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We improve upon the first Thom-Mather isotopy theorem for Whitney stratified sets. In particular, for the more general Bekka stratified sets we show that there is a local foliated structure with continuously varying tangent spaces, thus proving the smooth version of the Whitney fibering conjecture. A regular wing structure is also shown to exist locally, for both Whitney and Bekka stratifications. The proofs involve integrating carefully chosen controlled distributions of vector fields. As an application of our main theorem we show the density of the subset of strongly topologically stable mappings in the space of all smooth quasi-proper mappings between smooth manifolds, an improvement of a theorem of Mather.

1. Introduction. The most important and widely used result in stratification theory is the first Thom-Mather isotopy theorem dating from 1969-70, which implies in particular that every Whitney stratification is locally topologically trivial. The proof involves finding controlled lifts of vector fields from a given stratum X onto a tubular neighbourhood and showing they are integrable. At the time these controlled lifts were not known to be continuous. However continuous controlled lifts were shown to exist by Shiota $[\mathbf{Sh}]_2$, Bekka $[\mathbf{Be}]_2$, and the second author **[Pl]** twenty years ago. In 1993 the third author conjectured that the local foliation obtained in the isotopy theorem (with leaves of the same dimension as X) can be chosen to have continuously varying tangent planes, and Sullivan pointed out to us that this implies the smooth version of the (analytic) Whitney fibering conjecture (see below). In this paper we prove this smooth Whitney fibering conjecture with the weaker hypothesis that the stratification be Bekka (c)-regular, a condition *equivalent* to the existence of continuous controlled lifting of vector fields, and which is implied by Whitney regularity. Moreover we show the existence of foliations by regular wings for Whitney and Bekka stratifications. We thus strengthen significantly the first Thom-Mather isotopy theorem.

We now give a historical presentation of the Whitney conjecture and related results.

In his famous paper of 1965 **[Wh]** H. Whitney proposed a local fibering property around points of a complex analytic variety. More precisely he conjectured that every complex analytic variety V admits a stratification such that a neighbourhood U of each point is fibered by copies of the intersection of U with the stratum M containing the point. He asked also that the fibers be holomorphic manifolds and that their tangent spaces vary continuously as nearby points approach X (see section 2 for a precise formulation).

Note that if one does not require the continuity of tangent spaces to the fibers then the first Thom-Mather isotopy theorem **[Th]**, $[\mathbf{Ma}]_{1,2}$ suffices to prove the smooth version of Whitney's conjecture.

In 1989 R. Hardt and D. Sullivan gave a proof of a similar conclusion for holomorphic varieties but again without the essential continuity of the tangent spaces to the fibers **[HS]**.

From 1993 the first author studied the possibility of obtaining the analogous property in the case of smooth real stratified spaces in his thesis under the direction of the third author, who conjectured that this property be true for Whitney (b)-regular stratifications.

The smooth version of the Whitney conjecture was necessary to be able to use the notion of *semidifferentiability* introduced in $[\mathbf{Mu}]_1$ and $[\mathbf{MT}]_4$ with the aim of obtaining the preservation of regularity of a substratified space of a stratification after a deformation by stratified isotopy $[\mathbf{MPT}]_{1,2}$, so as to show the conjectured representation of homology by Whitney stratified cycles $[\mathbf{Go}]_{1,2}$ (an open problem since 1981) and also to approach the unsolved conjecture of Thom on the existence of Whitney triangulations and cellulations of Whitney stratified sets (preliminary work on this problem dating from 1994-96 remained in a manuscript form and will be treated in a forthcoming paper $[\mathbf{MT}]_5$). It improves moreover the first Thom-Mather isotopy theorem by ensuring *horizontally-*C^1 regularity ($[\mathbf{MT}]_{3,4}$, §5 Theorems 10 and 11).

The first and third authors began a collaboration on this research with the second author, whose book of 1995 **[PW]** with C. T. C. Wall introduced the notion of E-tame retractions, as retractions whose fibers are foliations having an analogous continity property. More precisely, with the aim of proving that multi-transversality with respect to a given partition in submanifolds of a jet space is a sufficient condition for strong C^0-stability, Wall and the second author introduced various regularity conditions for retractions $r : M \to N$ between two smooth manifolds : the *Tame, Very tame and Extremely tame retractions.* These last, the E-tame retractions, were characterized by the fact that the foliations defined by their fibres are of class $C^{0,1}$. This property in a stratified context for a local "horizontal" retraction $\pi' : \pi_X^{-1}(U_{x_0}) \to \pi_X^{-1}(x_0)$ is equivalent to a real $C^{0,1}$ version of the conclusion of the Whitney fibering conjecture (see §8 Remark 10 or $[\mathbf{MT}]_4$ §4.3 for details). These tame retractions were studied by Wall and the second author **[PW]**, and also Feragen **[Fe]** who found particular cases where retractions can be glued.

In 2007, P. Berger, in his Ph.D. thesis supervised by J.-C. Yoccoz, with the aim of generalizing some fundamental results of Hirsch-Pugh-Shub to stratifications of laminations, needed the smooth version of the Whitney fibering conjecture to study the persistence of normally expanded Whitney stratifications **[Ber]**

In 2014, A. Parusinski and L. Paunescu **[PP]** constructed for a given germ of complex or real analytic set a stratification satisfying a strong trivialization (called *arc-wise analytic*) property locally along each stratum and then proved the original Whitney fibering conjecture in the real and complex, local analytic and global algebraic cases.

In this paper we prove a result which implies the smooth version of Whitney's conjecture. This is that any Bekka (c)-regular stratification satisfies the smooth version of the Whitney properties. Recall that (c)-regularity is strictly weaker than Whitney's (b)-regularity $[\mathbf{Be}]_1$, and that every complex analytic variety admits a (b)-regular stratification (hence (c)-regular) **[Wh]**. More generally every subanalytic set admits a (b)-regular stratification **[Hi]**, **[DW]**, **[Ha]**, **[LSW]**, as does every definable subset in an o-minimal structure **[VM]**, **[Loi]**, **[NTT]**. Thus the smooth version of Whitney's conjecture holds for these classes of sets.

The contents of the paper are as follows.

In section 2, we present the Whitney fibering conjecture as stated in the original paper of H. Whitney **[Wh]**.

In section 3, we review some important classes of regular stratifications: the abstract stratified sets of Thom and Mather **[Th]**, $[\mathbf{Ma}]_{1,2}$, and the (c)-regular stratifications of Karim Bekka $[\mathbf{Be}]_1$, and we recall the first isotopy theorem which holds for these classes.

In section 4, we introduce the notion of controlled foliation on a stratification by two strata $X < Y$ in $\mathbb{R}^n$ and prove a theorem, for gluing these foliations, whose methods will be frequently used in the proof of Theorem 3 in section 5.

In section 5, we recall in §5.1 some important properties of the stratified topological triviality map obtained by using continuous canonical lifted frame fields **[MT]**$_{2,3,4}$. In §5.2 we recall some useful properties of the controlled frame fields tangent to the horizontal leaves of this trivialization map. Then in §5.3 we prove a two strata version of our main result, Theorem 3 (and Theorem 4) stating that the smooth version of the Whitney fibering conjecture holds for every stratum X with $depth_\Sigma(X) = 1$ of a (c)-regular stratification.

In section 6 under the same hypotheses as Theorems 3 and 4 we construct a local wing structure for Bekka (c)-regular (Theorem 5) and Whitney (b)-regular (Theorem 6) stratifications for every stratum X with depth 1. These results will play an important role in the proof of our main theorem in section 7.

In section 7 we prove our main theorems. First we use the notion of conical chart (Definition 10) and the local wing structure of section 6 to prove the conclusions of the smooth Whitney fibering conjecture for any stratum X of a (c)-regular stratification $\mathcal{X} = (A, \Sigma)$ having arbitrary depth (Theorem 7). Then we use Theorem 7 to extend the wing structure Theorems 5 and 6 of section 6 to a stratum of arbitrary depth (Theorem 8). Step 4 of Theorem 7 also completes some details not given in Theorems 2 and 3 of **[MT]**$_2$ of the existence of continuous controlled lifting of vector fields in the general case with more than 3 strata.

In section 8 we apply Theorem 7 to the results of **[MT]**$_{1,3,4}$, where we introduced the notion of horizontally-C^1 stratified controlled morphism $f : \mathcal{X} \to \mathcal{X}'$ (Definition 11) to prove that the flows of the continuous controlled lifted vector fields to a stratum X have a horizontally-C^1 regularity, stronger than C^0-regularity, but weaker than C^1-regularity (Corollaries 6 and 7) and we deduce a horizontally-C^1 version of the first Thom-Mather Isotopy Theorem for a stratified proper submersion $f : \mathcal{X} \to M$ into a manifold (Theorems 9 and 10). Then using the finer notion of $\mathcal{F}$-semidifferentiability (Definition 12), we improve these results by stating an $\mathcal{F}$-semidifferentiable version of the first Thom-Mather Isotopy Theorem which extends the horizontally-C^1 convergence of the topological trivialisation of f to all points of strata Y' such that $X \leq Y' \leq Y$ (Theorems 11 and 12).

In section 9 we use our main Theorem 7 to prove Theorem 13 which gives a sufficient condition for a smooth map between two smooth manifolds to be strongly topologically stable. This in turn implies a long-awaited improvement to strong topological stability of a classical (1973) theorem of Mather **[Ma]**$_2$ stating that the subset of topologically stable mappings in the space of all smooth (quasi-)proper mappings between smooth manifolds is dense. The proof uses the relation between the conclusion of the smooth Whitney fibering conjecture and the existence of the tame retractions introduced and studied in **[PW]**, Chapter 9.

We thank Dennis Sullivan for drawing our attention to Whitney's original fibering conjecture and to his own work with Bob Hardt on this conjecture **[HS]**. We thank also Edmond Fedida, Etienne Ghys, Pierre Molino, David Spring and the late Bill Thurston for useful discussions.

2. The Whitney fibering conjecture.

In his famous article *Local Properties of Analytic Varieties* **[Wh]**, after introducing the well-known (a) and (b)-regularity conditions and showing that *"Every (real or com-*

plex) analytic variety V admits a (b)-regular stratification", H. Whitney gave the following definition:

Definition 1. A stratification Σ of an analytic variety V will be considered *"good"* if each point $p_0 \in V$ admits a neighbourhood U_0 in V having a foliation $\mathcal{H}_{p_0} = \{F(q)\}_q$ obtained in the following way. Let M be the stratum of Σ containing p_0, $M_0 := M \cap U_0$ and $N_0 := (T_{p_0}M)^\perp \cap U_0$ (where $\perp$ means the orthogonal complement in the ambient space).

Then U_0 is homeomorphic to $M_0 \times N_0$ through a map $\phi : M_0 \times N_0 \to U_0$, $\phi = \phi(p, q)$ satisfying the following properties:

i) ϕ is analytic in $p \in M_0$ and continuous in $q \in N_0$;
ii) $\mathcal{H}_{p_0} = \{F(q)\}_q$ is exactly the foliation $\{M_q := \phi(M_0 \times \{q\})\}_{q \in N_0}$ induced by ϕ;
iii) every restriction $\phi_{|M_0 \times q} : M_0 \times \{q\} \to F(q)$ to a leaf of $\mathcal{H}_{p_0}$ is a biholomorphism;
iv) both restrictions $\phi_{|M_0 \times \{q_0\}} = id_{M_0}$ and $\phi_{|\{p_0\} \times N_0} = id_{N_0}$ are the identity.

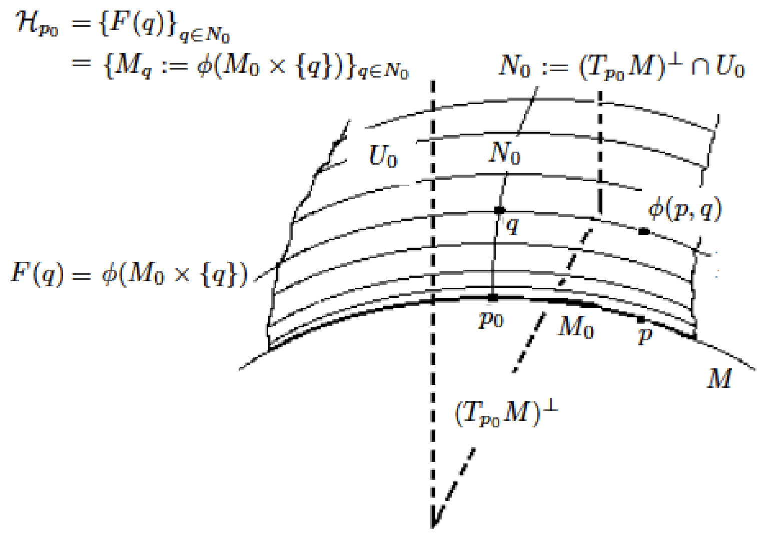

Figure 1

Whitney's definition of 1965 was a *precursor* of the idea that suitable regular stratifications have the property of local topological triviality, an idea completely clarified in the years 1969-70 by the famous Thom-Mather first isotopy theorem **[Ma]**$_{1,2}$ **[Th]**.

Whitney called such a map ϕ a *semianalytic fibration (for Σ) near p_0* and remarked that an analytic variety V does not have (in general) a stratification admitting near each point an analytic fibration. He gave the celebrated counterexemple (the *four lines family*)

$$V := \{(x, y, z) \in \mathbb{C}^3 \mid xy(y - x)\big(y - (3 + t)x\big) = 0\,\}$$

and stated the following conjecture:

Whitney fibering conjecture. *"Every analytic variety V has a stratification admitting in each point $p_0 \in V$ a semianalytic fibration".*

Whitney comments furthermore that *". . . a stratification satisfying the conjecture (possibly with further conditions on ϕ) would probably be sufficient for all needs".*

Remark 1. Whitney also states in a commentary that every stratification Σ of V with such a semianalytic fibration near a point p_0 is automatically (a)-regular at all points of the neighbourhood $U_0 \cap M$ of p_0 in M, because the properties of ϕ imply the convergence of the tangent planes to the leaves of the foliation $\mathcal{H}_{p_0} = \{F(q)\}_{q \in N_0}$:

$$(L_p): \qquad \lim_{z \to p} T_z F(z) = T_p M .$$

So, for Whitney, (a)-regularity is a consequence of the existence of such a *semi-analytic fibration.* In fact he wrote that in a local analysis in which M_0 is identified with the $(x_1, \dots, x_d)$-plane ($d = \dim M$), for each stratum $M_j > M_0$, *"any fiber $F(q)$, with $q \in M_j$ sufficiently near to p_0 is near $F(p_0) = M_0$ .... and $F(q)$ is expressed by holomorphic functions $x_i = f_i(x_1, \dots, x_d)$, $i = 1, \dots, n-d$. These functions are small throughout M_0; hence their partial derivatives are small in a smaller neighbourhood of p_0.*
Since $F(q) \subseteq M_j$ if $q \in M_j$, this clearly implies the condition (a)".

However, this argument is not valid in general as A. du Plessis explained in a conference in 2005 at the CIRM.

Definition 2. Because the limit condition (L_p) : $\lim_{z \to p} T_z F(z) = T_p M$ is very important for us we have previously redefined it in a more general C^1-real context as the *(a)-regularity of a local controlled foliation* $[\mathbf{Mu}]_1$, $[\mathbf{MT}]_4$ and (by abuse of language) we will refer to it as the *smooth version of the Whitney fibering conjecture*. In this paper we do not seek any (re)-stratification but our aim is to prove the conclusions of the conjecture of Whitney for each point x of every stratum X of a fixed arbitrary Bekka (c)-regular stratification $\mathcal{X}$ with C^1 strata.

Let us mention some work on the fibering conjecture.

Whitney proved ([**Wh**], §12) that the conjecture holds for every analytic hypersurface V of $\mathbb{C}^n$ for all points in $(n-2)$-strata after restratification of V.

Later, in 1983, Hardt [**Ha**] indicated a possible solution of the problem in the real analytic case and in 1988, Hardt and Sullivan [**HS**] treated the problem for complex algebraic varieties. The conclusion obtained by Hardt and Sullivan [**HS**] is weaker than that proposed by Whitney, in that they did not obtain the condition $\lim_{z \to p} T_z F(z) = T_p M$, i.e. the *(a)-regularity of the foliation $\mathcal{H}_{p_0}$.*

Very recently, A. Parusinski and L. Paunescu [**PP**], using a slightly stronger version of Zariski equisingularity constructed for a given germ of complex or real analytic set, a stratification satisfying a strong (real arc-analytic with respect to all variables and analytic with respect to the parameter space) trivialization property along each stratum (the authors call such trivializations *arc-wise analytic*). Then using a generalization of Whitney Interpolation they prove the original Whitney fibering conjecture in the real and complex, local analytic and global algebraic cases. Because Zariski equisingularity implies Whitney regularity in the complex case, their hypothesis is stronger than the (c)-regularity used in the present paper, while their conclusion is also stronger.

We conclude this section by recalling the following globalization problem ([**Wh**] §9):

Problem. *"May one fibre a complete neighbourhood of any stratum ?"*
That is :
Can one find a global stratified foliation of a complete neighbourhood of any stratum ?

We will not deal with this problem, but we just remark that without restratifying the smaller stratum it cannot have a solution in general. In fact, as P. Berger wrote to us (see his Ph.D. Thesis p. 60 or [**Ber**] p. 38) by considering the stratification of two strata

$M = S^2 \times \{(0,0)\} < S^2 \times (\mathbb{R}^2 - \{(0,0)\})$ of $S^2 \times \mathbb{R}^2$, a global foliation of a neighbourhood of $S^2 \times \{(0,0)\}$ in $S^2 \times (\mathbb{R}^2 - \{(0,0)\})$ cannot exist, because otherwise, starting from an arbitrary non-zero vector tangent to S^2 one could define by holonomy a continuous non-zero vector field on the whole of S^2 which cannot exist.

3. Bekka (c)-regular stratified spaces.

We recall that a *stratification* of a topological space A is a locally finite partition Σ of A into C^1 connected manifolds (called the *strata* of Σ) satisfying the *frontier condition* : if X and Y are disjoint strata such that X intersects the closure of Y, then X is contained in the closure of Y. We write then $X < Y$ and $\partial Y = \overline{Y} - Y$ so that $\overline{Y} = Y \sqcup \left(\sqcup_{X<Y} X\right)$ and $\partial Y = \sqcup_{X<Y} X$ ($\sqcup$ = disjoint union).

The pair $\mathcal{X} = (A, \Sigma)$ is called a *stratified space* with *support* A and *stratification* Σ. The union of the strata of dimension $\leq k$ is called the *k-skeleton*, denoted by A_k, inducing a stratified space $\mathcal{X}_k = (A_k, \Sigma_{|A_k})$.

A *stratified map* $f : \mathcal{X} \to \mathcal{X}'$ between stratified spaces $\mathcal{X} = (A, \Sigma)$ and $\mathcal{X}' = (B, \Sigma')$ is a continuous map $f : A \to B$ which sends each stratum X of $\mathcal{X}$ into a unique stratum X' of $\mathcal{X}'$, such that the restriction $f_X : X \to X'$ is smooth. We call such a map f a *stratified homeomorphism* if f is a global homeomorphism and each f_X is a diffeomorphism.

A stratified vector field on $\mathcal{X}$ is a family $\zeta = \{\zeta_X\}_{X \in \Sigma}$ of vector fields, such that ζ_X is a smooth vector field on the stratum X.

Extra conditions may be imposed on the stratification Σ, such as to be an *abstract stratified set* in the sense of Thom-Mather [**GWPL**], [**Ma**]$_{1,2}$, [**Ve**] or, when A is a subset of a C^1 manifold, to satisfy conditions (a) or (b) of Whitney [**Ma**]$_{1,2}$, [**Wh**], or (c) of K. Bekka [**Be**]$_1$.

Definition 3. (Thom and Mather) Let $\mathcal{X} = (A, \Sigma)$ be a stratified space.

A family $\mathcal{F} = \{(\pi_X, \rho_X, T_X)\}_{X \in \Sigma}$ is called *a system of control data)* for $\mathcal{X}$ if for each stratum X we have that:

1) T_X is a neighbourhood of X in A (called a *tubular neighbourhood of* X);
2) $\pi_X : T_X \to X$ is a continuous retraction of T_X onto X (called *projection on* X);
3) $\rho_X : T_X \to [0, \infty[$ is a continuous function such that $X = \rho_X^{-1}(0)$ (called the *distance function from* X)

and, furthermore, for every pair of adjacent strata $X < Y$, by considering the restriction maps $\pi_{XY} = \pi_{X|T_{XY}}$ and $\rho_{XY} = \rho_{X|T_{XY}}$ on the subset $T_{XY} = T_X \cap Y$, we have that :

5) the map $(\pi_{XY}, \rho_{XY}) : T_{XY} \to X \times [0, \infty[$ is a smooth submersion (it follows in particular that $\dim X < \dim Y$);
6) for every stratum Z of $\mathcal{X}$ such that $Z > Y > X$ and for every $z \in T_{YZ} \cap T_{XZ}$ the following *control conditions* are satisfied :
   i) $\pi_{XY}\pi_{YZ}(z) = \pi_{XZ}(z)$ (called the *π-control condition*),
   ii) $\rho_{XY}\pi_{YZ}(z) = \rho_{XZ}(z)$ (called the *ρ-control condition*).

In what follows we will pose $T_X(\epsilon) = \rho_X^{-1}([0, \epsilon))$, $\forall\, \epsilon \geq 0$, and without loss of generality will assume $T_X = T_X(1)$ [**Ma**]$_{1,2}$, [**GWPL**].

If A is Hausdorff, locally compact and admits a countable basis for its topology, the pair $(\mathcal{X}, \mathcal{F})$ is called an *abstract stratified set.* Since one usually works with a unique system of control data of $\mathcal{X}$, in what follows we will omit $\mathcal{F}$.

If $\mathcal{X}$ is an abstract stratified set, then A is metrizable and the tubular neighbourhoods $\{T_X\}_{X\in\Sigma}$ may (and will always) be chosen such that: "$T_{XY} \neq \emptyset$ if and only if $X < Y$, or $X > Y$ or $X = Y$" (see **[Ma]**$_1$, page 41 and following).

Let $f : \mathcal{X} \to \mathcal{X}'$ be a stratified map between two abstract stratified sets and fix two systems of control data $\mathcal{F} = \{(T_X, \pi_X, \rho_X)\}_{X\in\Sigma}$ and $\mathcal{F}' = \{(T_{X'}, \pi_{X'}, \rho_{X'})\}_{X'\in\Sigma'}$ respectively of $\mathcal{X}$ and $\mathcal{X}'$. The map f is called *controlled (with respect to $\mathcal{F}$ and $\mathcal{F}'$)* if when $X < Y$ there exists $\epsilon > 0$ such that for all $y \in T_{XY}(\epsilon) = T_X(\epsilon) \cap Y$ the following two *control conditions* hold :

$$\begin{cases} \pi_{X'Y'} f_Y(y) = f_X \pi_{XY}(y) & \text{(the } \pi\text{-}\textit{control condition} \text{ for } f) \\[2ex] \rho_{X'Y'} f_Y(y) = \rho_{XY}(y) & \text{(the } \rho\text{-}\textit{control condition} \text{ for } f). \end{cases}$$

Similarly, a stratified vector field $\zeta = \{\zeta_X\}_{X\in\Sigma}$ is *controlled (with respect to $\mathcal{F}$)* if the following two *control conditions* hold:

$$\begin{cases} \pi_{XY*}(\zeta_Y(y)) = \zeta_X(\pi_{XY}(y)) & \text{(the } \pi\text{-}\textit{control condition} \text{ for } \zeta) \\[2ex] \rho_{XY*}(\zeta_Y(y)) = 0 & \text{(the } \rho\text{-}\textit{control condition} \text{ for } \zeta). \end{cases}$$

The notion of system of control data of $\mathcal{X}$, introduced by Mather in **[Ma]**$_{1,2}$, is the fundamental tool allowing one to obtain good extensions of vector fields.

In fact, we have **[Ma]**$_{1,2}$, **[GWPL]** :

Proposition 1. *If $\mathcal{X}$ is an abstract stratified set with C^2 strata, every vector field ζ_X defined on a stratum X of $\mathcal{X}$ admits a stratified (π,ρ)-controlled lifting $\zeta_{T_X} = \{\zeta_Y\}_{Y\geq X}$ defined on a tubular neighbourhood T_X of X.*

Moreover, if ζ_X admits a global flow $\{\phi_t : X \to X\}_{t\in\mathbb{R}}$, then such a lifting ζ_{T_X} admits again a global flow $\{\phi_{T_X\,t} : T_X \to T_X\}_{t\in\mathbb{R}}$, and every $\phi_{T_X\,t} : T_X \to T_X$ is a stratified, continuous and (π,ρ)-controlled homeomorphism. □

Definition 4. (K. Bekka 1991). A stratified space (A,Σ) in $\mathbb{R}^n$ is called (c)-*regular* if, for every stratum $X \in \Sigma$, there exists an open neighbourhood U_X of X in $\mathbb{R}^n$ and a C^1 function $\rho_X : U_X \to [0,\infty[$, such that $\rho_X^{-1}(0) = X$, and such that its stratified restriction to the *star* of $\mathcal{X}$:

$$\rho_X \;\; : \;\; Star(X) \cap U_X \to [0,\infty[\qquad \text{is a Thom map,}$$

where $Star(X) = \cup_{Y\geq X}$ and the stratification on $Star(X) \cap U_X$ is induced by Σ.

The (c)-regularity of Bekka states exactly that for every pair of adjacent strata $X < Y$:

$$\lim_{y\to x} T_y \rho_{XY}^{-1}(\epsilon) \;=\; \lim_{y\to x} \ker \rho_{XY*y} \;\supseteq\; T_x X \quad \text{ for every } \quad x \in X \quad (\epsilon = \rho_X(y))\,.$$

Remark 2. A Bekka (c)-regular stratified space $\mathcal{X} = (A,\Sigma)$ admits a system of control data $\{(\pi_X, \rho_X, T_X)\}_{X\in\Sigma}$ in which for each stratum $X \in \Sigma$, $T_X = U_X \cap A$, and π_X, ρ_X are restrictions of C^1 maps defined on U_X **[Be]**$_1$. Thus (c)-regular stratifications admit a structure of abstract stratified set and so Proposition 1 holds for them.

We underline moreover that in this case, for each vector field ζ_X on a stratum X of A, the stratified (π,ρ)-controlled lifting $\zeta_{T_X} = \{\zeta_Y\}_{Y\geq X}$ defined on a tubular neighbourhood T_X of X may be chosen to be *continuous* **[Be]**$_1$, **[Pl]**, **[MT]**$_{2,3}$, **[Sh]**$_{1,2}$. Finer results may be obtained by using the notion of *canonical distribution* introduced in **[MT]**$_{1,3,4}$, which allow us to obtain a stronger regularity for the lifted flow $\{\phi_{T_X,t} : T_X \to T_X\}_{t\in\mathbb{R}}$ (see §8).

A *canonical distribution* $\mathcal{D}_X := \{\mathcal{D}_{XY} : T_X(1) = \sqcup_{Y\geq X} T_{XY}(1) \to \mathbb{G}_l(TM)\ \}_{Y\geq X}$ associated to a l-stratum X of $A \subseteq M$ is a stratified l-subbundle of $\sqcup_{Y\geq X} \ker \rho_{XY*y} \subseteq TM$, such that each restriction $\pi_{XY*y|} : \mathcal{D}_{XY}(y) \to T_x X$ is an isomorphism ($x = \pi_{XY}(y)$) and which is continuous : $\lim_{z\to y} \mathcal{D}_X(z) = \mathcal{D}_{XY}(y)$, for every $Y \geq X$ and every $y \in Y$.

In the context of (c)-regular stratifications, the canonical distributions are characterized by the property that for each vector field ξ_X defined on X there exists a *canonical* stratified continuous (π,ρ)-controlled extension $\xi = \{\xi_{XY}\}_{Y\geq X}$ to $\mathcal{D}_X$ of ξ_X and given by:

$$\xi_{XY}(y) \ := \ \pi^{-1}_{XY*y|\mathcal{D}_{XY}(y)}(\xi_X(x)) \ = \ \mathcal{D}_{XY}(y) \cap \pi^{-1}_{XY*y}(\xi_X(x)) \qquad \text{for every} \quad Y \geq X.$$

The first and the third author proved the existence of canonical distributions for (c)-regular stratifications in **[MT]**$_2$ (Theorem 4). In this paper we only consider canonical distributions defined in the stratification induced by $\mathcal{X} = (A,\Sigma)$ on a neighbourhood W of $x \in X$ in A. Moreover W will be always taken (*"π_X-fibre"*) of the type $W = \pi_X^{-1}(U)$ with U a domain of a chart of X near x.

We recall now the most important properties of lifting of vector fields on such regular stratifications and the most useful relations between them :

i) the condition "to be a Thom-Mather abstract stratified set" implies the existence of controlled lifting of vector fields **[Ma]**$_{1,2}$;

ii) Bekka's (c)-regularity is *characterized* by the existence of (π,ρ)-controlled and *continuous* lifting of vector fields **[Be]**$_1$, **[Pl]**, **[MT]**$_{2,3}$, **[Sh]**$_{1,2}$, and implies the property *"to be a Thom-Mather abstract stratified set"* **[Ma]**$_{1,2}$. Moreover (c)-regular stratifications admit systems of control data whose maps $\{(\pi_X,\rho_X) : T_X \to X \times [0,\infty[\ \}_X$ are C^1 **[Be]**$_1$. Bekka's (c)-regular stratifications have been used notably in **[Si]** to prove a Poincaré-Hopf index theorem for radial stratified vector fields, in **[Nad]** to study Morse theory and tilting sheaves on Schubert stratifications, and in **[RD]** to provide a sufficient condition for the existence of a real Milnor fibration.

Finally we recall the following important facts :

a) Whitney (b)-regularity implies (c)-regularity **[Be]**$_1$, **[Tr]**$_1$;

b) every abstract stratified set admits a (b)-regular embedding **[Na]**, **[Te]**, and even **[No]** a subanalytic (w)-regular and hence (b)-regular **[Kuo]**, **[Ve]** embedding in some $\mathbb{R}^N$;

c) abstract stratified sets admit triangulations, smooth in the sense of Goresky **[Go]**$_3$;

d) the first isotopy theorem of Thom-Mather holds for all the kinds of stratification considered above, using the (claimed) properties of stratified lifting of vector fields.

The first isotopy theorem of Thom-Mather applied to a projection map $\pi_X : T_X \to X$ on the stratum X can be stated as follows :

Theorem 1. *Let $\mathcal{X} = (A,\Sigma)$ be an abstract stratified closed subset in $\mathbb{R}^n$ with C^2-strata, X a stratum of $\mathcal{X}$ and $x_0 \in X$ and U_{x_0} a domain of a chart near x_0 of X.*

For every frame field $(v_1(x),\ldots,v_l(x))$ of U_{x_0} ($l = \dim X$) having a global flow, the (π,ρ)-controlled liftings $(v_1(z),\ldots,v_l(z))$ on $\pi_X^{-1}(U_{x_0})$ have global flows $(\phi_1,\ldots,\phi_l)$ and the map

$$\begin{array}{rcl} H \ = H_{x_0} \ : \quad U_{x_0} \times \pi_X^{-1}(x_0) & \longrightarrow & \pi_X^{-1}(U_{x_0}) \\ (t_1,\ldots,t_l,z_0) & \longmapsto & \phi_l(t_l,\ldots,\phi_1(t_1,z_0)\ldots) \end{array}$$

is a stratified homeomorphism, a diffeomorphism on each stratum of $U_{x_0} \times \pi_X^{-1}(x_0)$. $\square$

4. Gluing controlled foliations by generating frame fields.

The proof of a smooth version of the Whitney fibering conjecture for (c)-regular stratifications having two strata $X < Y$ that we will give in section 5 (Theorem 3) requires a careful analysis of the properties of the local foliations $\mathcal{H}_x$ induced by a topological trivialization H obtained using *continuous* (π,ρ)-controlled lifting of vector fields $[\mathbf{MT}]_4$ from X to Y. It is moreover strongly based on a careful gluing of foliations locally defined on neighbourhoods of each $y \in Y$ in which the tangent planes to the foliations are very close to the canonical distribution $\mathcal{D}_X$.

So, with the aim of reproducing the essential properties and situations of the local foliations $\mathcal{H}_x$, in this section we introduce the notion of *controlled* l-dimensional foliation and we prove Theorem 2 allowing us to glue together controlled foliations using adapted partitions of unity.

Although the statements of this section are not directly used in the proof of Theorem 4, the methods of the proof used in the gluing in Theorem 3, Corollary 1 and Remark 3 contain all the basic ingredients that allow us to prove the smooth two strata version of the Whitney fibering conjecture for (c)-regular stratifications. Thus we include this short section.

Definition 5. Let $\mathcal{X}$ be a (c)-regular stratification in $\mathbb{R}^n$, X a stratum of $\mathcal{X}$, $x_0 \in X$, U an open set of X, $W = \pi_X^{-1}(U)$, $W^\epsilon = W \cap T_X(\epsilon)$ stratified by $\sqcup_{Y \geq X} W \cap T_{XY}(\epsilon)$.

Let $\mathcal{F} = \{M_z\}_{z \in W^\epsilon}$, with M_z the unique leaf of $\mathcal{F}$ containing z, be an l-dimensional foliation of W^ϵ. We say that $\mathcal{F} = \{M_z\}_{z \in W^\epsilon}$ is a *controlled l-foliation* of W^ϵ if it is *stratified* (i.e. each leaf is contained in a unique stratum of W^ϵ) and there exists a stratified controlled frame field $(w_1, \dots, w_l)$ *of* W^ϵ *generating* $\mathcal{F}$. That is :

i) $(w_1, \dots, w_l)$ is tangent to $\mathcal{F}$, hence $\forall\, y \in M_z$, $T_y M_z = T_y\mathcal{F} = [(w_1(y), \dots, w_l(y)]$;

ii) $(w_1, \dots, w_l)$ is a (π,ρ)-controlled lifting of a frame field $(u_1, \dots, u_l)$ of U ;

iii) all Lie brackets $[w_i, w_j] = 0$, for every $i \neq j = 1, \dots, l$.

In this case for every stratum $Y \geq X$ and every $y \in Y$ with $x = \pi_X(y)$ we have:

a) If $(w_1, \dots, w_l)$ generates a controlled foliation $\mathcal{F}$, it defines an integrable l-distribution $\mathcal{D}(y) = [w_1(y), \dots, w_l(y)] = T_y\mathcal{F}$ of W^ϵ tangent to $\mathcal{F}$ and whose integral l-manifolds are exactly the leaves of $\mathcal{F}$.

b) By the ρ-control condition, each leaf $M_y \in \mathcal{F}$ is contained in the level hypersurface $\rho_{XY}^{-1}(\eta)$. with $\eta = \rho_{XY}(y)$;

c) By the π-control condition, $\pi_{XY*y}(w_i(y)) = u_i(x)$, each restriction $\pi_{XY*y} : T_yY \to T_xX$ with $x = \pi_X(y)$ is an isomorphism of vector spaces and $w_i(y) = T_yM_y \cap \pi_{XY*y}^{-1}(u_i(x))$.

Moreover each leaf M_y of $\mathcal{F}$, is transverse to the fiber $\pi_X^{-1}(x_0)$, $\forall\, x_0 \in U$ and meets it in a unique point $y_0 := M_y \cap \pi_X^{-1}(x_0)$ so the foliation $\mathcal{F}$ can be reparametrised by $\mathcal{F} := \{M_{y_0}\}_{y_0 \in \pi^{-1}(x_0)}$.

