strategy when simulating jump by jump was exhibited. We developed and tested a biasing strategy for a three-component heated-room system. Our importance sampling method has shown good performance, dividing the variance by a factor 25 500.

Appendix A. Proof that the measure ζ is σ-finite

Remember that we defined the σ-algebra $\mathcal{S}$ on the set of the possible values of $\big(Z_{S_k}, T_k\big)_{k\leq N}$ as the σ-algebra generated by the sets in $\bigcup_{n\in\mathbb{N}^*} \mathcal{B}\Big(\Big\{\big(z_{s_k}, t_k\big)_{k\leq n} \in (E\times\mathbb{R}_+^*)^n, \sum_{i=0}^{n} t_i = t_f\Big\}\Big)$. The measure ζ is defined by: :

$$B\in\mathcal{S},\quad \zeta\big(\Theta^{-1}(B)\big) = \int\limits_{(z_k,t_k)_{k\leq n}\in B} d\delta_{t_n^*}(t_n)\; d\nu_{z_n^-}(z_n)\; d\mu_{t^*_{z_{n-1}}}(t_{n-1})\; ...\; d\nu_{z_1^-}(z_1)\; d\mu_{t^*_{z_o}}(t_0) \quad \text{(A.1)}$$

Let $B_n = \Big\{\big(z_{s_k}, t_k\big)_{k\leq n} \in (E\times\mathbb{R}_+^*)^n, \sum_{i=0}^{n} t_i = t_f\Big\}$. Then $\Theta^{-1}(B_n)$ is the set of possible trajectories with n jumps, and the sets B_n for $n\in\mathbb{N}^*$ form a partition of the set of all possible trajectories. Note that $B_n \subseteq (E\times[0,t_f])^n$, so

$$\begin{aligned}\zeta\big(\Theta^{-1}(B_n)\big) &\leq \zeta(\Theta^{-1}\big((E\times[0,t_f])^n\big)\\ &\leq \int\limits_{(E\times[0,t_f])^n} d\delta_{t_n^*}(t_n)\; d\nu_{z_n^-}(z_n)\; d\mu_{t^*_{z_{n-1}}}(t_{n-1})\; ...\; d\nu_{z_1^-}(z_1)\; d\mu_{t^*_{z_o}}(t_0)\end{aligned}$$

We suppose that the ν_{z^-} are bounded, $\exists M>0, \forall z^-\in\overline{E},\ \nu_{z^-}(E)<M$. Under this assumption, we have:

$$\begin{aligned}\zeta\big(\Theta^{-1}(B_n)\big) &\leq M\int\limits_{(E\times[0,t_f])^{n-1}} d\mu_{t^*_{z_{n-1}}}(t_{n-1})\; ...\; d\nu_{z_1^-}(z_1)\; d\mu_{t^*_{z_o}}(t_0)\\ &\leq M\int\limits_{(E\times[0,t_f])^{n-2}}\int_E\int_{[0,t_f]} d\mu_{t^*_{z_{n-1}}}(t_{n-1})\; d\nu_{z^-_{n-1}}(z_{n-1})\; ...\; d\nu_{z_1^-}(z_1)\; d\mu_{t^*_{z_o}}(t_0)\\ &\leq M(t_f+1)\int\limits_{(E\times[0,t_f])^{n-2}}\int_E d\nu_{z^-_{n-1}}(z_{n-1}) d\mu_{t^*_{z_{n-2}}}(t_{n-2})\; ...\; d\nu_{z_1^-}(z_1)\; d\mu_{t^*_{z_o}}(t_0)\\ &\leq M^2(t_f+1)\int\limits_{(E\times[0,t_f])^{n-2}} d\mu_{t^*_{z_{n-2}}}(t_{n-2}) d\nu_{z^-_{n-2}}(z_{n-2})\; ...\; d\nu_{z_1^-}(z_1)\; d\mu_{t^*_{z_o}}(t_0)\end{aligned}$$

By recurrence we get that $\zeta\big(\Theta^{-1}(B_n)\big)\leq M^n(t_f+1)^n$, which proves that ζ is σ-finite.

