Orbital counting for some convergent groups

Marc Peigné, Samuel Tapie, Pierre Vidotto

Counting for some convergent groups

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Marc Peigné [(1)], Samuel Tapie [(2)] & Pierre Vidotto [(3)]

Abstract. We present examples of geometrically finite manifolds with pinched negative curvature, whose geodesic flow has infinite non-ergodic Bowen-Margulis measure and whose Poincaré series converges at the critical exponent δ_Γ. We obtain an explicit asymptotic for their orbital growth function. Namely, for any $\alpha \in]1, 2[$ and any slowly varying function $L : \mathbb{R} \to (0, +\infty)$, we construct N-dimensional Hadamard manifolds (X, g) of negative and pinched curvature, whose group of oriented isometries admits convergent geometrically finite subgroups Γ such that, as $R \to +\infty$,

$$N_\Gamma(R) := \#\{\gamma \in \Gamma \,;\, d(o, \gamma \cdot o) \leq R\} \sim C_\Gamma \frac{L(R)}{R^\alpha}\, e^{\delta_\Gamma R},$$

for some constant $C_\Gamma > 0$.

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1 Introduction

We fix $N \geq 2$ and consider a N-dimensional Hadamard manifold X of negative, pinched curvature $-B^2 \leq K_X \leq -A^2 < 0$. Without loss of generality, we may assume $A \leq 1 \leq B$. Let Γ be a *Kleinian group* of X, i.e. a discrete, torsionless group of isometries of X, with quotient $\bar{X} = \Gamma\backslash X$.

This paper is concerned with the fine asymptotic properties of the *orbital function* :

$$v_\Gamma(\mathbf{x}, \mathbf{y}; R) := \sharp\{\gamma \in \Gamma / d(\mathbf{x}, \gamma \cdot \mathbf{y}) \leq R\}$$

for $\mathbf{x}, \mathbf{y} \in X$, which has been the subject of many investigations since Margulis' [10] (see also Roblin's book [14]). First, a simple invariant is its *exponential growth rate*

$$\delta_\Gamma = \limsup_{R\to\infty} \frac{1}{R} \log(v_\Gamma(\mathbf{x}, \mathbf{y}; R)).$$

The exponent δ_Γ coincides also with the *exponent of convergence* of the *Poincaré series* associated with Γ :

$$P_\Gamma(\mathbf{x}, \mathbf{y}, s) := \sum_{\gamma\in\Gamma} e^{-sd(\mathbf{x},\gamma\cdot\mathbf{y})}, \qquad \mathbf{x}, \mathbf{y} \in X.$$

Thus, it is called the *Poincaré exponent* of Γ δ_Γ. It coincides with the topological entropy of the geodesic flow $(\phi_t)_{t\in\mathbb{R}}$ on the unit tangent bundle of $\bar{X}$, restricted to its non-wandering set. It equals also the Hausdorff dimension of the *radial limit set* $\Lambda(\Gamma)^{rad}$ of Γ with respect to some natural metric on the

[1]Marc Peigné, LMPT, UMR 7350, Faculté des Sciences et Techniques, Parc de Grandmont, 37200 Tours, France – email : peigne@univ-tours.fr

[2]Samuel Tapie, Laboratoire Jean Leray 2 rue de la Houssinière - BP92208 44322 NANTES CEDEX 3 France – email : samuel.tapie@univ-nantes.fr

[3]Pierre Vidotto, Laboratoire Jean Leray 2 rue de la Houssinière - BP92208 44322 NANTES CEDEX 3 France – email : pierre.vidoto@univ-nantes.fr

boundary at infinity ∂X of X. Recall that any orbit $\Gamma \cdot \mathbf{x}$ accumulates on a closed subset $\Lambda(\Gamma)$ of the geometric boundary ∂X of X, called the *limit set* of Γ; this set contains 1, 2 or infinitely many points and one says that Γ is non elementary when Λ_Γ is infinite. A point $x \in \Lambda_\Gamma$ is said to be *radial* when it is approached by orbit points in some M-neighborhood of any given ray issued from x, for some $M > 0$).

The group Γ is said to be *convergent* if $P_\Gamma(\mathbf{x}, \mathbf{y}, \delta_\Gamma) < \infty$, and *divergent* otherwise. Divergence can also be understood in terms of dynamics as, by Hopf-Tsuju-Sullivan theorem, it is equivalent to ergodicity and total conservativity of the geodesic flow with respect to the Bowen-Margulis measure m_Γ on the non wandering set of $(\phi_t)_{t\in\mathbb{R}}$ in the unit tangent bundle $T^1\bar{X}$ (see again [14] for a complete account and a definition of m_Γ and for a proof of this equivalence).