It follows that if U is a domain of a chart $\varphi : U \to \mathbb{R}^l$ of X and $u_i := \varphi_*^{-1}(E_i)$, the liftings $(w_1, \dots, w_l)$ tangent to $\mathcal{F}$ of $(u_1, \dots, u_l)$ are controlled and generate $\mathcal{F}$. Moreover if each u_i has a complete flow ϕ_i its controlled lifting w_i has a complete flow too $[\mathbf{Ma}]_{1,2}$.

Then by denoting $\psi_1, \dots, \psi_l$ respectively the flows of $w_1, \dots, w_l$, the map

$$H = H_{x_0} : U \times \pi_X^{-1}(x_0) \equiv \mathbb{R}^l \times \pi_X^{-1}(x_0) \to W^\epsilon \ , \quad H(t_1, \dots, t_l, z_0) = \psi_l(t_l, \dots, \psi_1(t_1, z_0)..) \,,$$

defines, for every $x_0 \in U$, a local topological trivialization of $\pi_X : T_X \to X$ whose images $\{M_z = H(\mathbb{R}^l \times \{z\})\}_{z \in W^\epsilon}$ are the leaves of $\mathcal{F}$. Thus $\mathcal{F} = \left\{H(\mathbb{R}^l \times \{z_0\})\right\}_{z_0 \in \pi_X^{-1}(x_0)}$ $[\mathbf{Ma}]_{1,2}$ and moreover $w_i(y) = H_{*y}(E_i)$ for every $i = 1, \dots, l$ (see §5.2, Proposition 3).

Conversely if $H = H_{x_0}$ is the map in the statement of Theorem 1, it is obvious that $\mathcal{F} = \big\{ H(\mathbb{R}^l \times \{z_0\}) \big\}_{z_0 \in \pi_X^{-1}(x_0)}$ is a controlled foliation *generated by* H.

Thus controlled foliations (near $x_0 \in X$) and controlled topological trivializations of $\pi_X : T_X \to X$ (near x_0) correspond bijectively and they generate in this way the same mathematical concept.

Definition 6. Let $\mathcal{F} = \{M_z\}_{z \in p^{-1}(x)}$ be a controlled l-foliation of W^ϵ.

We will say that $\mathcal{F}$ *is (a)-regular over U (or over W^ϵ)* if it satisfies the consequences of the Whitney fibering conjecture :

$$\lim_{z \to a} T_z M_z = T_a M_a \qquad \text{for every} \quad a \in U \quad (\text{resp. } a \in W^\epsilon)\,.$$

In general, if $\mathcal{D}$ is a distribution of l-planes tangent to Y and transverse to $\ker \pi_{XY*}$ (so each $\pi_{XY*y} : \mathcal{D}(y) \to X$ is an isomorphism) for every $\delta > 0$ we say that $\mathcal{F}$ *is δ-close* to $\mathcal{D}$ if

$$||\, w_i(y) - v_i(y) \,|| \;\le\; \delta \qquad \forall\, y \in T_{XY}(\epsilon) \quad \text{and} \quad \forall\, i = 1, \dots, l$$

where $(v_1, \dots, v_l)$ denote the *canonical* liftings of the $(E_1, \dots, E_l)$ on $\mathcal{D}$.

We will apply this notion to the canonical distribution $\mathcal{D}_X$ which is continuous on X.

Theorem 2. *Let $\mathcal{X}$ a (c)-regular stratification in $\mathbb{R}^n$ having $X = \mathbb{R}^l \times 0^m$ as stratum.*

Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be two controlled l-foliations of $T_X(\epsilon)$ generated respectively by frame fields $(w_1^1, \dots, w_l^1)$ and $(w_1^2, \dots, w_l^2)$ such that $w_l^1 = w_l^2$ and let $a < b \in \mathbb{R}$.

Then $\mathcal{F}_1$ and $\mathcal{F}_2$ can be glued in a controlled l-foliation $\mathcal{F} := \mathcal{F}_1 \vee \mathcal{F}_2$ of $T_X(\epsilon)$ such that :

$$\mathcal{F}_1 \vee \mathcal{F}_2 \;=\; \begin{cases} \mathcal{F}_1 \quad on \quad U_1 \;:=\; T_X(\epsilon) \cap (]-\infty, a\,[\, \times \mathbb{R}^{n-1}) \\[2ex] \mathcal{F}_2 \quad on \quad U_2 \;:=\; T_X(\epsilon) \cap (]\,b, +\infty[\, \times \mathbb{R}^{n-1})\,. \end{cases}$$

Moreover, if $\mathcal{F}_1$ and $\mathcal{F}_2$ are both δ-close to a distribution $\mathcal{D}$ then $\mathcal{F}$ is δ-close to $\mathcal{D}$ too.

Proof. We will define $\mathcal{F}$ through a generating frame field $(w_1, \dots, w_l)$ which we will construct by decreasing induction. We start by defining the vector field $w_l := w_l^1 = w_l^2$.

For $i = l-1$ take a partition of unity $\{\alpha, \beta : \mathbb{R} \to [0,1]\}$ subordinate to the open covering $\big\{]-\infty, b\,[, \,]\,a, +\infty[\big\}$ of $\mathbb{R}$ and extend it to a "partition of unity" of $T_X(\epsilon)$, $\mathcal{P}_{l-1} = \{\alpha_{l-1}(y), \beta_{l-1}(y)\}$ subordinate to the open covering $\{U_1, U_2\}$ of $T_{XY}(\epsilon)$ which is constant along the fiber $\pi_X^{-1}(0)$ (hence along each fiber $\pi_X^{-1}(x)$) and along the trajectories of $w_l = w_l^1 = w_l^2$ (we call it *adapted* to $\{U_1, U_2\}$).

Then we define the stratified vector field :

$$w_{l-1}(y) \;:=\; \alpha_{l-1}(y) w_{l-1}^1(y) \;+\; \beta_{l-1}(y) w_{l-1}^2(y)$$

for every stratum $Y > X$? One finds that the Lie bracket $[w_{l-1}, w_l]$ satisfies :

$$\begin{aligned} [w_{l-1}(y), w_l(y)] \;&=\; [\alpha_{l-1} w_{l-1}^1(y), w_l(y)] \;+\; [\beta_{l-1} w_{l-1}^2(y), w_l(y)] \\ &= \Big(\alpha_{l-1\,*y}(w_l(y)) \cdot w_{l-1}^1(y) \quad + \quad \alpha_{l-1}(y)[w_{l-1}^1(y), w_l(y)] \Big) \;+ \\ &+ \Big(\beta_{l-1\,*y}(w_l(y)) \cdot w_{l-1}^2(y) \quad + \quad \beta_{l-1}(y)[w_{l-1}^2(y), w_l(y)] \Big) \;=\; 0 \end{aligned}$$

where $\alpha_{l-1\,*y}(w_l(y)) = \beta_{l-1\,*y}(w_l(y)) = 0$ because α_{l-1} and β_{l-1} are constant along the trajectories of w_l and $[w^1_{l-1}(y), w_l(y)] = [w^2_{l-1}(y), w_l(y)] = 0$ because $(w^1_1, \ldots, w^1_l)$ and $(w^2_1, \ldots, w^2_l)$ are generating frame fields respectively of $\mathcal{F}_1$ and $\mathcal{F}_2$ with $w^1_l = w^2_l = w_l$.

It is convenient to explain explicitly the next inductive step.

For $i = l-2$ we consider a partition of unity $\mathcal{P}_{l-2} = \{\alpha_{l-2}(y), \beta_{l-2}(y)\}$ subordinate to $\{U_1, U_2\}$ which is constant this time along each fiber $\pi_X^{-1}(x)$ and along the whole of each integral surface generated by the 2-frame (w_{l-1}, w_l). Then define

$$w_{l-2}(y) = \alpha_{l-2}(y) w^1_{l-2}(y) + \beta_{l-2}(y) w^2_{l-2}(y\,).$$

As above we find that

$$[w_{l-2}(y), w_l(y)] = [w_{l-2}(y), w_{l-1}(y)] = 0\,.$$

For an arbitrary $i = 2, \ldots, l-1$, after having constructed the vector fields $w_l, \ldots, w_i$ whose Lie brackets are zero, then the inductive $(i-1)$-step to construct w_{i-1} can be obtained by considering a partition of unity $\mathcal{P}_{i-1} = \{\alpha_{i-1}(y), \beta_{i-1}(y)\}$ of $\mathbb{R}^n$ subordinate to $\{U_1, U_2\}$ which is constant along each fiber $\pi_X^{-1}(x)$ and along the trajectories of all vector fields $w_l, \ldots, w_i$, so constant along each integral manifold generated by $(w_i, \ldots, w_{l-1}, w_l)$, and defining

$$w_{i-1} \;=\; \alpha_{i-1} w^1_{i-1} \;+\; \beta_{i-1} w^2_{i-1}\,.$$

In this way the frame field $(w_1, \ldots, w_l)$ obtained at the end of the induction will satisfy

$$[w_i, w_j] \;=\; 0 \quad , \quad \forall\, i, j = 1, \ldots, l\,.$$

Moreover for every stratum $Y \geq X$ and $y \in Y$ by the (π_X, ρ_X)-control conditions :

$$\begin{cases} \pi_{XY*y}(w_i(y)) = \alpha_i(y)\pi_{XY*y}(w^1_i(y)) + \beta_i(y)\pi_{XY*y}(w^2_i(y)) = \alpha_i(y)\cdot E_i + \beta_i(y)\cdot E_i = E_i \\ \rho_{XY*y}(w_i(y)) = \alpha_i(y)\rho_{XY*y}(w^1_i(y)) + \beta_i(y)\rho_{XY*y}(w^2_i(y)) = \alpha_{l-1}(y)\cdot 0 + \beta_{l-1}(y)\cdot 0 = 0 \end{cases}$$

and similarly for every $X < Y < Z$, $z \in Z$, $y = \pi_{YZ}(z)$, $x = \pi_X(z)$, using the (π_{YZ}, ρ_{YZ})-control conditions and that α_i, β_i are constant along the fiber $\pi_X^{-1}(x)$ one finds :

$$\begin{cases} \pi_{YZ*z}(w_i(z)) = \alpha_i(z)\pi_{YZ*z}(w^1_i(z)) + \beta_i(z)\pi_{YZ*z}(w^2_i(z)) = \alpha_i(y)w^1_i(y) + \beta_i(y)w^2_i(y) = w_i(y) \\ \rho_{YZ*z}(w_i(y)) = \alpha_i(z)\rho_{YZ*z}(w^1_i(z)) + \beta_i(z)\rho_{YZ*z}(w^2_i(z)) = \alpha_{l-1}(y)\cdot 0 + \beta_{l-1}(y)\cdot 0 = 0\,. \end{cases}$$

Thus $(w_1, \ldots, w_l)$ is a controlled frame field with generates a controlled foliation $\mathcal{F}$.

If moreover $\mathcal{F}_1$ and $\mathcal{F}_2$ are both δ-close to an l-distribution $\mathcal{D}$ of Y then, with the canonical liftings $(v_1, \ldots, v_l)$ of $(E_1, \ldots, E_l)$ on $\mathcal{D}$ (see Definition 6), one has :

$$||w^1_i(y) - v_i(y)|| \leq \delta \quad \text{ and } \quad ||w^2_i(y) - v_i(y)|| \leq \delta\,, \qquad \forall\, i = 1, \ldots, l$$

hence for every $i = 1, \ldots, l$ one also finds :

$$||w_i(y) - v_i(y)|| = \alpha_i(y)||w^1_i(y) - v_i(y)|| + \beta_i(y)||w^2_{i-1} - v_i(y)|| \leq \big(\alpha_i(y) + \beta_i(y)\big)\cdot\delta \;=\; 1\cdot\delta \;=\; \delta$$

so that the foliation $\mathcal{F} := \mathcal{F}_1 \vee \mathcal{F}_2$ generated by $(w_1, \ldots, w_l)$ is δ-close to $\mathcal{D}$ too. $\square$

Geometric meaning. The fact that the (π, ρ)-control conditions are satisfied by $(w_1, \ldots, w_l)$ after the "convex-gluing" of the (π, ρ)-controlled frame fields $(w_1^1, \ldots, w_l^1)$ and $(w_1^2, \ldots, w_l^2)$ means that $(w_1, \ldots, w_l)$ modifies with respect to $(w_1^1, \ldots, w_l^2)$ and $(w_1^2, \ldots, w_l^2)$ only the components along each link $Lk(x, \eta) = \pi_X^{-1}(x) \cap \rho_X^{-1}(\eta)$, $\forall\, x \in \mathbb{R}^l$ and $\forall\, \eta \in]0, \epsilon[$.

With essentially the same proof as in Theorem 2 one has :

Corollary 1. *Let $h_i : \mathbb{R}^n \to \mathbb{R}^n$ be the linear diffeomorphism which permutes x_1 with x_i and fixes all other coordinates and consider the open covering of $T_{XY}(\epsilon)$ defined by :*

$$U_1^i \;:=\; T_{XY}(\epsilon) \cap h_i(]-\infty, a[\times \mathbb{R}^{n-1}) \qquad \text{and} \qquad U_2^i := T_{XY}(\epsilon) \cap h_i(]b, +\infty[\times \mathbb{R}^{n-1})\,.$$

If $i \in \{2, \ldots, l-1\}$ and $\mathcal{F}_1^i$ and $\mathcal{F}_2^i$ are controlled foliations of $T_{XY}(\epsilon)$ then Theorem 2 holds again replacing U_1 and U_2 by U_1^i and U_2^i so as to glue together $\mathcal{F}_1^i$ and $\mathcal{F}_2^i$. $\square$

Remark 3 On the contrary, for $i = l$, Theorem 2 cannot be used directly to glue together two controlled foliations $\mathcal{F}_1^l$ and $\mathcal{F}_2^l$ of U_1^l and U_2^l, because if $w_l^1 = w_l^2$, then a partition of unity subordinate to $\{U_1^l, U_2^l\}$ <u>cannot</u> be taken constant along the trajectories of $w_l^1 = w_l^2$.

Remark also that for a stratification of two strata $X < Y$, one needs to prove only the (π_{XY}, ρ_{XY})-control condition so it is not necessary to use a partition of unity constant along each fiber of π_X as we did in Theorem 2.

The techniques of Theorem 2 and Corollary 1 will be used in various steps of our proof of the smooth version of the Whitney fibering conjecture (Theorem 3) while the difficulty explained in Remark 3 above will appear in step 3 of the proof.

5. The smooth Whitney fibering conjecture: the depth 1 case.

In this section we prove a result (Theorem 3), which gives a positive answer to the smooth version of the Whitney fibering conjecture for a stratification $\mathcal{X} = (A, \Sigma)$ and a stratum X for which $depth_\Sigma(X) = 1$ such that Σ is Bekka (c)-regular $[\mathbf{Be}]_1$ over X.

Theorem 3. *Let $\mathcal{X} = (A, \Sigma)$ a smooth stratified subset of $\mathbb{R}^n$, (c)-regular over X.*

Then for every stratum X of $depth_\Sigma(X) = 1$ and for every $x_0 \in X$ and every stratum $Y > X$ there exists a neighbourhood W of x_0 in $X \cup Y$ and a controlled foliation $\mathcal{H} = \{M'_y\}_y$ of W whose leaves M'_y are smooth manifolds of dimension $l = \dim X$ diffeomorphic to $X \cap W$, with $M'_x = X \cap W$, $\forall x \in X$, and such that :

$$\lim_{y \to x} T_y M'_y \;=\; T_x M'_x \;=\; T_x X\,, \qquad \text{for every} \;\; x \in X \cap W\,.$$

Remark 4. In Theorem 3 we study a (c)-regular stratification with smooth (C^∞) strata and obtain a foliation which is C^∞ off X. If the stratification has C^1 strata then there is a C^1 diffeomorphism making all strata C^∞ $[\mathbf{Tr}]_2$ so we can apply the C^∞ result and then by pullback obtain a foliation with C^1 leaves with the required properties.

Before proving Theorem 3 in §5.1 we describe local regularity of the stratified topological triviality map H_{x_0} and some of its important properties when H_{x_0} is obtained by integrating *continuous canonical* lifted frame fields $[\mathbf{MT}]_{2,3,4}$. This brings us in §5.2 to a finer analysis of some new properties of the frame fields tangent to the horizontal leaves $\mathcal{H}_{x_0}$ defined by this topological trivialization.

We will use below statements and notations introduced in section 3.

5.1. Local topological triviality obtained from continuous controlled lifted frame fields.

Let $\mathcal{X}$ be a stratification in a Euclidian space $\mathbb{R}^k$, (c)-regular over a stratum X of $\mathcal{X}$ of dimension l, $x_0 \in X$ and $U_{x_0} \cong \mathbb{R}^l$ a neighbourhood of x_0 in X as in Theorem 1.

In a local analysis we can suppose $x_0 = 0^n \in \mathbb{R}^n$, $U_{x_0} = \mathbb{R}^l \times 0^m$ and $\pi_X : T_X \to \mathbb{R}^l \times 0^m$ the canonical projection such that the topological trivialization "with origin $x_0 = 0$" of the projection π_X can be written as

$$\begin{array}{rcccc} H \; = H_{x_0} & : & \mathbb{R}^l \times \pi_X^{-1}(x_0) & \longrightarrow & \pi_X^{-1}(U_{x_0}) \subseteq \mathbb{R}^n \\ & & (t_1, \ldots, t_l, z_0) & \longmapsto & \phi_l(t_l, \ldots, \phi_1(t_1, z_0) \ldots) \end{array}$$

where $\forall i \leq l$, ϕ_i is the flow of the lifted vector field $v_i(y)$, and thanks to (c)-regularity ([**Be**]$_1$ [**Pl**]), we can choose each $v_i(y)$ to be the <u>*continuous*</u> lifting of the standard vector fields E_i of $X = \mathbb{R}^l \times \{0^m\}$, in a canonical distribution $\mathcal{D}_X = \{\mathcal{D}_X(y)\}_{y \in \pi_X^{-1}(U_{x_0})}$ induced from X on the strata $Y > X$ [**MT**]$_{2,3,4}$. We also will identify $\pi_{XY}^{-1}(U_{x_0})$ with Y.

Remark 5. Although the canonical distribution $\mathcal{D}_X$, its spanning canonical lifted vector fields $v_1, \ldots, v_l$ and their flows $\phi_1, \ldots, \phi_l$ do not depend on the "starting point" x_0, in contrast the trivialization H_{x_0}, defined by a fixed and *a priori non-commuting* order of composition of the flows $\phi_1, \ldots, \phi_l$, depends strongly on the induced controlled foliations with a "starting point" x_0. In fact the non-commutativity of the flows $\phi_1, \ldots, \phi_l$ is the crucial point of our problem : if $\mathcal{D}_X$ is involutive, then the (a)-regularity of a local controlled foliation $\mathcal{H}_{x_0}$ holds ([**MT**]$_4$ and [**Mu**]$_1$ Chap. II §5) so $\mathcal{H}_{x_0}$ satisfies the conclusions of the smooth version of the Whitney fibering conjecture (see section 2).

In particular if $x = (\tau_1, \ldots, \tau_l)$ and $z \in \pi_X^{-1}(x)$ is the image $z = H_{x_0}(\tau_1, \ldots, \tau_l, z_0)$ with $z_0 \in \pi_{XY}^{-1}(x_0)$ then :

$$y \;=\; H_x(t_1, \ldots, t_l, z) \;=\; \phi_l(t_l, \ldots, \phi_1(t_1, z)..) \;=\; \phi_l(t_l, \ldots, \phi_1(t_1,\; (\phi_l(\tau_l, \ldots, \phi_1(\tau_1, z_0)..)..)$$

is a priori different from the image (obtained by commuting the flows ϕ_i) :

$$\phi_l(t_l + \tau_l, \ldots, \phi_1(t_1 + \tau_1, z_0)) \;=\; H_{x_0}(t_1 + \tau_1, \ldots, t_l + \tau_l, z_0)\,.$$

Let $Y > X$.

The stratified homeomorphism H (a C^∞-diffeomorphism on each stratum) induces a controlled foliation of dimension l

$$\mathcal{H}_{x_0} := \left\{ M_{z_0} = H(\mathbb{R}^l \times z_0) \right\}_{z_0 \in \pi_{XY}^{-1}(x_0)}$$

of the submanifold $\pi_{XY}^{-1}(U_{x_0})$ of Y.

For every $y \in Y$ let us denote by M_y the leaf of $\mathcal{H}_{x_0}$ containing y, so that $M_y = M_{y_0}$ when $y = H(t_1, \ldots, t_l, y_0)$ and $y_0 \in \pi_{XY}^{-1}(x_0)$.

We will see in Proposition 3, that writing $\forall\, i = 1, \ldots, l$, $w_i(y) := H_{*(t_1,\ldots,t_l,y_0)}(E_i)$, the frame field $(w_1, \ldots, w_l)$ is the unique (π, ρ)-controlled lifting on the foliation $\mathcal{H}_{x_0}$ (not necessarily continuous) of the frame field $(E_1, \ldots, E_l)$ of X generating $\mathcal{H}_{x_0}$ (see [**MT**]$_4$, [**Mu**]$_1$ Chap. II, §5).

Now H being smooth on Y, the $w_1, \ldots, w_l$ are smooth too on Y, but these vector fields are not necessarily continuous on X, i.e. <u>we do not know</u> whether $\lim_{y \to x} w_i(y) = E_i$ for $x \in X$!

This means that by using the canonical continuous liftings $v_1, \dots, v_l$ on $\mathcal{D}_{XY}$, the foliation H_{x_0} induced by the topological trivialization of Thom-Mather, does not in general give a positive answer to the smooth version of the Whitney fibering conjecture (the second and third authors gave an explicit counterexample in 1994).

5.2. Some useful properties of the w_i and of their flows.

We explain below a property of the vector fields v_i (and w_i) and of their flows ϕ_i (and ψ_i), important in the proof of the smooth Whitney fibering conjecture.

The vector fields $w_1, \dots, w_l$ satisfy obviously :

$$w_i(z_0) = H_{*(0,\dots,0,z_0)}(E_i) = v_i(z_0), \qquad \forall\, z_0 \in \pi_{XY}^{-1}(x_0) \quad \text{and} \quad \forall\, i = 1, \dots, l.$$

That is for every $i \leq l$, w_i coincides on the fiber $\pi_{XY}^{-1}(x_0)$ with the continuous lifting v_i **[MT]**$_4$ which satisfies $\lim_{y \to x \in X} v_i(y) = E_i$, for every $i = 1, \dots, l$ (but again this does not imply that $\lim_{y \to x} w_i(y) = E_i$ for $x \in X$!).

Now if $y = H(t_1, \dots, t_l, y_0)$, then $M_y = H(\mathbb{R}^l \times \{y_0\})$ is a leaf of the foliation $\mathcal{H}_{x_0}$ and

$$T_y M_y = H_{*(t_1,\dots,t_l,y_0)}(\mathbb{R}^l \times \{0\}) = [w_1(y), \dots, w_l(y)]\,.$$

On the other hand $H_{|Y}$ being C^∞, for each $z_0 \in \pi_{XY}^{-1}(x_0)$ we have that :

$$(L_{z_0}): \qquad \lim_{y \to z_0} T_y M_y \;=\; T_{z_0} M_{z_0} \;=\; [w_1(z_0), \dots, w_l(z_0)] \;=\; [v_1(z_0), \dots, v_l(z_0)]\,.$$

Lemma 1. *For every $y_0 \in \pi_{XY}^{-1}(x_0)$, denoting $Q_{0^l}(\delta) =]-\delta, \delta[^l$, the family*

$$\Big\{ H(Q_{0^l}(\delta) \times J_{y_0}) \quad \Big| \quad \delta \in]0,1[\ , \ \ J_{y_0} \text{ a neighbourhood of } y_0 \text{ in } \pi_{XY}^{-1}(x_0) \ \Big\}$$

is a fundamental system of neighbourhoods of y_0 in Y.

Proof. Exercise. □

From (L_{z_0}) for every $\epsilon > 0$, there is a relatively compact open neighbourhood V_{z_0} of z_0 in $\pi_{XY}^{-1}(U_{x_0})$ such that

$$(1) \quad : \qquad ||w_i(y) - v_i(y)|| < \epsilon \qquad \forall\, i = 1, \dots, l \ \text{ and } \ \forall\, y \in V_{z_0}\,.$$

By Lemma 1, we can take $V_{z_0} = H_{x_0}(Q_{0^l}(\delta_{z_0}) \times J_{z_0})$ so that

$$I_{x_0} \;:=\; pr_{\mathbb{R}^l \times 0^l}(V_{z_0}) \;=\; x_0 \,+\, Q_{0^l}(\delta_{z_0}) \;=\,]-\delta_{z_0}\,,\, +\delta_{z_0}[^l$$

is a relatively compact open neighbourhood of $x_0 = 0$ in $\widetilde{U} := [-1,1]^l \times \{0\}^m$ which is a *cube* of $\mathbb{R}^l \times \{0^l\}$ centered in x_0.

Definition 7. We will refer to the property (1) by saying that :

The foliation $\mathcal{H}_{x_0} = \{H_{x_0}(Q_{0^l}(\delta_{z_0}) \times \{z'\})\}_{z' \in J_{z_0}}$ on V_{z_0} is ϵ-close to the canonical distribution $\mathcal{D}_X = \{\mathcal{D}_{XY}(y)\}_{y \in Y}$.

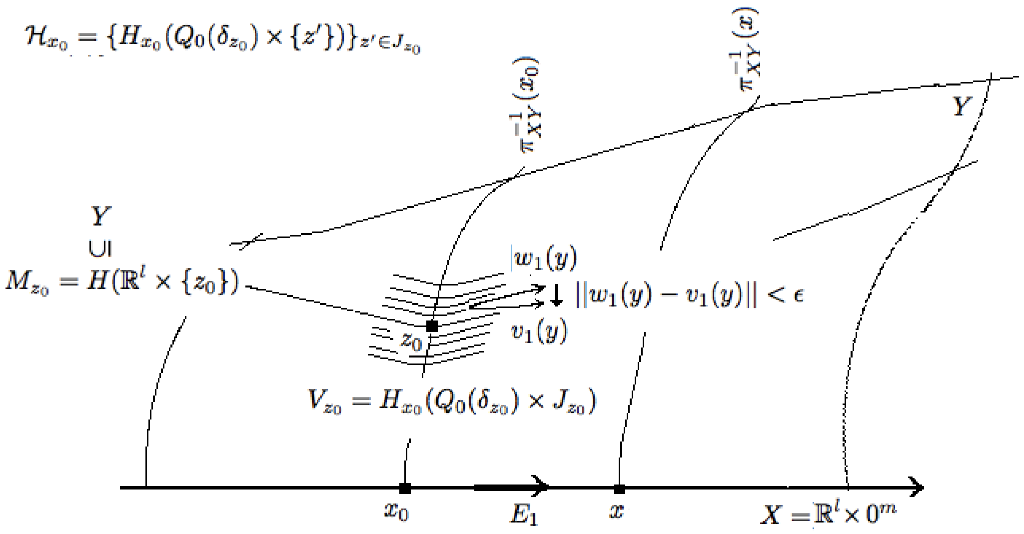

Figure 2

To analyse the difference between $v_i(y)$ and $w_i(y) = H_{*(t_1,\ldots,t_l,y_0)}(E_i)$ we introduce the following notation.

Notation. With every $y = H(t_1, \ldots, t_l, y_0) \in Y$ we associate the chain $y_0 \cdots y_i \cdots y_l = y$ defined starting from y_0 on the leaf $M_y = H(y_0 \times \mathbb{R}^l)$ of the foliation $\mathcal{H}_{x_0}$ as follows :

$$\begin{cases} y_0 & = H(0^l, y_0) \\ y_1 & = H(t_1, 0^{l-1}, y_0) = \phi_1(t_1, y_0) \\ y_2 & = H(t_1, t_2, 0^{l-2}, y_0) = \phi_2(t_2, \phi_1(t_1, y_0)) \\ \cdots & \quad \cdots\cdots\cdots\cdots\cdots \\ y_i & = H(t_1, \ldots, t_i, 0^{l-i}, y_0) = \phi_i(t_i, \ldots, \phi_1(t_1, y_0) \ldots) \\ \cdots & \quad \cdots\cdots\cdots\cdots\cdots \\ y_l & = H(t_1, \ldots, t_l, y_0) = \phi_l(t_l, \ldots, \phi_i(t_i, \ldots, \phi_1(t_1, y_0) \ldots) \ ; \end{cases}$$

so that :

$$y_1 = \phi_1(t_1, y_0), \quad y_2 = \phi_2(t_2, y_1), \quad \ldots \quad y_i = \phi_i(t_i, y_{i-1}), \quad \ldots \quad y_l = \phi_l(t_l, y_{l-1}) = y \ .$$

In the proposition below, $\forall \tau \in \mathbb{R}$, we let $\phi_i^\tau : Y \to Y$ be the diffeomorphism of Y defined by $\phi_i^\tau(y) = \phi_i(\tau, y)$.

Proposition 2. *For every* $y = H(t_1, \ldots, t_l, y_0) \in Y$,

$$w_i(y) = \phi^{t_l}_{l\ *y_{l-1}} \circ \cdots \circ \phi^{t_{i+1}}_{i+1\ *y_i}\big(v_i(y_i)\big) \qquad , \qquad \forall\, i = 1, \ldots, l-1 \ .$$

Proof. As $y = H(t_1, \ldots, t_l, y_0)$ it follows that :

$$w_i(y) \ := \ H_{*(t_1,\ldots,t_l,y_0)}(E_i) \ = \ \frac{\partial}{\partial \tau_i} H(\tau_1, \ldots, \tau_l, y_0)\Big|_{(\tau_1,\ldots,\tau_l)=(t_1,\ldots,t_l)}$$

$$= \ \frac{\partial}{\partial \tau_i}\Big|_{(\tau_1,\ldots,\tau_l)=(t_1,\ldots,t_l)} \phi_l(\tau_l, \ldots, \phi_i(\tau_i, \ldots, \phi_1(\tau_1, y_0) \ldots)$$

$$= \left.\frac{\partial}{\partial \tau_i}\right|_{\tau_i=t_i} \phi_l^{t_l} \circ \cdots \circ \phi_{i+1}^{t_{i+1}} \circ \phi_i^{\tau_i}(y_{i-1})$$

$$= \phi_l^{t_l}{}_{*y_{l-1}} \circ \cdots \circ \phi_{i+1}^{t_{i+1}}{}_{*y_i} \left(\left.\frac{\partial}{\partial \tau_i}\right|_{\tau_i=t_i} \phi_i(\tau_i, y_{i-1}) \right)$$

$$= \phi_l^{t_l}{}_{*y_{l-1}} \circ \cdots \circ \phi_{i+1}^{t_{i+1}}{}_{*y_i} \Big(v_i\big(\phi_i(t_i, y_{i-1})\big) \Big)$$

$$= \phi_l^{t_l}{}_{*y_{l-1}} \circ \cdots \circ \phi_{i+1}^{t_{i+1}}{}_{*y_i} \big(v_i(y_i)\big) . \quad \Box$$

For every $y \in Y$ and leaf M_y, denote ${\pi_{XY}}^{-1}(x_0) = S^0 \subseteq S^1 \subseteq \cdots \subseteq S^l = M_y$ the chain of "coordinate subspaces" S^i of Y containing all points of the type $y = y_i$:

$$S^i := H\big(\mathbb{R}^i \times 0^{l-i} \times {\pi_{XY}}^{-1}(x_0)\big) =$$

$$= \Big\{ y = H(t_1, \ldots, t_i, 0^{l-i}, y_0) \ \big| \ y_0 \in {\pi_{XY}}^{-1}(x_0) \ , \ t_1, \ldots, t_i \in \mathbb{R} \Big\} .$$

Then every S^i is a submanifold of dimension $i + (k - l)$ of Y, where $k = \dim Y$ ($k - l = \dim {\pi_{XY}}^{-1}(x_0)$), and one has :

Corollary 2. *For every $i = 1, \ldots, l$, the vector field $w_i(y)$ coincides with the lifting $v_i(y)$ in the canonical distribution $\mathcal{D}_{XY}(y)$ on all points of the submanifold S^i :*

$$w_i(y) = v_i(y) \quad , \quad \forall\, y \in S^i .$$

In particular $\forall\, i = 1, \ldots, l$, the flow ψ_i of w_i coincides with the flow ϕ_i of v_i on $S^i \times \mathbb{R}$.

Proof. If a point $y = H(t_1, \ldots, t_l, y_0)$ coincides with the corresponding y_i then necessarily $t_{i+1} = \cdots = t_l = 0$ and also $y = y_l = y_{l-1} = \cdots = y_i$.

For every $j = i + 1, \ldots, l$, since $t_j = 0$ the flows satisfy $\phi_j^{t_j} = \phi_j^0 = 1_Y$ and $\phi_{j*y_j}^{t_j} = 1_{Y*y_j} = 1_{T_{y_j}Y}$ and so by the previous proposition one finds :

$$w_i(y) = \phi_l^{t_l}{}_{*y_{l-1}} \circ \cdots \circ \phi_{i+1}^{t_{i+1}}{}_{*y_i} \big(v_i(y_i)\big) = v_i(y_i) = v_i(y) . \quad \Box$$

Corollary 2 allows us to better estimate the difference $u_i(y) := v_i(y) - w_i(y)$: it increases for i decreasing, being zero for $i = l$ and maximal when $i = 1$. This is a consequence of the nature of the definition of the trivialization H,

$$H(t_1, \ldots, t_l, y_0) = \phi_l(t_l, \ldots, \phi_i(t_i, \ldots, \phi_1(t_1, y_0) \ldots)$$

because of which any vector $w_i(y)$ whose index i is more to the left of the formula relating $w_i(y_i)$ to $v_i(y_i)$ occurs in the " perturbation " of the extra differential $\phi_{i+1}^{t_i+1}{}_{*y_i}$ compared with the previous pair $w_{i+1}(y_{i+1})$, $v_{i+1}(y_{i+1})$.

Thus since $S^l = Y$ and $S^0 = \pi_{XY}^{-1}(x_0)$, the vector fields $w_1(y), \ldots, w_l(y)$ satisfy :

$$\begin{cases} w_l(y) = v_l(y) & \text{on} \quad S^l = Y \\ \ldots \;\; = \;\; \ldots & \\ w_i(y) = v_i(y) & \text{on} \quad S^i = H\big(\mathbb{R}^i \times 0^{l-i} \times {\pi_{XY}}^{-1}(x_0)\big) \\ \ldots \;\; = \;\; \ldots & \\ w_1(y) = v_1(y) & \text{on} \quad S^1 = H\big(\mathbb{R}^1 \times 0^{l-1} \times \pi_{XY}^{-1}(x_0)\big) . \end{cases}$$

This explains why the order of the index $i = 1, \dots, l$ that we have chosen to define the topological trivialization H is significant!