Appendix B. Optimal intensity's expression, and some properties of U^*

Appendix B.1. Proof of equality (26)

Let $z^- \in \delta E$ and $s \in [0, t_f]$. Remember that equality (26) states that

$$U^-\big(\Phi_z(t_z^*), s + t_z^*\big) = \lim_{t \nearrow t_z^*} U^*\big(\Phi_z(t), s + t\big)$$

We note T the time until the next jump after the trajectory has reached the state $Z_{s+t} = \phi_z(t)$

$$\begin{aligned}
U^*\big(\Phi_z(t), s+t\big) &= \mathbb{E}\big[\mathbb{1}_{\mathcal{A}}(\mathbf{z})\big|Z_{s+t} = \phi_z(t)\big] \\
&= \mathbb{E}\Big[\mathbb{E}\big[\mathbb{1}_{\mathcal{A}}(\mathbf{z})\big|Z_{T+s+t}\big]\Big|Z_{s+t} = \phi_z(t)\Big] \\
&= \mathbb{E}\Big[(\mathbb{1}_{T<t^*_{\Phi_z(t)}} + \mathbb{1}_{T=t^*_{\Phi_z(t)}})U^*(Z_{T+s+t}, s+t+T)\Big|Z_{s+t} = \phi_z(t)\Big] \\
&= \int_0^{t^*_{\Phi_z(t)}} U^-(\Phi_{\Phi_z(t)}(u), s+t+u)\lambda_{\Phi_z(t)}(u)\exp\big[-\Lambda_{\Phi_z(t)}(u)\big]du \\
&\quad + \exp\big[-\Lambda_{\Phi_z(t)}(t^*_{\Phi_z(t)})\big]\int_E K_{z^-}(z^+)U^*(z^+, s+t+t^*_{\Phi_z(t)})d\nu_{z^-}(z^+) \\
&\quad \text{where } z^- = \Phi_{\Phi_z(t)}(t^*_{\Phi_z(t)}) \\
U^*\big(\Phi_z(t), s+t\big) &= \int_t^{t_z^*} U^-(\Phi_z(u), s+u)\lambda_z(u)\exp\big[-\Lambda_{\Phi_z(t)}(u-t)\big]du \\
&\quad + \exp\big[-\Lambda_{\Phi_z(t)}(t_z^*-t)\big]\int_E K_{z^-}(z^+)U^*(z^+, s+t_z^*)d\nu_{z^-}(z^+) \\
&\quad \text{where } z^- = \Phi_z(t_z^*)
\end{aligned}$$

so $U^*\big(\Phi_z(t), s+t\big) = o(1) + (1+o(1))U^-\big(\Phi_z(t_z^*), s+t_z^*\big)$ as $t \to t_z^*,\ t < t_z^*$.

Appendix B.2. Proof of equality (25)

We have seen in the proof above that

$$\begin{aligned}
U^*\big(\Phi_z(t), s+t\big) &= \int_t^{t_z^*} U^-(\Phi_z(u), s+u)\lambda_z(u)\exp\big[-\Lambda_{\Phi_z(t)}(u-t)\big]du \\
&\quad + \exp\big[-\Lambda_{\Phi_z(t)}(t_z^*-t)\big]\int_E K_{z^-}(z^+)U^*(z^+, s+t_z^*)d\nu_{z^-}(z^+) \\
&= \int_t^{t_z^*} U^-(\Phi_z(u), s+u)\lambda_z(u)\exp\big[-\Lambda_z(u)\big]\exp\big[+\Lambda_z(t)\big]du \\
&\quad + \exp\big[-\Lambda_z(t_z^*)\big]\exp\big[+\Lambda_z(t)\big]\int_E K_{z^-}(z^+)U^*(z^+, s+t_z^*)d\nu_{z^-}(z^+) \\
&= \frac{1}{\exp\big[-\Lambda_z(t)\big]}\int_{[t,t_z^*]} U^-(\Phi_z(u), s+u)\Big(\lambda_z(u)\Big)^{\mathbb{1}_{t<t_z^*}}\exp\big[-\Lambda_z(u)\big]d\mu_z(t)
\end{aligned}$$

This last equality allows to transform (24) in (25).

