A probabilistic rate theory connecting kinetics to thermodynamics

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Abstract.

Kinetics and thermodynamics are largely disconnected in current theories because Arrhenius activation energies (E_a) have strictly no influence on equilibrium distributions. A first step towards the incorporation of rate theories in thermodynamics is the identification of the pre-exponential term of the Arrhenius equation as an entropic quantity. The second step examined here is the possible contribution of E_a in equilibrium landscapes. Interestingly, this possibility exists if envisioning the energetic exponential term of rates constants as the probability that the energy of the reactant is sufficient for the transition. This radically new approach which encompasses Maxwell-Boltzmann distributions as particular cases, solves inconsistencies in previous theories, for instance on the role of temperature in kinetics and thermodynamics. These probabilistic rate constants are then reintroduced in dynamic systems to provide them with the two distinct facets of time: the time step and the time arrow.

Keywords: Kinetics; equilibrium; heat capacity; enthalpy; entropy; Gibbs free energy.

Notes to readers: The unitless statistical entropy usually written H will be renamed here $\mathcal{S}$ to avoid confusion with enthalpy. The letter C will be used for heat capacity but not for concentration.

1 Introduction

Rate constants are the drivers of all dynamic systems and are widely used in physical chemistry and modeling studies. By the energetic restrictions they impose on chemical transitions, rate constants prevent our organized world from falling directly into its state of weaker free energy and maximal entropy. Yet the profound nature of these constants remains obscure and the main focus of this study is to clarify their energetic component. After multiple attempts of description, its most common expression remains the exponential factor of Arrhenius [1, 2]. In its classical version (without tunneling effects),

the empirical Arrhenius equation

$$k = A\, \mathrm{e}^{-\frac{E_a}{k_B T}} \tag{1}$$

where k_B is the Boltzmann constant and T is the temperature, includes (i) a so-called preexponential factor A, containing the elementary frequency and configurational restrictions, and (ii) a unitless exponential function containing an energetic barrier called activation energy E_a. In the present paper, the exponential factor will be reformulated as the probability to reach a threshold energy, as initially proposed but incompletely appraised in [3]. This simple hypothesis will lead to major changes in thermodynamic relationships.
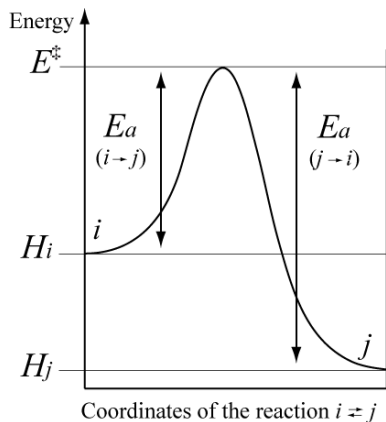
Figure 1. Representation of the energetic barrier according to the Arrhenius principle. The barrier is supposed to restrict reaction kinetics but to have no influence on equilibrium ratios. $E^{\ddagger}$ is a fixed energy threshold necessary for crossing the barrier in both directions and H are the mean enthalpies of reaction of the particles.

Using the notations of Fig.1, the Arrhenius approach reads

$$k_{ij} = A_{ij}\, \mathrm{e}^{-(E^{\ddagger}-H_i)/k_B T} \tag{2a}$$

and for the reverse reaction

$$k_{ji} = A_{ji}\, \mathrm{e}^{-(E^{\ddagger}-H_j)/k_B T} \tag{2b}$$

Note that the activation energy E_a of Eq.(1) corresponds to the difference $E^\ddagger - H$, related to $\Delta G^\ddagger$ in the activated complex theory of Eyring [4]. In equilibrium the forward and reverse fluxes equalize such that

$$n_i\, k_{ij} = n_j\, k_{ji} \qquad (2c)$$

giving

$$\left(\frac{n_i}{n_j}\right)_{eq} = \frac{k_{ji}}{k_{ij}} = K_{ji} = \frac{A_{ji}}{A_{ij}}\, \mathrm{e}^{(E^\ddagger - H_i - E^\ddagger + H_j)/k_B T}$$
$$= \frac{A_{ji}}{A_{ij}}\, \mathrm{e}^{-(H_i - H_j)/k_B T} = \frac{A_{ji}}{A_{ij}}\, \mathrm{e}^{-\Delta H/k_B T} \qquad (2d)$$

$E^\ddagger$ disappears. The equilibrium ratio is independent of the energetic barrier and depends only on the difference of energy between the reactants. These currently admitted relationships can be naively questioned.

2 Some concerns and alternative proposal

2.1 On the universality of $k_B T$ as a mean energy

$1/k_B T$ is introduced in statistical mechanics as a constant called β corresponding to a Lagrangian multiplier or a distribution parameter, and finally connected to temperature by analogy with classical thermodynamics. $k_B T$ is concretely useful here to adimension the exponent of the Arrhenius exponential and is usually interpreted as the mean particle energy. This interpretation holds in the kinetic theory of ideal gases but is much less clear for complex chemical systems in which the mean reactional energies are precisely different for each type of particle.

2.2 Insensitivity of dynamic equilibria to activation energies

The accepted conclusion that the presence and the height of energetic barriers do not participate to equilibrium ratios between the reactants, can appear not completely intuitive because one could expect that increasing the height of the barrier exponentially amplifies the differential capacity of the reactants to jump over, including in equilibrium. In potential energy landscapes, the montains regulate the populations but their crests are uninhabited unstable states. By contrast in the transition state theory, the activated complexes reside on the crests but their energy levels do not influence the distributions.

2.3 A non-probabilistic exponential law

The exponential term of the Arrhenius law has certain apparences of a probability, but as explained in the following section, as it stands it can not correspond to the exponential distribution. The purpose of the present study is precisely to examine the theoretical consequences of the hypothesis that it is a probability of sufficient energy for the transition. This simple hypothesis detailed later leads to the novel formulas

$$P(\text{sufficient energy}) = \mathrm{e}^{-\frac{E^\ddagger}{H}} = \mathrm{e}^{-\left(1+\frac{E_a}{H}\right)} \qquad (3a)$$

which gives the equilibrium relationship

$$\ln\left(\frac{n_i}{n_j}\right)_{eq} = \ln\frac{A_{ji}}{A_{ij}} + \left(1 - \frac{H_i}{H_j}\right)\frac{E^\ddagger}{H_i}$$
$$= \ln\frac{A_{ji}}{A_{ij}} + \left(1 - \frac{H_i}{H_j}\right)\left(1 + \frac{E_a}{H_i}\right) \qquad (3b)$$

whose structure is completely different from the usual result

$$\ln\left(\frac{n_i}{n_j}\right)_{eq} = \ln\frac{A_{ji}}{A_{ij}} + \frac{H_j - H_i}{k_B T} = \frac{G_j - G_i}{k_B T} \qquad (4)$$

In Eq.(4), the pre-exponential terms A disappear from the latter form containing Gibbs free energies G. This modification results from the identification of the pre-exponential factor as an entropic parameter, which can then be integrated in the Gibbs free energies. This pertinent operation will also be used here and applied to Eq.(3b), but let us first explain the entropic meaning of the pre-exponential factor. This factor can be defined as $A = \tau^{-1}\Omega^{-1}$ where τ^{-1} is the universal thermal frequency (discussed later), giving their time^{-1} unit to the rate constants, and Ω^{-1} is the reciprocal of the number of possible configurations of the reactant, one of which allowing the reaction to proceed, as simply illustrated in Fig.2. As a consequence, $\ln(A_{ji}/A_{ij})$ can be identified, after elimination of the identical frequencies, as the difference of statistical entropies

$$\ln\frac{A_{ji}}{A_{ij}} = \ln\frac{\Omega_{ij}}{\Omega_{ji}} = \mathcal{S}_{ij} - \mathcal{S}_{ji}$$

Note the inversion of the suffixes between the ratios of A and Ω. In addition, if a given reactant is involved in several reactions, one can assume that it is not necessary to precise the nature of the product in reactional entropy, because it is expected to be the same for all the reactions starting from this reactant. Indeed, even if each reaction requires a specific reactional configuration, this configuration belongs the same set of total configurations Ω of this reactant. Converting configurational into thermodynamic entropy through the Boltzmann formula $\mathcal{S} = S/k_B$, allows to detail Eq.(4) step-by-step

$$\ln\left(\frac{n_i}{n_j}\right)_{eq} = \ln\frac{A_j}{A_i} + \frac{H_j - H_i}{k_B T}$$

$$= \ln\frac{\Omega_i}{\Omega_j} + \frac{H_j - H_i}{k_B T}$$

$$= \frac{S_i - S_j}{k_B} + \frac{H_j - H_i}{k_B T}$$

$$= \frac{H_j - H_i - T(S_j - S_i)}{k_B T}$$

$$= \frac{(H_j - TS_j) - (H_i - TS_i)}{k_B T}$$

$$= \frac{G_j - G_i}{k_B T}$$

Such a fusion of entropy and enthalpy into free energy will be retained here for whole systems but not for individual reactants, because molecular entropy and enthalpy play different roles in reactions, which are described by different probabilities (Fig.2).
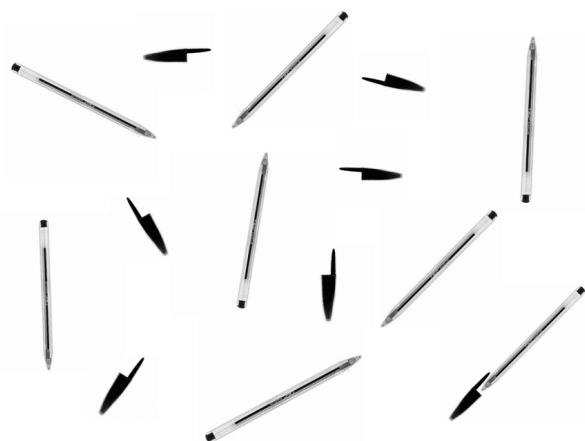

Figure 2. Randomly moving pens and caps simply illustrate the concept of reactional entropy. The set of their equiprobable relative arrangements (which would be much larger in 3D) is noted Ω, out of which only one, of probability $1/\Omega$, allows the assembly of the caps and the pens, as shown at the bottom right. But even when conveniently positioned, the pen and the cap can stably associate only if the energy of their collision is sufficient. This second condition is quantified from 0 to 1 in the energetic term of rate constants defined here. The configurations illustrated in this figure with separate components, can also be conceived for internal domains of the same macromolecule.