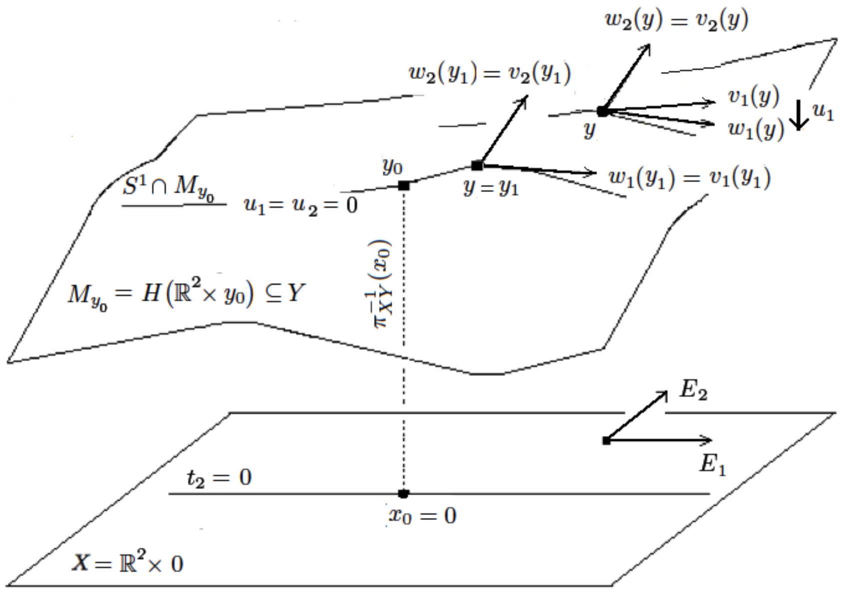

Figure 3

The vector fields $\{w_i(y)\}_{i=1,\dots,l}$ are characterised by the following property :

Proposition 3. *Every vector field $w_i(y)$ is the unique (π, ρ)-controlled lifting of the standard vector field E_i of X tangent to the leaves of the foliation $\mathcal{H}_{x_0}$.*

Proof. See **[MT]**$_4$, §5.1 Lemma 3. □

5.3. Proof of the smooth version of the Whitney fibering conjecture for (c)-regular stratifications of depth 1.

Let $\mathcal{X} = (A, \Sigma)$ be a (c)-regular stratification, X an l-stratum of $\mathcal{X}$ and $x_0 \in X$.

Since by hypothesis $depth_\Sigma(X) = 1$, a sufficiently small neighbourhood of x_0 in A meets only finitely many strata $\{Y_i^{k_i} > X\}_{i=1}^r$ **[Ma]**$_{1,2}$ which are of dimension $\dim Y_i^{k_i} = k_i \geq l+1 > \dim X$, and the closures of these strata intersect (near X) only in X. Therefore it will be sufficient to prove Theorem 3 for only one stratum Y. So we prove:

Theorem 4. *Suppose that $X \cup Y$ is a smooth stratified Bekka (c)-regular closed subset of $\mathbb{R}^n$ having only two smooth strata $X < Y$. Then $X \cup Y$ satisfies the smooth version of the Whitney fibering conjecture.*

I.e. for every $x_0 \in X$ there exists a neighbourhood $W = \pi_X^{-1}(U)$ of x_0 in $X \cup Y$ where U is a domain of a local chart near x_0 in X and a controlled foliation $\mathcal{H} = \{M'_y\}_{y \in W}$ of W whose leaves M'_y are smooth manifolds diffeomorphic to $X \cap W$, with $M'_x = X \cap W$, $\forall x \in X$ and such that :

$$\lim_{y \to x} T_y M'_y = T_x M'_x = T_x X \,, \qquad \text{for every} \quad x \in X \cap W \,.$$

Proof. Let $l = \dim X$, $k = \dim Y$ and let x_0 be a point of X and U an open neighbourhood of x_0 in X diffeomorphic to $\mathbb{R}^l$.

When $l = 1$, (c)-regularity, which implies the existence of controlled continuous lifting of vector fields on X, is enough to ensure the existence of the foliation $[\mathbf{Be}]_1$.

So we assume $l \geq 2$.

When $k = l + 1$, the level hypersurfaces of $\rho_{XY} = \rho_{X|Y}$ intersect Y in leaves of a foliation satisfying the conclusion of the theorem again by (c)-regularity $[\mathbf{MT}]_4$. Thus we will assume $k \geq l + 2 \geq 4$.

The problem being local, we can write $x_0 = 0^n$, $X \equiv U \equiv \mathbb{R}^l \times 0^m$ $(m = n - l)$, and $Y \equiv T_{XY} \equiv \pi_{XY}^{-1}(U)$ where the projection $\pi_{XY} : T_{XY} = T_X \cap Y \to X$ is the restriction $pr_{1|Y} : T_{XY} \to X$ of the first projection onto X ; in particular $\pi_{XY}^{-1}(x_0) \subseteq 0^l \times \mathbb{R}^m \subseteq \mathbb{R}^n$.

Consider the standard basis $\{E_i\}_{i=1}^l$ of $\mathbb{R}^l \times 0^m$ and the topological trivialization "of origin x_0" of the projection π_{XY}:

$$\begin{array}{rcccc} H = H_{x_0} & : & \mathbb{R}^l \times \pi_{XY}^{-1}(x_0) & \longrightarrow & Y = T_{XY} \subseteq \mathbb{R}^n \\ & & (t_1, \ldots, t_l, z_0) & \longmapsto & \phi_l(t_l, \ldots, \phi_1(t_1, z_0) \ldots) \end{array}$$

where $\forall i \leq l$, ϕ_i is the flow of the vector field v_i which is the *continuous* lifting of E_i in a canonical distribution $\mathcal{D}_X = \{\mathcal{D}_{XY}(y)\}_{y \in Y}$ induced from X on Y $[\mathbf{MT}]_{2,3}$, using (c)-regularity.

As $X \cup Y$ is (c)-regular, there exists $\epsilon > 0$ such that the map $(\pi_{XY}, \rho_{XY}) : T_{XY}(\epsilon) \to \mathbb{R}^l \times [0, \infty[$ is a proper submersion and, possibly making a change of scale, we suppose $\epsilon = 1$. Then if we consider the compact neighbourhood $\widetilde{U} = [-1, 1]^l \times 0^m$ of $x_0 = 0^k$ in X, its preimage $W := \pi_X^{-1}(\widetilde{U})$ via the projection $\pi_X : T_X(1) \to X$ is a compact subset of $T_X(1) \equiv X \cup Y$.

We apply the arguments of sections 5.1 and 5.2 for x_0 and for $y_0 \in \pi_{XY}^{-1}(x_0)$ to each point $x \in \widetilde{U}$ and each $z \in \pi_{XY}^{-1}(x)$. Every $x \in \widetilde{U}$ will be thus "the origin" of a new topological trivialization H_x obtained using the same continuous lifted vector fields $v_1, \ldots, v_l$ by composing their flows in the same order, but taking x as origin. This will define for every $x \in \widetilde{U}$ a foliation

$$\mathcal{H}_x = \left\{ M_z^x = H_x(\mathbb{R}^l \times z) \right\}_{z \in \pi_{XY}^{-1}(x)}$$

and a (π, ρ)-controlled frame field $(w_1^x, \ldots, w_l^x)$ generating the foliation $\mathcal{H}_x$, such that $\forall\, x \in \widetilde{U}$ and $\forall\, z \in \pi_{XY}^{-1}(x)$:

$$(L_z) : \qquad \lim_{y \to z} T_y M_y^x \;=\; \lim_{y \to z} T_y M_z^x \;=\; [w_1^x(z), \ldots, w_l^x(z)] = [v_1(z), \ldots, v_l(z)] = \mathcal{D}_{XY}(z)\,,$$

where $\mathcal{D}_{XY}(y)$ tends to $[E_1, \ldots, E_l] = \mathbb{R}^l \times 0^m = T_x X$, as $y \to x$, by (c)-regularity.

Suppose now that $\dim X = l = 2$. Later in the proof we will treat the general case.

With this hypothesis $X = \mathbb{R}^2 \times 0^m$ and by the results of section 5.1 and 5.2 we have the (π, ρ)-controlled continuous lifted frame field (v_1, v_2) on $\mathcal{D}_{XY}$, a frame field $(w_1^{x_0}, w_2^{x_0})$ tangent to the foliation $\mathcal{H}_{x_0}$ and for each $x \in \widetilde{U}$ a frame field (w_1^x, w_2^x) tangent to the foliation $\mathcal{H}_x$ such that :

1) for every $x \in X$: $\quad w_2^{x_0} = v_2 = w_2^x$;

2) for every $y = H_{x_0}(t_1, t_2, y_0)$, with $y_0 \in \pi_{XY}^{-1}(x_0)$, by $y_1 = \phi_1(t_1, y_0)$ we have:

$$w_1^{x_0}(y) \;=\; \phi_{2*y_1}^{t_2}(v_1(y_1))\,,$$

and similarly for every $z = H_x(t_1, t_2, z_0)$, with $z_0 \in \pi^{-1}_{XY}(x)$, by setting $z_1 = \phi_1(t_1, z_0)$ we have :

$$w^x_1(z) \ = \ \phi^{t_2}_{2*z_1}(v_1(z_1))\,;$$

3) by (L_z) applied to each $z \in \pi^{-1}_{XY}(x)$, for every $z \in W$ and for every $\epsilon > 0$, there is a relatively compact open neighbourhood V_z of z in W such that

$$(3)_z: \qquad ||w^x_1(y) - v_1(y)|| < \epsilon \qquad \text{for all} \quad y \in V_z\,.$$

By Lemma 1, we can choose every V_z to be of the type

$$V_z = H_x(Q_{0^2}(\delta_z) \times J_z) \quad \text{where} \quad x = \pi_{XY}(z) = (\tau_1, \tau_2)\,,$$

(see Lemma 1 in §5.2 for the definitions of $Q_{0^2}(\delta_z)$ and J_z) and :

$$I_x \ := \ pr_{\mathbb{R}^2\times 0^m}(V_z) \ = \ x \ + \ Q_{0^2}(\delta_z) \ = \ (\tau_1, \tau_2) \ + \ ]-\delta_z, \ +\delta_z[^2$$

is a relatively compact open neighbourhood of x in $\widetilde{U} = [-1, 1]^2 \times 0^m$ and a *square* of $\mathbb{R}^2 \times 0^m$ centered in x with edges of size $2\delta_z$ depending on z, and J_z is a relatively compact open neighbourhood of z in $\pi^{-1}_{XY}(x)$.

In this way, the foliation (called again) $\mathcal{H}_x = \{H_x(Q_{0^2}(\delta_z) \times z')\}_{z'\in J_z}$ on V_z will be ϵ-close to the canonical distribution $\mathcal{D}_X = \{\mathcal{D}_{XY}(y)\}_{y\in Y}$.

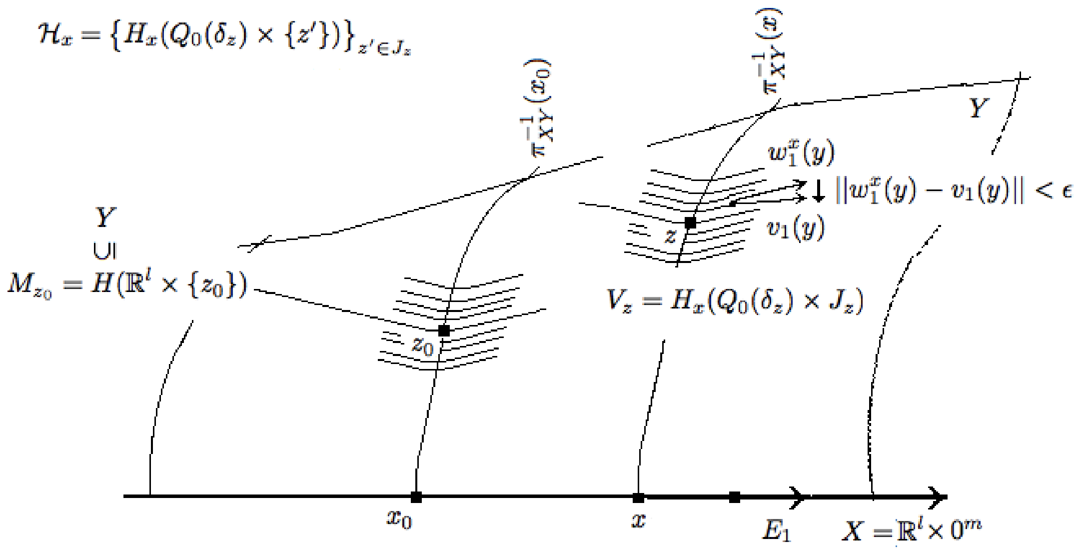

Figure 4

For every $n \in \mathbb{N}^*$ let

$$F_n(x_0) \ := \ (\pi_{XY}, \rho_{XY})^{-1}(\{x_0\} \times [\tfrac{1}{n+2}, \tfrac{1}{n}]) \ = \ \pi^{-1}_{XY}(x_0) \cap \rho^{-1}_{XY}([\tfrac{1}{n+2}, \tfrac{1}{n}])$$

and A_n be the compact cylindrical set of $\pi^{-1}_{XY}([-1, 1]^2)$:

$$A_n = (\pi_{XY}, \rho_{XY})^{-1}([-1, 1]^2 \times [\tfrac{1}{n+2}, \tfrac{1}{n}]) = \pi^{-1}_{XY}([-1, 1]^2) \cap \rho^{-1}_{XY}([\tfrac{1}{n+2}, \tfrac{1}{n}])\,.$$

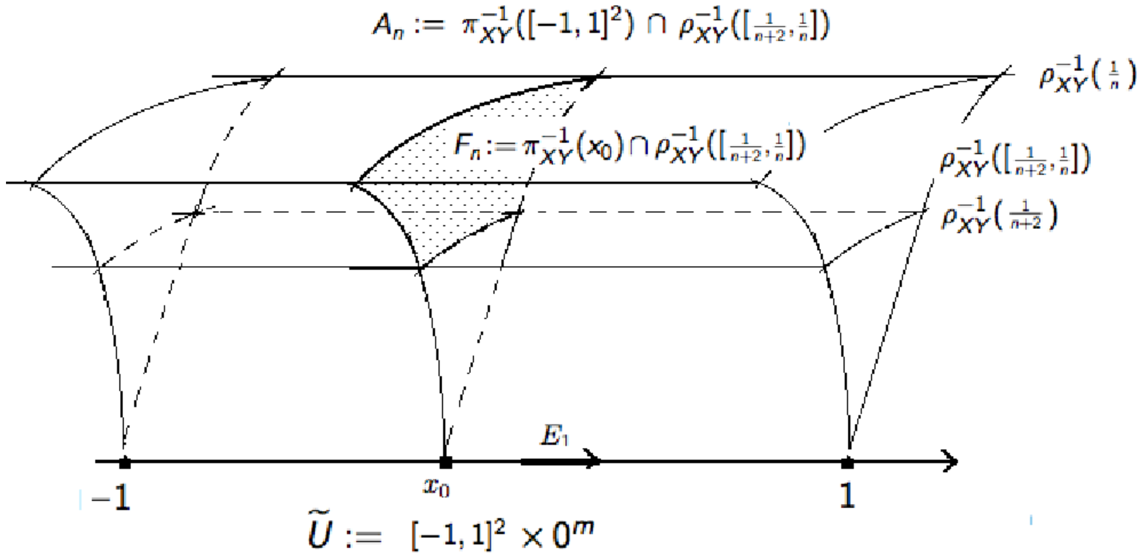

Figure 5

Then $\qquad A_n = H_{x_0}([-1,1]^2 \times F_n(x_0)) \qquad$ and $\qquad F_n(x_0) = \pi_{XY}^{-1}(x_0) \cap A_n$.

We now make more precise the geometric properties that we require of the neighbourhoods I_x of x in U and the neighbourhoods J_z of z in $\pi_{XY}^{-1}(x)$.

Consider for every $n \in \mathbb{N}^*$ and every $z \in A_n \subseteq Y$, the topological trivialization $H_x : \mathbb{R}^2 \times \pi_{XY}^{-1}(x) \to Y$, where $x = \pi_{XY}(z)$.

Fix $\epsilon = \frac{1}{n}$ and for every $z \in A_n$ neighbourhoods $V_z = H_x(Q_{0^2}(\delta_z) \times J_z)$ such that the previous property $(3)_z$, $||w_1^x(y) - v_1(y)|| < \frac{1}{n}$ holds $\forall\, y \in V_z$.

Since the set $\mathcal{S}_n := \{ V_z \mid z \in A_n \}$ is an open covering of A_n, and A_n is compact, there exists a finite subset $P_n := \{z_1, \ldots, z_{q_n}\}$ of points of A_n such that the open finite subcovering

$$\mathcal{C}_n := \left\{ V_{z_j} = H_{x_j}(Q_{0^2}(\delta_{z_j}) \times J_{z_j}) \;\middle|\; z_j \in P_n \right\}, \quad \text{where} \quad x_j = \pi_{XY}(z_j),$$

covers A_n.

Recall that since every local trivialization H_{x_j} is π-controlled, $\pi_{XY*y}(w_i^{x_j}(y)) = E_i$, for V_{z_j}, $z_j \in A_n$. Moreover, since H_{x_j} is ρ-controlled, $\rho_{XY*y}(w_i^{x_j}(y)) = 0$, so that the vector fields $w_i^{x_j}(y)$ have only components (apart from the E_i) along the tangent space to a link of the fiber $\pi_{XY}^{-1}(\pi_{XY}(y))$:

$$L(y) := (\pi_{XY}, \rho_{XY})^{-1}((\pi_{XY}(y), \rho_{XY}(y))) = \pi_{XY}^{-1}(\pi_{XY}(y)) \cap \rho_{XY}^{-1}(\rho_{XY}(y))$$

which is a compact $(k-3)$-submanifold of $\pi_{XY}^{-1}(\pi_{XY}(y))$.

We will prove that this finite open covering $\mathcal{C}_n$ of A_n by open sets $\{V_{z_j}\}_{j \le q_n}$, which are nicely foliated and $\frac{1}{n}$-close to $\mathcal{D}_X$, open sets $\{V_{z_j}\}_{j \le q_n}$, can be used to define a foliation defined on the whole annulus A_n, and this foliation is again $\frac{1}{n}$-close to $\mathcal{D}_{XY}$.

Let η_n^0 be the Lebesgue number of the open covering $\mathcal{C}_n$ so that every subset of A_n of diameter $< \eta_n^0$ is contained in at least one of the sets V_{z_j}, for some j.

For every $x_j = (t_1^j, t_2^j) \in [-1,1]^2$, the map $H_j := H(x_j, \cdot) : \pi_{XY}^{-1}(x_0) \to \pi_{XY}^{-1}(x_j)$ is a homeomorphism, so every open set J_{z_j} of the fiber $\pi_{XY}^{-1}(x_j)$ determines by preimage an open set $J_{z_j}^0 = \phi_1(-t_1^j,\, \phi_2(-t_2^j,\, J_{z_j}))$ of the fiber $\pi_{XY}^{-1}(x_0)$, such that $H(x_j \times J_{z_j}^0) = J_{z_j}$.

In this way we obtain an open covering $\{\, J_{z_j}^0 \,\}_{j \le q_n}$ of $F_n(x_0) = \pi_{XY}^{-1}(x_0) \cap A_n$ whose Lebesgue number will be denoted by η_n^1.

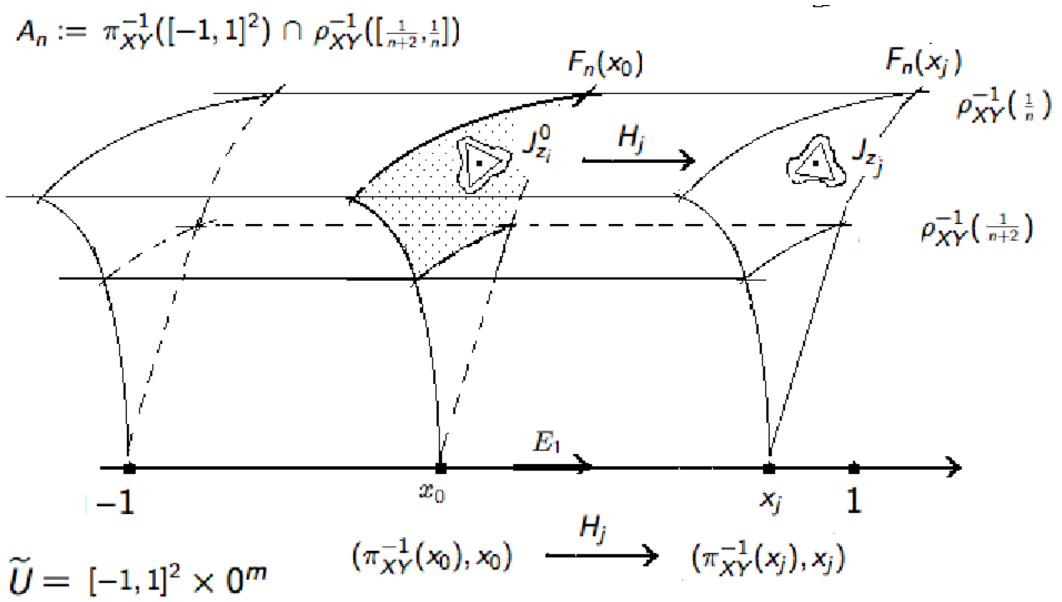

Figure 6

We let

$$\eta_n = \min \{ \eta_n^0 \,,\ \eta_n^1 \,,\ \ \frac{1}{2} \} \,.$$

Now because each trivialization H_x is defined ($[\mathbf{Ma}]_{1,2}$, $[\mathbf{MT}]_4$) by the formula

$$H_x(t_1, t_2, y_0) = \phi_2(t_2, \phi_1(t_1, y))) \,, \qquad y \in \pi_X^{-1}(x)$$

where the maps $\{\phi_i\}_{i \le 2}$ are the smooth flows of the smooth vector fields $\{v_i\}_{i \le 2}$ on Y, it follows that each H_x is smooth on Y and in particular locally Lipschitz on Y. Hence for every $j \le q_n$, the trivializations $H_{x_0 | A_n}$ and $H_{x_j | A_n}$ restricted to the compact set A_n are globally Lipschitz (at least with respect to the geodesic arc-length metric).

For $s_n \in \mathbb{N}^*$ such that $\delta := \frac{1}{s_n} < \eta_n$ define $\Sigma = \{-s_n, \ldots, -1, 0, 1, \ldots, s_n - 1\}$ and consider the closed coverings $\big\{ Q_i(\delta) := Q_i = [i\delta, (i+1)\delta] \big\}_{i \in \Sigma}$ of $[-1,1]$. Then

$$[-1,1] \quad = \quad \bigcup_{i \in \Sigma} Q_i(\delta) \quad =$$

$$= \ [-s_n\delta, -(s_n-1)\delta] \ \cup \ \ldots \ \cup \ [-\delta, 0] \ \cup \ [0, \delta] \ \cup \ \ldots \ \cup \ [(s_n-1)\delta, s_n\delta] \,.$$

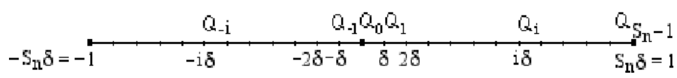

Figure 7

This covering induces the (paving) covering by closed cubes of $[-1,1]^2$:

$$\left\{ Q_{(i_1,i_2)}(\delta) = Q_{i_1}(\delta) \times Q_{i_2}(\delta) \right\}_{(i_1,i_2)\in\Sigma^2} \qquad \text{so that :} \qquad [-1,1]^2 \;=\; \bigcup_{(i_1,i_2)\in\Sigma^2} Q_{(i_1,i_2)}(\delta)$$

that we order following the lexicographic order of Σ^2.

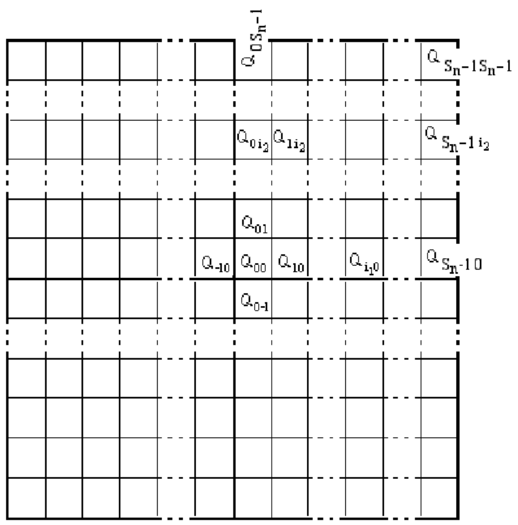

Figure 8

By (c)-regularity $(\pi_{XY}, \rho_{XY}) : T_{XY}(1) \to X \times]0,1[$ is a proper submersion, hence :

$$F_n(x_0) \;=\; \pi_{XY}^{-1}(x_0) \cap A_n \qquad \text{is a compact } (k-2)\text{-submanifold with boundary}$$

and we can choose a triangulation $T_n(x_0)$ of $F_n(x_0)$ which induces a covering by open cells of $F_n(x_0)$: $\left\{ N(\sigma) \;\middle|\; \sigma \in T_n(x_0) \right\}$ where $N(\sigma)$ denotes the simplicial neighbourhood of each simplex $\sigma \in T_n(x_0)$ **[ST]**.

Let us denote by $T_n^r(x_0)$ the r-th barycentric subdivision of the triangulation $T_n(x_0)$ of $F_n(x_0) = \pi_{XY}^{-1}(x_0) \cap A_n$ and consider for each closed simplex $\sigma \in T_n^r(x_0)$ its simplicial neighbourhood $N(\sigma)$ in $T_n^r(x_0)$.

Because for $\underline{i} := (i_1, i_2) \in \Sigma^2$ and $\sigma \in T_n^r(x_0)$,

$$\lim_{\delta\to 0} \max_{\underline{i}\in\Sigma^2} \, diam\, Q_{\underline{i}}(\delta) = 0\,, \qquad\qquad \lim_{r\to\infty} \max_{\sigma\in T_n^r(x_0)} \, diam\; N(\sigma) = 0$$

and $H = H_{x_0}$ is a Lipschitz map on the compact set A_n, we have :

$$\lim_{(\delta,r)\to(0,+\infty)} \max_{\sigma\in T_n^r(x_0)} \, diam\; H_{x_0}\big(Q_{\underline{i}}(\delta) \,\times\, N(\sigma)\big) \;=\; 0\,.$$

There exists then $\delta > 0$ small enough and $r \in \mathbb{N}$ big enough such that for each $\sigma \in T_n^r(x_0)$ the diameter of $H\left(Q_{\underline{i}}(\delta) \,\times\, N(\sigma)\right)$ is smaller than η_n^0, so that it is contained in an open set $V_{z_j} = H_{x_j}(Q_{0^2}(\delta_{z_j}) \times J_{z_j})$, $j = j(\underline{i}, \sigma)$ of the covering $\mathcal{C}_n$.

In particular via the projection π_X we also have :

$$Q_{\underline{i}}(\delta) \;=\; \pi_X\big(H\left(Q_{\underline{i}}(\delta) \,\times\, N(\sigma)\right)\big) \subseteq \pi_X(V_{z_j}) = x_j \,+\, Q_{0^2}(\delta_{z_j}) = I_{x_j}\,.$$

Fix such a pair (δ, r) and $\forall\,(\underline{i},\sigma) \in \Sigma^2 \times T_n^r(x_0)$ fix (only) one such index $j = j(\underline{i},\sigma)$.

Then the open sets $V_{z_j} = H_{x_j}(Q_{0^2}(\delta_{z_j}) \times J_{z_j})$ foliated by the local foliations $\mathcal{H}_{x_j} = \big\{M_z^{x_j} := H_{x_j}\big(Q_{0^2}(\delta_{z_j}) \times \{z\}\big)\big\}_{z \in J_{z_j}}$, which are $\frac{1}{n}$-close to the canonical distribution $\mathcal{D}_{XY}$, cover the "box-images" $H\,\big(Q_{\underline{i}}(\delta) \times N(\sigma)\big)$ which cover A_n :

$$A_n \;=\; H\big([-1,1]^2 \times F_n(x_0)\big) \quad = \quad H\bigg(\bigcup_{\underline{i}\in\Sigma^2} Q_{\underline{i}}(\delta) \;\times \bigcup_{\sigma\in T_n^r(x_0)} N(\sigma)\bigg)$$

$$= \bigcup_{\underline{i}\in\Sigma^2} \;\bigcup_{\sigma\in T_n^r(x_0)} H\Big(Q_{\underline{i}}(\delta) \;\times\; N(\sigma)\Big)\,.$$

The remainder of the proof will take five steps.

In Steps 1, 2, 3, we will show how the local foliations $\{\mathcal{H}_{x_j}\}_{j\le q_n}$ can be glued together on open sets containing these box-images, hence covering A_n and defining a foliation of A_n which is again $\frac{1}{n}$-close to the distribution $\mathcal{D}_{XY}$.

Then in Step 4) we will glue together foliations of the $\{A_n\}_n$ to obtain a foliation of an open set covering $\pi_X^{-1}([-1,1]^2)$ and finally in Step 5 we complete the proof of Theorem 4 by proving the general case in which $\dim X$ is arbitrary.

Step 1 : For every $\underline{i} = (i_1, i_2) \in \Sigma^2$, there exists a (π,ρ)-controlled frame field generating a controlled foliation $\mathcal{H}_{\underline{i}}$, $\frac{1}{n}$-close to $\mathcal{D}_{XY}$, of an open neighbourhood of :

$$\pi_{XY}^{-1}\big(Q_{\underline{i}}\big) \cap A_n \;=\; H\,\bigg(\,Q_{\underline{i}} \,\times \bigcup_{\sigma\in T_n^r(x_0)} N(\sigma)\bigg)\,, \qquad \textit{where :} \qquad Q_{\underline{i}} := Q_{\underline{i}}(\delta)\,.$$

Fix $\underline{i} \in \Sigma^2$ and let us write $P_n(\underline{i}) := \{z_j \in P_n \,|\, \exists\,\sigma \text{ such that } j = j(\underline{i},\sigma)\}$ and remark that :

$$\pi_{XY}^{-1}\big(Q_{\underline{i}}\big) \cap A_n \quad = \quad H\,\Big(\,Q_{\underline{i}} \,\times\, F_n(x_0)\Big)$$

$$\subseteq \bigcup_{\sigma\in T_n^r(x_0)} H\,\Big(\,Q_{\underline{i}} \,\times\, N(\sigma)\Big) \;\subseteq\; \bigcup_{z_j\in P_n(\underline{i})} V_{z_j}\,.$$

Now $p = \delta\cdot\underline{i} \in Q_{\underline{i}}$ is the first vertex.

For every $j \in P_n(\underline{i})$ the partial map $H_{x_j|} := H_{x_j}(p - x_j, \cdot)\;:\; \pi_{XY}^{-1}(x_j) \;\longrightarrow\; \pi_{XY}^{-1}(p)$ is a homeomorphism on each image so that

$$J_{z_j}^p \;:=\; V_{z_j} \,\cap\, \pi_{XY}^{-1}(p) \;=\; H_{x_j}(p - x_j\,, J_{z_j})\,, \qquad\qquad (\text{note that }\; J_{z_j} = V_{z_j} \,\cap\, \pi_{XY}^{-1}(x_j)\,)$$

defines an open set of $\pi_{XY}^{-1}(p)$ and it is easy to see that the family $\{J_{z_j}^p\}_{z_j\in P_n(\underline{i})}$ is an open covering of $F_n(p) := \pi_{XY}^{-1}(p) \cap A_n$.

Let $\mathcal{P} = \big\{\alpha_j : J_{z_j}^p \to [0,1] \;\big|\; z_j \in P_n(\underline{i})\,\big\}$ be a smooth partition of unity subordinate to the covering $\{J_{z_j}^p\}_{z_j\in P_n(\underline{i})}$ of $F_n(p)$.

Because each leaf $M_z^{x_j}$ (with $z \in J_{z_j}$) of the controlled foliation $\mathcal{H}_{x_j} = \{M_z^{x_j}\}_{z\in J_{z_j}}$ meets the fiber $F_n(p)$ in a unique point :

$$z_j^p \;:=\; M_z^{x_j} \cap \pi_{XY}^{-1}(p) \;=\; H_{x_j}(p - x_j, z)\,,$$

it follows that to every point $y = H_{x_j}(t_1, t_2, z) \in V_{z_j} = H_{x_j}(Q_{0^2}(\delta_{z_j}) \times J_{z_j})$ corresponds a unique point z_j^p which is the "horizontal projection" of $z = H_{x_j}(-t_1, -t_2, y)$ and y on $\pi_{XY}^{-1}(p)$.

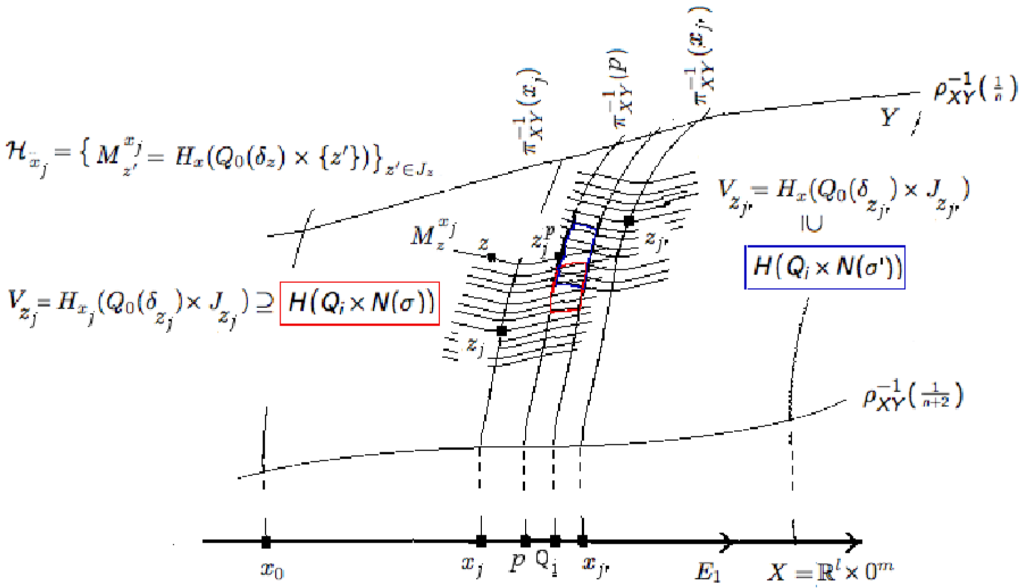

Figure 9

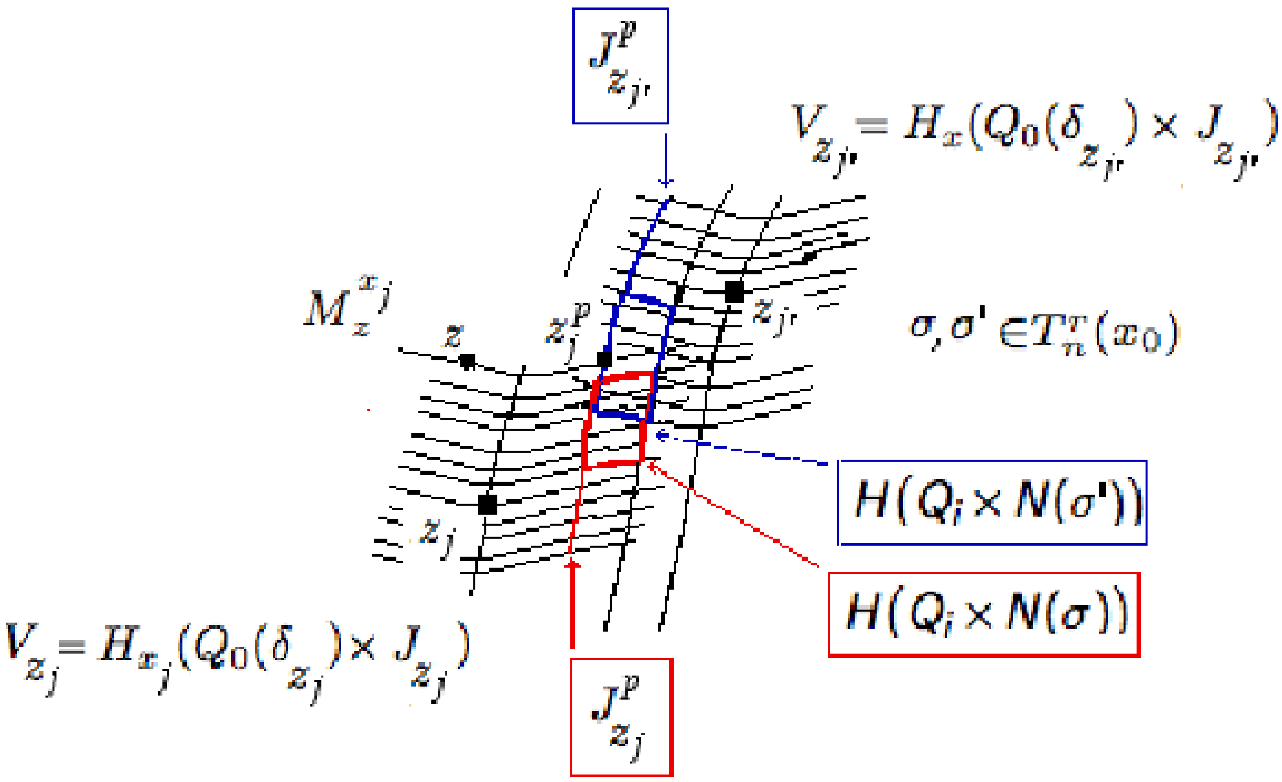

Figure 10 : Zoom of the red and blue boxes of Figure 9

This allows us to define an "adapted" partition of unity, $\tilde{\mathcal{P}}$ subordinate to the covering $\{V_{z_j}\}_{z_j\in P_n(\underline{i})}$ by extending it *constant along each leaf of* $\mathcal{H}_{x_j}$.