Appendix B.3. Proof of equality (27)

Let $z \in E$ and $s \in [0, t_f]$. Remember that equality (27) states that if the functions $u \to U^{\text{-}}\big(\Phi_z(u), s+u\big)$ and $u \to \lambda_z(v)$ are continuous almost everywhere on $[0, t_z^*)$, then almost everywhere

$$\frac{\partial U^*\big(\Phi_z(v), s+v\big)}{\partial v} = U^*\big(\Phi_z(v), s+v\big)\lambda_z(v) - U^{\text{-}}\big(\Phi_z(v), s+v\big)\lambda_z(v)$$

We note T the time until the next jump after the trajectory has reached $Z_s = z$. For $0 \leq h < t_z^*$, we define $\tau = \min(h, T)$.

$$\begin{aligned}
U^*(z,s) &= \mathbb{E}\big[\mathbb{1}_{\mathcal{A}}(\mathbf{Z})\big|Z_s = z\big] \\
&= \mathbb{E}\Big[\mathbb{E}\big[\mathbb{1}_{\mathcal{A}}(\mathbf{Z})\big|Z_{s+\tau}\big]\Big|Z_s = z\Big] \\
&= \mathbb{E}\Big[(\mathbb{1}_{\tau=h} + \mathbb{1}_{\tau<h})\mathbb{E}\big[\mathbb{1}_{\mathcal{A}}(\mathbf{Z})\big|Z_{s+\tau}\big]\Big|Z_s = z\Big] \\
&= \mathbb{E}\Big[\mathbb{1}_{T=h}\,\mathbb{E}\big[\mathbb{1}_{\mathcal{A}}(\mathbf{Z})\big|Z_{s+h} = \Phi_z(h)\big]\Big|Z_s = z\Big] \;+\; \mathbb{E}\Big[\mathbb{1}_{T<h}\,\mathbb{E}\big[\mathbb{1}_{\mathcal{A}}(\mathbf{Z})\big|Z_{s+T}\big]\Big|Z_s = z\Big] \\
&= U^*(\phi_z(h), s+h)\;\mathbb{E}\big[\mathbb{1}_{T=h}\big|Z_s = z\;\big] \;+\; \mathbb{E}\Big[\mathbb{1}_{T<h}\,U^*(Z_{s+T}, s+T)\Big|Z_s = z\Big] \\
&= U^*(\phi_z(h), s+h)\;\exp\big[-\Lambda_z(h)\big] \\
&\qquad + \int_0^h \int_E K_{\Phi_z(u)}(z^+)U^*(z^+, s+u)d\nu_{\Phi_z(u)}(z^+)\lambda_z(u)\exp\big[-\Lambda_z(u)\big]du
\end{aligned}$$

As $\lambda_z(.)$ is continuous almost everywhere we have that almost everywhere :

$$\begin{aligned}
U^*(z,s) &= U^*(\phi_z(h), s+h)\,(1 - \lambda_z(0)h + o(h)) \\
&\qquad + \int_0^h U^{\text{-}}(\Phi_z(u), s+u)\lambda_z(u)\exp\big[-\Lambda_z(u)\big]du
\end{aligned}$$

As $u \to U^{\text{-}}(\phi_z(u), s+u)\lambda_z(u)$ is continuous almost everywhere, and we can do a Taylor approximation of the integral, which gives :

$$U^*(z,s) - U^*(\phi_z(h), s+h) = -\lambda_z(0)\,h\,U^*(\phi_z(h), s+h) \;+\; h\,U^{\text{-}}(z,s)\lambda_z(0) + o(h)$$

So $u \to U^*(\phi_z(u), s+u)$ is right-continuous almost everywhere. Therefore $U^*(\phi_z(h), s+h) = U^*(z,s) + o(1)$, and we get :

$$\frac{U^*(z,s) - U^*(\phi_z(h), s+h)}{h} = -\lambda_z(0)\,U^*(z,s) \;+\; U^{\text{-}}(z,s)\lambda_z(0) + o(1)$$

Making h tends to zero we get that $u \to U^*(\phi_z(u), s+u)$ has a right-derivative in zero. Applying the same kind of reasoning in state $\Phi_z(-h)$ instead of z, we would find that the left-derivative exists and is equal to the right-derivative. So for almost every state $z \in E$,

$$\left(\frac{\partial U^*\big(\Phi_z(v), s+v\big)}{\partial v}\right)_{v=0} = U^*\big(\Phi_z(0), s+0\big)\lambda_z(0) - U^{\text{-}}\big(\Phi_z(0), s+0\big)\lambda_z(0)$$

Applying the same reasoning in a state $\Phi_{z_o}(v)$ instead of z and using the additivity of the flow, we get that almost everywhere:

$$\forall z_o \in E, v > 0, \quad \frac{\partial U^*\big(\Phi_{z_o}(v), s+v\big)}{\partial v} = U^*\big(\Phi_{z_o}(v), s+v\big)\lambda_{z_o}(v) - U^{\text{-}}\big(\Phi_{z_o}(v), s+v\big)\lambda_{z_o}(v)$$

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