Eq.(3b) contrasts with Eq.(4) in that it introduces a role for activation energies in equilibrium distributions. Large E_a do not inverse but accentuate the differences determined in both cases by the relative energy levels. This point results from a mathematical interpretation of the exponential factor.

3 A new probabilistic interpretation of the exponential factor

3.1 The exponential distribution

The exponential component of the Arrhenius law and of the Boltzmann distribution, can be simultaneously derived from the exponential distribution, which is basically mathematical. According to this law, the probability that a particle from an homogeneous population with an average number of energy quanta per particle $\langle\mathcal{E}\rangle$, reaches a fixed threshold number $\mathcal{E}^{\ddagger}$, is

$$P(\mathcal{E} \geqslant \mathcal{E}^{\ddagger}) = e^{-\frac{\mathcal{E}^{\ddagger}}{\langle\mathcal{E}\rangle}} \qquad (5)$$

Since $\mathcal{E}^{\ddagger}$ is a temperature-independent threshold and $\langle\mathcal{E}\rangle$ is temperature dependent, temperature remains at the denominator of the exponent, but an important difference with current rate equations is the absence of $k_B T$. $k_B T$ is generally considered as the mean single particle energy, acceptable as a mean kinetic energy for purely kinetic gases, but not for all heterogeneous systems containing different kinds of energies. An other problem even more critical with the Arrhenius equation of Eq.(1), is that in absence of energetic barrier ($E_a = 0$), the exponential is equal to 1, suggesting that reactions are no longer energetically restricted but only affected by the pre-exponential factor. In other words, the mean particle energy $k_B T$ would be fully sufficient for all the transitions. This result is clearly inconsistent with the view of $k_B T$ as a mean energy because of a basic principle: the average particle energy in a homogeneous population is no way reached by all the particles. Indeed, even when $E_a = 0$, that is to say when the barrier corresponds to the mean particle energy, not all the particles have a sufficient energy. In the new interpretation, $E_a = 0$ (equivalent to $E^{\ddagger} = H$ in Fig.1 and to $\mathcal{E}^{\ddagger} = \langle\mathcal{E}\rangle$) predicts a probability of $1/e$ (0.37).

3.2 The rule for particle interconversion

The present approach also fundamentally differs from the classical one in equilibrium. The relative populations of interconvertible species are determined not only by the difference of their mean energies, but also by the barriers. Interconversion between particles of different nature i and j is possible when either a particle i or a particle j, has accumulated a minimal number of energy quanta, denoted below by $\mathcal{E}^{\ddagger}_{i,j}$, which is the same for the conversions $i \rightarrow j$ and $j \rightarrow i$. For a particle of type i, the probability of this accumulation is simply given by the exponential distribution $P(X_i \geqslant \mathcal{E}^{\ddagger}_{i,j}) = e^{-\mathcal{E}^{\ddagger}_{i,j}/\langle\mathcal{E}_i\rangle}$ [3]. As the same holds for the particles of type j, when $\langle\mathcal{E}_i\rangle > \langle\mathcal{E}_j\rangle$, the particles of type i would switch more

probably than the reverse. As a consequence, the system spontaneously evolves until the reciprocal fluxes equalize. The numbers of particles in the sub-populations i and j necessarily adjust such that

$$n_i \, A_{ij} \, \mathrm{e}^{-\mathcal{E}_{i,j}^{\ddagger}/\langle\mathcal{E}_i\rangle} = n_j \, A_{ji} \, \mathrm{e}^{-\mathcal{E}_{i,j}^{\ddagger}/\langle\mathcal{E}_j\rangle} \qquad (6a)$$

or

$$\left(\frac{n_i}{n_j}\right)_{eq} = \frac{A_{ji}}{A_{ij}} \mathrm{e}^{\mathcal{E}_{i,j}^{\ddagger}\left(\frac{1}{\langle\mathcal{E}_i\rangle} - \frac{1}{\langle\mathcal{E}_j\rangle}\right)} \qquad (6b)$$

Returning to the correspondence between the unitless pre-exponential term and entropy, this equation takes the elegant form

$$\ln\left(\frac{n_i}{n_j}\right)_{eq} = \overbrace{\mathcal{S}_i - \mathcal{S}_j}^{\text{Entropic term}} + \overbrace{\mathcal{E}_{i,j}^{\ddagger}\left(\frac{1}{\langle\mathcal{E}_i\rangle} - \frac{1}{\langle\mathcal{E}_j\rangle}\right)}^{\text{Energetic term}} \qquad (7)$$

Eq.(7) can also be written with the activation energy $\mathcal{E}_{a(i\to j)} = \mathcal{E}_{i,j}^{\ddagger} - \langle\mathcal{E}_i\rangle$ as follows

$$\begin{aligned}
\ln\left(\frac{n_i}{n_j}\right)_{eq} &= \mathcal{S}_i - \mathcal{S}_j + \left(1 - \frac{\langle\mathcal{E}_i\rangle}{\langle\mathcal{E}_j\rangle}\right)\frac{\mathcal{E}_{i,j}^{\ddagger}}{\langle\mathcal{E}_i\rangle} \\
&= \mathcal{S}_i - \mathcal{S}_j + \left(1 - \frac{\langle\mathcal{E}_i\rangle}{\langle\mathcal{E}_j\rangle}\right)\left(1 + \frac{\mathcal{E}_a}{\langle\mathcal{E}_i\rangle}\right)
\end{aligned} \qquad (8)$$

4 Reconnection of the energetic parameter of rate constants to Boltzmann's energy distribution

4.1 Starting from the geometric distribution

One of the main goals of present proposal is to unify kinetic and thermodynamic rules under the same law, which can be derived from a simple discrete treatment. Mix E white balls and N black balls, E and N being positive integers, into a large bag and draw them at random. The number of balls E between two balls N, written $\mathcal{E}$, is supposed to correspond to the number of energy quanta in a particle, whose mean number is $\langle\mathcal{E}\rangle = E/N$. Simple probabilities say

$$P(\mathcal{E} \geqslant \mathcal{E}^{\ddagger}) = \left(\frac{E}{E + N}\right)^{\mathcal{E}^{\ddagger}} = \left(\frac{\langle\mathcal{E}\rangle}{1 + \langle\mathcal{E}\rangle}\right)^{\mathcal{E}^{\ddagger}} \qquad (9a)$$

For $\mathcal{E}^{\ddagger}$ and $\langle\mathcal{E}\rangle$ large and of the same order of magnitude, this discrete geometric distribution can be approximated as the continuous exponential distribution.

$$P(\mathcal{E} \geqslant \mathcal{E}^{\ddagger}) = \mathrm{e}^{-\frac{\mathcal{E}^{\ddagger}}{\langle\mathcal{E}\rangle}} \qquad (9b)$$

If we no longer reason in term of threshold but of exact value, we have

$$\begin{aligned}
P(\mathcal{E} = \mathcal{E}^{\ddagger}) &= \left(\frac{E}{E + N}\right)^{\mathcal{E}^{\ddagger}}\left(\frac{N}{E + N}\right) \\
&= \left(\frac{\langle\mathcal{E}\rangle}{1 + \langle\mathcal{E}\rangle}\right)^{\mathcal{E}^{\ddagger}}\frac{1}{1 + \langle\mathcal{E}\rangle}
\end{aligned} \qquad (10a)$$

Switching once again to the continuous distribution, in the thermodynamic limit

$$\begin{aligned}
P(\mathcal{E} = \mathcal{E}^{\ddagger}) &\sim \mathrm{e}^{-\frac{\mathcal{E}^{\ddagger}}{\langle\mathcal{E}\rangle}}\left(1 - \mathrm{e}^{-\frac{1}{\langle\mathcal{E}\rangle}}\right) \\
&= \frac{\mathrm{e}^{-\mathcal{E}^{\ddagger}/\langle\mathcal{E}\rangle}}{\sum_{j=0}^{\infty}\mathrm{e}^{-j/\langle\mathcal{E}\rangle}}
\end{aligned} \qquad (10b)$$

which is, when multiplying the dimensionless energies by the energy unit $\mathfrak{q}$ and when $\langle\mathcal{E}\rangle\,\mathfrak{q} = k_B T$, the Boltzmann probability law where the denominator can be understood as the sum of all possible microstates, called the partition function. It was considered by Feynman as the summit of statistical mechanics [5], but is recovered in two lines from the exponential distribution. As a consequence, the relative subpopulations of energies $\mathcal{E}_1$ and $\mathcal{E}_2$ are given by the ratio of two Eq.(10b)

$$\frac{P_{(\mathcal{E}=\mathcal{E}_1)}}{P_{(\mathcal{E}=\mathcal{E}_2)}} = \mathrm{e}^{\frac{\mathcal{E}_2 - \mathcal{E}_1}{\langle\mathcal{E}\rangle}} \qquad (10c)$$

Note that the approximations used above are based on the following property of the exponential function

$$\lim_{\varepsilon\to 0}(1 + \varepsilon\,x)^{1/\varepsilon} = \mathrm{e}^x$$

which can be written

$$\lim_{q\to 1}(1 + (1 - q)\,x)^{1/(1-q)} = \mathrm{e}^x$$

that is the q-exponential on which rate theories are also built [6, 7].