That is we define : $\tilde{\mathcal{P}} \;=\; \{\tilde{\alpha}_j \,:\, V_{z_j} \;\longrightarrow\; [0,1]\,\}_{z_j\in P_n(\underline{i})}$ as follows :

$$\tilde{\alpha}_j \;:\; V_{z_j} \longrightarrow [0,1] \qquad \text{by} \qquad \tilde{\alpha}_j(y) = \alpha_j(z_j^p) \qquad\quad \text{where} \quad y \in M_z^{x_j} \cap F_n(p)\,.$$

Now we use this partition of unity to glue together the (π,ρ)-controlled frame fields $\{(w_1^{x_j}, w_2^{x_j})\}_j$ of the foliations $\mathcal{H}_{x_j}$ to define on the open set

$$\bigcup_{z_j\in P_n(\underline{i})} V_{z_j} \quad\supseteq\quad \bigcup_{\sigma\in T_n^r(x_0)} H\Big(Q_{\underline{i}} \,\times\, N(\sigma)\Big) \quad\supseteq\quad \pi_{XY}^{-1}\big(Q_{\underline{i}}\big)\cap A_n$$

the new frame field :

$$W_1^{\underline{i}}(y) \;=\; \sum_{z_j\in P_n(\underline{i})} \tilde{\alpha}_j(y)\cdot w_1^{x_j}(y) \qquad\qquad \text{and} \qquad\qquad W_2^{\underline{i}} \;=\; v_2\,.$$

Then the Lie bracket :

$$\big[\,W_1^{\underline{i}}\,,\; W_2^{\underline{i}}\,\big](y) \;=\; \Big[\,\sum_{j\in P_n(\underline{i})} \tilde{\alpha}_j w_1^{x_j}\,,\; v_2\,\Big](y)$$

$$= \sum_{z_j\in P_n(\underline{i})} \Big(\tilde{\alpha}_{j*y}(v_2(y))\cdot w_1^{x_j}(y) \;+\; \tilde{\alpha}_j(y)\cdot[w_1^{x_j}, w_2^{x_j}](y)\Big) \;=\; \sum_{z_j\in P_n(\underline{i})} (0+0) = 0$$

where each $\tilde{\alpha}_{j*y}(v_2(y)) = 0$ because the $\tilde{\alpha}_j$ are constant along the trajectories of v_2 and each $[w_1^{x_j}, v_2](y) = [w_1^{x_j}, w_2^{x_j}](y) = 0$ because $(w_1^{x_j}, w_2^{x_j})$ is a generating frame field of the foliation $\mathcal{H}_{x_j}$.

On the other hand, each $(w_1^{x_j}, w_2^{x_j})$ being π_{XY}-controlled, we have :

$$\pi_{X*y}\big(W_1^{\underline{i}}(y)\big) \;=\; \pi_{X*y}\big(\sum_j \tilde{\alpha}_j(y) w_1^{x_j}(y)\big) \;=\; \sum_j \tilde{\alpha}_j(y)\cdot \pi_{X*y}\big(w_1^{x_j}(y)\big) \;=\; 1\cdot(E_1) \;=\; E_1$$

and similarly each $(w_1^{x_j}, w_2^{x_j})$ being ρ_X-controlled, we have :

$$\rho_{X*y}\big(W_1^{\underline{i}}(y)\big) \;=\; \rho_{X*y}\big(\sum_j \tilde{\alpha}_j(y) w_1^{x_j}(y)\big) \;=\; \sum_j \tilde{\alpha}_j(y)\cdot \rho_{X*y}\big(w_1^{x_j}(y)\big) \;=\; \sum_j \tilde{\alpha}_j(y)\cdot 0 \;= 0\,.$$

Thus the frame field $(W_1^{\underline{i}}, W_2^{\underline{i}}) = (W_1^{\underline{i}}, v_2)$ is (π,ρ)-controlled too.

This means, in particular, that the partition of unity modifies *only* the components of $W_1^{\underline{i}}(y)$ along the tangent space to the link $L(y)$ of the π_{XY}-fibre containing y.

Finally, each $\mathcal{H}_{x_j}$ being $\frac{1}{n}$-close to $\mathcal{D}_{XY}$ one finds :

$$||W_1^{\underline{i}}(y) - v_1(y)|| = \Big|\Big|\; \sum_j \tilde{\alpha}_j(y)\big(w_1^{x_j}(y) - v_1(y)\big)\;\Big|\Big|$$

$$\leq\; \sum_j \tilde{\alpha}_j(y)\cdot ||w_1^{x_j}(y) - v_1(y)\,|| \;\;\leq\;\; \sum_j \tilde{\alpha}_j(y)\cdot\Big(\frac{1}{n}\Big) \;=\; 1\cdot\Big(\frac{1}{n}\Big) \;=\; \frac{1}{n}\,.$$

Hence the (π,ρ)-controlled frame field $(W_1^{\underline{i}}, W_2^{\underline{i}}) = (W_1^{\underline{i}}, v_2)$ generates a new foliation $\mathcal{H}_{\underline{i}}$ on an open set containing $\pi_{XY}^{-1}(Q_{\underline{i}}) \cap A_n$ on which it is $\frac{1}{n}$-close to the canonical distribution $\mathcal{D}_{XY}$.

We will denote by $H_{\underline{i}} : \mathbb{R}^2 \times \pi_{XY}^{-1}(p) \to Y$ the induced topological trivialization obtained by composing the flows of such frame fields. Then $H_{\underline{i}}$ also generates $\mathcal{H}_{\underline{i}}$ because $\mathcal{H}_{\underline{i}}$ is involutive as proved above.

Step 2 : For each fixed i_2, there exists a controlled foliation $\mathcal{H}_{i_2}$, $\frac{1}{n}$-close to $\mathcal{D}_{XY}$ on an open neighbourhood of :

$$\bigcup_{i_1 \in \Sigma} \pi_{XY}^{-1}(Q_{\underline{i}}) \cap A_n \;=\; \pi_{XY}^{-1}\big([-1,1] \times Q_{i_2}\big) \cap A_n\,.$$

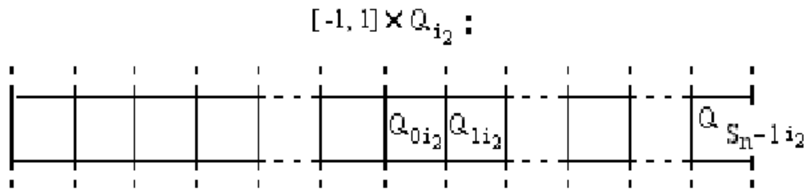

Figure 11

Fix an $i_2 \in \Sigma$ and, for every $i_1 \in \Sigma$, consider the controlled foliation $\mathcal{H}_{\underline{i}}$ obtained in step 1 with generating frame field $(W_1^{\underline{i}}, W_2^{\underline{i}}) = (W_1^{\underline{i}}, v_2)$.

We will show how the controlled foliations $\mathcal{H}_{(0,i_2)}$ and $\mathcal{H}_{(1,i_2)}$ glue together to give a new foliation $\mathcal{H}_{((0,1),i_2)} := \mathcal{H}_{(0,i_2)} \vee \mathcal{H}_{(1,i_2)}$ of an open neighbourhood of

$$\Big(\pi_{XY}^{-1}\big(Q_{(0,i_2)}\big) \cup \pi_{XY}^{-1}\big(Q_{(1,i_2)}\big)\Big) \cap A_n \;=\; \pi_{XY}^{-1}\big([0,2\delta] \times Q_{i_2}\big) \cap A_n\,.$$

Let α be a smooth decreasing function :

$$\alpha \;:\; [0,2\delta] \to [0,1] \quad \text{such that} \quad \alpha(t) \;=\; \begin{cases} 1 & \text{if} \quad t \in [0, \frac{1}{2}\delta] \\ \\ 0 & \text{if} \quad t \in [\frac{3}{2}\delta, 2\delta]\,. \end{cases}$$

Then α can be extended to a map defined on a neighbourhood of $\pi_{XY}^{-1}\big([0,2\delta] \times Q_{i_2}\big) \cap A_n$ which is constant on the trajectories of v_2. That is for every $y = H_{(0,i_2)}(t_1, t_2, y_0)$ we define:

$$\tilde{\alpha} \;:\; \pi_{XY}^{-1}\big([0,2\delta] \times Q_{i_2}\big) \cap A_n \longrightarrow [0,1] \quad , \quad \tilde{\alpha}(y) = \tilde{\alpha}\big(H_{(0,i_2)}(t_1,t_2,y_0)\big) \;=\; \alpha(t_1)\,.$$

Now consider the vector field

$$W_1^{((0,1),i_2)}(y) \;=\; \tilde{\alpha}(y) \cdot W_1^{(0,i_2)}(y) \;+\; \big(1 - \tilde{\alpha}(y)\big) \cdot W_1^{(1,i_2)}(y)\,,$$

where the verifications that the Lie bracket $[W_1^{((0,1)\,,\,i_2)},\; v_2\,](y) \;=\; 0$ and that $W_1^{((0,1),i_2)}$ is a (π,ρ)-controlled vector field $\frac{1}{n}$-close to $\mathcal{D}_X$ are similar to and simpler than those seen in step 1.

Continuing in this way, after a finite number of steps we obtain a vector field $W_1^{i_2}(y)$ defined on a neighbourhood of $\pi_{XY}^{-1}\big([-1,1] \times Q_{i_2}\big) \cap A_n$.

Remark 6. In the construction of the final vector field $W_1^{i_2}(y)$ of step 2, for example when we glue $W_1^{((0,1),i_2)}(y)$ to $W_1^{(2,i_2)}(y)$, the new partition of unity will act only for values of $t_1 \in [\frac{3}{2}\delta, \frac{5}{2}\delta]$ so as to give a vector field $W_1^{((0,1,2),i_2)}(y)$ defined on $\pi_{XY}^{-1}\big([0,3\delta] \times Q_{i_2}\big) \cap A_n$ and satisfying :

$$W_1^{((0,1,2),i_2)}(y) \;=\; \begin{cases} W_1^{((0,1),i_2)}(y) & \text{for} \quad t_1 \in [0, \frac{3}{2}\delta] \\ \\ W_1^{(2,i_2)}(y) & \text{for} \quad t_1 \in [\frac{5}{2}\delta, 3\delta]\,. \end{cases}$$

Hence this second gluing is in a set <u>disjoint</u> from the set in which we did the first gluing and this ensures that $||W_1^{((0,1,2),i_2)}(y) - v_1(y)|| \leq \frac{1}{n}$.

This argument holding for all successive gluing one obtains a final vector field :

$$W_1^{i_2}(y) \;:=\; W_1^{((-s_n,\dots,s_{n-1}),i_2)}(y) \qquad \text{satisfying} \qquad ||W_1^{i_2}(y) - v_1(y)|| \leq \frac{1}{n}\,. \qquad \square$$

Note that the final commuting frame field $(W_1^{i_2}, v_2)$ generates the controlled foliation claimed in step 2 :

$$\mathcal{H}_{i_2} \;:=\; \mathcal{H}_{(-s_n,i_2)} \vee \,\dots \vee\, \mathcal{H}_{(0,i_2)} \vee\, \dots\, \vee \mathcal{H}_{(s_n-1,i_2)}$$

which is $\frac{1}{n}$-close to $\mathcal{D}_X$ on an open neighbourhood of $\pi_{XY}^{-1}\big([-1,1] \times Q_{i_2}\big) \cap A_n$.

We will denote by H_{i_2} the induced topological trivialization which is obtained by composing the flows of this frame field $(W_1^{i_2}, v_2)$ and which generates $\mathcal{H}_{i_2}$.

Step 3 : There exists a controlled foliation $\mathcal{F}_n$ and its (π,ρ)-controlled frame field (W_1, v_2) which is $\frac{1}{n}$-close to $\mathcal{D}_{XY}$ on an open neighbourhood of :

$$\pi_{XY}^{-1}\big([-1,1]^2\big) \cap A_n \;=\; \bigcup_{i_2 \in \Sigma} \pi_{XY}^{-1}\big([-1,1] \times Q_{i_2}\big) \cap A_n\,.$$

Let us fix $i_2 \in \{0,1\}$. We will show below how the foliations $\mathcal{H}_0$ and $\mathcal{H}_1$ and their generating frame fields (W_1^0, v_2) and (W_1^1, v_2) glue together to obtain a convenient foliation $\mathcal{H}_0 \vee \mathcal{H}_1$ of an open neighbourhood of

$$\pi_{XY}^{-1}\big([-1,1] \times Q_0\big) \cup \pi_{XY}^{-1}\big([-1,1] \times Q_1\big) \cap A_n \;=\; \pi_{XY}^{-1}\big([-1,1] \times [0,2\delta]\big) \cap A_n\,.$$

Let α be the smooth decreasing function of step 2. This time we cannot extend α to be constant along the t_2-trajectories (see §4, Remark 3) so we extend it to be constant on the t_1-trajectories.

Define a map on a neighbourhood of $\pi_{XY}^{-1}\big([-1,1] \times [0,2\delta]\big) \cap A_n$ by setting for every $y = H_1(t_1, t_2, y_0)$:

$$\beta \;:\; \pi_{XY}^{-1}\big([-1,1] \times [0,2\delta]\big) \cap A_n \longrightarrow [0,1] \quad , \quad \beta(y) = \beta(H_1(t_1,t_2,y_0)) = \alpha(t_2)\,.$$

Define a vector field W_1 by :

$$W_1(y) \;:=\; \beta(y) \cdot W_1^0(y) + \big(1 - \beta(y)\big) \cdot W_1^1(y)$$

for which the verification that it is (π,ρ)-controlled is similar and simpler than in step 1.

It is easy to see that W_1 is again $\frac{1}{n}$-close to v_1 :

$$||W_1(y) - v_1(y)|| \;\leq\; \beta(y) \cdot ||W_1^0(y) - v_1(y)|| \;+\; \big(1 - \beta(y)\big) \cdot ||W_1^1(y) - v_1(y)\,||$$

$$\leq\; \beta(y) \cdot \frac{1}{n} \;+\; \big(1 - \beta(y)\big) \cdot \frac{1}{n} \;=\; \frac{1}{n}\,.$$

However unfortunately this time the Lie bracket

$$\big[W_1\,,\, v_2\big](y) \;=\; \big[\;\beta \cdot W_1^0 + \big(1 - \beta\big) \cdot W_1^1\,,\; v_2\big](y)$$

$$= \Big(\beta_{*y}(v_2(y)) \cdot W_1^0(y) \;+\; \beta(y) \cdot [W_1^1,\, v_2](y)\Big) \;-\Big(\beta_{*y}(v_2(y)) \cdot W_1^0(y) \;+\; \beta(y) \cdot [W_1^1,\, v_2](y)\Big)$$

$$= \;\beta_{*y}(v_2(y)) \cdot W_1^0(y) \;+\; 0 \;-\beta_{*y}(v_2(y)) \cdot W_1^1(y) \;+\; 0$$

$$= \;\beta_{*y}(v_2(y)) \cdot \big(W_1^0(y) - W_1^1(y)\big)$$

is not zero in general.

But W_1 is (π, ρ)-controlled, and $\frac{1}{n}$-close to the lifting v_1 on $\mathcal{D}_{XY}$ so that we can use the flow ψ_1 of W_1 and the flow ϕ_2 of v_2 to define the desired new foliation $\mathcal{H}_0 \vee \mathcal{H}_1$ on a neighbourhood of $\pi_{XY}^{-1}\big([-1,1] \times [0, 2\delta]\big) \cap A_n$ as follows.

Define :

$$K \;:\; \Big([-1,1] \times [0, 2\delta]\Big) \cap F_n(x_0) \;\longrightarrow\; \pi_{XY}^{-1}\big([-1,1] \times [0, 2\delta]\big)$$

$$(t_1, t_2, y_0) \;\longrightarrow\; \phi_2(t_2,\, \psi_1(t_1, y_0))\,.$$

where ϕ_2 is the flow of v_2. It is easy to verify that the foliation

$$\mathcal{H}_0 \vee \mathcal{H}_1 := \Big\{ K\big([-1,1] \times [0, 2\delta] \times \{y_0\}\big) \;\Big\}_{y_0 \in F_n(x_0)}$$

has a generating frame field $(\widetilde{W}_1, v_2)$ where

$$\widetilde{W}_1(y) :=: \widetilde{W}_1^{(0,1)}(y) := K_{*(t_1, t_2, y_0)}(E_1) \;=\; \phi_{2*y_1}^{t_2}(W_1(y_1))\,.$$

Recall that $W_1(y) := \beta(y) \cdot W_1^0(y) + \big(1 - \beta(y)\big) \cdot W_1^1(y)$, $\widetilde{W}_1$ satisfies :

$$(*) : \qquad ||\widetilde{W}_1(y) - v_1(y)|| \;=\; \big|\big|\phi_{2*y_1}^{t_2}(W_1(y_1)) - v_1(y)\big|\big|$$

$$= \;\Big|\Big|\phi_{2*y_1}^{t_2}\Big(\beta(y) \cdot W_1^0(y_1) \;+\; \big(1 - \beta(y)\big) \cdot W_1^1(y_1)\Big) - v_1(y)\Big|\Big|$$

$$\leq\; \beta(y) \cdot \big|\big|\phi_{2*y_1}^{t_2}\big(W_1^0(y_1)\big) - v_1(y)\big|\big| + \big(1 - \beta(y)\big) \cdot \big|\big|\phi_{2*y_1}^{t_2}\big(W_1^1(y_1)\big) - v_1(y)\big|\big|$$

$$\leq\; \beta(y) \cdot \big|\big|W_1^0(y) - v_1(y)\big|\big| \;+\; \big(1 - \beta(y)\big) \cdot \big|\big|W_1^1(y) - v_1(y)\big|\big|\,.$$

By $(*)$ and the inequalities obtained at the end of Remark 6 in step 2 for $W_1^{i_2}(y)$ for every $i_2 = -s_n, \ldots, s_{n-1}$ applied to $W_1^0(y)$ and $W_1^1(y)$ we deduce that

$$||\widetilde{W}_1(y) - v_1(y)|| \;\leq\; \beta(y) \cdot \frac{1}{n} \;+\; \big(1 - \beta(y)\big) \cdot \frac{1}{n} \;=\; \frac{1}{n}\,.$$

This proves that the vector field $\widetilde{W}_1(y) = \widetilde{W}_1^{(0,1)}$ is again $\frac{1}{n}$-close to $\mathcal{D}_X$.

At the second gluing, we define a vector field :

$$\widetilde{W}_1^{(0,1,2)}(y) \;:=\; K^{(0,1,2)}_{*(t_1,t_2,y_0)}(E_1) \;=\; \phi^{t_2}_{2*y_1}(W_1^2(y_1))$$

which satisfies :

$$||\widetilde{W}_1^{(0,1,2)}(y) - v_1(y)|| \leq \frac{1}{n} \qquad \text{exactly as in (*) of Step 3,}$$

with a formal repetition of the inequalities $(*)$ in which we replace $\widetilde{W}_1^{(0,1)}(y) = \widetilde{W}_1(y)$ by $\widetilde{W}_1^{(0,1,2)}(y)$ etc . , and at the end this time using that :

$$||\widetilde{W}_1^{(0,1)}(y) - v_1(y)|| \;\leq\; \frac{1}{n} \quad \text{and} \quad ||W_1^2(y) - v_1(y)|| \;\leq\; \frac{1}{n}\,.$$

Continuing in this way, after $2s_n - 1$ steps we define a vector field

$$u_1^n(y) := \widetilde{W}_1^{(-s_n,\ldots,s_{n-1})} \qquad \text{on a neighbourhood of} \qquad \pi_{XY}^{-1}\big([-1,1]\times[-1,1]\big) \cap A_n$$

such that the frame field (u_1^n, v_2) is (π,ρ)-controlled and generates the desired controlled foliation

$$\mathcal{F}_n \;:\quad \mathcal{H}_{-s_n} \vee \ldots \vee \mathcal{H}_0 \vee \mathcal{H}_1 \vee \ldots \vee \mathcal{H}_{s_n - 1}$$

which, with the same arguments as in Remark 6, where this time we glue the foliations $\mathcal{H}_j$ along the i_2-direction instead of the i_1-direction, one checks to be $\frac{1}{n}$-close to the canonical distribution $\mathcal{D}_{XY}$.

Step 4 : There exists a controlled foliation on a neighbourhood $\pi_{XY}^{-1}\big([-1,1]^2\big)$ of x_0 in $X \sqcup Y$ and this proves Theorem 4 for $l = 2$.

In the previous step for every $n \in \mathbb{N}^*$, we constructed a controlled foliation $\mathcal{F}_n$ and its generating (π,ρ)-controlled frame field (u_1^n, v_2) which is $\frac{1}{n}$-close to $\mathcal{D}_X$ on an open neighbourhood of the solid annulus $\pi_{XY}^{-1}\big([-1,1]^2\big) \cap A_n$.

We prove now that all foliations of the sequence $\{\mathcal{F}_n\}_{n\in\mathbb{N}^*}$ glue together to give a final controlled foliation $\mathcal{H}$ defined on the whole of $\pi_{XY}^{-1}\big([-1,1]^2\big) \equiv W = Y$ satisfying the smooth version of the Whitney fibering conjecture.

Fix $n \geq 1$ and consider the two controlled foliations :

$$\begin{cases} \mathcal{F}_n & \text{with generating frame } (u_1^n, v_2)\,, \qquad \frac{1}{n}\text{-close to } \mathcal{D}_X\,, \quad \text{on } A_n \subseteq \rho_{XY}^{-1}\big([\frac{1}{n+2}, \frac{1}{n}]\big) \\ \text{and} & \\ \mathcal{F}_{n+1} & \text{with generating frame } (u_1^{n+1}, v_2)\,, \ \frac{1}{n+1}\text{-close to } \mathcal{D}_X\,, \quad \text{on } A_{n+1} \subseteq \rho_{XY}^{-1}\big([\frac{1}{n+3}, \frac{1}{n+1}]\big). \end{cases}$$

Let α_1 be a smooth increasing function,

$$\alpha_1 : [\tfrac{1}{n+3}, \tfrac{1}{n}] \to [0,1] \qquad \text{such that} \qquad \alpha_1(t) \;=\; \begin{cases} 0 & \text{if} \quad t \in [\frac{1}{n+3}\,,\,\frac{1}{n+2}] \\ & \\ 1 & \text{if} \quad t \in [\frac{1}{n+1}\,,\,\frac{1}{n}]\,. \end{cases}$$

By using the function α_1 we will glue together the foliations $\mathcal{F}_n$ and $\mathcal{F}_{n+1}$ along their intersection $A_n \cap A_{n+1} \subseteq \rho_{XY}^{-1}\big([\frac{1}{n+2}, \frac{1}{n+1}]\big)$ without changing them in $\rho_{XY}^{-1}\big([\frac{1}{n+3}, \frac{1}{n+2}] \cup [\frac{1}{n+1}, \frac{1}{n}]\big)$.

Consider the vector field $w_1^{n+1} \ : \ A_n \cup A_{n+1} \to \mathbb{R}^k$ defined by :

$$w_1^{n+1}(y) \ = \ \gamma(y) \cdot u_1^n(y) \ + \ (1-\gamma(y)) \cdot u_1^{n+1}(y) \,, \qquad \text{where} \qquad \gamma(y) = \alpha_1 \circ \rho_{XY}(y) \,,$$

which coincides with $u_1^n(y)$ for $y \in \rho_{XY}^{-1}([\frac{1}{n+1}, \frac{1}{n}])$.

We have :

$$[w_1^{n+1}, v_2](y) \ = \ \Big(\gamma_{*y}(v_2(y)) \cdot u_1^n(y) \ + \ \gamma(y) \cdot [u_1^n, v_2](y) \Big)$$

$$- \Big(\gamma_{*y}(v_2(y)) \cdot u_1^{n+1}(y) \ + \ \gamma(y) \cdot [u_1^{n+1}, v_2](y) \Big) = 0 - 0 = 0$$

where $\gamma_{*y}(v_2(y)) = 0$ because $\gamma(t)$ is constant along all the trajectories of v_2 and each $[u_1^n, v_2](y) = [u_1^{n+1}, v_2](y) = 0$ because (u_1^n, v_2) and (u_1^{n+1}, v_2) are two generating frame fields respectively of the foliations $\mathcal{F}_n$ and $\mathcal{F}_{n+1}$.

Hence, the frame field (w_1^{n+1}, v_2) defines a new controlled foliation $\mathcal{F}_n \vee \mathcal{F}_{n+1}$ on $A_n \cup A_{n+1}$ for which it is easy to verify that (w_1^{n+1}, v_2) is (π, ρ)-controlled and coincides with $\mathcal{F}_n$ on the upper part $A_n \cap \rho_{XY}^{-1}([\frac{1}{n+1}, \frac{1}{n}])$ of A_n .

Moreover $\mathcal{F}_n \vee \mathcal{F}_{n+1}$ is $\frac{1}{n}$-close to $\mathcal{D}_{XY}$ in $A_n \cup A_{n+1}$:

$$||w_1^{n+1}(y) - v_1(y)|| \ \leq \ \ \gamma(y) \cdot ||u_1^n(y) - v_1(y)|| \ + \ (1-\gamma(y)) \cdot ||u_1^{n+1}(y) - v_1(y) \, ||$$

$$\leq \ \gamma(y) \cdot \frac{1}{n} \ + \ (1-\gamma(y)) \cdot \frac{1}{n+1} \ \leq \ \frac{1}{n} \,.$$

Using this way of gluing inductively the foliations of the sequence $\{\mathcal{F}_n\}_{n \geq 1}$ starting from $\mathcal{H}_1 := \mathcal{F}_1 \vee \mathcal{F}_2$ we define an "increasing" sequence of controlled foliations $\{\mathcal{H}_n\}_{n \geq 1}$:

$$\mathcal{H}_n := \Big(\big((\mathcal{F}_1 \vee \mathcal{F}_2 \big) \vee \ldots \vee \mathcal{F}_n \Big) \vee \mathcal{F}_{n+1} \ \text{ of the annular region } \ \ A_1 \cup \ldots \cup A_{n+1} \ \subseteq \ \rho_{XY}^{-1}([\tfrac{1}{n+3} \, , \ 1])$$

where $\mathcal{H}_n$ coincides with $\mathcal{H}_{n-1}$ on $\rho_{XY}^{-1}([\frac{1}{n} \, , \ 1])$ $\frac{1}{n-1}$-close to $\mathcal{D}_X$ on A_n.

In this way the restrictions $\mathcal{H}'_n \ := \ \mathcal{H}_{n|\rho_{XY}^{-1}([\frac{1}{n} \, , \ 1])}$ define an increasing sequence of controlled foliations with each $\mathcal{H}'_n$ which is $\frac{1}{n-1}$-close to $\mathcal{D}_X$ on A_n.

Taking on the whole of $W = \pi_{XY}^{-1}([-1,1]^2 \cap \rho_{XY}^{-1}(]0,1])$ the foliation union $\mathcal{H}' = \cup_{n=1}^{\infty} \mathcal{H}'_n$ and using that $\lim_{y \to x \in X} \mathcal{D}_{XY}(y) = T_x X$ for every $x \in X$, by (c)-regularity $[\mathbf{MT}]_2$, we conclude that

$$\lim_{y \to x \in X} T_y \mathcal{H} \ = \ \lim_{y \to x \in X} \mathcal{D}_{XY}(y) \ = \ T_x X \,.$$

Step 5 : The general case of $\dim X = l \geq 2$.

The proof of Theorem 4 when $\dim X = l > 2$ can be obtained directly by a formal repetition of the steps 1 to 4 of the proof of the case $\dim X = 2$ where the paving by squares $\big\{ Q_{\underline{i}}(\delta) := Q_{i_1} \times Q_{i_2} \big\}_{\underline{i} \in \Sigma^2}$ of $[-1,1]^2$ is replaced by a paving by l-cubes $\{Q_{\underline{i}}(\delta) := Q_{i_1} \times \ldots \times Q_{i_l}\}_{\underline{i} \in \Sigma^l}$ of $[-1,1]^l$ using a multi-index $\underline{i} = (i_1, \ldots, i_l)$ and all essential ideas and techniques are adapted to a bigger dimension.

However this would be long and formally heavy so we give a shorter inductive proof.

Theorem 4 was proved in Step 4 for $\dim X = 2$.

Let $\dim X = l > 2$ and suppose Theorem 3 is true for all X' such that $\dim X' = l-1$.

Let $Y > X$. In a local analysis we suppose as usual $X = \mathbb{R}^l \times 0^{n-l}$ and $x_0 = 0^n \in X$.

For every $t \in [-1,1]$, let $X_t = \mathbb{R}^{l-1} \times \{t\} \times 0^{n-l}$ and $Y_t = \pi_{XY}^{-1}(X_t)$, then $X_t < Y_t$ is a (c)-regular stratification with control data $(\pi_{X_t}, \rho_{X_t}) : X_t \cup Y_t \to X_t \times [0,1]$ which are the restriction of the control data $(\pi_X, \rho_X) : X \cup Y \to X \times [0,1]$ of $X \cup Y$.

The stratification $X_t \cup Y_t$ satisfies the inductive hypothesis so we assume for it all results obtained in the previous steps $1, \ldots, 4$, starting from a topological trivialisation H_{0_t} of origin $0_t := (0^{l-1}, t)$ and a canonical distribution $\mathcal{D}_{X_tY_t}(y) = [v_1(y), \ldots, v_{l-1}(y)]$ generated by the first $l-1$ coordinates of the frame field which is a continuous controlled canonical lifting $(v_1(y), \ldots, v_l(y))$ of the standard frame field $(E_1, \ldots, E_l)$, and which generates the canonical distribution $\mathcal{D}_{XY}(y) = [v_1(y), \ldots, v_l(y)]$ of X.

By the inductive hypothesis every pair of strata $X_t < Y_t$ admits an (a)-regular $(l-1)$-foliation $\mathcal{H}_t = \{M_{y_{0_t}} := H_t(\mathbb{R}^{l-1} \times \{y_{0_t}\})\}_{y_{0_t} \in \pi_X^{-1}(0_t)}$ of Y_t obtained from a trivialisation

$$H_t : \mathbb{R}^{l-1} \times \pi_{X_t}^{-1}\{0_t\} \to Y_t \qquad \text{where} \qquad y_{0_t} \in \pi_{X_t}^{-1}(0_t)\,.$$

Following our proof in step 4, by induction every foliation $\mathcal{H}_t$ is $\frac{1}{n}$-close to $\mathcal{D}_{X_tY_t}$ in the annulus $A_{n+1,t} := A_{n+1} \cap Y_t$.

Let $y_{l-1,t}$ denote an arbitrary point of Y_t.

For every $t \in [-1,1]$ the frame field $(u_1^t(y), \ldots, u_{l-1}^t(y))$ defined by

$$u_i^t(y_{l-1,t}) \;:=\; H_{t*(t_1,\ldots,t_{l-1},y_{0_t})}(E_i) \qquad \text{for every} \qquad i = 1, \ldots, l-1$$

is, by Proposition 3, §5.2, the unique commuting (π_{X_t}, ρ_{X_t})-controlled frame field tangent to $\mathcal{H}_t$, generating $T_{y_{l-1,t}}\mathcal{H}_t$ and is $\frac{1}{n}$-close to $\mathcal{D}_{X_tY_t}$ (and $\mathcal{D}_{XY}$) in the annulus $A_{n+1,t}$ and continuous on X_t (step 4).

Moreover one can write ([**Mu**]$_1$, Chap 2, §5.2 Prop. 1) :

$$\begin{aligned} H_t \;:\; \mathbb{R}^{l-1} \times \pi_{X_t}^{-1}\{0_t\} &\longrightarrow \pi_{X_t}^{-1}(X_t) \equiv Y_t \\ (t_1, \ldots, t_{l-1}, y_{0_t}) &\longmapsto y_{l-1,t} = \psi_{l-1}^t(t_{l-1}, \ldots, \psi_1^t(t_1, y_{0_t}) \ldots) \end{aligned}$$

where $(\psi_1^t, \ldots, \psi_{l-1}^t)$ are the commuting flows of the frame field $(u_1^t, \ldots, u_{l-1}^t)$.

