# Modification of Moreau-Jean's Scheme for Energy Conservation in Inelastic Impact Dynamics

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*Summary*. This contribution suggests a modification of Moreau-Jean's scheme to obtain an energy conserving formulation for frictionless vibro-impact problems involving an inelastic impact law. In this approach, the velocities of the non-contacting masses are modified in order to balance the amount of energy dissipated during impact. This correction is shown to be useful to avoid dissipation of energy through the simulation of a two-degree-of-freedom vibro-impact system involving sticking phases. It is found that the modification yields an energy conserving scheme for inelastic impact dynamics. The objective of this modification is to find families of periodic solutions of discrete systems involving sticking phases, which is not possible with the current approach due to the numerical dissipation.

## Vibro-impact system

The system of interest consists in a two degrees-of-freedom (dof) vibro-impact system as depicted in Figure 1. The

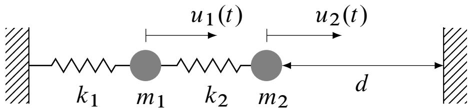


**Figure 1:** Investigated two-degree-of-freedom vibro-impact system

equations describing the dynamics of the system read

$$\mathbf{M}\ddot{\mathbf{u}} + \mathbf{K}\mathbf{u} = \mathbf{r} \tag{1a}$$

$$\mathbf{u}(0) = \mathbf{u}_0, \quad \dot{\mathbf{u}}(0) = \dot{\mathbf{u}}_0 \tag{1b}$$

$$u_2(t) \leq d, \quad R(t) \leq 0, \quad (u_2(t) - d)R(t) = 0, \quad \forall t \tag{1c}$$

where

$$\mathbf{M} = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \;;\quad \mathbf{K} = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \;;\quad \mathbf{u}(t) = \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} \;;\quad \mathbf{r}(t) = \begin{pmatrix} 0 \\ R(t) \end{pmatrix} \tag{2}$$

The corresponding masses are $m_1$ and $m_2$ and the stiffness of the springs are $k_1$ and $k_2$. Quantities $u_j$, $\dot{u}_j$ and $\ddot{u}_j$ respectively represent the displacement, velocity and acceleration of mass $j = 1, 2$. The gap $d$ separates the obstacle and the second mass at rest. Note that the first mass is not subjected to any unilateral contact condition. The reaction force of the wall on mass 2 is defined by $R(t)$. To ensure well-posedness of the problem, information about the impact must be incorporated. In this work, the information is added via a Newton impact law which introduces a restitution of coefficient $0 \leq e \leq 1$ and reads $\dot{u}_2(t^+) = -e\dot{u}_2(t^-)$ if $u_2(t) = d$ and $\dot{u}_2(t^-) \geq 0$. For a purely inelastic impact law with $e = 0$, as selected in this work, the system is expected to have "sticking" phases where the mass in contact could remain stuck on the rigid obstacle. The problem is then numerically integrated in time via Moreau-Jean's scheme. Since an inelastic impact law introduces dissipation of energy, it is proposed to modify the numerical scheme in order to enforce total energy conservation which is crucial for the computation of periodic solutions [3].

## Moreau-Jean's scheme

Moreau-Jean's scheme is based on Moreau's sweeping process [2, 4]. Numerical time integration is performed on an interval $(t^i, t^{i+1}]$ of length $h$ as follows

$$\mathbf{M}\big(\dot{\mathbf{u}}^{i+1} - \dot{\mathbf{u}}^i\big) + h\mathbf{K}\mathbf{u}^{i+\theta} = \mathbf{r}^{i+1}, \tag{3a}$$

$$\mathbf{u}^{i+1} = \mathbf{u}^i + h\dot{\mathbf{u}}^{i+\theta}, \tag{3b}$$

$$\text{if } u_2^i \geq d, \quad 0 \leq \dot{u}_2^{i+1} + e\dot{u}_2^i \perp R^{i+1} \leq 0 \tag{3c}$$

with $\theta \in [0, 1]$. Vectors $\mathbf{u}^i$ and $\dot{\mathbf{u}}^i$ store quantities $u_n^i \approx u_n(t^i)$ and $\dot{u}_n^i \approx \dot{u}_n(t^i)$ which are respectively the discretized displacement and velocity of mass $n$ at time step $t^i$ for $n = 1, 2$. The energy conservation and numerical dissipation properties of this approach are reported in [1]. The most conservative scheme is obtained for $\theta = 1/2$ which is considered in the remainder. Accordingly, the total energy balance for the unforced and non-damped case writes

$$\Delta\mathcal{E} = \mathcal{E}^{i+1} - \mathcal{E}^i = \dot{u}_2^{i+1/2} R^{i+1} \tag{4}$$

This equation states that the numerical dissipation of the total energy is due to the impulse generated by the contact condition on the second mass. This dissipated energy governs a modification in the velocity $\dot{u}_1^{i+1}$ of the first mass during contact closure in the numerical time integration in order to enforce conservation of energy. Note that if the amount of dissipated energy is known, such modification can be applied with any $\theta$.