4

4.2 Simplified Maxwell-Boltzmann distribution

When $\langle \mathcal{E} \rangle$ corresponds to $k_B T$, with energy units, the structure of Eq.(10b) corresponds exactly to that of the celebrated Boltzmann's energy distribution, valid for example for ideal gases. The ideal monoatomic gas is the perfect example of a system with a maximal entropy but whose individual components have an energy stritly proportional to temperature. Inversely, most real systems have a certain degree of organization (non-maximal entropy) but are made of molecular components with a certain degree of entropy, which will be examined later. Let us first deepen the analogies between the exponential probability distribution and Maxwell-Boltzmann results. The probability density function (PDF) of the exponential distribution of particle energies is

$$f(\mathcal{E}) = \frac{1}{\langle \mathcal{E} \rangle} \, e^{-\frac{\mathcal{E}}{\langle \mathcal{E} \rangle}}$$
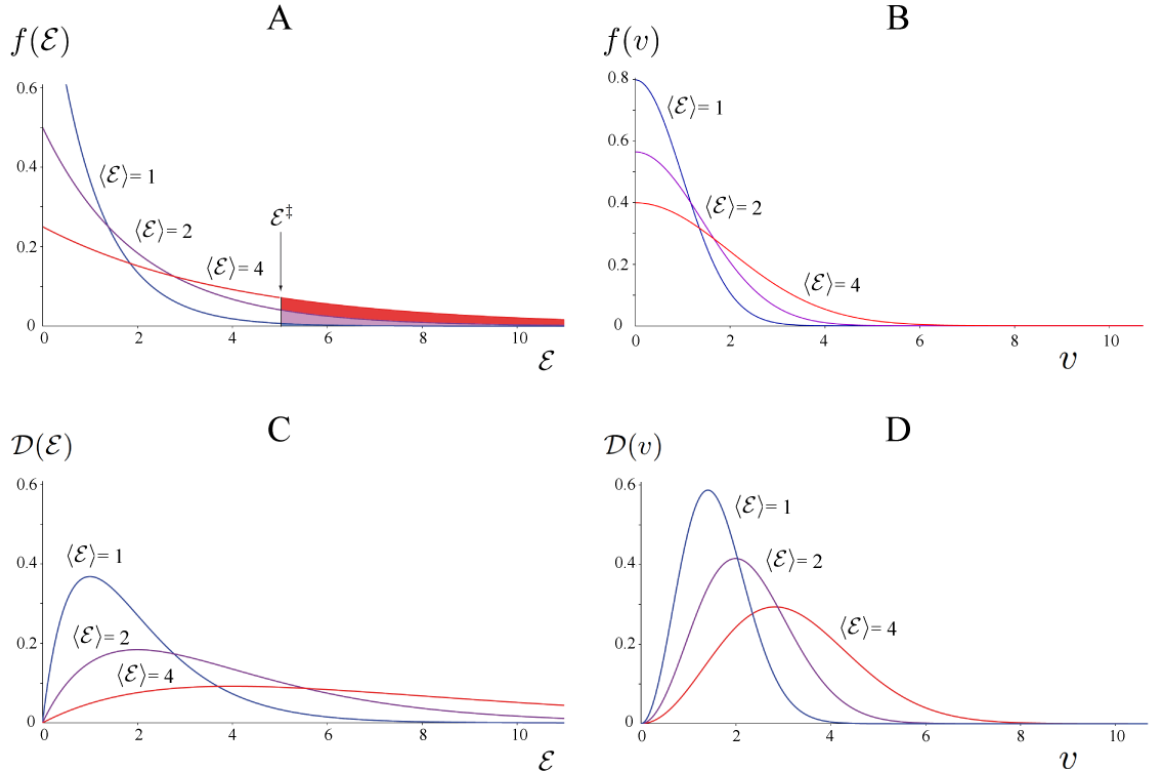
represented in Fig.3A.

Figure 3. Fractional distributions of: (**A**) particle energetic states, (**B**) particle velocities, (**C**) energy density, and (**D**) velocity density. Curves drawn with arbitrary units and a mass of 1 unit. The fractions of particles with energy higher than $\mathcal{E}^{\ddagger}$ correspond to the colored surfaces in panel A, quantified in Eq.(15a). They increase with $\langle \mathcal{E} \rangle$, itself increasing with temperature. This result meets the traditional interpretation of the role of temperature on rate constants, which supposes that particle energies increase with temperature, contrary to Gibbs free energies. This scheme does not hold for the activated complex theory in which the threshold energy is not fixed, but is itself temperature-dependent.

Given that the nth moment of this distribution is $E[X^n] = \langle \mathcal{E} \rangle^n \, n!$, its typical fluctuation is simply

$$\sigma_{\mathcal{E}} = \sqrt{E[X^2] - E[X]^2} = \langle \mathcal{E} \rangle$$

and a variance of $\langle \mathcal{E} \rangle^2$ is acceptable without breaking equilibrium. For a monoatomic ideal gas, the energy content of a particle is exclusively translational. Assuming an implicit energy unit $\mathfrak{q}$ (in joules) such that $\langle \tilde{\mathcal{E}} \rangle = \langle \mathcal{E} \rangle \mathfrak{q} = k_B T$, a particle of energy $\tilde{\mathcal{E}}_i$ has a velocity v_i such that $\tilde{\mathcal{E}}_i = \frac{1}{2} m v_i^2$, giving the distribution of particle velocities, after normalization,

$$f(v) = \sqrt{\frac{2m}{\pi \langle \tilde{\mathcal{E}} \rangle}} \, e^{-\frac{mv^2}{2\langle \tilde{\mathcal{E}} \rangle}}$$

represented in Fig.3B and which has a mean value of $\bar{v} = \int_0^{\infty} v f(v) dv = \sqrt{2 \langle \tilde{\mathcal{E}} \rangle / \pi m}$. The particles which have an energy content $\mathcal{E}_i$ constitute a fraction of the system n_i/N and bring together in the system a fraction of energy $f(\mathcal{E}_i) \, \mathcal{E}_i$. The most represented particles are, according to the exponential as well as Boltzmann distri-

butions, the low-energy and low-mobility particles, which poorly contribute to the total energy of the system, thus generating a peak in the energy density curve at the level of the largest product $\mathcal{E}_i\ f(\mathcal{E}_i)$. The normalized expression of this curve is the density of the Γ_2 function (shape shown in Fig.3C).

$$\mathcal{D}(\mathcal{E}) = \gamma_2(\mathcal{E}) = \frac{\mathcal{E}}{\langle\mathcal{E}\rangle^2}\ \mathrm{e}^{-\frac{\mathcal{E}}{\langle\mathcal{E}\rangle}} \qquad (11)$$

This function can also be understood as the convolution of the densities of two independent exponential distributions of identical parameter, corresponding to the bottom and top boundaries of the energetic content. For exclusively translational particles, when substituting $\tilde{\mathcal{E}}_i = \frac{1}{2}mv_i^2$,

$$\mathcal{E}_i\ f(\mathcal{E}_i) = \frac{mv_i^2}{2\langle\tilde{\mathcal{E}}\rangle}\ \mathrm{e}^{-\frac{mv_i^2}{2\langle\tilde{\mathcal{E}}\rangle}}$$

whose integration between 0 and ∞ gives $\sqrt{\pi\langle\tilde{\mathcal{E}}\rangle/8m}$. Multiplying the latter equation by the inverse of this value directly yields the normalized velocity density function

$$\mathcal{D}(v) = \left(\frac{2}{\pi}\right)^{1/2}\left(\frac{m}{\langle\tilde{\mathcal{E}}\rangle}\right)^{3/2}v^2\ \mathrm{e}^{-\frac{mv^2}{2\langle\tilde{\mathcal{E}}\rangle}} \qquad (12)$$

which gives the bell-shaped curves shown in Fig.3D, with a mode at

$$v_{mp} = \sqrt{\frac{2\langle\tilde{\mathcal{E}}\rangle}{m}}$$

Therefore, we verify that the most probable values of energy and velocity densities (tips of the curves of Fig.3C and 3D respectively), naturally coincide through $\tilde{\mathcal{E}}_{mp} = \frac{1}{2}mv_{mp}^2$. The mean velocity density $\int_0^\infty v\mathcal{D}(v)dv$ is

$$\langle v\rangle = \sqrt{\frac{8\langle\tilde{\mathcal{E}}\rangle}{\pi m}}$$

When substituting $\langle\tilde{\mathcal{E}}\rangle = k_BT$, Eq.(12) is exactly the celebrated Maxwell velocity distribution, currently introduced in different and longer ways. Let us now consider more elaborate systems with different types of particles and energies.