Each map H_t extends in a natural way along the direction of the vector field v_l using its flow ϕ_l by setting

$$\begin{aligned} H^t \;:\; \mathbb{R}^{l} \times \pi_{X_t}^{-1}\{0_t\} &\longrightarrow \pi_{X}^{-1}(X) \equiv Y \\ (t_1, \ldots, t_{l-1}, t_l, y_{0_t}) &\longmapsto y = y_{l,t} \;:=\; \phi_l(t_l, \psi_{l-1}^t(t_{l-1}, \ldots, \psi_1^t(t_1, y_{0_t})) \ldots) \end{aligned}$$

and for every point $y_{0_t} \in \pi_{X_t}^{-1}(0_t)$ one has

$$H^t_{*(t_1,\ldots,t_{l-1},0,y_{0_t})}(E_i) \;=\; H_{t*(t_1,\ldots,t_{l-1},y_{0_t})}(E_i) \;=\; u_i^t(y_{l-1,t})\,, \qquad \forall\, i = 1, \ldots, l-1\,.$$

Let $\mathcal{H}^t = \{M_{y_{0_t}} := H^t(\mathbb{R}^l \times \{y_{0_t}\})\}_{y_{0_t} \in \pi_X^{-1}(0_t)}$ be the foliation defined by H^t. Then the frame field $(w_1^t, \ldots, w_l^t)$ defined by

$$w_i^t(y) = H^t_{*(t_1,\ldots,t_l,y_{0_t})}(E_i)\,, \qquad \forall\; i = 1, \ldots, l-1\,.$$

is the unique commuting (π_X, ρ_X)-controlled frame field tangent to $\mathcal{H}^t$, generating $T_y\mathcal{H}^t$ (Proposition 3, §5.2) and lifting $(E_1, \ldots, E_l)$ on the leaves of $\mathcal{H}^t$ and it coincides with the frame field $(u_1^t, \ldots, u_{l-1}^t, v_l)$ on every point $y_{l-1,t} = H_t(t_1, \ldots, t_{l-1}, y_{0_t}) \in Y_t$

For every $i, j = 1, \ldots, l-1$, since $[u_i^t, u_j^t] = 0$, the flows $\psi_{i\,a}^t$, $\psi_{j\,b}^t$ of u_i^t, u_j^tcommute for all times $a, b \in \mathbb{R}$, and so using the relation $\psi_{i\,a}^t \psi_{j\,b}^t = \psi_{j\,b}^t \psi_{i\,a}^t$ before differentiating (see **[Mu]**$_1$) for every $t \in [-1,1]$ and $y = H^t(t_1, \ldots, t_l, y_{0_t}) \in Y$ we obtain the equalities :

$$\begin{cases} w_l^t(y) = v_l(y) \\ \\ w_i^t(y) := H^t_{*(t_1,\ldots,t_l,y_{0_t})}(E_i) = \phi_{l\,t_l * y_{l-1}}(u_{i-1}^t(y_{l-1})) \quad \text{with the notation in §5.2 for } y_{l-1}. \end{cases}$$

By continuity of each $H^t_{*(t_1,\ldots,t_{l-1},0,y_{0_t})}$ on $X_t \times \pi^{-1}_{X_tY_t}(0_t)$, and since

$$H^t_{*(t_1,\ldots,t_{l-1},0,y_{0t})}(E_i) \ = \ w_i^t(y) \ = \ u_i^t(y)$$

for every $\epsilon > 0$ there exists an open neighbourhood W_t of $Y_t = \pi^{-1}_{X_tY_t}(X_t)$ such that

$$\big|\big|w_i^t(y) - u_i^t(y)\big|\big| < \epsilon\,, \qquad \text{i.e.} \quad T_y\mathcal{H}^t \text{ is } \epsilon\text{-close to } \mathcal{D}_X \qquad \text{for every } y \in W_t\,,$$

and moreover

$$\bigcup_{t\in[-1,1]} W_t \ \supseteq \ \bigcup_{t\in[-1,1]} \pi^{-1}_{X_tY_t}(X_t) \ = \ \pi^{-1}_{X_tY_t}\big(\bigcup_{t\in[-1,1]} X_t\big) \ \supseteq \ \pi^{-1}_{X_tY_t}([-1,1]^l) \ \equiv \ Y\,.$$

Then the family $\mathcal{S}^{n+1} := \{V_t := W_t \cap A_{n+1}\}_{t\in[-1,1]}$ is an open covering of the compact subset $A_{n+1} = \cup_{t\in[-1,1]} A_{n+1,t}$ of Y and there exists a finite subfamily $\{V_{t_j}\}_j$ covering A_{n+1}.

In a similar way as in the first part of the proof (before step 1), for $\epsilon = \frac{1}{n+1}$ there exists $\delta > 0$ and $s_n \in \mathbb{N}^*$ with $\delta := \frac{1}{s_n}$ such that we can obtain every V_{t_j} of the form :

$$V_{t_j} \ \supseteq \ H\Big([-1,1]^{l-1} \times Q_j \times \pi^{-1}_{X_{t_j}}(0_{t_j})\Big) \cap A_{n+1}$$

where $Q_j := [j\delta, (j+1)\delta]$ for every $j \in J_n := \{-s_n \ldots, 0, \ldots, s_{n-1}\}$ and $\cup_{J\in J_n} Q_j = [-1,1]$.

Following the same construction as in step 3, the foliations $\mathcal{H}^j := \mathcal{H}^{t_j}_{|V_{t_j}}$ with $j \in J_n$, induced by each $\mathcal{H}^{t_j}$ on V_{t_j}, glue together in a unique controlled foliation

$$\mathcal{F}^{n+1} \ := \quad \mathcal{H}^{-s_n} \vee \ldots \vee \mathcal{H}^{-1} \vee \ \mathcal{H}^0 \vee \mathcal{H}^1 \ \vee \ \ldots \ \vee \ \mathcal{H}^{s_{n-1}}$$

of an open set

$$\bigcup_{J\in J_n} V_{t_j} \ \supseteq \ \bigcup_{J\in J_n} H\Big([-1,1]^{l-1} \times \big(\bigcup_{J\in J_n} Q_j\big) \times \pi^{-1}_{X_{t_j}}(0_{t_j})\Big) \cap A_{n+1} \ =$$

$$= \ H\Big([-1,1]^l \times \pi_X^{-1}(0_{t_j})\Big) \cap A_{n+1} \ = \ \pi_X^{-1}\big([-1,1]^l\big) \cap A_{n+1}\,.$$

Moreover as in Remark 6, since each $\mathcal{H}^j = \mathcal{H}^{t_j}_{|V_{t_j}}$ is $\frac{1}{n}$-close to $\mathcal{D}_{XY}$ on V_{t_j}, the global foliation $\mathcal{F}^{n+1}$ of A_{n+1} is $\frac{1}{n}$-close to $\mathcal{D}_{XY}$ too on the open set $\cup_{j\in J_n} V_{t_j}$.

We obtain thus for every $n \in \mathbb{N}^*$ a controlled foliation $\mathcal{F}^{n+1}$ which is $\frac{1}{n}$-close to $\mathcal{D}_{XY}$.

At this point with formally the same proof as in step 4 one obtains an "increasing" sequence $\{\mathcal{K}_n\}_{n\geq 1}$ of foliations

$$\mathcal{K}_n := \Big(\big((\mathcal{F}^1 \vee \mathcal{F}^2) \vee \ldots \vee \mathcal{F}^n\big) \vee \mathcal{F}^{n+1} \quad \text{of the set} \quad A_1 \cup \ldots \cup A_{n+2} \subseteq \rho_{XY}^{-1}([\tfrac{1}{n+3}\,,\,1])$$

with $\mathcal{K}_n$ coinciding with $\mathcal{K}_{n-1}$ on $\rho_{XY}^{-1}([\frac{1}{n}\,,\,1])$ and $\frac{1}{n-1}$-close to $\mathcal{D}_X$ on A_n.

In this way the restrictions $\mathcal{K}'_n := \mathcal{K}_{n|\rho_{XY}^{-1}([\frac{1}{n}\,,\,1])}$ define an increasing sequence of controlled foliations with each $\mathcal{K}'_n$ $\frac{1}{n-1}$-close to $\mathcal{D}_X$ on A_n.

Finally, take on the whole of $W = \pi_{XY}^{-1}([-1,1]^l \cap \rho_{XY}^{-1}(]0,1])$ the foliation union $\mathcal{K}' = \cup_{n=1}^{\infty}\mathcal{K}'_n$. By (c)-regularity **[MT]**$_2$, the canonical distribution is continuous on X and we conclude that :

$$\lim_{\substack{y\to x\in X\\ y\in Y}} T_y\mathcal{K}' \;=\; \lim_{\substack{y\to x\in X\\ y\in Y}} \mathcal{D}_{XY}(y) \;=\; T_xX\,. \qquad \square$$

Corollary 3. *With the hypotheses of Theorem 4, the open l-foliated neighbourhood W of $\pi_X^{-1}(U_{x_0}) \cap T_X(1)$ may be chosen of type $\pi_X^{-1}(U') \cap T_X(1)$, where U' is the maximal domain of a chart near x_0 of X as a submanifold of $\mathbb{R}^n$.*

Proof. Let U' be a maximal domain of a chart $\phi : U' \xrightarrow{\equiv} \mathbb{R}^l \times \{0^m\}$ near $x_0 \in X$.

By the first Thom-Mather Isotopy Theorem there exists a topological trivialisation of $\mathcal{X}$ near x_0 :

$$H = H_{x_0} \;:\; \pi_{XY}^{-1}(\{x_0\}) \times U' \;\equiv\; \pi_{XY}^{-1}(\{x_0\}) \times \mathbb{R}^l \;\longrightarrow\; \pi_{XY}^{-1}(U')$$

having its values on the whole of $\pi_{XY}^{-1}(U')$.

Let us consider Theorem 4 proved for such a maximally defined map $H = H_{x_0}$.

In Theorem 4 we proved that starting from the compact set $[-1,1]^l \times \{0^m\} \subseteq \mathbb{R}^l \times \{0^m\}$, there exists a bounded neighbourhood W of 0^n in $\mathbb{R}^n$ containing the relatively compact set $\widetilde{U} = \pi_{XY}^{-1}([-1,1]^l \times \{0^m\}) \cap T_X(1)$, and there exists a controlled foliation $\mathcal{U}$ of $\widetilde{U}$ which is (a)-regular on all points of $[-1,1]^l \times \{0^m\}$, i.e. satisfying :

$$\lim_{y\to x} T_y\mathcal{U} = T_xX \qquad \text{for every} \quad x \in [-1,1]^l \times \{0^m\}\,.$$

Following the proof of Theorem 4 it is clear that the compact cube $[-1,1]^l \times \{0^k\}$ can be replaced by the bigger cube $U_n := [-n,n]^l \times \{0^m\}$. The same proof holds allowing us to find an l-foliation $\mathcal{U}_n$ of an open bounded neighbourhood W'_n of the relatively compact set $\widetilde{U}_n := \pi_{XY}^{-1}([-n,n]^l \times \{0^m\}) \cap T_X(1)$ which is (a)-regular on all points of U_n.

At this point the proof follows using Zorn's Lemma to give the existence of a maximal element of the set of all (a)-regular l-foliations each of whose domains contains a set of the sequance $\pi_{XY}^{-1}([-n,n]^l \times \{0^m\}) \cap T_X(1)$ with respect to an appropriate partial order relation.

However, we give a constructive proof as follows.

We prove by induction that the sequence of these (a)-regular controlled foliations $\{\mathcal{U}_n\}_n$ may be modified to a new sequence of (a)-regular controlled foliations $\{\mathcal{U}'_n\}_n$ in which each foliation $\mathcal{U}'_{n+1}$ defined on $\widetilde{U}_{n+1}$ is an extension of $\mathcal{U}_{n|\widetilde{U}_{n-1}}$.

Let $\mathcal{U}'_n$ be a foliation on $\widetilde{U}_n = \pi_X^{-1}([-n,n]^l \times \{0^m\}) \cap T_X(1)$ extending the restriction of $\mathcal{U}'_{n-1}$ to $\widetilde{U}_{n-2}$ and (a)-regular on all points of U_n.

Via a sequence of two gluings, using the same techniques as in the proofs of Step 2 and Step 3 in Theorem 4, we glue $\mathcal{U}_n$ and the restriction $\mathcal{U}_{n+1|U_{n+1}-U_{n-1}}$ and define a new l-foliation :

$$\mathcal{U}'_{n+1} \quad \text{which} \;: \quad \begin{cases} \text{coincides with } \mathcal{U}'(n) & \text{on} \;\; \widetilde{U}'_{n-1} \;; \\ \text{coincides with } \mathcal{U}_{n+1} & \text{on} \;\; \widetilde{U}_{n+1} - \widetilde{U}_n \;; \\ \text{is } (a)\text{-regular} & \text{on} \;\; U_{n+1}. \end{cases}$$

We have then an increasing sequence of (a)-regular controlled foliations : $\mathcal{U}'_n{}_{\big|\widetilde{U}_{n-1}}$ whose union is defined on the set

$$\bigcup_n \widetilde{U}_{n-1} \;=\; \pi_{XY}^{-1}\Big(\bigcup_n [-n,n]^l \times \{0^m\}\Big) \cap T_{XY}(1) = \pi_{XY}^{-1}\big(\mathbb{R}^l \times \{0^m\}\big) \cap T_{XY}(1)$$

and which is (a)-regular on all points of :

$$\bigcup_{n=1}^{+\infty} U_{n-1} \;=\; \bigcup_n [-n,n]^l \times \{0^m\} \;=\; \mathbb{R}^l \times \{0^m\} \;\equiv\; U'. \quad \square$$

6. Local regular wing structures.

In this section, we prove Theorems 5 and 6 in which we construct a local wing structure for Bekka (c)-regular and Whitney (b)-regular stratifications near every stratum X with $depth_\Sigma(X) = 1$.

These partial results (since $depth_\Sigma(X) = 1$) will play an important role in the proof of our main Theorem 7 of section 7 and will be extended to the general case of arbitrary depth as corollaries of Theorem 7.

Definition 8. Let $\mathcal{X} = (A, \Sigma)$ be a smooth (a)- or (c)- or (b)-regular stratification in $\mathbb{R}^n$, $X \in \Sigma$ and $x_0 \in X$.

One says that $\mathcal{X}$ admits a *local wing structure at (or near)* x_0 if there exists a system of control data $\mathcal{F} = \{(\pi_X, \rho_X, T_X)\}_{X\in\Sigma}$, a neighbourhood U_{x_0} of x_0 in X and $\epsilon > 0$ such that the stratified space $\big(\pi_X^{-1}(U_{x_0}) - U_{x_0}\big) \cap T_X(\epsilon)$ has a stratified foliation

$$\mathcal{W}_{x_0} = \big\{W_{y_0} \;\mid\; y_0 \in (\pi_X^{-1}(x_0) - \{x_0\}) \cap S_X(\epsilon)\big\}$$

such that for every stratum $Y > X$ and $y_0 \in Y$,

i) W_{y_0} is a C^∞-submanifold of $T_{XY}(\epsilon)$ containing y_0 ;

ii) $U_{x_0} \subseteq \overline{W_{y_0}}$ (frontier condition) ;

iii) the restriction $(\pi_{XY}, \rho_{XY})_{|W_{y_0}} : W_{y_0} \longrightarrow U_{x_0} \times\,]0,\epsilon[$ is a C^∞-diffeomorphism.

If these conditions hold, each stratified set $W_{y_0} \sqcup U_{x_0}$ is called a *local wing at* x_0 *in* Y.

The local wing structure $\mathcal{W}_{x_0}$ is called (a)*- or* (c)*- or* (b)*-regular* respectively if moreover :

iv) every pair of strata $U_{x_0} < W_{y_0}$ is (a)- or (c)- or (b)-regular.

If such conditions are satisfied we also say that $\mathcal{W}$ *is a local* (a)*- or* (c)*- or* (b)*-regular wing structure over* U (omitting x_0).

In 1976 $[\mathbf{Go}]_1$ Goresky introduced the following very useful notion :

Definition 9. Let $\mathcal{X} = (A, \Sigma)$ be an abstract stratified set, a family of maps

$$\Big\{ r^\epsilon_X : T_X(1) - X \to S_X(\epsilon) \Big\}_{X\in\Sigma\,,\,\epsilon\in]0,1[}\,,$$

is said to be *a family of lines for* $\mathcal{X}$ *(with respect to a given system of control data)* $\big\{(T_X, \pi_X, \rho_X)\big\}$ if for every pair of strata $X < Y$, the following properties hold :

1) every restriction $r^\epsilon_{XY} := r^\epsilon_{X|Y} : T_{XY} \;\longrightarrow\; S_{XY}(\epsilon)$ of r^ϵ_X is a C^1-map ;
2) $\pi_X \circ r^\epsilon_X = \pi_X$;
3) $r^{\epsilon'}_X \circ r^\epsilon_X = r^{\epsilon'}_X$;
4) $\pi_X \circ r^\epsilon_Y = \pi_X$;
5) $\rho_Y \circ r^\epsilon_X = \rho_Y$;
6) $\rho_X \circ r^\epsilon_Y = \rho_X$;
7) $r^{\epsilon'}_Y \circ r^\epsilon_X = r^\epsilon_X \circ r^{\epsilon'}_Y$.

In order to obtain his important theorem of triangulation of abstract stratified sets, Goresky $[\mathbf{Go}]_3$ proved that every abstract stratified set $\mathcal{X}$ admits a family of lines. Since (c)-regular $[\mathbf{Be}]_1$ and a fortiori (b)-regular $[\mathbf{Ma}]_{1,2}$ stratifications admit structures of abstract stratified sets , a family of lines exists for them.

We can now prove :

Theorem 5. *Let* $\mathcal{X} = (A, \Sigma)$ *be a closed smooth Bekka* (c)*-regular stratified subset of* $\mathbb{R}^n$*. Then for every stratum* X *of depth* 1*, each pair of strata* $X < Y$ *admits a local* (c)*-regular wing structure near every* $x_0 \in X$*.*

Proof. Since the pair of strata $X < Y$ is (c)-regular, by Theorem 3 there exists a neighbourhood U_{x_0} in X (which in a local analysis we identify with $\mathbb{R}^l \times 0 \subseteq \mathbb{R}^n$), and a local (a)-regular foliation $\mathcal{H}_{x_0} = \big\{M_{y_0} := H(\mathbb{R}^l \times y_0)\big\}_{y_0\in\pi^{-1}_{XY}(x_0)}$ corresponding to a stratified local topological trivialization :

$$\begin{array}{rccc} H: & \mathbb{R}^l \times \pi^{-1}_X(x_0) & \longrightarrow & \pi^{-1}_X(\mathbb{R}^l \times 0^m) \subseteq \mathbb{R}^n \\ & (t_1, \ldots, t_l, y_0) & \longmapsto & y := \phi_l(t_l, \ldots, \phi_1(t_1, y_0)\ldots) \end{array}$$

where $(\pi_{XY}, \rho_{XY}) : T_{XY} \to X \times]0, 1$ is the C^∞-submersion of a system of control data.

As $X \sqcup Y$ is (c)-regular it is an abstract stratified set $[\mathbf{Be}]_1$ and hence admits a family of lines $\big\{r^\epsilon_X : T_X(1) - X \to S_X(\epsilon)\big\}_{\epsilon\in]0,1[}$ $[\mathbf{Go}]_3$.

For every y_0 in the link $L(x_0, \epsilon) := S_X(\epsilon) \cap \pi^{-1}_{XY}(x_0)$ we consider the C^∞-arc

$$\gamma_{y_0} \;\;:\;\;]0, \epsilon[\longrightarrow \pi^{-1}_{XY}(x_0)\,, \qquad \gamma_{y_0}(s) = r^s_X(y_0)$$

which is a C^∞-diffeomorphism on its image and define the foliation

$$\mathcal{L}_{x_0} := \Big\{ L_{y_0} := \gamma_{y_0}(]0, \epsilon[) \Big\}_{y_0\in L(x_0,\epsilon)}$$

by 1-dimensional arcs of the fiber $\pi^{-1}_{XY}(x_0) \cap T_{XY}(\epsilon)$ parametrized in the link $L(x_0, \epsilon)$.

Since $\gamma_{y_0}(s) \subseteq S_X(s) = \rho_X^{-1}(s)$ and $\rho_X^{-1}(0) = X$, one has $\lim_{s\to 0} \gamma_{y_0}(s) = x_0$.
Hence each line L_{y_0} satisfies : $\{x_0\} \subseteq \overline{L_{y_0}}$.

For every $y_0 \in L(x_0, \epsilon)$, setting $W_{y_0} := H(\mathbb{R}^l \times L_{y_0})$ the family

$$\mathcal{W}_{x_0} \;:=\; \big\{W_{y_0}\big\}_{y_0 \in L(x_0,\epsilon)}$$

defines a foliation satisfying the local wing structure properties near x_0.

In fact, since H is a homeomorphism it is easy to see that $U_{x_0} \subseteq \overline{W_{y_0}}$ for every y_0.

Since H is a diffeomorphism on strata, every leaf W_{y_0} is a C^∞-submanifold of $T_{XY}(\epsilon)$ of dimension $(l+1)$.

Because $(\pi_{XY}, \rho_{XY}) : T_{XY} \to X \times]0,1[$ is a C^∞-submersion, its restriction to each leaf $(\pi_{XY}, \rho_{XY})_{|W_{y_0}} : W_{y_0} \to X \times]0, \epsilon[$ is a C^∞-diffeomorphism.

Finally, for every $x = (t_1, \ldots, t_l) \in U_{x_0} \equiv \mathbb{R}^l$ we have that, for every $y \in W_{y_0}$, there exists $s \in]0, \epsilon[$ such that $y = H\big((t_1, \ldots, t_l, \gamma_{y_0}(s))\big)$ and so that

$$W_{y_0} \;=\; H(\mathbb{R}^l \times L_{y_0}) \;\supseteq\; H\big(\mathbb{R}^l \times \gamma_{y_0}(s)\big) \;=\; M_{\gamma_{y_0}(s)}$$

and hence by (a)-regularity of the controlled foliation $\mathcal{H}_{x_0}$ one finds :

$$\lim_{y\to x} T_y W_{y_0} \;\supseteq\; \lim_{y\to x} T_y M_{\gamma_{y_0}(s)} \;\supseteq\; T_x X$$

which proves (a)-regularity of the pair of strata $U_{x_0} < W_{y_0}$ at every $x \in U_{x_0}$.

Finally by considering the distance function $\rho_{U_{x_0} W_{y_0}}$, the restriction of ρ_{XY}, each level hypersurface satisfies :

$$\rho^{-1}_{U_{x_0} W_{y_0}}(\epsilon) \;=\; \rho^{-1}_{XY}(\epsilon) \cap W_{y_0} \;=\; M_{y_0}$$

and hence (c)-regularity of $U_{x_0} < W_{y_0}$ follows by (a)-regularity of the foliation $\mathcal{H}_{x_0} = \{M_y\}_y$:

$$\lim_{y\to x} \rho^{-1}_{U_{x_0} W_{y_0}}(\epsilon) \supseteq \lim_{y\to x} T_y M_y = T_x X. \qquad \square$$

For a (b)-regular stratification, with the aim of constructing a corresponding (b)-regular wing structure near $x_0 \in X$, we cannot use an arbitrary Goresky family of lines because these lines are not necessarily (b)-regular over x_0. Fortunately this result holds if the lines are the integral curves of the gradient of the distance function ρ_X.

Theorem 6. *Let $\mathcal{X} = (A, \Sigma)$ be a closed smooth Whitney (b)-regular stratified subset of $\mathbb{R}^n$. For every stratum X of depth 1, each pair of strata $X < Y$ admits a local (b)-regular wing structure near every $x_0 \in X$.*

Proof. The pair of strata $X < Y$ being (b)-regular, it is (c)-regular too **[Be]**$_1$**, [Tr]**$_1$, hence by Theorem 3 there exists a neighbourhood U_{x_0} in X, which in a local analysis we identify with $\mathbb{R}^l \times 0^m \subseteq \mathbb{R}^n$, and there exists a local (a)-regular controlled foliation $\mathcal{H}_{x_0} = \big\{M_{y_0} := H(\mathbb{R}^l \times y_0)\big\}_{y_0 \in \pi^{-1}_{XY}(x_0)}$ obtained from the stratified local topological trivialization :

$$\begin{array}{rccc} H: & \mathbb{R}^l \times \pi_X^{-1}(x_0) & \longrightarrow & \pi_X^{-1}(\mathbb{R}^l \times 0^m) \subseteq \mathbb{R}^n \\ & (t_1, \ldots, t_l, y_0) & \longmapsto & y := \phi_l(t_l, \ldots, \phi_1(t_1, y_0)\ldots) \end{array}$$

where $\{(\pi_{XY}, \rho_{XY}) : T_{XY} \to X \times]0,1[\}$ is the C^∞-submersion of a system of control data.

Let us consider the distance function $\rho_{XY} : T_{XY}(1) \to X$ and on $T_{XY}(1)$, the vector field $v(y) := \nabla \rho_{XY}(y)$ and the integral flow $\phi : \mathbb{R} \times T_{XY}(1) \to T_{XY}(1)$ of v.

Since $\rho_{XY} : T_{XY}(1) \to X$ is a submersion, $\nabla \rho_{XY}(y) \neq 0 \ \forall\, y \in T_{XY}(1)$.

For every y in the link $\pi^{-1}_{XY}(x) \cap T_{XY}(1)$ we consider the C^∞-arc

$$\gamma_y \ : \ ]-\infty, 0[\longrightarrow \pi^{-1}_{XY}(x)\,, \qquad \gamma_y(s) = \phi(s, y)$$

which is a C^∞-diffeomorphism on its image.

For every $x \in X$ we define the foliation

$$\mathcal{L}_x := \Big\{ L_{y_0} := \gamma_{y_0}(]-\infty, 0[) \Big\}_{y_0 \in L(x,1)}$$

of the fiber $\pi^{-1}_{XY}(x) \cap T_{XY}(1)$ by arcs parametrized in the link $L(x,1) := \pi^{-1}_{XY}(x_0) \cap S_{XY}(1)$.

We write $L_y := L_{y_0}$ if $y = \gamma_{y_0}(s)$ is in the same trajectory $\gamma_{y_0}(]-\infty, 0[)$ as y_0.

Moreover, for every $s \in]-\infty, 0[$, by $\gamma_y(s) \subseteq S_X(s) = \underline{\rho_X^{-1}}(s)$ and $\rho_X^{-1}(0) = X$, one has $\lim_{s \to -\infty} \gamma_y(s) = x$ and hence each line L_y satisfies : $x \in \overline{L_y}$, with $x = \pi_{XY}(y)$.

For every $y \in T_{XY}(1)$, setting $W_y := H(\mathbb{R}^l \times L_y)$ the family

$$\mathcal{W}_{x_0} \ := \ \Big\{ W_y \ := \ H(\mathbb{R}^l \times L_y) \Big\}_{y \in T_{XY}(1)}$$

defines a foliation for which as in the (c)-regular case one proves it satisfies the local (a)-regular wing structure properties near x_0.

Recall now the following two useful properties of (b)-regularity at $x \in X < Y$ for two strata $X < Y$ of a stratification in $\mathbb{R}^n$:

i) $X < Y$ is (b)-regular at $x \in X$ if and only it is (a)- and (b^π)*-regular* (defined below) with respect to each C^∞-projection $\pi_X : T_X \to X$.

ii) $X < Y$ (b)-regular at $x \in X$ implies $[\mathbf{Ma}]_{1,2}$ that in local coordinates there exist control data (π_X, ρ_X) where π is the canonical projection $\pi(t_1, \dots, t_n) = (t_1, \dots, t_l, 0^m)$ and ρ the standard distance from $\mathbb{R}^l \times 0^m$, i.e. $\rho(t_1, \dots, t_n) = \sum_{i=l+1}^n t_i^2$.

By *i)* it remains to prove that $U_{x_0} < W_{y_0}$ is (b^π)-regular at each point $x \in U_{x_0}$.

Let us fix $x \in X$. To simply notations we identify $T_{XY}(1)$ and Y.

By definition of (b^π)-regularity at $x \in X$, we must prove that for every sequence $\{y_n\}_n \subseteq W_{y_0}$ such that $\lim_n y_n = x$ and both limits below exist in the appropriate Grassmann manifolds,

$$\lim_n T_{y_n} W_{y_0} \ = \ \sigma \ \in \ \mathbb{G}_n^{l+1} \qquad \text{and} \qquad \lim_n \overline{y_n \pi_X(y_n)} \ = \ L \ \in \ \mathbb{G}_n^1\,, \qquad \text{then} \qquad \sigma \ \supseteq \ L\,.$$

The Grassmann manifold $\mathbb{G}_n^{\dim Y}$ being compact, taking a subsequence if necessary we can suppose that $\lim_n T_{y_n} Y \ = \ \tau \ \in \ \mathbb{G}_n^{l+1}$.

By hypothesis $X < Y$ is (b)-regular and hence (b^π)-regular at $x \in X$ so that $\tau \ \supseteq \ L$.

Moreover by *ii)* we can assume that $\pi_X = \pi : \mathbb{R}^n \to \mathbb{R}^l \times 0^m$ and ρ_X is the standard distance $\rho(t_1, \dots, t_n) = \sum_{i=l+1}^n t_i^2$, so that $\nabla \rho_X(y) = 2(y - \pi_X(y))$ and the vectors generate the same vector space $[\nabla \rho_X(y)] = [y - \pi_X(y)]$.

For every $n \in \mathbb{N}$, let u_n be the unit vector $u_n := \frac{y_n - x_n}{||y_n - x_n||}$ where $x_n = \pi_X(y_n)$.

For every vector subspace $V \subseteq \mathbb{R}^n$, let $p_V : \mathbb{R}^n \to V$ be the orthogonal projection on V and let us consider the "distance" function defined by (**[Ve]**, **[Mu]**$_2$ §4.2) :

$$\begin{cases} \delta(u,V) \;=\; \inf_{v\in V}\; \|\, u-v\,\| \;=\|\, u-p_V(u)\| & \text{for every } u\in\mathbb{R}^n \\ \text{and} & \\ \delta(U,V) \;=\; \sup_{u\in U, \|\,u\,\|=1}\; \|\, u-p_V(u)\,\| & \text{for every subspace } U\subseteq\mathbb{R}^n\,. \end{cases}$$

Then by (b^π)-regularity of $X < Y$ it follows that :

$$\tau \;\supseteq\; L \quad\Longrightarrow\quad \lim_n\,[u_n] \;\subseteq\; \lim_n\; T_{y_n}Y \quad\Longrightarrow\quad \lim_n\; \delta([u_n],T_{y_n}Y) \;=\; 0\,.$$

Since ρ_{XY} is the restriction $\rho_{X|Y}$ of ρ_X to Y, every vector $\nabla\rho_{XY}(y_n)$ is the orthogonal projection $p_{T_{y_n}Y}(\nabla\rho_X(y_n))$ on $T_{y_n}Y$ of the vector $\nabla\rho_X(y_n)$ and we have :

$$T_{y_n}L_{y_n} \;=\; [\nabla\rho_{XY}(y_n)] \;=\; p_{T_{y_n}Y}(\nabla\rho_X(y_n)) = p_{T_{y_n}Y}([y_n-x_n]) = p_{T_{y_n}Y}([u_n])$$

by which, u_n being a unit vector of $[u_n]$, one deduces that :

$$\delta([u_n],\,T_{y_n}L_{y_n}) \;=\; \delta([u_n],\,p_{T_{y_n}Y}([u_n])) \;=\; \|\; u_n - p_{T_{y_n}Y}(u_n)\,\| \;\;=\delta([u_n],\,T_{y_n}Y)\,.$$

On the other hand $L_{y_n} \subseteq W_{y_0} \subseteq Y$, so $T_{y_n}L_{y_n} \subseteq T_{y_n}W_{y_0} \subseteq T_{y_n}Y$, and hence :

$$0 \;\le\; \lim_{y_n\to x}\delta([u_n],\,T_{y_n}W_{y_0}) \;\le\; \lim_{y_n\to x}\delta([u_n],\,T_{y_n}L_{y_n}) \;=\; \lim_{y_n\to x}\delta([u_n],\,T_{y_n}Y) \;=\; 0\,.$$

We deduce that $\lim_{y_n\to x}\delta([u_n],\,T_{y_n}W_{y_0}) \;=\; 0$ and this implies :

$$L \;=\; \lim_{y_n\to x}\overline{y_n\pi_X(y_n)} \;=\; \lim_{y_n\to x}[u_n] \;\subseteq\; \lim_{y_n\to x}T_{y_n}W_{y_0} \;=\; \sigma$$

which proves that $U_{x_0}\equiv\mathbb{R}^l\times 0^m < W_{y_0}$ is (b^π)-regular at x, for every $x\in U_{x_0}$. $\square$

7. Proof of the smooth Whitney fibering conjecture in the general case.

In this section we prove our main results. First we use the local wing structure of section 6 to prove the conclusion of the smooth Whitney fibering conjecture for a stratum X of a (c)-regular stratification $\mathcal{X}=(A,\Sigma)$ having arbitrary depth (Theorem 7) and then we use Theorem 7 to extend the wing structure Theorems 5 and 6 of section 6 to a stratum of arbitrary depth (Theorem 8).

The definition below will be useful in the proof of Theorem 7. A similar notion (Σ-chart) was introduced in **[Fer]**.

Definition 10. Let $\mathcal{X}=(A,\Sigma)$ be an abstract stratified set with a fixed system of control data $\mathcal{T}=\{(T_X,\pi_X,\rho_X)\}_{X\in\Sigma}$ and $X^l<Y^k$ adjacent strata of $\mathcal{X}$,

Let U be the domain of a chart $\varphi: U\subseteq X\to\mathbb{R}^l$, $(u_1,\ldots,u_l)$ the frame field defined by $u_i:=\varphi_*^{-1}(E_i)$ and $\mathbb{R}^k_+ := \mathbb{R}^{k-1}\times]0,+\infty[$ (so $\mathbb{R}_+ :=]0,+\infty[$).

We call *conical chart of Y over U* a chart $\tilde\varphi:\pi^{-1}_{XY}(U)\cap T_{XY}(\epsilon)\to\mathbb{R}^k_+$ of Y such that:

1) $\varphi\circ\pi_{XY}=p\circ\tilde\varphi$ where $p:\mathbb{R}^k\to\mathbb{R}^l$ is the canonical projection ;

2) $\forall\epsilon'\in]0,\epsilon[$ the restriction $\tilde\varphi_{\epsilon'}:\pi^{-1}_{XY}(U)\cap S_{XY}(\epsilon')\to\mathbb{R}^{k-1}\times\{\epsilon'\}$ is a chart of $S_{XY}(\epsilon')$;

3) $\tilde{\varphi}$ extends φ to the stratified homeomorphism

$$\varphi \sqcup \tilde{\varphi} \;:\; U \;\sqcup\; \left(\pi_{XY}^{-1}(U) \cap T_{XY}(\epsilon)\right) \;\longrightarrow\; \mathbb{R}^l \times 0^{k-l} \;\sqcup\; \mathbb{R}^k_+ \,.$$

Example 1. Let H be the topological trivialization of the projection $\pi_{XY} : T_{XY} \to X$:

$$\begin{array}{rcccc} H \;=H_{x_0} & : & U \times \pi_{XY}^{-1}(x_0) \cong \mathbb{R}^l \times \pi_{XY}^{-1}(x_0) & \longrightarrow & \pi_{XY}^{-1}(U) \subseteq \mathbb{R}^n \\ & & (t_1, \dots, t_l, y_0) & \longmapsto & \phi_l(t_l, \dots, \phi_1(t_1, y_0)..) \end{array}$$

where $\forall i \leq l$, ϕ_i is the flow of the vector field v_i which is the (π, ρ)-controlled lifting of u_i.