## Modification of the velocity to achieve energy conservation

Unavoidable energy dissipation during inelastic contact is circumvented by adding a scalar correction term $\beta^{i+1}$ to the velocity of the non-contacting masses when the contacting mass closes contact. The vector $\mathbf{I} = [1 \;\; 0]^\top$ defines which velocities will be corrected during time integration; for this case, only the first mass is corrected. The amount of correction in velocity is not known a priori and should be calculated for each time step when the contact is activated. The corrected conservation of energy is written as follows

$$\hat{\mathcal{E}}^{i+1} - \mathcal{E}^{i} = 0 \tag{5}$$

where $\hat{\mathcal{E}}^{i+1}$ is the total energy accounting for the corrected velocity. Invoking Equations (4) and (5), the correction in velocity can be calculated from the condition $\hat{\mathcal{E}}^{i+1} - \mathcal{E}^{i+1} + \dot{u}_2^{i+1/2} R^{i+1} = 0$ and the corrected total energy is

$$2\hat{\mathcal{E}}^{i+1} = \left(\dot{\mathbf{u}}^{i+1} + \mathbf{I}\beta^{i+1}\right)^\top \mathbf{M}\left(\dot{\mathbf{u}}^{i+1} + \mathbf{I}\beta^{i+1}\right) + \left(\mathbf{u}^{i+1}\right)^\top \mathbf{K}\mathbf{u}^{i+1} \tag{6}$$

with $2\mathbf{u}^{i+1} = 2\mathbf{u}^{i} + h\left(\dot{\mathbf{u}}^{i+1} + \mathbf{I}\beta^{i+1} + \dot{\mathbf{u}}^{i}\right)$. After basic manipulations, the quadratic equation to be solved for the correction reads

$$\frac{1}{2}\left(\mathbf{I}\beta^{i+1}\right)^\top \mathbf{M}\left(\mathbf{I}\beta^{i+1}\right) + \left(\mathbf{I}\beta^{i+1}\right)^\top \mathbf{M}\dot{\mathbf{u}}^{i+1} + \frac{h^2}{8}\left(\mathbf{I}\beta^{i+1}\right)^\top \mathbf{K}\left(\mathbf{I}\beta^{i+1}\right) + \frac{h}{2}\left(\mathbf{I}\beta^{i+1}\right)^\top \mathbf{K}\mathbf{u}^{i+1} + \dot{u}_2^{i+1/2} R^{i+1} = 0 \tag{7}$$

It has two solutions. It has been numerically observed that when the amount of correction is chosen as $\min|\beta^{i+1}|$ the system does not present infinite sticking phases nor a behavior involving purely elastic impacts.

## Vibro-impact responses with sticking contact phases

For the illustration of the capability of the proposed modification, consider a system with the following mechanical properties: $k_1 = k_2 = 1$ and $m_1 = m_2 = 1$. The initial conditions are $\mathbf{u}_0 = (0 \;\; 0)^\top$ and $\dot{\mathbf{u}}_0 = (10 \;\; 10)^\top$. The gap is $d = 1$. The results are depicted for both the standard and the modified Moreau-Jean's schemes using $h = 1/500$. Convergence has been observed in several numerical experiments: this is not included for the sake of brevity.

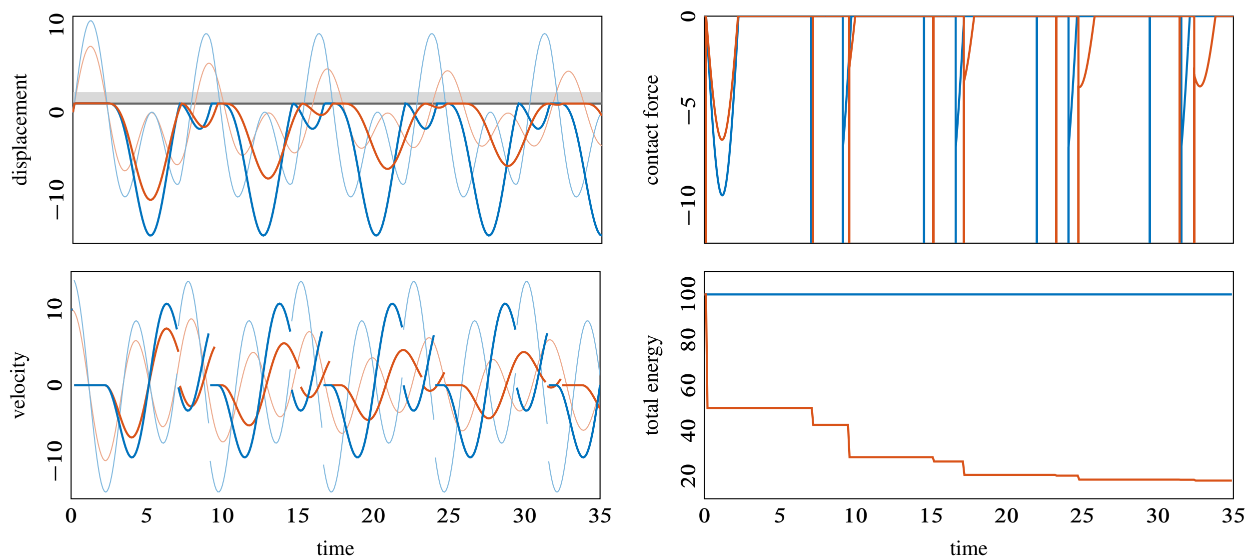


**Figure 2:** Vibro-impact dynamics with $e = 0$. Blue color corresponds to modified scheme and red color corresponds to standard scheme. For displacement and velocity plots: light colors depict mass 1 and dark colors depict mass 2.

The inclusion of the correction in velocity yields an energy conserving framework. Within the time integration, the result converges to a periodic solution involving sticking phases after certain impacts with the obstacle. Systems with more than two dof can be integrated as well. With this approach, continuous families of periodic orbits with sticking phases might be exhibited, but it requires further studies involving shooting methods, modal analysis and continuation techniques.

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