5   Comparative influences of temperature in the different treatments

5.1   Influence of temperature on rate constants

If the mean reactional energies are proportional to temperature ($H = CT$), the traditional shape of the Arrhenius plot is obtained with the present treatment, but through a different way. Increasing the temperature increases the thermal energy stored in the particles and responsible for their reactional capacity, thus reducing the activation energy $E_a = E^\ddagger - H$. By contrast using the conventional formula of Eq.(1), E_a is the constant steering coefficient of the Arrhenius plot [8], in spite of its frequently noticed dependence on temperature [9, 10], and the effect of temperature comes only from the entity k_BT.

5.2   Influence of temperature in equilibrium

5.2.1   Where is the temperature in the different equations?

Equilibrium constants are assumed to vary with temperature according to the equation derived from the van't Hoff relationship.

$$\ln K_{ji} = \frac{G_j - G_i}{k_BT} \qquad (13)$$

Writing $\Delta G = \Delta H - T\Delta S$ in this equation and supposing that H and S are temperature-independent, has the practical advantage to yield the expected van't Hoff plots linearly dependent of $1/T$. In the present treatment, the equilibrium constant remains a function of the energies. Using an energy threshold $E^\ddagger$ and a mean enthalpy H for a reactant or a group of reactants according to the Hess's law,

$$\ln K_{ji} = \mathcal{S}_i - \mathcal{S}_j + E^\ddagger\left(\frac{1}{H_i} - \frac{1}{H_j}\right) \qquad (14a)$$

showing that when the compounds have the same enthalpy, their equilibrium is governed by entropies. Using the activation energy E_a,

$$\ln K_{ji} = \mathcal{S}_i - \mathcal{S}_j + \left(1 - \frac{H_i}{H_j}\right)\left(1 + \frac{E_a(i \to j)}{H_i}\right) \qquad (14b)$$

Temperature is not apparent in these equations but hidden in the variables. The relative temperature-

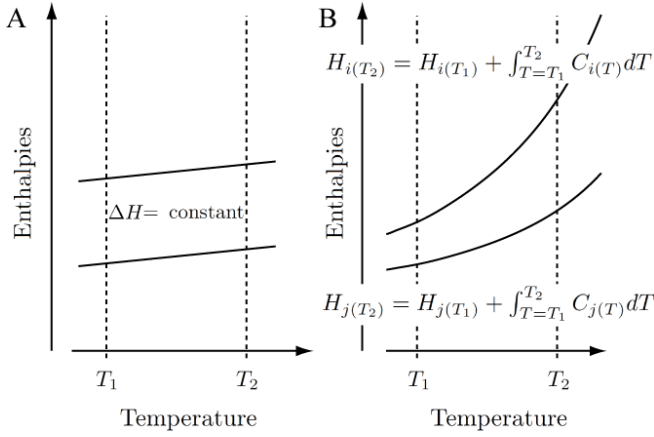dependence of the enthalpies regulate the equilibrium, as schematized in Fig.4.

Figure 4. Comparative interpretation of the role of temperature in van't Hoff plots. (**A**) In the classical interpretation, the heat capacities of the two reactants are always parallel and may vary or not. (**B**) In the present interpretation, temperature affects in a different manner the reactivity of both reactants because of their different heat capacities. As a consequence, ΔH depends on temperature. The variations of enthalpies are exaggerated in this scheme but note that even a faint divergence between the reactant enthalpies would have exponential consequences on the difference of waiting times between the back and forth reactions.

In the classical interpretation (Fig.4A), the enthalpies of the reactants are shown to increase in parallel with temperature, but note that even if these enthalpies simply remain constant, the traditional temperature-dependence of van't Hoff plots would still be obtained.

5.2.2 The kinetic formalism using Gibbs free energies

Temperature is intuitively expected to increase the reaction rates by increasing the energy of the reactants, but using free energies in the kinetic treatments would say the opposite. G always decreases with temperature, thereby increasing the difference $E^{\ddagger} - G$ and raising the height of the barrier. Using the so-called enthalpies of reaction H instead of Gibbs free energies, allows to circumvent this contradiction, but if H is indeed proportional to temperature in certain ranges of temperature, it no longer corresponds to G used in equilibrium. The energy landscape of Arrhenius is governed by enthalpies whereas thermodynamic landscapes include entropies. In this respect, the activated complex theory [4, 11] may be misleading because it describes the kinetic barrier in term of difference of Gibbs free energies. In this theory, the tip of the barrier is not a temperature-independent threshold but the energy of an activated complex that

itself depends on temperature, so that the temperature-dependence of the height of the barrier $\Delta G^{\ddagger}$ results from the relative dependences on temperature of the reactant (G) and of the activated complex ($G^{\ddagger}$). The activated complex is an atypical structure stably standing on a crest of the energetic landscape and with the strange property, when it is reached, to only evolve in the direction of the complete transition but can not turn back. Moreover, the forbidden transition is not the same depending of the starting reactant [3]. In the kinetic theory, temperature is the macroscopic manifestation of microscopic motions, that is re-scaled at the microscopic level through the Boltzmann constant, so that $k_B T$ is an average index of single particle motion. But temperature is also present in Gibbs free energy, not only in its most frequent expression $G = H - TS$, but also because H and S are themselves temperature-dependent [12, 13]. Raising temperature from T_1 to T_2 ($> T_1$) increases enthalpy by $H(T_2) - H(T_1) = C_P\,(T_2 - T_1)$ and entropy by $S(T_2) - S(T_1) = C_P \ln(T_2/T_1)$, where C_P is the heat capacity supposed constant at constant pressure.

5.3 Reintroducing temperature in reactional energies

5.3.1 Increasing particle energy favors reactions

Using the energy quanta approach, the dependence on temperature is conferred by the ratio $\mathcal{E}^{\ddagger}$ (temperature-independent) over $\langle \mathcal{E} \rangle$ (temperature-dependent). This is in fact less clear for the traditional Eq.(13) where both the numerator and the denominator are temperature-dependent. Reactions are facilitated when the energy of the reactants is higher. Possibly because this reasoning does not hold for Gibbs free energies, teachers prefer drawing the Maxwell velocity distribution to show that increasing the temperature increases the fractional population over the activation energy threshold. This view can be recovered probabilistically, without returning to the ideal gas, using the partial integration of the PDF of the exponential distribution.

$$P(\mathcal{E} \geqslant \mathcal{E}^{\ddagger}) = \int_{\mathcal{E}=\mathcal{E}^{\ddagger}}^{\infty} \frac{1}{\langle \mathcal{E} \rangle}\, \mathrm{e}^{-\frac{\mathcal{E}}{\langle \mathcal{E} \rangle}}\, d\mathcal{E}$$
$$= \mathrm{e}^{-\frac{\mathcal{E}^{\ddagger}}{\langle \mathcal{E} \rangle}} \tag{15a}$$

One recovers the energetic part of rate constants proposed here, given in Eq.(5) and illustrated by the colored surfaces in Fig.3A. Remarkably, Boltzmann probability can also be obtained by integration of the same function, but in a different manner.

$$P(\mathcal{E} = \mathcal{E}^{\ddagger}) = \int_{\mathcal{E}=\mathcal{E}^{\ddagger}}^{\mathcal{E}^{\ddagger}+1} \frac{1}{\langle \mathcal{E} \rangle}\, \mathrm{e}^{-\frac{\mathcal{E}}{\langle \mathcal{E} \rangle}}\, d\mathcal{E}$$
$$= \left(1 - \mathrm{e}^{-\frac{1}{\langle \mathcal{E} \rangle}}\right) \mathrm{e}^{-\frac{\mathcal{E}^{\ddagger}}{\langle \mathcal{E} \rangle}} \tag{15b}$$

The rational explanation of a role of temperature mediated by particle energies, corresponds exactly to the present model and to Eq.(15a). On the one hand, it can be understood for enthalpies but not for Gibbs free energies, but on the other hand, even when using enthalpies, the current and present approaches strikingly differ, as explained below.