If

$$h : V \subseteq S_{XY}(\epsilon) \longrightarrow \mathbb{R}^{k-l-1} \quad \text{is a chart of the link} \quad L_{XY}(x_0, \epsilon) := \pi_{XY}^{-1}(x_0) \cap S_{XY}(\epsilon)$$

and we consider the families of smooth open $\mathcal{L}_{x_0}$ arcs as in §6 (recall that $\gamma_{y_0}(1) = y_0$ and $\lim_{t \to 0} \gamma_{y_0}(t) = x_0$) :

$$\mathcal{L}_{x_0} = \left\{\gamma_{y_0} :]0,1[\to \pi_{XY}^{-1}(x_0)\right\}_{y_0 \in V} \quad \text{whose images are the lines} \quad \left\{L_{y_0} := \gamma_{y_0}(]0,1])\right\}_{y_0 \in V}$$

then the union $V' = \sqcup_{y_0 \in V} L_{y_0}$ fills $\pi_{XY}^{-1}(x_0) \cap T_{XY}(\epsilon)$ and the disjoint union of wings $\{W_{y_0} = H(U \times V')\}_{y_0 \in V}$:

$$U' = H(U \times V') \;:=\; \sqcup_{y_0 \in V} H(U \times L_{y_0})$$

is the domain of a conical chart $\tilde{\varphi}$ of Y over U defined by :

$$\begin{array}{rcl} \tilde{\varphi} \;:\; U' := H(U \times V') & \longrightarrow & U \,\times\, \mathbb{R}^{k-l-1} \times]0, \epsilon[\\ y = H(t_1, \dots, t_l, y_{0,t}) & \longmapsto & \tilde{\varphi}(y) := \big(\pi_{XY}(y), h(y_0), \rho_{XY}(y)\big) \end{array}$$

(where $y_{0,t} := \gamma_{y_0}(t)$) which satisfies :

$$\tilde{\varphi}_{*y}(v_i(y)) \;=\; \Big(\pi_{XY*y}(v_i(y)),\, h_{*y_0}(v_i(y_0)),\, \rho_{XY*y}(v_i(y))\Big) \;=\; \big(u_i(x),\, 0^{k-l-1},\, 0\big)\,.$$

Remark 7. With the same notation as in Example 1 one has :

i) If $(q_1, \dots, q_k)$ denotes the coordinate frame field induced by $\tilde{\varphi} : U' \to U \times \mathbb{R}^{k-l-1} \times]0, \epsilon[$ then for every $y = H(t_1, \dots, t_l, y_{0,t}) \in U'$ we have :

$$q_i(y) = \begin{cases} \tilde{\varphi}_{*y}^{-1}(u_i(x), 0^{k-l}) \;=\; v_i(y) & \text{for } i = 1, \dots, l \\ \tilde{\varphi}_{*y}^{-1}(E_i) \;=\; \gamma_{t*y}(h_{*y_0}^{-1}(E_i)) & \text{for } i = l+1, \dots, k-1 \\ \tilde{\varphi}_{*y}^{-1}(E_k) \;=\; (\gamma_{y_0})'(t) & \text{for } i = k\ ; \end{cases}$$

ii) If $\mathcal{A} := \{h_i : V_i \to \mathbb{R}^{k-l-1}\}_{i \in I}$ is an atlas of $L_{XY}(x_0, \epsilon)$, then

$$\cup_i V_i = L_{XY}(x_0, \epsilon) \quad \Rightarrow \quad \cup_i V_i' = \pi_{XY}^{-1}(x_0) \quad \Rightarrow \quad \cup_i U_i' = \pi_{XY}^{-1}(U) \cap T_{XY}(\epsilon)\,,$$

and hence

$$\tilde{\mathcal{A}} := \{\, \tilde{\varphi}_i : U'_i \to U_i \times]0,\epsilon[\times \mathbb{R}^{k-l-1} \,\}_{i\in I} \qquad \text{is an atlas of} \quad \pi^{-1}_{XY}(U) \cap T_{XY}(\epsilon).$$

Theorem 7 below implies that any Bekka (c)-regular stratification satisfies the conclusion of the smooth version of the Whitney fibering conjecture (see Corollary 4 below).

Theorem 7. *Let $\mathcal{X} = (A, \Sigma)$ be a smooth stratified Bekka (c)-regular subset of $\mathbb{R}^n$, X a stratum of $\mathcal{X}$, $x_0 \in X$ and U a domain of a chart of X near x_0.*

Then there exists a controlled foliation $\mathcal{F}_{x_0} = \{F_{z_0}\}_{z_0 \in \pi_X^{-1}(x_0)}$ of the neighbourhood $W = \pi_X^{-1}(U)$ in A of x_0 whose leaves F_{z_0} are smooth l-manifolds diffeomorphic to $X \cap W$, with $F_x = W \cap X = U$, $\forall\, x \in U$, such that for every stratum $X_j \geq X$, $X_j \cap W$ is a union of leaves and $\mathcal{F}_{x_0}$ satisfies:

$$(1): \qquad \lim_{z\to a} T_z F_z \;=\; T_a F_a \;\subseteq\; T_a X_j\,, \qquad \textit{for every} \quad a \in X_j \cap W\,,$$

and in particular for $X_j = X$:

$$(2): \qquad \lim_{z\to a} T_z F_z \;=\; T_a F_a \;=\; T_a X\,, \qquad \textit{for every} \quad a \in X \cap W = U\,.$$

Proof. We prove the theorem by induction on $s = depth_\Sigma X$.

In Theorems 3 and 4 we proved the statement when $s = 1$; this provides the start of the induction.

Let X be a stratum of $\mathcal{X}$ having $s \geq 2$.

A tubular neighbourhood T_X of X is naturally stratified by strata $X^i_j \geq X$ (with $dim X^i_j = i$ and $j \in J_i$) and if T_X is sufficiently small every two strata of the same dimension have disjoint tubular neighbourhoods $[\mathbf{Ma}]_{1,2}$. By interpreting all strata of the same dimension i as a unique (non connected) i-stratum of T_X we can suppose that T_X admits at most one stratum $X^i > X$ of each dimension $i > l = dim X$.

Hence it is sufficient to prove the theorem when T_X has a unique maximal chain of strata adjacent to X :

$$X = X_0 < X_1 < \cdots < X_{s-1} < X_s \qquad \text{with} \quad s \geq 2\,.$$

Let $\varphi : U \to \mathbb{R}^l$ be a chart of X near $x_0 \in X$ and set $Y := X_{s-1}$, $Z := X_s$.

The stratification $\mathcal{X}' = (A', \Sigma')$ obtained by removing from Σ all strata of dimension strictly bigger than $\dim X_{s-1}$ is obviously again (c)-regular with system of control data the family of restrictions $\{(\pi_{XA'}\,,\, \rho_{XA'})\}_{X\in\Sigma'}$ to A' of the control data $\{(\pi_X\,,\, \rho_X)\}_{X\in\Sigma}$, has Y as maximal stratum and $depth_{\Sigma'} X = s-1$ so by the inductive hypothesis the theorem holds for $\mathcal{X}'$ and there exists a controlled foliation $\mathcal{F}'_{x_0} := \{F'_y\}_{y \in \pi^{-1}_{XA'}(x_0)}$ of the neighbourhood $W' := \pi^{-1}_{XA'}(U) = W \cap A'$ of x in A' satisfying the limit properties (1) and (2) for every $j = 0, \ldots, s-1$.

Denote $T_X \cap A'$, by $T_{XA'}$ and $S_X(\epsilon) \cap A'$ by $S_{XA'}(\epsilon)$ the ϵ-sphere of X in A' induced by the control data $\{(\pi_{XA'}\,,\, \rho_{XA'})\}_{X\in\Sigma'}$ of $\mathcal{X}'$.

Let $(u_1, \ldots, u_l)$ be the frame field $u_i := \varphi_*^{-1}(E_i)$ induced by the chart φ and H the topological trivialization of the projection $\pi_X : T_X(1) \to X$,

$$\begin{array}{rcccl} H = H_X & : & U \times \pi_X^{-1}(x_0) \equiv \mathbb{R}^l \times \pi_X^{-1}(x_0) & \longrightarrow & \pi_X^{-1}(U) \subseteq \mathbb{R}^n \\ & & (t_1, \ldots, t_l, z_0) & \longmapsto & \phi_l(t_l, \ldots, \phi_1(t_1, z_0)..) \end{array}$$

where (as in §5.1) $\forall i = 1, \ldots, l$, ϕ_i is the stratified flow of the stratified vector field v_i which is the continuous controlled lifting in the stratified space $\pi_X^{-1}(U) = \sqcup_{j=0}^{s} \pi_{XX_j}^{-1}(U)$ of the coordinate vector field u_i defined on $U \subseteq X$.

In the following we identify $\pi_X^{-1}(U)$ with A, and $\pi_{XX_j}^{-1}(U)$ with $T_{XX_j}(1)$.

According to **[MT]**$_2$ (where the proof is given for the three strata case) for each vector field u on X a stratified continuous controlled vector field is obtained inductively by the following steps 1), ... 4) :

1) one lifts u to a continuous (π_{XZ}, ρ_{XZ})-controlled vector field u^{XZ} on $T_{XZ}(1)$ in A (using (c)-regularity) ;

2) by induction on depth there exists a continuous (π, ρ)-controlled stratified vector field, extension of u, $u^{A'} := \{u^{X_j}\}_{j=0}^{s-1}$ on the stratified space $T_{XA'}(1) = \sqcup_{j=0}^{s-1} T_{XX_j}(1)$ having Y as maximum stratum (again using (c)-regularity) ;

3) denoting d the usual distance of $\mathbb{R}^n$ and setting $y := \pi_{YZ}(z)$ for $z \in T_{YZ} \cap T_{XZ}$, one lifts u^Y to a continuous (π_{YZ}, ρ_{YZ})-controlled vector field u^{YZ} on $T_{YZ}(\epsilon) \cap T_{XZ}(1)$ in A where $T_{YZ}(\epsilon)$ is an open subset of $T_{YZ}(1)$ such that :

$$||u^{YZ}(z) - u^Y(y)|| < \min_{j \leq s-2} d(y, X_j) \qquad \text{(using } (c)\text{-regularity)}.$$

This restriction is important to obtain the continuity on each $X_j < Y$ (see **[MT]**$_2$) :

$$(*) : \quad \forall\, a \in \bigsqcup_{j=0}^{s-1} T_{XX_j}\,, \qquad \lim_{z \to a} u^{YZ}(z) = u^{X_j}(a), \qquad \text{i.e.} \qquad \lim_{z \to a} u^{YZ}(z) = u^{A'}(a)$$

and making a change of scale we can (as usual) suppose $\epsilon = 1$ and $T_{YZ}(\epsilon) = T_{YZ}(1)$.

4) one glues u^{XZ} and u^{YZ} by a partition of unity subordinate to the open covering $\mathcal{O} := \{O_1, O_2\}$ of $T_{XZ}(1)$ where :

$$O_1 := T_{XZ}(1) - \overline{T_{YZ}(1/2)} \qquad \text{and} \qquad O_2 := T_{XZ}(1) \cap T_{YZ}(1).$$

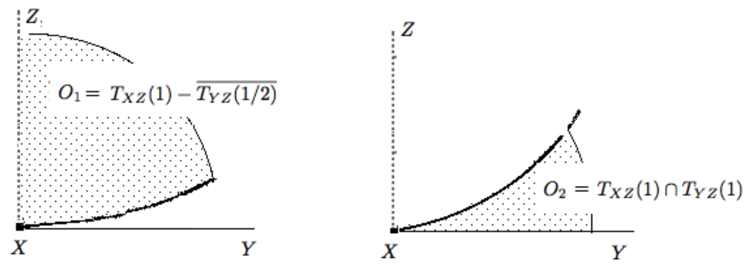

Figure 12

This gives a vector field $u^Z = u^{X_s}$ on $T_{XZ}(1)$, such that the final stratified vector field $u^A := \{u^{X_j}\}_{j=0}^{s}$ is a lifting of u, controlled with respect to all strata of $T_X(1)$, and it is continuous:

$$\lim_{z \to a} u^A(z) = u^{X_j}(a) = u^{XA'}(a)\,, \quad \text{for every} \quad a \in \bigsqcup_{j=0}^{s-1} \pi_{XX_j}^{-1}(U) \;\equiv\; \bigsqcup_{j=0}^{s-1} T_{XX_j}(1)\,.$$

This construction was already used to lift the continuous controlled frame field $(v_1, \ldots, v_l)$ defining the previous trivialization map H. We will re-apply and optimize the steps $1), \ldots 4)$ to lift the integrable frame field $(u_1, \ldots, u_l)$ to an integrable continuous (π, ρ)-controlled frame field $(w_1, \ldots, w_l)$ of

$$\pi_X^{-1}(U) = \pi_{XZ}^{-1}(U) \sqcup \pi_{XA'}^{-1}(U) \ = \ \pi_{XZ}^{-1}(U) \sqcup \sqcup_{j=1}^{s-1} \pi_{XX_j}^{-1}(U) \sqcup U$$

in which $\pi_{XZ}^{-1}(U)$ is the maximal stratum. Moreover we will do it in the general case of many strata, so the present proof also completes details omitted in **[MT]**$_2$.

Remark that, in the present case, each time that we lift an l-frame field tangent to an l-foliation, the lifted frame field will be integrable too.

To start the induction, we first lift $(u_1, \ldots, u_l)$ tangent to a canonical distribution $\mathcal{D}_X$, to obtain continuous controlled vector fields $(v_1, \ldots, v_l)$ defining the trivialization H above and thus give the induced controlled l-foliation (not necessarily (a)-regular) :

$$\mathcal{H} = \left\{M_y^{x_0}\right\}_{y \in \pi_X^{-1}(x_0)} \ = \ \left\{M_z^{x_0}\right\}_{z \in \pi_{XZ}^{-1}(x_0)} \bigsqcup \left\{M_y^{x_0}\right\}_{y \in \pi_{XA'}^{-1}(x_0)} .$$

Then we apply Theorem 3 and (resp.) the inductive hypothesis to the latter two sub-foliations of $\mathcal{H}$ to obtain two (a)-regular controlled l-foliations, whose leaves (with a slight abuse of notation) we denote again by $M_z^{x_0}$ and $M_y^{x_0}$:

$$\begin{cases} \mathcal{H}_{XZ} \ = \ \{M_z^{x_0}\}_{z \in \pi_{XZ}^{-1}(x_0)} & \text{of} \quad \pi_{XZ}^{-1}(U) \\ \mathcal{H}_{A'} \ = \ \{M_y^{x_0}\}_{y \in \pi_{XA'}^{-1}(x_0)} & \text{of} \quad \pi_{XA'}^{-1}(U) \end{cases}$$

and we denote their generating frame fields by

$$(w_1^{XZ}, \ldots, w_l^{XZ}) \qquad \text{and} \qquad (w_1^{A'}, \ldots, w_l^{A'}) .$$

Step 1 : Construction of a controlled l-foliation $\mathcal{H}_{XZ}$, (a)-regular over U.

Let $(w_1^{XZ}, \ldots, w_l^{XZ})$ be the unique continuous (π_X, ρ_X)-controlled l-frame field which is the lifting of $(u_1, \ldots, u_l)$ on the distribution $T\mathcal{H}_{XZ}$.

Since $T\mathcal{H}_{XZ}$ is integrable, $(w_1^{XZ}, \ldots, w_l^{XZ})$ is integrable too.

Using $(w_1^{XZ}, \ldots, w_l^{XZ})$ we can write the topological trivialization of origin x_0 of the projection $\pi_{XZ} : T_{XZ}(1) \to X$ by :

$$\begin{array}{rcll} H_{XZ} & : & U \times \pi_{XZ}^{-1}(x_0) \ \equiv \ \mathbb{R}^l \times \pi_{XZ}^{-1}(x_0) & \longrightarrow \qquad \pi_{XZ}^{-1}(U) \subseteq \mathbb{R}^n \\ & & (t_1, \ldots, t_l, z_0) & \longmapsto \quad \psi_l(t_l, \ldots, \psi_1(t_1, z_0) \ldots) \end{array}$$

where $\forall i \leq l$, ψ_i is the flow of the vector field w_i^{XZ}, and the controlled foliation generated is again the $\mathcal{H}_{XZ}$ which satisfies the Whitney fibering conjecture for $X < Z$ on U (Theorem 4).

Remark that the l^{th} vector field w_l^{XZ}, lift on $\pi_{XZ}^{-1}(U)$ of the l^{th} coordinate vector field u_l of U, remains <u>un-modified</u> during the construction in Theorem 4 because of its special position in the composition of the flows defining H_{XZ}. Hence :

$$(*)_l^{XZ} : \qquad w_l^{XZ}(z) = v_l^{XZ}(z) = v_l(z) \qquad \text{for every} \quad z \in T_{XZ}(1) .$$

Step 2 : Finding a controlled l-foliation $\mathcal{H}_{XA'}$, (a)-regular over A' with wing structure.

By induction, we consider the unique stratified continuous (π, ρ)-controlled l-frame field $(w_1^{A'}, \ldots, w_l^{A'})$ on the stratified distribution $T\mathcal{H}_{A'}$ which is the lifting of $(u_1, \ldots, u_l)$. Then $(w_1^{A'}, \ldots, w_l^{A'})$ is integrable on each stratum of A' since the distribution $T\mathcal{H}_{A'}$ is integrable on each stratum of A' and it generates the controlled foliation $\mathcal{H}_{A'}$.

By considering the trivialization map $H_{A'}$ of the projection $\pi_{XA'} : T_{XA'}(1) \to X$ we can write the leaves of the (a)-regular controlled foliation $\mathcal{H}_{A'}$ contained in $S_{XA'}(1) = S_X(1) \cap A'$ as:

$$\mathcal{H}_{S_{XA'}(1)} \;:=\; \left\{ M_{y_0}^{x_0} \;:=\; H_{A'}(U \times \{y_0\}) \right\}_{y_0 \in L_{XA'}(x_0,1)} ,$$

parametrized in the link $L_{XA'}(x_0, 1) := \pi_{XA'}^{-1}(x_0) \cap S_{XA'}(1)$ of x_0 in A'.

By Theorem 5 and induction, for the stratification $\mathcal{X}'$ in which $depth_{\Sigma'} X = s - 1$, there exists a (c)-regular wing structure of $U < \pi_{XY}^{-1}(U) \cap A'$ over U for every $Y > X$ and the union defines a wing structure :

$$\mathcal{W}_{x_0} \;:=\; \left\{ W_{y_0} \;:=\; H_{A'}(U \times L_{y_0}) \right\}_{y_0 \in L_{XA'}(x_0,1)}$$

where each $L_{y_0} := \gamma_{y_0}(]0, 1])$ is a C^∞-arc contained in the fiber $\pi_{XA'}^{-1}(x_0)$ so that W_{y_0} is parametrized by the link $L_{XA'}(x_0, 1)$ of x_0 in $\pi_{XA'}^{-1}(x_0)$ (as in Theorems 5 and 6).

In order to prove that the foliations $\mathcal{H}_{XZ}$ and $\mathcal{H}_{A'}$ can be glued together into a controlled foliation satisfying the statement of Theorem 7, consider the natural restrictions of $H_{A'}$ and $\mathcal{H}_{A'}$ to $T_{XY}(1) \equiv \pi_{XY}^{-1}(U)$, which is the maximal stratum of $A' \equiv \pi_{XA'}^{-1}(U)$:

$$H_Y := H_{A'|T_{XY}(1)} \qquad \text{and} \qquad \mathcal{H}_Y := \mathcal{H}_{A'|T_{XY}(1)} .$$

Using the restrictions $w_i^Y := w_{i\,|T_{XY}(1)}^{A'}$ for every $i = 1, \ldots, l$, the topological trivialization (of origin x_0) of the projection $\pi_{XY} : T_{XY}(1) \to X$ can be written as

$$\begin{array}{rclcl} H_{XY} & : & U \times \pi_{XY}^{-1}(x_0) \;\equiv\; \mathbb{R}^l \times \pi_{XY}^{-1}(x_0) & \longrightarrow & \pi_{XY}^{-1}(U) \subseteq \mathbb{R}^n \\ & & (t_1, \ldots, t_l, z_0) & \longmapsto & \psi_l(t_l, \ldots, \psi_1(t_1, z_0) \ldots) \end{array}$$

where $\forall i \leq l$, ψ_i is the flow of the vector field w_i^Y and whose foliation generated by H_{XY} is again the $\mathcal{H}_Y$ which satisfies, by induction the properties (1) and (2) in the statement of the Theorem for every two strata $\pi_{XX_j}^{-1}(U) < \pi_{XY}^{-1}(U)$.

Step 3 : Lifting $\mathcal{H}_{XA'}$ to a controlled l-foliation $\mathcal{H}_{XYZ}$ of $A' \sqcup T_{YZ}$, (a)-regular over A' through conical trivializations and having l-foliated wings.

To glue together the controlled foliations $\mathcal{H}_{XZ}$ and $\mathcal{H}_{A'}$ we first need to lift the integrable frame field $(w_1^Y, \ldots, w_l^Y)$ generating the foliation $\mathcal{H}_Y$ of $\pi_{XY}^{-1}(U)$ into a continuous (π, ρ)- controlled integrable frame field $(w_1^{YZ}, \ldots, w_l^{YZ})$ on $\pi_{YZ}^{-1}(\,\pi_{XY}^{-1}(U)\,)$.

Now we have diffeomorphisms

$$\pi_{XY}^{-1}(U) \;\cong\; U \;\times\; \pi_{XY}^{-1}(x_0) \;\cong\; U \times]0, 1[\times L_{XY}(x_0, 1) \,,$$

so in general $\pi_{XY}^{-1}(U)$ is not a domain of a chart of Y. Thus we cannot use a trivialization map H_{YZ} (of origin $y_0 \in \pi_{XY}^{-1}(x_0)$) to define such a lifting $(w_1^{YZ}, \ldots, w_l^{YZ})$ on $\pi_{YZ}^{-1}(\,\pi_{XY}^{-1}(U)\,)$.

On the other hand by writing $L_{XY}(x_0,1) = \cup_i V_i$ where each $V_i \cong \mathbb{R}^{k-l-1}$ ($k = \dim Y$) is a domain of a chart of $L_{XY}(x_0,1)$ we can write $\pi_{XY}^{-1}(U)$ via diffeomorphisms ($\cong$) as a union of domains of charts of $\pi_{XY}^{-1}(U)$:

$$\pi_{XY}^{-1}(U) \;\cong\; U \times]0,1[\, \times L_{XY}(x_0,1) \;\cong\; \cup_i \; U \times]0,1[\, \times V_i \;\cong\; \sqcup_i \; U \times]0,1[\, \times \mathbb{R}_i^{k-l-1}$$

and then we can use these domains to lift $(w_1^Y, \ldots, w_l^Y)$ and define $(w_1^{YZ}, \ldots, w_l^{YZ})$.

More precisely we write :

$$\pi_{XY}^{-1}(x_0) \;=\; \bigcup_{y_0 \in L_{XY}(x_0,1)} L_{y_0} \;=\; \bigcup_{y_0 \in \cup_i V_i} L_{y_0} \;=\; \bigcup_i \bigsqcup_{y_0 \in V_i} L_{y_0}\,,$$

so that

$$\pi_{XY}^{-1}(U) \;=\; H_{XY}\Big(U \times \pi_{XY}^{-1}(x_0)\Big) \;=\; \bigcup_i H_{XY}\Big(U \times \bigsqcup_{y_0 \in V_i} L_{y_0}\Big) \;=\; \bigcup_i H_{XY}\Big(U \times V_i'\Big) \;=\; \bigcup_i U_i'$$

where by definition, as in Example 1, for every i :

$$\begin{cases} V_i' \;:=\; \sqcup_{y_0 \in V_i} L_{y_0} \text{ is the domain of a } \textit{conical} \text{ chart of } \pi_{XY}^{-1}(x_0) \text{ over } x_0\,; \\ U_i' \;:=\; H_{XY}(U \times V_i') \cong \mathbb{R}^k \text{ is a domain of a } \textit{conical} \text{ chart } \tilde{\varphi} \text{ of } \pi_{XY}^{-1}(U) \text{ over } U. \end{cases}$$

This allows us to rewrite the foliation $\mathcal{H}_Y$ as a union of sub-foliations :

$$\mathcal{H}_Y \;=\; \bigcup_i \mathcal{H}_{U_i'}$$

with each

$$\mathcal{H}_{U_i'} \;:=\; \big\{ M_{y_{0,t}}^{x_0} = H_{XY}(U \times \{y_{0,t}\}) \,\big\}_{y_0 \in V_i\,,\; t \in]0,1]} \qquad \text{and} \qquad y_{0,t} = \gamma_{y_0}(t)$$

satisfying the properties of Remark 7. Hence the k-frame field induced by $\tilde{\varphi}$ on U_i' (see Remark 7) has for the first l coordinates exactly the frame field $\big(w_1^Y, \ldots, w_l^Y\big)$, lifted to the (a)-regular foliation $\mathcal{H}_Y$, and we can denote it by $\big(w_1^Y, \ldots, w_k^Y\big)$ where $k = \dim Y$.

We now complete *Step 3* by lifting the foliation $\mathcal{H}_{U_i'}$ of each U_i' and its generating integrable frame field $\big(w_1^Y, \ldots, w_l^Y\big)$ to $\pi_{YZ}^{-1}(U_i')$ via a trivialisation map $H_{U_i'Z}$.

In this way we obtain the desired controlled foliation and its integrable frame field foliation on

$$\cup_i \, \pi_{YZ}^{-1}(U_i') \;=\; \pi_{YZ}^{-1}(\,\cup_i U_i'\,) \;=\; \pi_{YZ}^{-1}\big(\pi_{XY}^{-1}(U)\big)\,.$$

For each $y_0^i \in V_i$ fix the point $p_0^i = y_{0,\frac{1}{2}}^i = \gamma_{y_0^i}(\frac{1}{2}) \in \pi_{XY}^{-1}(x_0) \subseteq V_i'$. Then $p_0^i \in U_i'$.

By Theorem 4 and (c)-regularity of the pair of strata $Y < Z$, using the coordinate frame field $(w_1^Y, \ldots, w_k^Y)$, induced by $\tilde{\varphi}$ on U_i', we have a topological trivialization of the projection $\pi_{YZ} : T_{YZ}(1) \to Y$ with p_0^i as origin :

$$K_{U_i'Z} \;:\; U_i' \times \pi_{YZ}^{-1}(p_0^i) \;\equiv\; \mathbb{R}^k \times \pi_{YZ}^{-1}(p_0^i) \;\longrightarrow\; \pi_{YZ}^{-1}(U_i')$$

and a k-foliation of the image $\pi_{YZ}^{-1}(U_i')$ induced by $K_{U_i'Z}$ defined by :

$$\mathcal{K}_{U_i'Z} \;:=\; \Big\{ N_{z_0}^{p_0^i} \;:=\; K_{U_i'Z}(U_i' \times \{z_0\}) \Big\}_{z_0 \in \pi_{YZ}^{-1}(p_0^i)}$$

such that :

$$\lim_{z\to y}\Big(w_1^{YZ}(z),\dots,w_k^{YZ}(z)\Big) \;=\; \Big(w_1^{Y}(y),\dots,w_k^{Y}(y)\Big) \qquad \text{for every} \quad y\in U_i'$$

which is hence (a)-regular over $U_i' < Z$:

$$\lim_{z\to y} T_z N_z^{p_0^i} \;=\; T_y N_y^{p_0^i} \;=\; T_y Y\,, \qquad \text{for every} \quad y\in U_i'\,.$$

The first l coordinates $(w_1^{YZ},\dots,w_l^{YZ})$ provide the continuous (π_{YZ},ρ_{YZ})-controlled lifting on the foliation $\mathcal{H}_{U_i'Z}$ (of $\pi_{YZ}^{-1}(U_i')$) of the frame field $(w_1^{XY},\dots w_l^{XY})$ (of $\pi_{XY}^{-1}(U)$), lifting of the frame field $(u_1,\dots u_l)$ (of U).

In particular, in the same way as for the previous property $(*)_l^{XZ}$, we have :

$$(*)_l^{YZ}: \qquad w_l^{YZ}(z) \;=\; v_l(z) \;=\; w_l^{XZ}(z)\,, \qquad \text{for every} \qquad z\in T_{YZ}(1).$$

Moreover since

$$U_i' \;=\; H_{XY}(U\times V_i') \;=\; \bigsqcup_{y_{0,t}^i\in V_i'} H_{XY}(U\times\{y_{0,t}^i\}) \;=\; \bigsqcup_{y_{0,t}^i\in V_i'} M^{x_0}_{y_{0,t}^i}\,,$$

it follows that each k-leaf of $\mathcal{H}_{U_i'Z}$ generated by the frame field $(w_1^{YZ},\dots,w_k^{YZ})$:

$$N_{z_0}^{p_0^i} \;:=\; K_{U_i'Z}(U_i'\times\{z_0\}) \;=\; \bigsqcup_{y_{0,t}^i\in V_i'} K_{U_i'Z}\big(M^{x_0}_{y_{0,t}^i}\times\{z_0\}\big)\,, \qquad \text{with} \qquad z_0\in\pi_{YZ}^{-1}(p_0^i),$$

is foliated by the family of l-leaves generated by the frame field $(w_1^{YZ},\dots,w_l^{YZ})$:

$$\mathcal{H}_{U_i'Z} := \Big\{F_{z_0}^{y_{0,t}^i} \;:=\; K_{U_i'Z}\big(M^{x_0}_{y_{0,t}^i}\times\{z_0\}\big)\Big\}_{y_{0,t}^i\in V_i',\; z_0\in\pi_{YZ}^{-1}(p_0^i)}\,.$$

Hence, by the property (*) : $\lim_{z\to a} w_i^{YZ}(z) = w_i^{A'}(a)$ for every $i\le l$, we find that for every $a\in\pi_{XA'}^{-1}(U)$, with $a\in\pi_{XX_j}^{-1}(U)$:

$$\lim_{\substack{z\to a\\ z\in T_{YZ}(1)}} T_z\mathcal{H}_{U_i'Z} \;=\; \lim_{\substack{z\to a\\ z\in T_{YZ}(1)}}\big[w_1^{YZ}(z),\dots,w_l^{YZ}(z)\big] \;=\; \big[w_1^{A'}(a),\dots,w_l^{A'}(a)\big]\subseteq T_aX_j$$

and in particular for every $a\in X_0=X$

$$\lim_{\substack{z\to a\\ z\in T_{YZ}(1)}} T_z\mathcal{H}_{U_i'Z} \;=\; \lim_{\substack{z\to a\\ z\in T_{YZ}(1)}}\big[w_1^{YZ}(z),\dots,w_l^{YZ}(z)\big] \;=\; \big[w_1^{A'}(a),\dots,w_l^{A'}(a)\big] = T_aX\,.$$

Remark 8. Making the same construction with two different charts (h_i,V_i), (h_j,V_j), one obtains open domains of conical charts U_i' and U_j' of Y such that we have (Remark 7):

a) if the frame fields of V_i and V_j coincide in the intersection $V_i\cap V_j$ then the k-foliations, of leaves $N_{z_0}^{p_0}$, $\mathcal{H}_{U_i'Z}$ and $\mathcal{H}_{U_j',Z}$ coincide in the intersection $\pi_{YZ}^{-1}(U_i'\cap U_j')$;

b) the two l-foliations $\mathcal{H}_{U_i'Z}$ and $\mathcal{H}_{U_j'Z}$ coincide in the intersection $\pi_{YZ}^{-1}(U_i'\cap U_j')$ and their frame field $(w_1^{YZ},\dots,w_l^{YZ})$ does not depend on U_i' and U_j'. $\square$

It follows that there exists an l-foliation union $\mathcal{H}_{YZ} := \cup_i \mathcal{H}_{U'_i Z}$ of $\pi^{-1}_{YZ}(\pi^{-1}_{XY}(U))$ which, by $\cup_i V'_i = \pi^{-1}_{XY}(x_0)$, can be written by

$$\mathcal{H}_{YZ} \;=\; \{F^{y^i_{0,t}}_{z_0}\}_{z_0 \in \pi^{-1}_{YZ}(y^i_{0,t})\,,\; y^i_{0,t} \in \cup_i V'_i} \;=\; \{F^{x_0}_{z_0}\}_{z_0 \in \pi^{-1}_{YZ}(\pi^{-1}_{XY}(x_0))}$$

such that for every $a \in X_j \subseteq A'$ one has :

$$\lim_{\substack{z \to a \\ z \in T_{YZ}(1)}} T_z \mathcal{H}_{YZ} \;=\; \lim_{\substack{z \to a \\ z \in T_{YZ}(1)}} \big[w_1^{YZ}(z), \dots, w_l^{YZ}(z)\big] \;=\; \big[w_1^{A'}(a), \dots, w_l^{A'}(a)\big] \subseteq T_a X_j$$

and in particular for $a \in X$

$$\lim_{\substack{z \to a \\ z \in T_{YZ}(1)}} T_z \mathcal{H}_{YZ} \;=\; \lim_{\substack{z \to a \\ z \in T_{YZ}(1)}} \big[w_1^{YZ}(z), \dots, w_l^{YZ}(z)\big] \;=\; \big[u_1(a), \dots, u_l(a)\big] = T_a X \,.$$

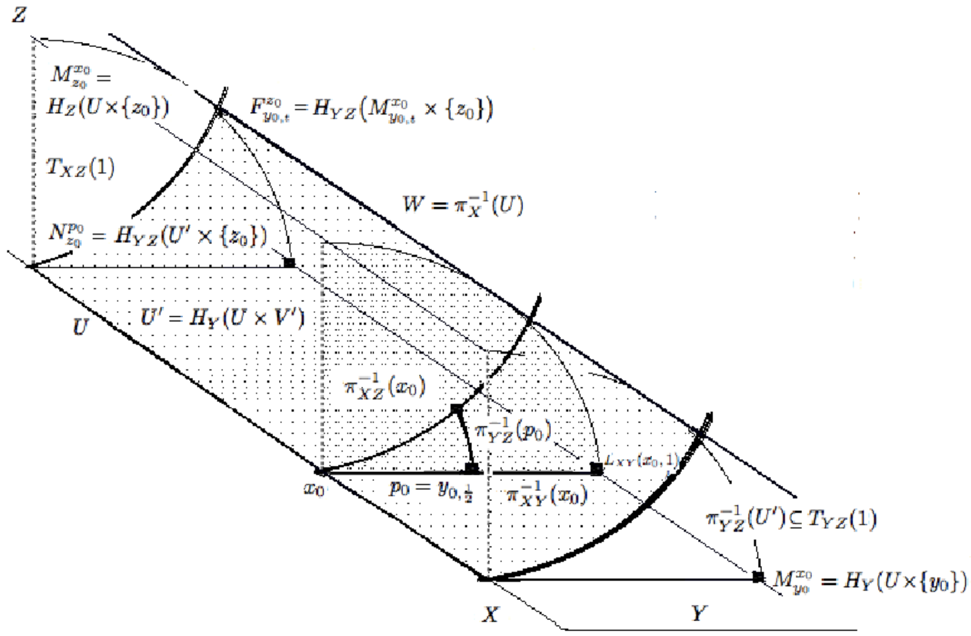

Figure 13

Step 4 : Constructing a controlled (a)-regular l-foliation $\mathcal{F}_{x_0}$ over W by gluing $\mathcal{H}_{XZ}$ and $\mathcal{H}_{XYZ}$.

We now define a new frame field $(w_1^Z, \dots, w_l^Z)$ of $\pi^{-1}_{XZ}(U)$ by gluing together, by an adapted "partition of unity", the two integrable continuous l-controlled frame fields:

$$\begin{cases} \big(w_1^{XZ}, \dots, w_l^{XZ}\big) & \text{generating the foliation } \mathcal{H}_{XZ} \text{ on } \pi^{-1}_{XZ}(U) \equiv T_{XZ}(1)\;, \\ \text{and} & \\ \big(w_1^{YZ}, \dots, w_l^{YZ}\big) & \text{generating the foliation } \mathcal{H}_{YZ} \text{ on } \pi^{-1}_{YZ}(\pi^{-1}_{XY}(U)) \equiv T_{XZ}(1) \cap T_{YZ}(1)\;. \end{cases}$$

Note that each l-leaf $M^{x_0}_{z_0}$ of $\mathcal{H}_{XZ}$ meets the fiber $\pi^{-1}_{XZ}(x_0)$ in the unique point z_0 and similarly each l-leaf $F^{x_0}_{z_0}$ of $\mathcal{H}_{YZ}$ meets the fiber $\pi^{-1}_{XZ}(x_0)$ in the unique point z_0.