5.3.2 Temperature increases the reactional energy of the particles

The dimensionless exponents of the two competing structures of rate constants compared here, contain energies in the numerator and denominator, but beyond this common property, the principles are different. In the usual form, the exponent is $H/k_B T$, whereas in the present one, it is $E^{\ddagger}/H$. In the former ratio, the numerator and the denominator both depend on temperature, whereas in the latter one, only the denominator is temperature-dependent. In other words, the traditional formulas still work even when H are independent of temperature, in contradiction with the interpretation of the role of temperature through particle energies. Thermal energy increases the speed of molecules, important for the reactional efficacy of collisions, but more generally the heat capacity of macromolecules describes how much energy can be stored by a molecule in its internal vibrations or rotations, which are often the triggering factors of chemical transformations, particularly for biological macromolecules whose heat capacity is clearly not negligible. The heat capacity of a monoatomic gas is purely translational, but that of more complex particles include the other forms of energy. In the new exponent proposed here, the denominator represents the temperature-dependent reactional energy conceived in all its forms: translational, rotational and vibrational. Since all these forms define the macroscopic notion of temperature, the reactional propensity is naturally expected to follow temperature. This property precisely characterizes enthalpy, which obeys the thermodynamic relation $\partial H(T)/\partial T = C_p$ whose integration with the postulated initial value $H = 0$ at $T = 0$, simply gives $H \propto CT$. The variation of enthalpies with temperature is in fact less simple in practice. It is calculated using Kirchhoff's law in term of a reference state different from $T = 0$ [14]. In addition, when temperature is far from 298 K, the temperature-dependence of C_P should be introduced using empirical expressions found in tables like $C_P = \alpha + \beta T + \gamma T^{-2} + ...$ Absolute enthalpies are refractory to experimental measurements contrary to differences of enthalpy calculable by calorimetry. Hence for simplicity in the following equations, the proportionality between H and T will be retained as a gross approximation for theoretical elementary reactants. Introducing

this definition of enthalpy in Eq.(14) gives

$$\ln K_{ji} \sim \mathcal{S}_i - \mathcal{S}_j + \frac{E^{\ddagger}_{i,j}}{T}\left(\frac{1}{C_i} - \frac{1}{C_j}\right) \qquad (16)$$

Where C_i and C_j are the average heat capacities of the interconvertible reactants weighted by the stoichiometric coefficients. Strikingly, Eq.(16) can yield typical van't Hoff plots. It also shows that at very high temperature, the equilibrium constant becomes essentially a matter of entropy, as expected. Considering the exponential term of rate constants as a probability implies profound conceptual changes but moderate apparent changes in practice, which just impose to reinterpret the experimental plots, as schematized in Fig.5.
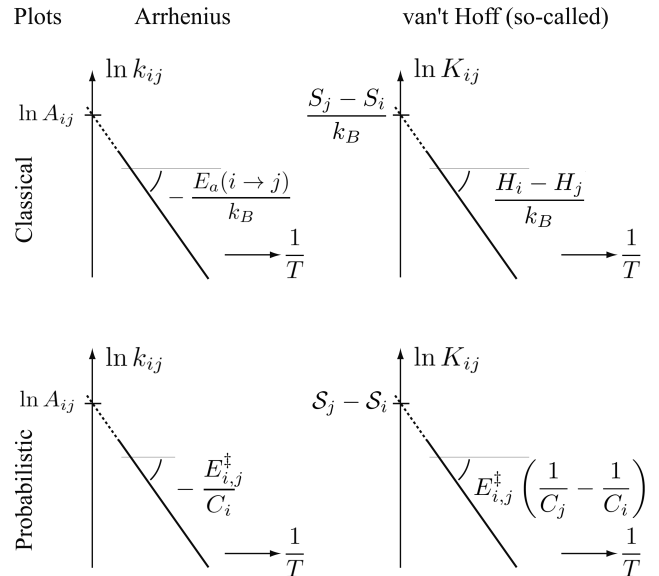

Figure 5. Comparative interpretations of the Arrhenius and van't Hoff plots between the classical and the present treatments of the energetic exponential. The meaning of the entropy intersect is unchanged but that of the slopes is completely different.

The most striking difference is that in the classical interpretation of the van't Hoff plot, the temperature-dependence of enthalpies is denied, whereas the present probabilistic model is fundamentally based on calorimetric enthalpies. The enthalpies of some mineral substances, solid and liquid, were indeed shown approximately independent of temperature up to 200 degrees, but this behaviour is likely to not hold for all reactional moieties, like biological macromolecules in solution. The simplistic dependence on temperature depicted in Fig.5 should be further refined in these very approximate equations, using Kirchhoff's law, setting the standard values H_0 at $T = 298K$, and taking into account the dependence on temperature of heat capacities, entropies and

8

enthalpies according to the basic thermodynamic differentials

$$\frac{\partial S(T)}{\partial T} = \frac{C_p}{T}$$

$$\frac{\partial H(T)}{\partial T} = C_p$$

Moreover, certain ranges of temperature should not be considered in case of phase transition or of inactivation of certain reactants. For example, proteins are denatured over 320 K and freezed below 270 K.

5.3.3   Deviations from the expected plots

Experimental Arrhenius plots are rarely the straight lines expected from the Arrhenius equation [$\ln k = \ln A - \frac{E_a}{R}\frac{1}{T}$], but can have convex or concave shapes, explained by tunneling effects and other hypotheses [9, 15, 16]. Deviations are also the rule for the so-called van't Hoff plots in equilibrium. Moreover, the two types of enthalpy: (i) calorimetric and (ii) determined from the analysis of experimental plots using the currently admitted functions, are always different [12, 13]. Reintroducing the temperature-dependence of enthalpies and entropies might account for certain deviations [17, 18], but more fundamentally, a difference of enthalpy between the interconvertible reactants is synonymous to a heat of reaction, that itself reflects a difference of heat capacity.

5.4   Giving a calorimetric meaning to the Boltzmann constant

Heat capacity is assumed to be closely related to energy fluctuations. We have seen that the typical fluctuation of the exponential distribution is $\sigma_\mathcal{E} = \langle \mathcal{E} \rangle$. Considering it as proportional to temperature through an absolute heat capacity C, gives $\sigma_\mathcal{E} = CT$, to be compared with the classical result of statistical mechanics $\sigma_E = \sqrt{C_V k_B}\ T$ obtained using $\langle E \rangle = U = -\frac{1}{Z}\left(\partial Z/\partial \beta\right)_{N,V}$, $\beta = 1/(k_B T)$ and $C_V = (\partial U/\partial T)_{N,V}$. This comparison suggests a correspondence between the Boltzmann constant and a heat capacity. This correspondence is also supported by the proportionality, in the classical limit and following the Dulong-Petit law, between heat capacity and Boltzmann constant in simple systems like ideal gases. In constant volume, the heat capacity of ideal gases is $C_V = \frac{n}{2}k_B$, where n is the number of degrees of freedom. Thus, ideal monoatomic gas particles translating in a constant volume have a mean kinetic energy

$$C_V T = \frac{3}{2}\langle \tilde{\mathcal{E}} \rangle = \tilde{\mathcal{E}}_{\mathrm{rms}} = \frac{1}{2}\ m v_{\mathrm{rms}}^2$$

where $v_{\mathrm{rms}} = \sqrt{3\langle \tilde{\mathcal{E}} \rangle/m}$ is the root mean square velocity, obtained using velocity vectors split over the 3 spatial coordinates.

Volume changes regulate pressure for systems of mutually colliding particles like gases, but for macromolecules in solution, pressure is essentially due to microscopic collisions with the solvent, water, that is itself incompressible and whose cellular content automatically adjusts to the cellular volume. Heat capacity defines the extent of heat aborbed or released upon temperature change, according to $\Delta Q = C\Delta T$. But at constant temperature if for some reason the heat capacity evolves, then a release or uptake of heat is also expected such that $\Delta Q = T\Delta C$, thus making a clear connection between changes of heat capacity and of thermodynamic entropy, the latter one including in addition the configurational information of statistical entropy such that $S = C\mathcal{S}$. Under this postulate, heat capacities would become the main ingredients of the thermodynamic constants. For the equilibrium constant,

$$\ln K_{ji} = \left(\frac{S_i}{C_i} - \frac{S_j}{C_j}\right) + \frac{E_{i,j}^\ddagger}{T}\left(\frac{1}{C_i} - \frac{1}{C_j}\right) \qquad (17)$$

and for the kinetic constant,

$$\ln k_{ij} = -\ln\tau - \frac{1}{C_i}\left(S_i + \frac{E_{i,j}^\ddagger}{T}\right) \qquad (18)$$

Differentiation of Eq.(17) with respect to temperature gives

$$\frac{d\ln K_{ji}}{dT} = (C_i - C_j)\ \frac{E_{i,j}^\ddagger}{C_i C_j}\ \frac{1}{T^2} \qquad (19)$$

which is interesting since Eq.(19) has the structure of the original equation of van't Hoff, based on the heat developped by the reaction [19]. Using conventional Boltzmann entropies $S = k_B\mathcal{S}$, differentiation of Eq.(16) gives

$$\frac{d\ln K_{ji}}{dT} = (C_i - C_j)\left(\frac{1}{k_B T} + \frac{E_{i,j}^\ddagger}{C_i C_j T^2}\right) \qquad (20)$$

5.5   Extension of the detailed balance to activation energies

To the properties of the hypothetical rate constants listed above, is added a complication to satisfy the detailed balance rule linking the different rates constants of a system. The traditional form of the energetic exponential obviously complies with this rule since the total difference of energy of a global reaction made of several elementary steps is simply the sum of energy gaps, with the automatic elimination of activation energies, as illustrated in Eq.(2d). Things are less simple with the new form since

9

the activation energies should now take part to the detailed balance relationship. Consider for example a triangular circuit. The compounds a and b can interconvert either directly, with an activation threshold $E_{a,b}^{\ddagger}$, or indirectly through an intermediate c. Assuming that the reactional enthalpy and entropy of every reactant is the same for all its transformations, writing $E_{a,c}^{\ddagger}$ the activation threshold between a and c, and $E_{c,b}^{\ddagger}$ between c and b, the detailed balance relation

$$\frac{k_{ab}}{k_{ba}}\frac{k_{bc}}{k_{cb}}\frac{k_{ca}}{k_{ac}} = 1 \tag{21a}$$

implies for the energies

$$\frac{E_{a,c}^{\ddagger} - E_{a,b}^{\ddagger}}{H_a} + \frac{E_{a,b}^{\ddagger} - E_{b,c}^{\ddagger}}{H_b} + \frac{E_{b,c}^{\ddagger} - E_{a,c}^{\ddagger}}{H_c} = 0 \tag{21b}$$

generalizable for all the cycles of the system, with any number of components n

$$\sum_{i=1}^{n} \frac{E_{i,i-1}^{\ddagger} - E_{i,i+1}^{\ddagger}}{H_i} = 0 \ \text{ with } \ E_{1,0}^{\ddagger} = E_{n,n+1}^{\ddagger} = E_{1,n}^{\ddagger} \tag{21c}$$

These relationships with circular permutations are mutually compatible for all the cycles of a reticulated system. In this new treatment, the activation energies are now envisioned as integral components of the system, whereas in the classical theory, they were extra-parameters whose values can be arbitrary.

6 Rebuilding discrete thermodynamics

Matter was discretized in the visionary theory of Boltzmann and the same was extended to energy by Planck [20]. Since handling energy as a number of quanta facilitates the formulation of basic equations, let us apply again this recipe to the rudiments of thermodynamics and then reintroduce the new kinetic rules proposed above. The concept of entropy long proved fruitful to begin statistical physics.

6.1 Index of uncertainty for a population

Two criteria are sufficient for defining this index equivalent to entropy. It should be: (i) a function of the number of possible configurations (microstates) of the system's components and (ii) additive for uncorrelated systems.
(i) The different possible configurations of a macroscopic system (total number written Ω), are invisible and definitely not measurable in practice, but the number of these theoretical "snapshots" can be deduced statistically. It reflects the uncertainty of the system, a notion closely related to missing information and disorganization. For example, a book is an organized object in

which the page 189 is located between the pages 187 and 191 with a good degree of confidence, but if the pages are torn and thrown down, this probability strongly decreases because the number of possible arrangements of the pages increases.
(ii) The other requirement for an index of disorganization is its additivity for independent systems. This can also be easily conceived: For each microstate of a system 1, all the states of a system 2 are possible, so the total number of states of the global system $1 + 2$ is the product $\Omega_1 \Omega_2$. Obviously the function satisfying

$$f(\Omega_1 \Omega_2) = f(\Omega_1) + f(\Omega_2) \tag{22}$$

is the logarithmic function $(\ln \Omega)$. We will also define a single particle average index $\mathcal{S} = \dfrac{1}{N} \ln \Omega$. Concretely, the total number of configurations of a system made of N noninteracting components falling into r categories is

$$\Omega = \frac{N!}{n_1! n_2! \dots n_r!} \tag{23}$$

Using the Stirling approximation $x! \sim x^x \, \mathrm{e}^{-x} \, \sqrt{2\pi x}$ [21] of which a rough approximation is $x! \sim x^x$,

$$\Omega \sim \frac{N^N}{n_1^{n_1} n_2^{n_2} \dots n_r^{n_r}} = \left[\left(\frac{n_1}{N}\right)^{n_1} \left(\frac{n_2}{N}\right)^{n_2} \dots \left(\frac{n_r}{N}\right)^{n_r} \right]^{-1} \tag{24}$$

so that

$$\ln \Omega = -\sum_{i=1}^{r} \ln \left(\frac{n_i}{N}\right)^{n_i} \tag{25}$$

giving a single particle average index of

$$\mathcal{S} = -\frac{1}{N} \sum_{i=1}^{r} n_i \ln \left(\frac{n_i}{N}\right) = -\sum_{i=1}^{r} p_i \ln p_i \tag{26}$$

with $p_i = n_i/N$ and $\sum_i n_i = N$. With respect to these definitions, the so-called Boltzmann entropy (S_B) and Gibbs/Shannon entropy (S_G) are not equivalent, but related through $S_G = S_B/N$. S_G/k_B corresponds to $\mathcal{S}$ whereas S_B/k_B corresponds to $N\mathcal{S}$.

But for deriving a pure theory, using an approximation, as acceptable as it can be, is not satifactory. To bypass the need for the Stirling approximation, it is of some interest to reconsider Eq.(24), which could simply mean that every fractional state n_i/N is occupied n_i times, so that Ω is simply the reciprocal of the probability of occupancy. Based on the same principle, another equation can be designed by supposing that the system can evolve.

6.2 Evolving system

In a first-order network, r particle states obey a linear evolution system, which has a unique stable solution written n_i^{eq}. A new state function allows to describe the occupation of the equilibrium proportions $\left(\frac{n_i^{eq}}{N}\right)^{n_i}$:

$$\mathcal{U} = -\frac{1}{N}\sum_{i=1}^{r} n_i \ln\left(\frac{n_i^{eq}}{N}\right) = -\sum_{i=1}^{r} p_i \ln p_i^{eq} \qquad (27)$$

$\mathcal{U}$ is an index of the mean gap between the current and the equilibrium distributions. This gap can incite the system to evolve, provided it is higher than its degree of disorganization $\mathcal{U} > \mathcal{S}$. The difference $\mathcal{F} = \mathcal{U} - \mathcal{S}$ is

$$\mathcal{F} = -\sum_{i=1}^{r} p_i \ln p_i^{eq} + \sum_{i=1}^{r} p_i \ln p_i$$
$$= \sum_{i=1}^{r} p_i \ln\left(\frac{p_i}{p_i^{eq}}\right) \qquad (28)$$

$\mathcal{F}$ will prove particularly interesting upon derivation.

6.2.1 Introducing time

To concretize the notion of system evolution and of interchangeability between single particle states, one should introduce a temporal ingredient, in the form of rate constants k examined previously of unit time^{-1}. As these rates depend on the temperature, the system will be considered closed but not isolated and embedded in a thermal bath (NVT ensemble). As the different states can be interchanged two by two, a simple two-state system can be used to derive the proportions of particles in states i and j (or for a single particle, the probability to be at a given moment in state i or j). Let us first derive $\mathcal{S}$.

$$\frac{d\mathcal{S}}{dt} = -\left(p_i \ln p_i + p_j \ln p_j\right)'$$
$$= (k_{ij}p_i - k_{ji}p_j)(1 + \ln p_i)$$
$$+ (k_{ji}p_j - k_{ij}p_i)(1 + \ln p_j) \qquad (29a)$$
$$= (k_{ij}p_i - k_{ji}p_j)\ln\left(\frac{p_i}{p_j}\right)$$

which can be generalized to the r nodes

$$\frac{d\mathcal{S}}{dt} = \sum_{i,j} (k_{ij}p_i - k_{ji}p_j)\ln\left(\frac{p_i}{p_j}\right) \qquad (29b)$$

Eq.(29) is not particularly illuminating because it can be either positive or negative; but things are more in-

teresting when deriving $\mathcal{F}$. Taking two interconvertible nodes i and j,

$$\mathcal{F}'(i,j) = \left[p_i \ln\left(\frac{p_i}{p_i^{eq}}\right) + p_j \ln\left(\frac{p_j}{p_j^{eq}}\right)\right]'$$
$$= (k_{ij}p_i - k_{ji}p_j)\left[\ln\left(\frac{p_j}{p_j^{eq}}\right) - \ln\left(\frac{p_i}{p_i^{eq}}\right)\right]$$
$$= -(k_{ij}p_i - k_{ji}p_j)\,\ln\left(\frac{p_i}{p_j}\frac{p_j^{eq}}{p_i^{eq}}\right)$$

and considering that $k_{ij}p_i^{eq} = k_{ji}p_j^{eq}$,

$$= -(k_{ij}p_i - k_{ji}p_j)\,\ln\left(\frac{k_{ij}p_i}{k_{ji}p_j}\right)$$

$$(30a)$$

which can be written in term of flux ($J = nk$) for the r nodes,

$$\frac{d\mathcal{F}}{dt} = -\sum_{i,j} (J_{ij} - J_{ji})\,\ln\left(\frac{J_{ij}}{J_{ji}}\right) \qquad (30b)$$

Contrary to Eq.(29), Eq.(30) can be only zero or negative. For this reason, it remarkably illustrates the free energy dissipation and can naturally be identified with the arrow of time [22, 23, 24]. The discrete statistical functions described above can be reconnected to traditional thermodynamics by identifying $k_B T \mathcal{F}$ with the Gibbs free energy of the system, $k_B T \mathcal{U}$ with the internal energy and $k_B \mathcal{S}$ with the entropy. An increase of entropy (towards equipartition) logically decreases free energy and the possibility to extract some work. Further extensions of this discrete approach to thermodynamics are described in [25, 26]. Now let us look at Eq.(30) which has the particularity to bring together the two different aspects of time.