We define $(w_1^Z, \dots, w_l^Z)$ by decreasing induction on $i = l \geq \dots \geq 1$.

For $i = l$, define $w_l := v_l$ so that it coincides with w_l^{XZ} and w_l^{YZ}.

Let $i = l-1$, and consider the open covering $\mathcal{O} := \{O_1, O_2\}$ of $\pi_{XZ}^{-1}(x_0)$ defined by :

$$\begin{cases} O_1 \ := \ \pi_{XZ}^{-1}(x_0) \cap \Big(T_{XZ}(1) - \overline{T_{YZ}(1/2)}\Big) \,, \\ O_2 \ := \ \pi_{XZ}^{-1}(x_0) \cap \Big(T_{XZ}(1) \cap T_{YZ}(1)\Big) \ \equiv \ \pi_{YZ}^{-1}\big(\pi_{XY}^{-1}(x_0)\big) \end{cases}$$

and let $P_{l-1} := \{\alpha,\, \beta\}$ be a partition of unity subordinate to the open covering $\mathcal{O}$.

The open covering $\mathcal{O}$ and the partition of unity P_{l-1} on the fiber $\pi_{XZ}^{-1}(x_0)$ induce an open covering $\mathcal{O}\,' := \big\{O_1'\,,\ O_2'\big\}$ of $\pi_{XZ}^{-1}(U)$ where

$$\begin{cases} O_1' \ := \ H_{XZ}(U \times O_1) \ = \ \bigsqcup_{z_0 \in O_1} H_{XZ}(U \times \{z_0\}) \ = \ \bigsqcup_{z_0 \in O_1} M_{z_0}^{x_0} \\ O_2' \ := \ \bigcup_i K_{U_i' Z}(U_i' \times O_2) \ = \ \bigcup_i \bigsqcup_{z_0 \in O_2} K_{U_i' Z}(M_{y_{0,t}}^{x_0} \times \{z_0\}) \ = \ \bigsqcup_{z_0 \in O_2} F_{z_0}^{x_0} \end{cases}$$

(see Remark 8), and a partition of unity $\mathcal{P}_{\updownarrow - \infty} := \{\alpha_{\updownarrow - \infty},\, \beta_{\updownarrow - \infty}\}$ of $\pi_{XZ}^{-1}(U)$ subordinate to $\mathcal{O}\,'$ is obtained extending $\{\alpha,\, \beta\}$ in a constant way along each trajectory of w_l and $\mathcal{P}_{\updownarrow - \infty}$ is thus *adapted* to $\{O_1', O_2'\}$ in the sense of the proof of Theorem 2 of §4.

To simplify notation we will denote

$$w_i := w_i^Z\,, \qquad w_i^1 := w_i^{XZ} \quad \text{and} \quad w_i^2 := w_i^{YZ} \qquad \forall\, i = 1, \dots, l\,.$$

Let w_{l-1} be the vector field defined by :

$$w_{l-1}(z) \ := \ \alpha_{l-1}(z) w_{l-1}^1(z) \ + \ \beta_{l-1}(z) w_{l-1}^2(z)\,.$$

With formally the same calculation as in the proof of Theorem 2 of §4 we have that the Lie bracket $[w_{l-1}, w_l]$ satisfies :

$$\begin{aligned} [w_{l-1}(y), w_l(y)] \ &= \ [\alpha_{l-1} w_{l-1}^1(y), w_l(y)] \ + \ [\beta_{l-1} w_{l-1}^2(y), w_l(y)] \\ &= \Big(\alpha_{l-1\,*y}(w_l(y)) \cdot w_{l-1}^1(y) \quad + \quad \alpha_{l-1}(y)[w_{l-1}^1(y), w_l(y)]\Big) \ + \\ &+ \Big(\beta_{l-1\,*y}(w_l(y)) \cdot w_{l-1}^2(y) \quad + \quad \beta_{l-1}(y)[w_{l-1}^2(y), w_l(y)]\Big) \ = \ 0 \end{aligned}$$

where $\alpha_{l-1\,*y}(w_l(y)) = \beta_{l-1\,*y}(w_l(y)) = 0$ since α_{l-1} and β_{l-1} are constant along the trajectories of w_l and $[w_{l-1}^1(y), w_l(y)] = [w_{l-1}^2(y), w_l(y)] = 0$ since $(w_1^1, \dots, w_l^1)$ and $(w_1^2, \dots, w_l^2)$ are generating frame fields respectively of $\mathcal{H}_{XZ}$ and $\mathcal{H}_{YZ}$ with $w_l^1 = w_l^2 = w_l$.

At this point the definition by induction of w_i for $i < l-1$ is obtained in the same formal way as in Theorem 2 of §4 and this completes the inductive step.

Therefore, by gluing the l-frame field $(w_1^1, \dots, w_l^1)$ generating $\mathcal{H}_{XZ}$ together to the l-frame field $(w_1^2, \dots, w_l^2)$ generating $\mathcal{H}_{YZ}$, we obtain a final integrable frame field $(w_1, \dots, w_l)$ on $T_{XZ} \equiv \pi_{XZ}^{-1}(U)$:

$$w_i(z) \ = \ \alpha_i(z) \cdot w_i^1(z) \ + \ \beta_i(z) \cdot w_i^2(z) \qquad \text{for every } \ i = 1, \dots, l$$

such that :

$$(w_1,\dots,w_l) \;=\; \begin{cases} (w_1^1,\dots,w_l^1) = (w_1^{XZ},\dots,w_l^{XZ}) & \text{on} \quad T_{XZ}(1) - T_{YZ}(1) \\ (w_1^2,\dots,w_l^2) = (w_1^{YZ},\dots,w_l^{YZ}) & \text{on} \quad T_{XZ}(1) \cap T_{YZ}(1/2) \;. \end{cases}$$

Hence :

$$(w_1,\dots,w_l) \qquad \text{generates the foliation} \qquad \begin{cases} \mathcal{H}_{XZ} & \text{on} \quad T_{XZ}(1) - T_{YZ}(1) \\ \mathcal{H}_{YZ} & \text{on} \quad T_{XZ}(1) \cap T_{YZ}(1/2) \;. \end{cases}$$

Lemma 2. *The frame field $(w_1,\dots,w_l)$ satisfies the following :*

i) $(w_1,\dots,w_l)$ is a (π,ρ)-controlled extension of $(w_1^{A'},\dots,w_l^{A'})$ on $\pi_X^{-1}(U)$;

ii) $(w_1,\dots,w_l)$ extends continuously $(w_1^{A'},\dots,w_l^{A'})$ on $\pi_X^{-1}(U)$.

Proof of i). By inductive hypothesis all the (π,ρ)-conditions are satisfied for the strata $X \le X_j \le Y$ of the stratification $\mathcal{X}' = (A',\Sigma')$ in which $depth_{\Sigma'} X = s-1$.

Thus it will be sufficient to prove the control condition for strata $X \le X_j < Z$, with $j = 1,\dots,s-1$. Let us fix then a point $z \in T_{X_jZ}$. We have :

$$\pi_{X_jZ*}\big(w_i(z)\big) = \alpha(z)\cdot\pi_{X_jZ*}\big(w_i^1(z)\big) + \beta(z)\cdot\pi_{X_jY*}\pi_{YZ*}\big(w_i^2(z)\big)$$

then since by construction $w_i^1(z) = w_i^{XZ}(z)$ is a π_{X_jZ}-controlled lifting of $w_i^{A'}(y)$ on $T_{X_jZ}(1)$, and $w_i^2(z) = w_i^{YZ}(z)$ is a π_{YZ}-controlled lifting of $w_i^Y(y)$ on $T_{YZ}(1)$, this is also equal to:

$$= \alpha(z)\cdot w_i^{A'}(\pi_{X_jZ}(z)) + \beta(z)\cdot\pi_{X_jY*}w_i^Y(\pi_{YZ}(z))$$

and since by induction $w_i^Y(y) = w_i^{A'}(y)$ is a π_{X_jY}-controlled lifting on $T_{X_jY}(1)$,

$$= \alpha(z)\cdot w_i^{A'}(\pi_{X_jZ}(z)) + \beta(z)\cdot w_i^{A'}(\pi_{X_jY}\pi_{YZ}(z))$$

$$= \alpha(z)\cdot w_i^{A'}(\pi_{X_jZ}(z)) + \beta(z)\cdot w_i^{A'}(\pi_{X_jZ}(z))$$

$$= [\alpha(z)+\beta(z)]\cdot w_i^{A'}(\pi_{X_jZ}(z)) = w_i^{A'}(\pi_{X_jZ}(z))\,.$$

Similarly, to prove the ρ-control condition we write :

$$\rho_{X_jZ*}(w_i(z)) = \alpha(z)\cdot\rho_{X_jZ*}(w_i^1(z)) + \beta(z)\cdot\rho_{X_jZ*}(w_i^2(z))$$

$$= \alpha(z)\cdot\rho_{X_jZ*}(w_i^{XZ}(z)) + \beta(z)\cdot\rho_{X_jY*}\pi_{YZ*}(w_i^{YZ}(z))$$

$$= \alpha(z)\cdot\rho_{X_jZ*}(w_i^{XZ}(z)) + \beta(z)\cdot\rho_{X_jY*}(w_i^Y(\pi_{YZ}(z))) \;=\; 0 \,+\, 0 \,=\, 0$$

since $\rho_{X_jZ*}(w_i^{XZ}(z)) = 0$ by construction and $\rho_{X_jY*}(w_i^Y(y)) = 0$ by induction.

Proof of ii). Let X_j be a stratum, $X \le X_j \le Y$ and $a \in T_{XX_j}(1) \subseteq X_j \subseteq A'$.

There are essentially three cases :

Case 1) : $X_j = Y$, $a \in$ $T_{XY}(1)$.

In this case in a sufficiently small neighbourhood of $y \in Y$ contained in A, for every $z \in T_{YZ}(1/2)$ by construction we have $(\alpha(z), \beta(z)) = (0,1)$, so $w_i(z) = w_i^{YZ}(z)$ and then :

$$(**) : \qquad \lim_{z \to a} w_i(z) = \lim_{z \to a} w_i^{YZ}(z) = w_i^{Y}(a) = w_i^{A'}(a).$$

Thus w_i is a continuous extension of $w_i^{A'}$ at each $a \in Y$.

Case 2) : $X_j = X < Y$, $a \in X$.

We write :

$$\begin{aligned} w_i(z) - w_i^{A'}(a) &= \alpha(z)\big(w_i^1(z) - w_i^{A'}(a)\big) + \beta(z)\big(w_i^2(z) - w_i^{A'}(a)\big) \\ &= \alpha(z)\big(w_i^{XZ}(z) - w_i^{A'}(a)\big) + \beta(z)\big(w_i^{YZ}(z) - w_i^{A'}(x_j)\big) \end{aligned}$$

with w_i^{XZ} the continuous lifting on $T_{XZ}(1)$ of u_i and by induction $u_i = w^{A'}_{i\,|\,X}$, so we find :

$$\lim_{z \to a} \alpha(z) \cdot \big(w_i(z) - w_i^{A'}(a)\big) = \lim_{z \to a} \alpha(z)\big(w_i^{XZ}(z) - w_i^{A'}(a)\ \big) = 0\,.$$

Moreover as in $(**)$ we also have :

$$\lim_{z \to a} \beta(z) \cdot \big(w_i^{YZ}(z) - w_i^{A'}(a)\big) = 0 \qquad \text{since} \quad \beta(z) \in [0,1]$$

and so :

$$\lim_{z \to a} w_i(z) - w_i^{A'}(a) = 0\,, \qquad \text{i.e. } w_i(z) \text{ extends continuously } w_i^{A'} \text{ at } a \in X.$$

Case 3) : $X < X_j < Y$, $a \in\ T_{XX_j}(1) \subseteq X_j$.

In this case $d(a, X) > 0$, so $a \in \overline{T_{YZ}} - \overline{X}$, hence by construction

$$\lim_{z \to a} (\alpha(z), \beta(z)) = (0,1) \qquad \text{and the proof follows as in Case 2.}$$

We deduce the continuity of each w_i on every stratum X_j of $T_X(1) = \sqcup_{j=1}^{s} T_{XX_j}(1)$:

$$\lim_{z \to a} w_i(z) = w_i^{A'}(a). \quad \square$$

By *ii)*, denoting again (with a slight abuse of notation) $w_i = w_i^A$ the stratified vector field extension $w_i^A := w_i^{A'} \sqcup w_i^Z = w_i^{A'} \sqcup w_i$, it follows easily that the stratified foliation $\mathcal{F}_{x_0}$ generated by the stratified continuous (π, ρ)-controlled frame field $(w_1, \ldots, w_l)$:

$$\mathcal{F}_{x_0} := \big\{F_z^{x_0}\big\}_{z \in \pi_X^{-1}(x_0)} \qquad \text{defined by} \qquad F_z^{x_0} := [w_1(z), \ldots, w_l(z)]$$

satisfies for every stratum X_j such that $X \leq X_j \leq Z$ and for every $a \in X_j \cap W = \pi_{XX_j}^{-1}(U)$:

$$\lim_{z \to a} T_z F_z^{x_0} = \lim_{z \to a} \big[w_1(z), \ldots, w_l(z)\big] = \big[w_1^{A'}(a), \ldots, w_l^{A'}(a)\big] \subseteq T_a X_j$$

and in particular for every $a \in U = W \cap X$:

$$\lim_{z \to a} T_z F_z^{x_0} = \lim_{z \to a} \big[w_1(a), \ldots, w_l(a)\big] = \big[u_1(a), \ldots, u_l(a)\big] = T_a X\,.$$

We conclude then that the controlled foliation $\mathcal{F}_{x_0}$ generated by the frame field $(w_1,\ldots,w_l)$ satisfies all the properties in the statement of the Theorem. $\square$

Corollary 4. *Every analytic variety or subanalytic set or definable set in an o-minimal structure satisfies the smooth version of the Whitney fibering conjecture.*

Prof. Since analytic varieties, subanalytic sets, and definable sets admit Whitney stratifications (**[Ve], [Hi]** and **[Loi], [NTT]**) and Whitney regularity implies (c)-regularity **[Be]**$_1$ **[Tr]**$_1$, the proof follows from Theorem 7. $\square$

We generalize now Theorems 5 and 6 of section 6 to a stratum X of arbitrary depth.

Theorem 8. *Let $\mathcal{X}=(A,\Sigma)$ be a Bekka (c)- (resp. Whitney (b))-regular stratification. Let X be a stratum of $\mathcal{X}$, $x_0\in X$ and U a domain of a chart near x_0 of X.*

Then $\mathcal{X}$ admits a (c)- (resp. (b)-) regular wing structure $\mathcal{W}_{x_0}=\{W_{z_0}\}_{z_0\in L(x_0,\epsilon)}$ on $W=\pi_X^{-1}(U)$ over U such that for every stratum $Y>X$, $Y\cap W$ is a union of wings, and moreover $\mathcal{W}_{x_0}$ satisfies :

$$(3): \qquad \lim_{z\to y} T_zW_{z_0} \;=\; T_yW_y \;\subseteq\; T_yY\,. \qquad \textit{for every} \;\; y\in Y\cap W\,.$$

Proof. Let $\mathcal{X}'=(A',\Sigma')$ be the stratification induced by $\mathcal{X}$ on $A':=\pi_X^{-1}(x_0)\cap T_X(1)$:

$$\pi_X^{-1}(x_0) \;=\; \bigsqcup_{X\le Y}\pi_{XY}^{-1}(x_0)\,.$$

By (c)-regularity, as in Theorems 5 and 6, $\mathcal{X}'$ admits a natural stratified foliation by lines

$$\mathcal{L}_{x_0} := \Big\{L_{z_0}:=\gamma_{z_0}(]0,1[)\Big\}_{z_0\in L(x_0,1)}$$

satisfying $\{x_0\}\subseteq\overline{L_{z_0}}$ where we suppose as usual (after a change of scale) $\epsilon=1$.

By (c)-regularity and Theorem 7 there exists a trivialization of $W:=\pi_X^{-1}(U)$,

$$\begin{array}{llcl} H: & U\times\pi_X^{-1}(x_0)\equiv\mathbb{R}^l\times\pi_X^{-1}(x_0) & \longrightarrow & W=\pi_X^{-1}(U)\,,\\ & (t_1,\ldots,t_l,z_0) & \longmapsto & z:=\phi_l(t_l,\ldots,\phi_1(t_1,z_0)\ldots) \end{array}$$

whose induced "horizontal" foliation

$$\mathcal{F}_{x_0}=\{F_{z_0}:=H(U\times\{z_0\})\}_{z_0\in\pi_X^{-1}(x_0)} \qquad \text{is globally } (a)\text{-regular over } U.$$

Hence, we define the global family of wings over U, as in Theorems 5 and 6, by :

$$\mathcal{W}_{x_0} \;:=\; \Big\{W_{z_0} \;=\; H\big(U\times L_{z_0}\big)\;\Big\}_{z_0\in L(x_0,1)}$$

such that each wing W_{z_0} satisfies :

$$W_{z_0} \;:=\; H\big(U\times L_{z_0}\big) \;=\; \bigsqcup_{s\in]0,1[} H\big(U\times\{\gamma_{z_0}(s)\}\big) \;\supseteq\; H\big(U\times\{\gamma_{z_0}(s)\}\big) \;=\; F_{\gamma_{z_0}(s)}\,.$$

The proofs follow as in Theorems 5 and 6 since, by the global (a)-regularity at $x \in U$ of the foliation $\mathcal{F}_{x_0}$, this time we can write :

$$\lim_{z \to x} T_z W_{z_0} \supseteq \lim_{z \to x} T_z F_{\gamma_{z_0}(s)} \supseteq T_x X .$$

This proves (a)-regularity at every $x \in U$ of the strata $U < W_{z_0}$ and this for every wing $W_{z_0} \subseteq W = \sqcup_{X \leq Y} \pi_{XY}^{-1}(U)$ and so

$$\mathcal{W}_{x_0} := \{W_{z_0}\}_{z_0 \in L(x_0,1)}$$

is a foliation by wings satisfying the (a)- and (c)-regular wing properties over U.

If $\mathcal{X}$ is (b)-regular, (b^π)-regularity of $U < W_{y_0}$ follows exactly as in Theorem 6.

To show that the foliation of wings $\mathcal{W}_{x_0}$ satisfies the limit property (3), we have to specify more carefully the stratified foliation of lines

$$\mathcal{L}_{x_0} := \Big\{ L_{z_0} := \gamma_{z_0}(]0,1[) \Big\}_{z_0 \in L(x_0,1)} .$$

By (c)-regularity, using the theorem of continuous lifting of vector fields $[\mathbf{MT}]_2$ we can obtain the continuity on $\pi_X^{-1}(U) - U = \cup_{X<Y} \pi_{XY}^{-1}(U)$ of the stratified vector field $\gamma'_{z_0}(t) = \{\gamma'_{z_0\, XY}(t)\}_{Y \geq X}$. Hence by denoting $y_0 = \pi_{YZ}(z_0) \in Y$, we have :

$$(*) : \qquad \lim_{z \to y} T_z L_{z_0} = T_y L_{y_0} \qquad \text{for every} \quad y \in Y,\ Y \geq X .$$

Let us fix a stratum $Y > X$ and remark that, with the same notation as in the proof of Theorem 7, for every stratum $Z > Y$ the neighbourhood $T_{YZ}(1/2)$ is foliated by the family of k-leaves

$$\mathcal{H}_{U'_i Z} := \Big\{ N_{z_0}^{p_0^i} := K_{U'_i Z}(U'_i \times \{z_0\}) \Big\}_{z_0 \in \pi_{YZ}^{-1}(p_0^i)} \qquad \text{with} \quad p_0^i := y^i_{0,\frac{1}{2}} ,\ \ y_0^i \in V_i$$

which is (a)-regular over U'_i : i.e. satisfies the limit property (1) of Theorem 7.

Moreover each leaf $N_{z_0}^{p_0^i}$ of $\mathcal{H}_{U'_i Z}$ is the continuous lifting of $U'_i \subseteq \pi_{XY}^{-1}(U)$ and is generated by the frame field $(w_1^{YZ}, \ldots, w_k^{YZ})$ where (Remark 7) for every

$$z = \gamma_{z_0}(t) \in \pi_{YZ}^{-1}(U'_i) \cap T_{YZ}(1/2) \subseteq \pi_{XZ}^{-1}(U) \cap T_{YZ}(1/2)$$

we have :

$$[w_{l+1}^{YZ}(z)] = [\gamma'_{z_0}(t)] = T_z L_{z_0} \qquad \text{with} \quad z_0 \in S := S_{XZ}(1) \cap T_{YZ}(1/2) .$$

By property $(*)$ above, for every $z_0 \in T_{YZ}(1/2)$ and $y_0 = \pi_{YZ}(z_0)$, the line L_{z_0} is the continuous lifting on $T_{YZ}(1/2)$ of the line L_{y_0}, and hence at the level of the wings :

$$(**) : \quad z_0 \in L(x_0,1) \cap T_{YZ}(1/2) \implies W_{z_0} \text{ is the continuous lifting of } W_{y_0} .$$

On the other hand, for every $z_0 \in \pi_{YZ}^{-1}(p_0^i)$, we have that (see Theorem 7 for the notation)

$$N_{z_0}^{p_0^i} := K_{U'_i Z}(U'_i \times \{z_0\}) = \bigsqcup_{y^i_{0,t} \in V'_i} K_{U'_i Z}\big(M^{x_0}_{y^i_{0,t}} \times \{z_0\}\big) ,$$

so it is foliated by the family of l-leaves generated by the frame field $(w_1^{YZ}, \dots, w_l^{YZ})$:

$$\begin{aligned} N_{z_0}^{p_0^i} &:= H_{U_i' Z}(U_i' \times \{z_0\}) = \bigsqcup_{y_0^i \in V_i} \bigsqcup_{t \in]0,1[} K_{U_i' Z}\Big(M^{x_0}_{y^i_{0,t}} \times \{z_0\}\Big) \\ &= \bigsqcup_{y_0^i \in V_i} K_{U_i' Z}\Big(\bigsqcup_{t \in]0,1[} H_{XY}(U \times \{y_{0,t}^i\}) \times \{z_0\}\Big) \\ &= \bigsqcup_{y_0^i \in V_i} K_{U_i' Z}\Big(H_{XY}\Big(U \times (\bigsqcup_{t \in]0,1[} \{y_{0,t}^i\})\Big) \times \{z_0\}\Big) \\ &= \bigsqcup_{y_0^i \in V_i} K_{U_i' Z}\Big(H_{XY}(U \times L_{y_0^i}) \times \{z_0\}\Big) \\ &= \bigsqcup_{y_0^i \in V_i} K_{U_i' Z}\big(W_{y_0^i} \times \{z_0\}\big) = \bigsqcup_{z_0 \in \pi_{YZ}^{-1}(V_i)} W_{z_0} . \end{aligned}$$

In conclusion for every $z_0 \in S \subseteq T_{YZ}(1/2)$ each k-leaf $N_{z_0}^{p_0^i}$ of $T_{YZ}(1/2)$ is foliated by the sub-family of $(l+1)$-wings $\{W_{z_0}\}_{z_0 \in \pi_{YZ}^{-1}(V_i)}$ and so for every $y \in Y \cap W$ one has :

$$\begin{aligned} \lim_{z \to y} T_z W_{z_0} &= \lim_{\substack{z \to y \\ z \in T_{YZ}(1/2)}} [w_1^{YZ}(z), \dots, w_{l+1}^{YZ}(z)] \\ &= [w_1^{XY}(y), \dots, w_{l+1}^{XY}(y)] = T_y W_y \subseteq T_y Y . \quad \square \end{aligned}$$

Corollary 5. *Every analytic variety or subanalytic set or definable set in an o-minimal structure admits a stratification Σ in which for every stratum X and every U domain of a chart of X there exists a local (b)-regular wing structure over U.*

Proof. Since analytic varieties, subanalytic sets and respectively definable sets admit Whitney stratifications (**[Ve]**, **[Hi]**, respectively **[VM]**, **[Loi]**, **[NTT]**) the proof follows from Theorems 7 and 8. $\square$

8. $\mathcal{F}$-semidifferentiable first Thom-Mather Isotopy Theorems.

The first Thom-Mather isotopy theorem is the most important result in stratification theory. For a stratified submersion $f : \mathcal{X} \to M$ defined on a (c)-regular stratification and into a manifold M, it provides, for every $m_0 \in M$ a stratified isomorphism $H_{m_0} : U_{m_0} \times f^{-1}(m_0) \to f^{-1}(U_{m_0})$ which is a diffeomorphism on each stratum of $\mathcal{X}$ but globally only a homeomorphism and which cannot be made in general C^1, since by real examples similar to the Whitney counterexample the complex *four lines family* (see section 2), one easily see that H cannot be made in general C^1.

In this section we give a significant application of Theorem 7 by showing for (c)- and hence (b)-regular stratifications $\mathcal{X}$, as consequences of the fact that $\mathcal{X}$ satisfies the smooth Whitney fibering conjecture, that this isomorphism H_{m_0} can be made to satisfy two regularity conditions between C^0- and C^1-regularity : *horizontally-C^1* and the finer condition *$\mathcal{F}$-semidifferentiability.*

In this section $\mathcal{X} = (A, \Sigma)$ will be a (c)-regular stratification of a closed subset A in a manifold M, X an l-stratum of $\mathcal{X}$, $x_0 \in X$,

$$H = H_{x_0} : U_{x_0} \times \pi_X^{-1}(x_0) \to \pi_X^{-1}(U_{x_0}) , \qquad H(t_1, \dots, t_l, y_0) = \phi_l(t_l, \dots, \phi_1(t_1, y_0)..)$$

the topological trivialization of the projection $\pi_X : T_X(1) \to X$ over a neighbourhood $U_{x_0} \subseteq X$ of X defined by composition of flows $\phi_1, \ldots, \phi_l$ of continuous lifted controlled vector fields $u_1, \ldots, u_l$, and $\mathcal{H} = \{M_y = \{H(U_{x_0} \times \{y_0\})\}_{y_0 \in \pi_X^{-1}(x_0)}$ the stratified controlled (a)-regular l-foliation of $W := \pi_X^{-1}(U_{x_0})$ defined by H.

Such a stratified l-foliation of W, (a)-regular on U_{x_0}, exists by Theorem 7 and all foliations that we consider in this section will be of this type.

First we describe some results on the regularity of the flows of the continuous lifted vector fields to $\mathcal{H}$, which are significant because they imply an improvement of the regularity of the trivialization H. Then we obtain horizontally-C^1 and $\mathcal{F}$-semidifferentiable versions of the first Thom-Mather Isotopy Theorem.

These results were initially announced under the hypothesis of the existence of an (a)-regular foliation without proof in $[\mathbf{MT}]_{1,3}$, then proved in $[\mathbf{MT}]_4$. By Theorem 7 they apply to all strata X of a (c)-regular stratification. The proofs are contained in $[\mathbf{MT}]_4$.

8.1. Horizontally-C^1 morphisms and the first Thom-Mather Isotopy Theorem.

In $[\mathbf{MT}]_{1,3,4}$ we introduce the notions of horizontally-C^1 stratified maps $f : \mathcal{X} \to \mathcal{X}'$.

Definition 11. Let $f : \mathcal{X} \to \mathcal{X}'$ be a stratified morphism between two (c)-regular stratifications $\mathcal{X} = (A, \Sigma)$ and $\mathcal{X}' = (A', \Sigma')$ in smooth manifolds M and (resp.) N, X an l-stratum of $\mathcal{X}$ and $x \in X$. For each $X \in \Sigma$ let X' be the stratum of Σ' containing $f(X)$.

We say that f is *horizontally-C^1 at $x \in X$* if there exists a (local) canonical l-distribution $\mathcal{D}_X = \{\mathcal{D}_{XY}\}_{Y \geq X}$ defined on a neighbourhood W of x in A, such that for each stratum $Y > X$ (so $Y' \geq X'$), the restriction $f_{Y*|\mathcal{D}_{XY}} : \mathcal{D}_{XY} \to TY'$ extends continuously the differential $f_{X*} : TX \to TX'$.

That is for every sequence $\{(y_n, v_n)\}_n \subseteq \cup_{y \in Y}\{y\} \times \mathcal{D}_{XY}(y)$:

$$\lim_{n\to\infty} (y_n, v_n) = (x, v) \in TX \quad \Longrightarrow \quad \lim_{n\to\infty} f_{Y*y_n}(v_n) = f_{X*x}(v) \ .$$

This makes sense because by the frontier condition, $X \subseteq \overline{Y} \subseteq M$, and (a)-regularity implies that $TX \subseteq \overline{TY}$ and $TX' \subseteq \overline{TY'}$ in TM and (resp.) TN.

Remark 9. *Every controlled map $f : \mathcal{X} \to M$ into a manifold M is horizontally-C^1.*

Proof. Since M is a manifold, its system of control data reduces to the identity map and for every $Y \geq X$ the π_X-control condition for f becomes $f_Y = f_X \circ \pi_{XY}$.

Hence $f_{Y*} = f_{X*} \circ \pi_{XY*}$ and so $\lim_n f_{Y*y_n}(v_n) = f_{X*x}(v)$ follows since $f_X : X \to M$ and $\pi_{XY} : T_{XY} \to X$ are C^1. $\square$

Continuous controlled lifting of vector fields plays an important role in studying horizontally-C^1 regularity. In fact, if a vector field ξ_X is lifted to a stratified continuous (π, ρ)-controlled vector field $\xi = \{\xi_Y\}_{Y \geq X}$ on a neighborhood T_X of X in A, then assuming the existence of an integrable canonical distribution $\mathcal{D}_X$ the lifted flow $\phi = \cup_{Y \geq X} \phi_Y$ on T_X is a horizontally-C^1 extension of ϕ_X ($[\mathbf{MT}]_4$, Theorem 4).

An arbitrary (local) canonical distribution $\mathcal{D}_X$ is not integrable in general. However, for a stratum X of a (c)-regular stratification $\mathcal{X}$, by Theorem 7 we can consider as canonical distribution $\mathcal{D}_X = T\mathcal{H}$ the distribution tangent to a local (a)-regular foliation and we find:

Corollary 6. *Let $\mathcal{D}_X = T\mathcal{H}$ be the canonical distribution tangent to a stratified l-foliation (a)-regular near $x_0 \in U \subseteq X$, ξ_X a smooth vector field on X and $\xi = \{\xi_Y\}_{Y \geq X}$ its continuous controlled lifting tangent to $\mathcal{H} = \{M_y\}_{y \in W}$.*

Then the flow $\phi_t = \{\phi_{Y\,t} : Y \to Y\}_{Y \geq X}$ *to a fixed* $t \in \mathbb{R}$ *of* ξ *is horizontally-*C^1 *on* U.

Proof. Theorem 4 of [**MT**]$_4$. □

With the usual hypotheses and notations and by identifying $U_{x_0} \equiv \mathbb{R}^l \times 0^m$ and the frame field $(u_1, \ldots, u_l)$ of U_{x_0} with the standard frame field $(E_1, \ldots, E_l)$, we have :

Corollary 7. *Let* H *be the local topological trivialization of* $\pi_X : T_X(1) \to X$ *obtained by lifting the vector fields* $\{u_i \equiv E_i\}_{i=1}^l$ *tangent to a stratified* l*-foliation* $\mathcal{H}$ *of* $W = \pi_X^{-1}(U_{x_0})$ *satisfying the smooth Whitney fibering conjecture.*

The following properties hold and are equivalent conditions :

1) The stratified l*-foliation* $\mathcal{H} = \{M_y\}_{y \in W}$ *of* $W = \pi_X^{-1}(U_{x_0})$ *is* (a)*-regular on the neighbourhood* U_{x_0} *of* $x_0 \in X$.

2) The topological trivialization homeomorphism of the projection $\pi_X : T_X(1) \to X$,

$$H \ : \ U_{x_0} \times \pi_X^{-1}(x_0) \to \pi_X^{-1}(U_{x_0})\,, \qquad H(t_1, \ldots, t_l, y_0) = \phi_l(t_l, \ldots, \phi_1(t_1, y_0)..)$$

is horizontally-C^1 *on* U_{x_0}.

3) $\displaystyle\lim_{(t_1,\ldots,t_l,y_0)\to x} H_{*(t_1,\ldots,t_l,y_0)}(E_i) = E_i\,, \qquad \forall\, x \in U_{x_0} \equiv \mathbb{R}^l \times 0^m\,,$ *and* $\forall\, i = 1, \ldots, l\,.$

4) The controlled liftings $w_1, \ldots, w_l$ *tangent to the foliation* $\mathcal{H} = \{M_y\}_{y \in W}$ *of the vector fields* $E_1, \ldots, E_l$ *are continuous on* U_{x_0} *and have horizontally-*C^1 *flows on* U_{x_0}.

5) The controlled lifting ξ *tangent to the foliation* $\mathcal{H} = \{M_y\}_{y \in W}$ *of every vector field* ξ_X *on* X *is continuous over* U_{x_0} *and has a horizontally-*C^1 *flow on* U_{x_0}.

Proof. The equivalence of the properties 1), ..., 5) is proved in Theorem 8 in [**MT**]$_4$.

Now $\mathcal{H}$ satisfies the smooth Whitney fibering conjecture, $\lim_{z \to x} T_x\mathcal{H} = T_xX,\ \forall\, x \in U_{x_0}$ and so is (a)-regular on U_{x_0}. Hence property 1) holds and properties 2), ... 5) hold too. □

We also have the following equivalence with the theory of *E-tame retractions* of du Plessis-Wall [**PW**].

Remark 10. *Let* π' *be the stratified "horizontal" projection :*

$$\pi' \ : \ \pi_X^{-1}(U_{x_0}) \ \longrightarrow \ \pi_X^{-1}(x_0)\,, \qquad \pi'(y) = y_0 = M_{y_0} \cap \pi_X^{-1}(x_0)\,, \quad \forall\, y \in M_{y_0}.$$

The following conditions are equivalent :

1) The foliation $\mathcal{H}$ *is* (a)*-regular on* U_{x_0}.

2) The stratified horizontal projection π' *satisfies the* (a_f) *condition of Thom on* U_{x_0}.

3) The stratified horizontal projection π' *is an* E*-tame retraction.*

Proof. 1) ⇔ 2) is elementary. 2) ⇔ 3) is Proposition 4, of §8, Chapter II [**Mu**]$_1$. □

Corollary 6 also holds for such general morphisms :

Theorem 9. *Let* $f : \mathcal{X} \to \mathcal{X}'$ *be a stratified controlled morphism between two* (c)*-regular spaces* $\mathcal{X}$ *and* $\mathcal{X}'$, X *a stratum of* $\mathcal{X}$, $x_0 \in X$ *and* $x_0' = f(x_0) \in X'$.