6.2.2 Time's arrow and time steps

The arrow of time envisioned as the dissipation of free energy in Eq.(30) has a clear thermodynamic origin out of equilibrium, but even in full equilibrium conditions, the rates constants keep their time units and the particles indefinitely continue to move and transform. This sort of time, completely reversible contrary to the time's arrow, corresponds to the elementary time steps related to another field of physics: quantum physics. Statistical and quantum physics are intimately and necessarily related because enumerating configurations would be infinite in continuous space and time. Accordingly, the quantum of time can be defined as the time necessary to cross the length unit below which successive configurations cannot be distinguished, because of the uncertainty principle. This value can be obtained in multiple ways including

in the context of the Maxwell-Boltzmann system. For a single particle, the length unit is the thermal wavelength of de Broglie $\lambda = h/p = h/\sqrt{2\pi m k_B T}$ and the mean particle velocity determined previously is $\bar{v} = \sqrt{2k_B T/\pi m}$. The ratio gives the minimal time step

$$\tau = \frac{\lambda}{\bar{v}} = \frac{h}{2k_B T} \approx 8 \times 10^{-14} \text{ s at } T = 300 \text{ K} \quad (31)$$

whose reciprocal is the maximal frequency close to $k_B T/h$, considered by Eyring as the most remarkable achievement of his theory [27], and which was already present in the equation of Herzfeld [28]. In the rate constants, this maximal frequency τ^{-1} is more or less decreased by entropic and energetic restrictions.

The arrow of time is a source of endless debates, particularly interesting when applied to the universe taken as a whole. The free energy of a closed system should necessarily diminish within the limits of residual fluctuations. Then, whence comes the initial free energy of our universe which does not seem very organized in its youth? Did previously latent forces release free energy during the physico-chemical maturation of the universe? or do we live in a local fluctuation in a giant universe in equilibrium? according to an elegant but contested proposal by Boltzmann [29].

6.3 Introducing discrete energy

The frequencies k_{ij} introduced in Section 6.2 remain abstract and their nature should be precised. In addition to their temporal parameter, the rate constants include an energy parameter which can be different in the different k. Before envisioning reactional systems, the spreading of energy will be first described in simple systems without chemical transformations. Instead of distributing particles into "energy levels", energy units are distributed over particles. Things appear clearer when energy is handled discretely. Perhaps influenced by what Boltzmann said him in 1891: "I see no reason why energy shouldn't also be regarded as divided atomically", Planck proposed in the founder article which opened the way to quantum physics [20], to discretize energy into quanta of value $h\nu$, written more generally $\mathfrak{q}$ below.

6.3.1 Homogeneous system

The mean energy of thermal systems can be recovered in a discrete manner from the second law. Returning to the white balls supposed to correspond to energy quanta in Section.4, the number of ways to distribute E objects in N boxes is

$$\Omega = \frac{(N + E - 1)!}{(N - 1)! \, E!} \quad (32)$$

which gives, using the Stirling approximation, a single particle average entropy of

$$\mathcal{S} = \frac{1}{N} \ln \Omega = \left(1 + \frac{E}{N}\right) \ln \left(1 + \frac{E}{N}\right) - \frac{E}{N} \ln \left(\frac{E}{N}\right) \quad (33a)$$

When the system contains a single category of particles, the ratio E/N is equivalent to the mean number of energy quanta per particle $U/\mathfrak{q}$ [20].

$$\mathcal{S} = \frac{S}{k_B} = \left(1 + \frac{U}{\mathfrak{q}}\right) \ln \left(1 + \frac{U}{\mathfrak{q}}\right) - \frac{U}{\mathfrak{q}} \ln \left(\frac{U}{\mathfrak{q}}\right) \quad (33b)$$

Temperature can now be introduced using the fundamental entropy equation.

$$\frac{dS}{dU} = \frac{1}{T} \quad (34)$$

according to which integration of Eq.(33b) yields

$$\frac{k_B}{\mathfrak{q}} \ln \left(1 + \frac{\mathfrak{q}}{U}\right) = \frac{1}{T} \quad (35a)$$

and

$$\frac{U}{\mathfrak{q}} = \langle \mathcal{E} \rangle = \frac{1}{e^{\mathfrak{q}/k_B T} - 1} \quad (35b)$$

The reasoning can be circular in statistical thermodynamics and these results can also be recovered in a reverse way starting from the partition function, as did Einstein [30]. The ratio of Planck ($\mathfrak{q} = h\nu$) over Boltzmann entities, turns to be the logarithm of the inverse proportion of energy quanta in the mixture.

$$\frac{h\nu}{k_B T} = -\ln \frac{E}{N + E} = \ln \left(1 + \frac{1}{\langle \mathcal{E} \rangle}\right) \quad (36)$$

At the thermal scale, $\mathfrak{q}$ is negligible compared to $k_B T$ and the expansion of Eq.(35b) gives a mean number (unitless) of energy quanta

$$\langle \mathcal{E} \rangle \sim \frac{k_B T}{\mathfrak{q}} \quad (37a)$$

and a mean energy (joules) per particle of

$$\langle \mathcal{E} \rangle \, \mathfrak{q} = U \sim k_B T \quad (37b)$$

6.3.2 Heterogeneous system

When the system is composite and made of r different categories of particles which have different mean energies

$\langle \mathcal{E}_i \rangle$, we have to fill r boxes with $N = \sum_i n_i$ particles and $E = \sum_i n_i \langle \mathcal{E}_i \rangle$ energy quanta

$$\Omega = \prod_{i=1}^{r} \frac{(n_i + E_i - 1)!}{(n_i - 1)! \, E_i!} \tag{38}$$

giving, for large populations and using the Stirling approximation, an entropy of

$$\ln \Omega = \sum_{i=1}^{r} n_i \left[\left(1 + \frac{E_i}{n_i} \right) \ln \left(1 + \frac{E_i}{n_i} \right) - \frac{E_i}{n_i} \ln \left(\frac{E_i}{n_i} \right) \right] \tag{39}$$

Keeping N and E constant, this entropy is lower than that of an homogeneous system, showing that the categorization of objects is a first step towards organization. As the derivative of a sum is the sum of the derivatives,

$$\langle \mathcal{E}_i \rangle = \frac{E_i}{n_i} = \frac{1}{e^{\mathfrak{q}i/k_B T} - 1} \tag{40}$$

First-order particle interconversion

To render this heterogeneous system dynamic, one should introduce the possibility of interconversion between the particles of the different kinds in the form of rate constants. The rule for particle interconversion has been established in Eq.(8) using the exponential law. For unimolecular transconversions which do not modify the total number of particles, let us abandon the ratios of pre-exponential factors. The single particle energy averaged over the $r(r-1)/2$ couples of categories in the system is

$$\langle \mathcal{E}_m \rangle = \frac{E}{N} = \frac{\sum_i \sum_{j \neq i} \langle \mathcal{E}_i \rangle \, e^{-\frac{\mathcal{E}_{i,j}^{\ddagger}}{\langle \mathcal{E}_i \rangle}}}{\sum_i \sum_{j \neq i} e^{-\frac{\mathcal{E}_{i,j}^{\ddagger}}{\langle \mathcal{E}_i \rangle}}} \tag{41}$$

in which the inexisting interchanges are characterized by $\mathcal{E}_{i,j}^{\ddagger} = \infty$. For a system of purely enthalpic components, one can define the transition rate constants

$$k_{ij} = \tau^{-1} e^{-\mathcal{E}_{i,j}^{\ddagger}/\langle \mathcal{E}_i \rangle} \tag{42a}$$

$$k_{ji} = \tau^{-1} e^{-\mathcal{E}_{i,j}^{\ddagger}/\langle \mathcal{E}_j \rangle} \tag{42b}$$

where τ is the time step and $\langle \mathcal{E}_i \rangle$ is the mean energy of the particles of type i, both depending on temperature through Eq.(31) and Eq.(40) respectively.

For two categories n_1 and n_2, the relationships linking energy and particle numbers are

$$\frac{n_1}{N} = \left(1 + e^{\mathcal{E}_{1,2}^{\ddagger} \left(\frac{1}{\langle \mathcal{E}_1 \rangle} - \frac{1}{\langle \mathcal{E}_2 \rangle} \right)} \right)^{-1} \tag{43a}$$

$$\frac{n_2}{N} = \left(1 + e^{\mathcal{E}_{1,2}^{\ddagger} \left(\frac{1}{\langle \mathcal{E}_2 \rangle} - \frac{1}{\langle \mathcal{E}_1 \rangle} \right)} \right)^{-1} \tag{43b}$$

$$\frac{E_1}{E_2} = \frac{\langle \mathcal{E}_1 \rangle}{\langle \mathcal{E}_2 \rangle} e^{\mathcal{E}_{1,2}^{\ddagger} \left(\frac{1}{\langle \mathcal{E}_1 \rangle} - \frac{1}{\langle \mathcal{E}_2 \rangle} \right)} \tag{43c}$$

and the mean particle energy in the system is

$$\langle \mathcal{E}_m \rangle = \frac{\langle \mathcal{E}_1 \rangle \, e^{-\frac{\varepsilon_{1,2}^{\ddagger}}{\langle \mathcal{E}_1 \rangle}} + \langle \mathcal{E}_2 \rangle \, e^{-\frac{\varepsilon_{1,2}^{\ddagger}}{\langle \mathcal{E}_2 \rangle}}}{e^{-\frac{\varepsilon_{1,2}^{\ddagger}}{\langle \mathcal{E}_1 \rangle}} + e^{-\frac{\varepsilon_{1,2}^{\ddagger}}{\langle \mathcal{E}_2 \rangle}}} \tag{43d}$$

Circular equilibria

Three categories of particles are sufficient to establish the famous detailed balance relationship of Wegscheider linking the constants of a cycle [31]. The relation

$$\frac{n_1}{n_2} \frac{n_2}{n_3} \frac{n_3}{n_1} = 1 \tag{44a}$$

is obviously always true, in particular at equilibrium, so the constants are necessarily related through

$$\frac{k_{12}}{k_{21}} \frac{k_{23}}{k_{32}} \frac{k_{31}}{k_{13}} = 1 \tag{44b}$$

The activation energies, which were completely arbitrary in previous theories, are now mutually connected through

$$\frac{\mathcal{E}_{1,3}^{\ddagger} - \mathcal{E}_{1,2}^{\ddagger}}{\langle \mathcal{E}_1 \rangle} + \frac{\mathcal{E}_{1,2}^{\ddagger} - \mathcal{E}_{2,3}^{\ddagger}}{\langle \mathcal{E}_2 \rangle} + \frac{\mathcal{E}_{2,3}^{\ddagger} - \mathcal{E}_{1,3}^{\ddagger}}{\langle \mathcal{E}_3 \rangle} = 0 \tag{45}$$

Circular equilibria are particularly relevant with respect to catalysts. Contrary to the widespread representation of catalysis as a deformation of the energy landscape along the coordinates of a single transition, it should rather be envisioned as a circuit, at least triangular and letting unchanged the uncatalysed single step reaction.

6.4 Possible source of confusion for modeling the relationships between fractional populations and energy

The enthalpic particles of the toy systems described above are sufficient to visualize the important differences in the description of the ratios of particles, depending on whether they belong to the same category or not. For a homogeneous system of mean energy $\langle \mathcal{E}_1 \rangle$ in which energy diffuses without barriers between particles n_1, the

ratio of particles with energies $\mathcal{E}_a$ and $\mathcal{E}_b$ follows, according to Eq(10c),

$$n_1(a)/n_1(b) = \mathrm{e}^{\frac{\mathcal{E}_a - \mathcal{E}_b}{\langle \mathcal{E}_1 \rangle}} \qquad (46)$$

which is a familiar Boltzmann relation when setting $\langle \mathcal{E}_1 \rangle = k_B T$, whereas interchangeable particles of different categories with different mean energies $\langle \mathcal{E}_1 \rangle$ and $\langle \mathcal{E}_2 \rangle$, are related through

$$n_1/n_2 = \mathrm{e}^{\mathcal{E}_{1,2}^{\ddagger} \left(\frac{1}{\langle \mathcal{E}_1 \rangle} - \frac{1}{\langle \mathcal{E}_2 \rangle} \right)} \qquad (47)$$

Eq.(46) is valid for a single phase in the pioneer statistical theories, applicable for instance to energy spreading between gas particles, but the object of this manuscript is precisely to propose that this form can not be extended to chemical reactions. The next tasks will be to implement these rules from first order to second order networks, to introduce particle entropies, to extend results to entropic open systems in steady state like biochemical systems, for modeling the fundamental ingredients of life: nonlinearity, retroactions and multistability.

7 Conclusions

The present probabilistic treatment based on the exponential distribution does not invalidate the Maxwell-Boltzmann distribution, but encompasses it as a particular case of homogeneous system with temperature-dependent particle energies. Hence, there is no reason to not extend this approach to the energy-dependence of particle conversions, because rate constants are integral constituents of dynamic systems. The central and original assumption of statistical physics was the modeling of randomness, but randomness is precisely described in the most fundamental way by the exponential law. In fact, the equations of Arrhenius and Boltzmann are two facets of the same law of randomness, unified in the present theory. This is perfectly illustrated by Eqs.(9) and (10) respectively: the first one is the probability that $\mathcal{E} \geqslant \mathcal{E}^{\ddagger}$ and the second one that $\mathcal{E} = \mathcal{E}^{\ddagger}$. These two probabilities can also be obtained by different integrations of the exponential distribution PDF (Eq.(15a) and Eq.(15b) respectively). To further enlighten this point, Boltzmann probability can be simply obtained by subtraction using the exponential distribution only, as follows

$$\begin{aligned} P(\mathcal{E} = j) &= P(\mathcal{E} \geqslant j) - P(\mathcal{E} \geqslant j+1) \\ &= \mathrm{e}^{-j/\langle \mathcal{E} \rangle} - \mathrm{e}^{-(j+1)/\langle \mathcal{E} \rangle} \\ &= \mathrm{e}^{-j/\langle \mathcal{E} \rangle} \left(1 - \mathrm{e}^{-1/\langle \mathcal{E} \rangle} \right) \qquad (48) \\ &= \frac{\mathrm{e}^{-j/\langle \mathcal{E} \rangle}}{\sum_{n=0}^{\infty} \mathrm{e}^{-n/\langle \mathcal{E} \rangle}} \end{aligned}$$

The relationships established here are simply derived from the single postulate that the exponential term of rate constants has the status of a probability, in line with the profoundly probabilistic spirit of Boltzmann's theory and with the intuition of Maxwell: "The true logic of this world is in the calculus of probabilities". In this respect, the rate constant defined here is entirely probabilistic

$$k = \tau^{-1} \times P(\text{configuration is OK}) \times P(\text{energy is OK})$$

which simply says that the frequency of a transformation follows the probability of favorable conjunctions of configurational and energetic conditions, which is

$$k = \frac{1}{\tau} \times \frac{1}{\Omega} \times \mathrm{e}^{-\mathcal{E}^{\ddagger}/\langle \mathcal{E} \rangle}$$

leading to the equilibrium relationship

$$\ln K_{ji} = \mathcal{S}_{ij} - \mathcal{S}_{ji} + \mathcal{E}_{i,j}^{\ddagger} \left(\frac{1}{\langle \mathcal{E}_i \rangle} - \frac{1}{\langle \mathcal{E}_j \rangle} \right)$$

where the temperature-dependence is mediated by the variables. The mean reactional energy $\langle \mathcal{E} \rangle$ can logically be identified with enthalpy, but Gibbs free energies are not appropriate for these definitions because the entropic and energetic parts are involved in independent probabilities. If the starting probabilistic postulate is wrong, the present study would remain a theoretical exercise, but is however appealing owing to: (i) its mathematical relevance; (ii) its capacity to recover the Maxwell-Boltzmann distribution and to unify kinetics and thermodynamics; (iii) the recovery of the van't Hoff relation between equilibrium constants and temperature and (iv) the restoration of the link between the reactional propensity and the heat capacity, which is ignored in the traditional interpretation of the slope of van't Hoff plots as $\Delta H/k_b$. In the traditional formulas, the enthalpies are not responsible for the dependence on temperature that is due only to the division of enthalpies by $k_B T$. By contrast, it is proposed here that enthalpies directly mediate this dependence. Strangely, this view is found in textbooks explaining the dependence of reactions on temperature using populational energy distributions, but without realizing that this explanation contradicts the general formula where the particle energies are at the numerator of the exponent. But although profoundly restructured, the thermodynamic relationships derived from this approach yield resembling behaviours, so that the comparison between the previous and new treatments is more a question of interpretation of the experimental plots than a true discrimination. In the field of rate theories, Laidler already noticed that, surprisingly, the widely different rate equations proposed in the past can give reasonably good fit to the same experimental data [2]. Precise experimental validations in this field are hindered by numerous problems, including the knowledge of absolute enthalpies, the complex temperature-dependence of heat capacities, the interferences with tunnel effects

and the narrow range of temperature testable in practice. Moreover, experimental and calculated values are rarely in good agreement [14] and strikingly, calorimetric and van't Hoff enthalpies never coincide [12, 13], which is not a surprise considering that the enthalpy change for a reaction is responsible for a heat of reaction, whereas the heat capacity of enthalpies is ignored in the current van't Hoff plot approach. In the alternative view examined here, the entropies of the reactants are completely identical to the previous ones; the calorimetric enthalpies and the Hess's law are unchanged; the equilibrium constants remain defined by the ratios of interconvertible reactants concentrations in equilibrium and still satisfies the van't Hoff formula. Note in this respect that the original formula of van't Hoff was not the equation currently attributed to van't Hoff in texbooks, since van't Hoff did not use the Boltzmann's constant or the ideal gas constant, and his reasoning was fundamentally calorimetric [19], in line with the changes of heat capacity on which the present theory is based. In addition, van't Hoff indicated that he managed his equation for numerical applications and experimental controls only, but did not enter into the details of thermodynamics necessary for its demonstration [19]. The 19th century formula of Arrhenius also remains an equation of practical convenience, even if experimental data show that it is a poor quantitative predictor of reaction rates, particularly in high-dimensional, complex chemical kinetics. Attempts to rectify this form include transition state theory [4], diffusion from an energy well [32], reactive flux theory [33], and transition path theory [34]. The alternative approach proposed here is basically simpler since it is just the probability for a particle in a population to acquire a given level of energy.

The elucidation of the fundamental nature of rate constants is essential considering their widespread use from chemistry to systems modeling. Rate constants are the regulators of systems dynamics and as such allow to introduce the notion of time. If time proves so difficult to conceive, it is probably because it relies on two completely different pedestals belonging to different branches of physics: (i) the step of time, always at work, both out of and in equilibrium and originating from quantum physics and (ii) time arrow, existing only out of equilibrium and which is a populational phenomenon emerging from statistical physics [29]. Dynamic systems clarify this perception because they combine both aspects: the time step included in the preexponential component of the Arrhenius equation, and the time arrow describing the spontaneous evolution of a system of interconvertible particle states.

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