Let $\mathcal{H} = \{M_y\}_{y \in W}$ *and* $\mathcal{H}' = \{M_{y'}\}_{y' \in W'}$ *be two stratified* l*-foliations of the neighbourhoods* $W = \pi_X^{-1}(U_{x_0})$ *of* $x_0 \in X$ *in* A *and (resp.)* $W' = \pi_{X'}^{-1}(U'_{x_0'})$ *of* $x_0' \in X'$ *in* A'.

If $\mathcal{H}$ *and* $\mathcal{H}'$ *are* (a)*-regular on* U_{x_0} *and* $U'_{x_0'}$ *and if* $f : \mathcal{X} \to \mathcal{X}'$ *sends each leaf of* $\mathcal{H}$ *into a unique leaf of* $\mathcal{H}'$, *then* f *is horizontally-*C^1 *on* U_{x_0}.

Proof. Theorem 9 in **[MT]**$_4$. □

One deduces a horizontally-C^1 version of the first Thom-Mather Isotopy Theorem.

Theorem 10 (Horizontally-C^1 first Thom-Mather Isotopy Theorem).
Let $\mathcal{X} = (A, \Sigma)$ be a (c)-regular stratification, $X \in \Sigma$ a stratum of $\mathcal{X}$ $x_0 \in X$, U_{x_0} a domain of a chart near x_0 in X, $W = \pi_X^{-1}(U_{x_0})$ and $\mathcal{H} = \{M_y\}_{y \in W}$ a controlled l-foliation (a)-regular on U_{x_0} (which exists by Theorem 7).

Let $f : (A, \Sigma) \to M$ be a stratified proper submersion into a smooth m-manifold M.

For every $m_0 \in M$, and for every domain of a chart $U_{m_0} \equiv \mathbb{R}^m$ of M near m_0, the stratified homeomorphism of the topological trivialization of f :

$$H \;:\; U_{m_0} \times f^{-1}(m_0) \to f^{-1}(U_{m_0}) \;, \quad H(t_1, \ldots, t_m, a_0) = \phi_m(t_m, \ldots \phi_1(t_1, a_0))..)$$

is horizontally-C^1 on $U_{m_0} \times \left[f^{-1}(m_0) \cap U_{x_0}\right]$, and its inverse stratified homeomorphism:

$$G \;:\; f^{-1}(U_{m_0}) \to U_{m_0} \times f^{-1}(m_0) \;, \quad G(a) = \Big(f(a),\; \phi_1(-t_1, \ldots \phi_m(-t_m, a) \ldots)\Big)$$

is horizontally-C^1 on $f^{-1}(U_{m_0}) \cap U_{x_0}$.

Above $f(a) := (t_1, \ldots, t_m)$ and for all $i = 1, \ldots, m$, $\phi_1, \ldots, \phi_m$ are the flows of the continuous controlled lifted vector fields $v_1, \ldots, v_m$, such that $f_*(v_i) = E_i$, on $f^{-1}(U_{m_0})$ of the standard vector fields $E_1, \ldots, E_m \in \mathbb{R}^m \equiv U_{m_0}$.

Proof. Theorem 10 in **[MT]**$_4$. □

Corollary 8. *The topological trivialization K of the projection $\pi_X : T_X(1) \to X$ corresponding to the continuous, controlled, integrable frame field $(w_1, \ldots, w_l)$ tangent to the foliation $\mathcal{H}$ constructed in Theorem 7,*

$$K \;:\quad U \times \Big(\bigsqcup_{X \le Y} \pi_{XY}^{-1}(x_0)\Big) \quad\longrightarrow\quad \pi_X^{-1}(U) \;=\; \bigsqcup_{X \le Y} \pi_{XY}^{-1}(U)$$

is horizontally-C^1 on each stratum of $U \times \big(\sqcup_{X \le Y} \pi_{XY}^{-1}(x_0)\big)$ and its inverse stratified homeomorphism K^{-1} is horizontally-C^1 on each stratum $\pi_{XY}^{-1}(U)$ of the stratification $W = \pi_X^{-1}(U) = \sqcup_{X \le Y} \pi_{XY}^{-1}(U)$.

Proof. It follows by Theorem 10 applied to the projection $\pi_X : T_X(1) \to X$. □

8.2. $\mathcal{F}$-semidifferentiable morphisms and the first Thom-Mather Isotopy Theorem.

In this section we generalize the horizontally-C^1 regularity of section 8.1 through the notion of $\mathcal{F}$-semidiferentiability, a finer regularity condition for stratified morphisms.

We saw in §8.1 that (a)-regularity over a neighbourhood U_{x_0} of x_0 in X, of a foliation $\mathcal{H} = \{M_z\}_{z \in W}$ of $W = \pi_X^{-1}(U_{x_0})$ implies the horizontally-C^1 regularity over U_{x_0} of the stratified flows of continuous lifting of vector fields and of the topological trivialization maps. In a similar way we see here that (a)-regularity of $\mathcal{H}$ on the whole of W implies, for these stratified morphisms, an analogous and more complete regularity :

$$\lim_{z \to y'} f_{Z*z|T_z M_z} = f_{Y*y|T_y M_y} \,.$$

The notion of $\mathcal{F}$-semidifférentiability below refines horizontally-C^1 regularity.

Definition 12. Let $\mathcal{F} = \{F_z\}_z$ be an (a)-regular stratified $C^{1,0}$ l-foliation of an open set U of A, Y a stratum of $\mathcal{X}$ and $y \in Y$.

We say a morphism $f = \{f_Z\}_{Z\in\Sigma} : \mathcal{X} \to \mathcal{X}'$ is *$\mathcal{F}$-semidifférentiable at $y \in Y$* iff for every $\forall (y,v) \in TY$ and sequence $\{(z_n, v_n)\} \subseteq T_{z_n}\mathcal{F}$, with Z_n the stratum containing z_n we have :

$$\lim_n (z_n, v_n) \;=\; (y,v) \qquad \Longrightarrow \qquad \lim_n f_{Z_n * z_n}(v_n) \;=\; f_{Y*y}(v).$$

That is the differentials of $f_{|F_{z_n}}$ must converge to the differential of $f_{|F_y}$.

In an obvious way one defines the $\mathcal{F}$-semidifférentiability on a stratum X (or on $X \cap U$) and on $\mathcal{X}$ (or on U).

Remark 11. *Let $f : \mathcal{X} \to \mathcal{X}'$ be a stratified morphism, X a l-stratum of $\mathcal{X}$, $l = \dim \mathcal{F}$.*

Then $f : \mathcal{X} \to \mathcal{X}'$ is $\mathcal{F}$-semidifférentiable at $x \in X$ iff f is horizontally-C^1 at x with respect to the canonical distribution $\mathcal{D} = \mathcal{D}(z) = T_z\mathcal{F}$. $\square$

With the same hypotheses and notation as in section 8.1, the analogues of the results of 8.1 hold again for $\mathcal{F}$-semidifférentiability.

Remark 12. *Every controlled map $f : \mathcal{X} \to M$ into a manifold is $\mathcal{H}$-semidifférentiable.*

Corollary 9 below improves in a $\mathcal{F}$-semidifferentiable version the previous Corollary 7:

Corollary 9. *Let $\mathcal{X} = (A, \Sigma)$ be a (c)-regular stratification $X \in \Sigma$ a stratum of $\mathcal{X}$, $x_0 \in X$ and $U_{x_0} \equiv \mathbb{R}^l \times 0^m$ a domain of a chart of X near x_0 and $W = \pi_X^{-1}(U_{x_0})$.*

Let H be the local topological trivialization of π_X obtained by lifting the vector fields $\{u_i \equiv E_i\}_{i=1}^l$ tangent to a stratified controlled l-foliation $\mathcal{H}$ of $W = \pi_X^{-1}(U_{x_0})$ satisfying the Whitney fibering conjecture.

The following properties hold and are equivalent conditions :

1) The stratified l-foliation $\mathcal{H} = \{M_y\}_{y\in W}$ of W is (a)-regular on W.

2) The topological trivialization homeomorphism of the projection $\pi_X : T_X \to X$,

$$H \;:\; U_{x_0} \times \pi_X^{-1}(x_0) \to \pi_X^{-1}(U_{x_0}), \qquad y := H(t_1, \dots, t_l, y_0) = \phi_l(t_l, \dots, \phi_1(t_1, y_0)..)$$

is $\mathcal{H}$-semidifferentiable on W.

3) For every $y := H(t_1, \dots, t_l, y_0) \in \pi_{XY}^{-1}(U_{x_0}) \subseteq W$ one has :

$$\lim_{(t'_1, \dots, t'_l, z_0) \to (t_1, \dots, t_l, y_0)} H_{*(t'_1, \dots, t'_l, z_0)}(E_i) = w_i(y), \qquad \forall\,,\ \forall\, i = 1, \dots, l\,.$$

4) The controlled liftings $w_1, \dots, w_l$ tangent to the foliation $\mathcal{H} = \{M_y\}_{y\in W}$ of the vector fields $E_1, \dots, E_l$ are continuous on W and have $\mathcal{H}$-semidifferentiable flows on W.

5) The controlled lifting ξ tangent to $\mathcal{H} = \{M_y\}_{y\in W}$ of every vector field ξ_X on X is continuous on W and has an $\mathcal{H}$-semidifferentiable flow on W.

Proof. Similar to Corollary 7 ; see also Theorem 6 and Theorem 11 in **[MT]**$_4$. $\square$

In the same spirit as Theorem 9 we have :

Theorem 11. *Let $f : \mathcal{X} \to \mathcal{X}'$ be a stratified controlled morphism between two (c)-regular spaces $\mathcal{X}$ and $\mathcal{X}'$, X a stratum of $\mathcal{X}$, $x_0 \in X$ and $x'_0 = f(x_0) \in X'$.*

Let $\mathcal{H} = \{M_y\}_{y\in W}$ and $\mathcal{H}' = \{M_{y'}\}_{y'\in W'}$ be two stratified l-foliations of the neighbourhoods $W = \pi_X^{-1}(U_{x_0})$ of $x_0 \in X$ in A and (resp.) $W' = \pi_{X'}^{-1}(U'_{x'_0})$ of $x'_0 \in X'$ in A'.

If $\mathcal{H}$ and $\mathcal{H}'$ are (a)-regular on W and W' and if $f : \mathcal{X} \to \mathcal{X}'$ sends each leaf of $\mathcal{H}$ into a unique leaf of $\mathcal{H}'$ then f is $\mathcal{H}$-semidifferentiable on W.

Proof. See Theorem 9 and Theorem 12 **[MT]**$_4$. □

Theorem 12 below improves the Horizontally$-C^1$ first Thom-Mather Isotopy Theorem, adding regularity with respect to all strata, in an $\mathcal{F}$-semidifferentiable version :

Theorem 12. ($\mathcal{H}$-semidifferentiable first Thom-Mather Isotopy Theorem).

Let $\mathcal{X} = (A, \Sigma)$ be a (c)-regular stratification, $X \in \Sigma$ a stratum of $\mathcal{X}$, $x_0 \in X$, U_{x_0} a domain of a chart near x_0 in X, $W = \pi_X^{-1}(U_{x_0})$ and $\mathcal{H} = \{M_y\}_{y\in W}$ an l-foliation (a)-regular on W (which exists by Theorem 7).

Let $f : (A, \Sigma) \to M$ be a stratified proper submersion into a smooth m-manifold M.

For every $m_0 \in M$, and for every domain of a chart $U_{m_0} \equiv \mathbb{R}^m$ of M near m_0, the stratified homeomorphism of topological trivialization of f :

$$H \;:\; U_{m_0} \times f^{-1}(m_0) \to f^{-1}(U_{m_0}) \;, \quad H(t_1, \ldots, t_m, a_0) = \phi_m(t_m, \ldots \phi_1(t_1, a_0))..)$$

is $\mathcal{H}$-semidifferentiable on $U_{m_0} \times \big[f^{-1}(m_0) \cap W\big]$, and its inverse stratified homeomorphism:

$$G \;:\; f^{-1}(U_{m_0}) \to U_{m_0} \times f^{-1}(m_0) \;, \quad G(a) = \Big(f(a),\; \phi_1(-t_1, \ldots \phi_m(-t_m, a) \ldots)\Big)$$

is $\mathcal{H}$-semidifferentiable on $f^{-1}(U_{m_0}) \cap W$.

Above $f(a) := (t_1, \ldots, t_m)$ and for all $i = 1, \ldots, m$, $\phi_1, \ldots, \phi_m$ are the flows of the continuous controlled lifted vector fields $v_1, \ldots, v_m$, such that $f_*(v_i) = E_i$, on $f^{-1}(U_{m_0})$ of the standard vector fields $E_1, \ldots, E_m \in \mathbb{R}^m \equiv U_{m_0}$.

Proof. Similar to the proof of Theorem 10 using Theorem 11 instead of Theorem 9. See Theorem 13 **[MT]**$_4$. □

As in Corollary 8, for a (c)-regular stratification $\mathcal{X} = (A, \Sigma)$, $X \in \Sigma$, $x_0 \in X$, we find:

Corollary 10. *The topological trivialization K of the projection $\pi_X : T_X(1) \to X$ corresponding to the continuous, controlled, integrable frame field $(w_1, \ldots, w_l)$ tangent to the controlled foliation $\mathcal{H}$ constructed in Theorem 7,*

$$K \;:\; U \times \Big(\bigsqcup_{X\le Y} \pi_{XY}^{-1}(x_0)\Big) \;\longrightarrow\; \pi_X^{-1}(U) \;=\; \bigsqcup_{X\le Y} \pi_{XY}^{-1}(U)$$

is $\mathcal{F}$-semidifferentiable, with $\mathcal{F} = \{U \times \{z\}\}_{z\in\pi_X^{-1}(x_0)}$, at each point of $U \times \big(\sqcup_{X\le Y} \pi_{XY}^{-1}(x_0)\big)$ and its inverse stratified homeomorphism K^{-1} is $\mathcal{H}$-semidifferentiable at each point of the stratification $W = \pi_X^{-1}(U) = \sqcup_{X\le Y} \pi_{XY}^{-1}(U)$.

Proof. This is an immediate consequence of Theorem 12 applied to the projection $\pi_X : T_X(1) \to X$ of the system of control data of $\mathcal{X}$. □

9. Strong topological stability of smooth maps

In this section we use Theorem 7 to prove a result (Theorem 13) providing sufficient conditions for a smooth map between smooth manifolds to be strongly topologically stable (a notion recalled below).

For some definitions that we do not recall here, the reader can refer to **[PW]**.

Let N, P be smooth manifolds.
In what follows, we use the Whitney C^∞-topology on $C^\infty(N,P)$, and the Whitney C^0-topologies on the spaces of homeomorphisms $Homeo(N)$ and $Homeo(P)$.

Definition 13. Let f, $g : N \to P$ be two smooth maps.
One says that f and g are *topologically equivalent* if there exist homeomorphisms $h : N \to N$, $k : P \to P$ such that $g = k \circ f \circ h$, we write then $f \sim g$. This defines an equivalence relation in $C^\infty(N,P)$.

The smooth map $f : N \to P$ is called *topologically stable* if there is a neighbourhood W of f in $C^\infty(N,P)$ such that all $g \in W$ are topologically equivalent.

One says that $f \in C^\infty(N,P)$ is *strongly topologically stable* if there exists a neighbourhood W of f in $C^\infty(N,P)$ and a continuous map

$$(h,k) : W \longrightarrow Homeo(N) \times Homeo(P) \quad , \quad (h,k)(g) := (h(g), k(g))$$

such that $g = k(g) \circ f \circ h(g)$ for all $g \in W$.

Definition 14. Let $f : N \to P$ be a smooth map.
An *unfolding* of f is a triple $(F; i, j)$ where $F : N' \to P'$ is smooth and $j : P \to P'$ and $i : N \to N'$ are smooth embeddings such that $j \circ f = F \circ i$ with j transverse to F.

In this case one has :

i) since j is transverse to F, the set $(j \times F)^{-1}(\Delta) = \{(y,z) \in P \times N' \mid j(y) = F(x)\}$ is a smooth manifold and $(i,f) : N \to N' \times P$ defines a diffeomorphism of N onto the manifold $\{(x,y) \in N' \times P \mid j(y) = F(x)\}$;

ii) if (r,s) is a pair of smooth retractions for the embeddings (i,j) (i.e. $r \circ i = 1_N$ and $s \circ j = 1_P$), the following diagram is commutative :

$$\begin{array}{ccccc} N & \overset{i}{\hookrightarrow} & N' & \overset{r}{\longrightarrow} & N \\ f \downarrow & & \downarrow F & & \downarrow f \\ P & \overset{j}{\hookrightarrow} & P' & \overset{s}{\longrightarrow} & P \end{array}$$

and we call (r,s) a retraction from F to f and write $(r,s) : F \to f$.

Definition 15. Let $f : N \to P$ be a smooth map. A *(b)(resp. (c))-regular stratification of f* is a pair of (b)(resp. (c))-regular stratifications $(\mathcal{N}, \mathcal{P})$ of (N,P) such that $f : \mathcal{N} \to \mathcal{P}$ is a Thom map.

Definition 16. Let $f : N \to P$ be a smooth map and $\Sigma(f)$ the set of its critical points. One says that f is *quasi-proper* if there exists an open neighbourhood V of $f(\Sigma(f))$ in P such that the restriction $f_{|f^{-1}(V)} : f^{-1}(V) \to V$ is proper.

For more details see **[PW]**, §3.2. It is known in particular (see **[PW]**, 4.3.2 (iii)) that if f is strongly topologically stable, then f is quasi-proper. Now we study the converse.

In theorem 13 below $\mathcal{K}$ is the usual group of contact equivalences, introduced by Mather $[\mathbf{Ma}]_3$.

Theorem 13. *Let $f : N \to P$ be a quasi-proper smooth map of bounded $\mathcal{K}$-codimension.*

For each $y \in f(\Sigma(f))$ write $\Sigma_y = f^{-1}(y) \cap \Sigma(f)$, and $f_y : (N, \Sigma_y) \to (P, y)$ for the germ of f at Σ_y.

Suppose that for each $y \in f(\Sigma(f))$ there is a C^∞-stable unfolding $(F_y; i_y, j_y)$ of f_y which admits a (c)-regular stratification $(\mathcal{S}_y, \mathcal{T}_y)$ such that j_y is transverse to $\mathcal{T}_y$.

Then f is strongly topologically stable.

Before proving Theorem 13, let us see how it can be used to strengthen a theorem of Mather on topological stability.

For $n, p, \ell \in \mathbb{N}$, let $W^\ell(n,p) \subset J^\ell(n,p)$ be the set of ℓ-jets with $\mathcal{K}_e^r$-codimension $\geq \ell$ (see [**PW**] for more details).

For N a smooth n-manifold and P a smooth p-manifold, let $W^\ell(N,P)$ be the corresponding sub-bundle of the jet-bundle $J^\ell(N,P)$.

In $[\mathbf{Ma}]_2$ and $[\mathbf{Ma}]_4$ (see also [**GWPL**]) Mather proved that $J^\ell(N,P) - W^\ell(N,P)$ admits a canonical Whitney-regular stratification $\mathcal{A}^\ell(\mathcal{N},\mathcal{P})$, such that, if $f : N \to P$ is a proper smooth map of finite singularity type whose ℓ-jet avoids $W^\ell(N,P)$ and is multi-transverse to $\mathcal{A}^\ell(\mathcal{N},\mathcal{P})$, then f is topologically stable. *The improvement to strong topological stability remained an outstanding open problem that we are now able to solve.*

Using Theorem 13, we find:

Corollary 11. *Let $f : N \to P$ be a quasi-proper smooth map of finite singularity type whose ℓ-jet avoids $W^\ell(N,P)$ and is multi-transverse to $\mathcal{A}^\ell(\mathcal{N},\mathcal{P})$.*

Then f is strongly topologically stable.

Proof. Let V be an open neighbourhood of $f(\Sigma(f))$ such that $f_{|f^{-1}(V)} : f^{-1}(V) \to V$ is proper. Replacing N by $f^{-1}(V)$ and P by V, we may suppose that f is proper.

Because f is proper and of finite singularity type, f has a proper C^∞-stable unfolding $(F : N' \to P'; i, j)$. Then F has a canonical (b)-regular stratification $(\mathcal{S}, \mathcal{T})$, and, since f is multi-transverse to $\mathcal{A}^\ell(\mathcal{N},\mathcal{P})$ (see [**PW**], p. 5) , j is transverse to $\mathcal{T}$. Since $\mathcal{S}$ and $\mathcal{T}$ are (b)-regular, they are (c)-regular.

For any $y \in \Sigma(f)$ let $\Sigma_y = f^{-1}(y) \cap \Sigma(f)$. Taking germs at Σ_y and y and their images under i and j, respectively, shows that the hypotheses of the theorem are satisfied; so f is strongly topologically stable. $\square$

Corollary 11 has as an immediate consequence the next corollary which improves upon the classical density theorem of Mather ($[\mathbf{Ma}]_4$, [**GWPL**]) for topologically stable maps.

Corollary 12. *The space of strongly topologically stable maps is dense in the space of (quasi-)proper maps between two smooth manifolds.* $\square$

We will need the following definitions of tame and E-tame retractions, and of tame P-C^0-stability, extracted from [**PW**], Chapters 4 and 9.

Definition 17. Let M, N be C^k manifolds and let $i : M \to N$ be a C^k embedding. A retraction $r : N \to M$ for i is said to be *tame* if there exists a neighbourhood U of i in $C^\infty(M,N)$ such that for every $\phi \in U$, $r \circ \phi$ is a homeomorphism.

A retraction $r : N \to M$ for the C^k embedding $i : M \to N$ is said to be *extremely tame or E-tame* if there is a neighbourhood V of $i(M)$ in N such that the fibres of the restriction $r_V : V \to M$ are the leaves of a $C^{0,1}$ foliation and are all transverse to $i(M)$.

This is not the original definition of tame retraction given in **[PW]**, but it is equivalent to that definition by Proposition 9.3.11 of **[PW]**, and suggests a relation with the conclusion of the smooth Whitney fibering conjecture (cf. Remark 10). Note that E-tame retractions are tame.

Definition 18. A C^k map $f : N \to P$ is *tamely P-C^0-stable* (the P here signifies parametrized) if for every unfolding $(g; a, b)$ of f, there exists a tame retraction $(r, s) : g \to f$, i.e. a pair of retractions r, s, for a, b respectively, with s a tame retraction and making the following diagram commute :

$$\begin{array}{ccccc} N & \overset{a}{\hookrightarrow} & N' & \overset{r}{\to} & N \\ f \downarrow & & \downarrow g & & \downarrow f \\ P & \overset{b}{\hookrightarrow} & P' & \overset{s}{\to} & P. \end{array}$$

We recall the following important result in **[PW]**, 9.1.2(i) :

Theorem 14 *Let $f : N \to P$ be a quasi-proper smooth map with multi-germs of bounded $\mathcal{K}$-codimension, and suppose that f is locally tamely P-C^0-stable.*
Then f is strongly topologically stable. $\square$

Thus, to prove Theorem 13 it will be enough to prove Proposition 5 below.

Proposition 5. *Let $S_0 \subset N$ be a finite set, and let $f : (N, S_0) \to (P, y_0)$ be a map-germ of finite singularity type.*
Suppose that there is a C^∞-stable unfolding $(F : (N', S_0') \to (P', y_0'); i, j)$ of f such that F admits a (c)-regular stratification $F : \mathcal{N}' \to \mathcal{P}'$ with j transverse to $\mathcal{P}'$.
Then f is tamely P-C^0-stable.

Proof. We show first that there exists an E-tame (and hence tame) retraction-germ $(R, S) : F \to f$.

With respect to appropriate coordinate systems we can make the identifications

$$(N', S_0') \equiv (N \times \mathbb{R}^l, S_0 \times 0^l) \quad \text{and} \quad (P', y_0') \equiv (P \times \mathbb{R}^l, y_0 \times 0^l)$$

and view F as an $(\mathbb{R}^l, 0)$-level-preserving map-germ

$$F : (N \times \mathbb{R}^l, S_0 \times 0^l) \to (P \times \mathbb{R}^l, y_0 \times 0^l),$$

with i, j germs of the inclusions : $i(x) = (x, 0^l)$ and $j(y) = (y, 0^l)$.

We will construct the retraction $(R, S) : F \to f$ from F to f as in the diagram below:

$$\begin{array}{ccccc} (N, S_0) & \overset{i}{\hookrightarrow} & (N \times \mathbb{R}^l, S_0 \times 0^l) & \overset{R}{- - \to} & (N, S_0) \\ f \downarrow & & \downarrow F & & \downarrow f \\ (P, y_0) & \overset{j}{\hookrightarrow} & (P \times \mathbb{R}^l, y_0 \times 0^l) & \overset{S}{- - \to} & (P, y_0) \end{array}$$

where the retractions R, S will be defined by $(R, S) := (\pi_N \circ H^{-1}, \pi_P \circ K^{-1})$ and where (H, K) are germs of stratified homeomorphisms obtained as stratified trivialization maps and $\pi_N : N \times \mathbb{R}^l \to N$, $\pi_P : P \times \mathbb{R}^l \to P$ the natural projections, as in the following diagram :

$$\begin{array}{ccccc} (N \times \mathbb{R}^l, S_0 \times 0^l) & \overset{H^{-1}}{--\to} & (N \times \mathbb{R}^l, S_0 \times 0^l) & \xrightarrow{\pi_N} & (N, S_0) \\ F \downarrow & (1) & \downarrow f \times 1_{\mathbb{R}^l} & (2) & \downarrow f \\ (P \times \mathbb{R}^l, y_0 \times 0^l) & \overset{K^{-1}}{--\to} & (P \times \mathbb{R}^l, y_0 \times 0^l) & \xrightarrow{\pi_P} & (P, y_0) . \end{array}$$

If Z denotes the stratum of $\mathcal{P}'$ containing $x_0 = (y_0, 0^l) \in P \times \mathbb{R}^l$, since j is transverse to $\mathcal{P}'$ then Z is transverse to $P \times 0^l$ and $\dim Z \geq l$.

Let X be an l-submanifold of Z and $U \equiv \mathbb{R}^l$ a domain of a chart near x_0 of X, transverse to $P \times 0^l$ and having boundary $\partial U \equiv S^{l-1}$.

Then the projection $\pi : P \times \mathbb{R}^l \to \mathbb{R}^l$ restricts to a diffeomorphism-germ $\pi_{|X} : X \to \mathbb{R}^l$.

We refine the stratification $\mathcal{P}'$ of $P \times \mathbb{R}^l$ by replacing Z with the strata $\{Z - U, \partial U, U\}$ and we refine the stratification $\mathcal{N}'$ of $N \times \mathbb{R}^l$ by replacing any stratum S' mapped (necessarily submersively) to Z by F with strata $\{S' \cap F^{-1}(Z - U), F^{-1}(Z - \partial U), S' \cap F^{-1}(U)\}$.

It is easy to see that the stratifications $\mathcal{N}^*$, $\mathcal{P}^*$ so constructed define a (c)-regular stratification of $F : \mathcal{N}^* \to \mathcal{P}^*$, with $P \times 0^l$ transverse to $\mathcal{P}^*$.

Let $(u_1, \ldots, u_l)$ be coordinate frame fields of U, then by Theorem 7 there exists a lifted controlled frame field $(w_1, \ldots, w_l)$ in the open neighbourhood $W := \pi_X^{-1}(U)$ of x_0 in $P \times \mathbb{R}^l$ (where $\pi_X : T_X \to X$ is the projection on X of the system of control data of $\mathcal{P}^*$ which we can take such that $\pi_X^{-1}(x_0) \subseteq P \times 0^l$) whose flows $(\psi_1, \ldots, \psi_l)$ commute and define a trivialisation homeomorphism

$$K \ : \ \mathbb{R}^l \times \pi_X^{-1}(x_0) \to \pi_X^{-1}(U) = W \ , \quad K(t_1, \ldots, t_l, z_0) = \psi_l(t_l, \ldots \psi_1(t_1, z_0))..) =: z$$

such that, denoting $\pi_X(z) \equiv (t_1, \ldots, t_l) \in U$, its inverse stratified homeomorphism is

$$K^{-1} \ : \ \pi_X^{-1}(U) = W \to \mathbb{R}^l \times \pi_X^{-1}(x_0) \ , \quad K^{-1}(z) = \Big(\pi_X(z),\ \psi_1(-t_1, \ldots \psi_l(-t_l, z) \ldots)\Big)$$

and K and K^{-1} are both horizontally-C^1 (Theorem 10 and Corollary 8) and also $\mathcal{F}$-semidifferentiable (Theorem 12 and Corollary 10).

Hence the corresponding l-foliation $\mathcal{F}$ of W is (a)-regular, i.e. it has continuously-varying leaf tangent spaces, and respects the distance-functions of the strata of $\mathcal{P}^*$ (and hence respects the strata themselves) and is such that the leaves of $\mathcal{F}$ are mapped submersively by $\pi_X : W \to U$.

We can suppose that the stratifications $\mathcal{N}^*, \mathcal{P}^*$ admit a system of compatible tubular neighbourhoods with respect to F **[Ma]**$_{1,2}$; this yields integrable lifts $(\eta_1, \ldots, \eta_l)$ over F of $(w_1, \ldots, w_l)$, i.e. $F_*(\eta_i) = w_i$ for every $i = 1, \ldots, l$.

Denoting by $(\chi_1^t, \ldots, \chi_l^t)$ the flows at time t of $(\eta_1, \ldots, \eta_l)$ we have then :

$$F \circ \chi_i^t = \psi_i^t \circ F \quad \text{for every} \quad i = 1, \ldots, l, \quad \text{and} \quad t \in \mathbb{R}.$$

By considering the corresponding map H defined by the flows $(\chi_1^t, \ldots, \chi_l^t)$ it follows that the diagram below commutes :

$$\begin{array}{ccc} \mathbb{R}^l \times F^{-1}\big(\pi_X^{-1}(x_0)\big) \subseteq \mathbb{R}^l \times (N \times 0^l) & \overset{H}{\to} & W' := F^{-1}(W) \subseteq N \times \mathbb{R}^l \\ 1_{\mathbb{R}^l} \times F \downarrow & & \downarrow F \\ \mathbb{R}^l \times \pi_X^{-1}(x_0) \ \subseteq \ \mathbb{R}^l \times (P \times 0^l) & \overset{K}{\to} & W = \pi_X^{-1}(U) \subseteq P \times \mathbb{R}^l . \end{array}$$

I.e. $$F \circ H\,(t_1,\dots,t_l,z_0') = F\big(\chi_l(t_l,\dots\chi_1(t_1,z_0'))..\big) = \psi_l\big(t_l,\dots\psi_1(t_1,F(z_0'))..\big)$$
$$= K\big(t_1,\dots,t_l,F(z_0')\big) = K\circ(1_{\mathbb{R}^l}\times F)\,(t_1,\dots,t_l,z_0')\,.$$

Then, since $\pi_X^{-1}(x_0)\subseteq P\times 0^l$ and $F_{|N\times 0^l}$ acts as $f\times 1_{\mathbb{R}^l}$, one deduces the existence and the commutativity of the above diagram (1) (and obviously (2)).

Hence $(R,S) := (\pi_N\circ H^{-1}, \pi_P\circ K^{-1}) : F\to f$ is a retraction from F to f.

Moreover, since fibres of the retraction $S=\pi_P\circ K^{-1}$ are exactly the leaves of $\mathcal{F}$ then S is E-tame, and using 9.3.11 in **[PW]** $(\pi_N\circ H^{-1},\pi_P\circ K^{-1})$ is E-tame as claimed.

Observe that the retraction $S=\pi_P\circ K^{-1}$ coincides with the 'horizontal" projection of Remark 10 §8

$$\pi' : \pi_X^{-1}(U)\to\pi_X^{-1}(x_0)\,,\qquad \pi'(z) = M_z\cap\pi_X^{-1}(x_0)$$

whose fibres are the leaves of $\mathcal{F}$.

Now let $(g;a,b)$ be any unfolding of f with $g : N\times\mathbb{R}^q\to P\times\mathbb{R}^q$. In order to show that f is tamely P-C^0-stable, we must show that there is a tame retraction $(r,s) : g\to f$.

It will be enough to treat the case when g is C^∞-stable. For if g is not C^∞-stable, it admits a further unfolding $(g_1;a_1,b_1)$ with g_1 C^∞-stable; and if there is a tame retraction $(r_1,s_1) : g_1\to f$, then $(r_1\circ a_1, s_1\circ b_1)$ is a tame retraction : $g\to f$ (see **[PW]**, 9.3.22(ii)).

So we assume that g is C^∞-stable. The unfoldings $(g;a,b)$ and $(F;i,j)$ are then stably smoothly equivalent, i.e. for some $h\geq 0$ there is either :

(i) a smooth equivalence $(\Phi,\Psi) : F\times 1_{\mathbb{R}^h}\simeq g$ of unfoldings of f or

(ii) a smooth equivalence $(\Phi,\Psi) : g\times 1_{\mathbb{R}^h}\simeq F$ of unfoldings of f.

In case (i) there is an obvious smooth retraction of the unfolding $F\times 1_{\mathbb{R}^h}$ of F to g given by the projections $(p_{N\times\mathbb{R}^l}, p_{P\times\mathbb{R}^l})$; composing this with the tame retraction (R,S) gives a tame retraction $F\times 1_{\mathbb{R}^h}\to f$ (using **[PW]**, 9.3.22(i)); and then composing with (Φ,Ψ) gives an E-tame retraction $(r,s) : g\to f$.

In case (ii), composing the equivalence (Φ,Ψ) with the tame retraction (R,S) gives a tame retraction $(R\circ\Phi, S\circ\Psi) : g\times 1_{\mathbb{R}^h}\to f$. Composing with the unfolding inclusions then gives a tame retraction $(r,s) : g\to f$ (using **[PW]**, 9.3.22(ii), again). □

Remark. The more usual way to discuss stability of mappings is via multi-transversality to a submanifold partition of jet-bundles. In the book **[PW]** a central result is **[PW**, 9.1.1], showing that strong C^0-stability follows from transversality to what are called *civilised* submanifolds.

As on **[PW**, p.347], a $J^k\mathcal{K}$-invariant submanifold S of codimension m of the space $J^k(n,p) - W^k(n,p)$ satisfying the immersion condition is *civilised* if, for any isomorphism class of local algebras represented by a $J^k\mathcal{K}$-orbit contained in S, there exists a germ $f : (\mathbb{R}^m,0)\to(\mathbb{R}^{m+p-n},0)$ with this local algebra, a stable unfolding

$$\Big(F : (\mathbb{R}^{m+a},0)\to(\mathbb{R}^{m+p-n+a},0),\ 1_{\mathbb{R}^m}\times\mathbf{0},\ 1_{\mathbb{R}^{m+p-n}}\times\mathbf{0}\Big)\quad\text{of}\quad f,$$

and a V-tame retraction $(r,s) : F\to f$ such that $s^{-1}(0) = F\big((j^kF)^{-1}(S)\big)$ (see **[PW]**).

Arguing as in Proposition 5, the existence of the V-tame retraction required will be assured if there is a (c)-regular stratification of F such that the stratum of the target stratification containing the origin is $F\big((j^kF)^{-1}(S)\big)$.

The approach to civilisation in [**PW**, 9.6.6] is more technical, and relies on (weighted) homogeneity of the germs considered; (c)-regularity, particularly in weighted homogeneous situations, is likely to be much easier to establish. It seems likely, therefore, that the lists of canonical strata in [**PW**] can be considerably extended using (c)-regularity arguments instead. This will be reported on elsewhere.

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