The more general statement concerning the asymptotic behavior of $v_\Gamma(\mathbf{x}, \mathbf{y}; R)$ is due to Th. Roblin: if Γ is a non elementary, discrete subgroup of isometries of X with *non-arithmetic length spectrum*[4], then δ_Γ is a true limit and it holds, as $R \to +\infty$,

(i) if $\|m_\Gamma\| = \infty$ then $v_\Gamma(\mathbf{x}, \mathbf{y}; R) = o(e^{\delta_\Gamma R})$,

(ii) if $\|m_\Gamma\| < \infty$, then $v_\Gamma(\mathbf{x}, \mathbf{y}; R) \sim \frac{||\mu_{\mathbf{x}}||\cdot||\mu_{\mathbf{y}}||}{\delta_\Gamma\|m_\Gamma\|} e^{\delta_\Gamma R}$,

where $(\mu_{\mathbf{x}})_{\mathbf{x}\in X}$ denotes the family of Patterson δ_Γ-conformal densities of Γ, and m_Γ the Bowen-Margulis measure on $T^1\bar{X}$. Let us emphasize that in the second case, the group Γ is always divergent while in the first one it can be convergent.

In this paper, we aim to investigate, for a particular class of groups Γ, the asymptotic behavior of the function $v_\Gamma(\mathbf{x}, \mathbf{y}; R)$ when Γ is convergent. As far as we know, the only precise asymptotic for the orbital function of convergent groups holds for groups Γ which are normal subgroups $\Gamma \lhd \Gamma_0$ of a co-compact group Γ_0 for which the quotient Γ_0/Γ is isometric up to a finite factor to the lattice $\mathbb{Z}^k$ for some $k \geq 3$ [13]. The corresponding quotient manifold has infinite Bowen-Margulis measure; in fact, m_Γ is invariant under the action of the group of isometries of $\bar{X}$ which contains subgroups $\simeq \mathbb{Z}^k$.

The finiteness of m_Γ is not easy to establish excepted in the case of *geometrically finite groups* where there exists a precise criteria. Recall that Γ (or the quotient manifold $\bar{X}$) is said to be geometrically finite if its limit set $\Lambda(\Gamma)$ decomposes in the radial limit set and the Γ-orbit of finitely many *bounded parabolic points* $x_1, \ldots, x_\ell$, fixed respectively by some parabolic subgroups $P_i, 1 \leq i \leq \ell$, acting co-compactly on $\partial X \setminus \{x_i\}$; for a complete description of geometrical finiteness in variable negative curvature see [4]. Finite-volume manifolds $\bar{X}$ (possibly non compact) are particular cases of geometrically finite manifolds; in contrast, the manifolds considered in [13] are not geometrically finite.

For geometrically finite groups, the orbital functions v_{P_i} of the parabolic subgroups $P_i, 1 \leq i \leq \ell$, contain the relevant information about the metric inside the cusps, which in turn may imply m_Γ to be finite or infinite. On the one hand, it is proved in [6] that the divergence of the parabolic subgroups $P \subset \Gamma$ implies $\delta_P < \delta_\Gamma$, which in turn yields that Γ is divergent and $\|m_\Gamma\| < \infty$. On the other hand there exist geometrically finite groups with parabolic subgroups P satisfying $\delta_P = \delta_\Gamma$: we call such groups *exotic* and say that the parabolic subgroup P (or the corresponding cusps $\mathcal{C}$) is *dominant* when $\delta_P = \delta_\Gamma$. Let us emphasize that dominant parabolic subgroups of exotic geometrically finite groups Γ are necessarily convergent. However, the group Γ itself may as well be convergent or divergent; we refer to [6] and [12] for explicit constructions of such groups.

In this paper, we consider a Schottky product Γ of elementary subgroups $\Gamma_1, \ldots, \Gamma_{p+q}$, of isometries of X (see §3 for the definition) with $p + q \geq 3$. Such a group is geometrically finite. We assume that Γ is convergent; thus, by [6], it is exotic and possesses factors (say $\Gamma_1, \ldots, \Gamma_p, p \geq 1$) which are dominant parabolic subgroups of Γ. We assume that, up to the dominant factor $e^{\delta_\Gamma R}$, the orbital functions $v_{\Gamma_j}(\mathbf{x}, \mathbf{y}, \cdot)$ of these groups satisfy some asymptotic condition of polynomial decay at infinity. More precisely we have the

Theorem 1.1 *Fix $p, q \in \mathbb{N}$ such that $p \geq 1, p + q \geq 2$ and let Γ be a Schottky product of elementary subgroups $\Gamma_1, \Gamma_2 \ldots, \Gamma_{p+q}$ of isometries of a pinched negatively curved manifold X. Assume that the metric g on X satisfies the following assumptions.*

$\mathbf{H_1}$. *The group Γ is convergent with Poincaré exponent $\delta_\Gamma = \delta$.*

[4] It means that the set $\mathcal{L}(\bar{X}) = \{\ell(\gamma) \ ; \ \gamma \in \Gamma\}$ of lengths of closed geodesics of $\bar{X} = \Gamma\backslash X$ is not contained in a discrete subgroup of $\mathbb{R}$

$\mathbf{H_2}$. *There exist $\alpha \in]1,2[$, a slowly varying function L [5] and strictly positive constants $c_1, \ldots, c_p$ such that, for any $1 \le j \le p$ and $\Delta > 0$,*

$$\lim_{R\to+\infty} \frac{R^\alpha}{L(R)} \sum_{\substack{\gamma\in\Gamma_j \\ R\le d(\mathbf{o},\gamma\cdot\mathbf{o})<R+\Delta}} e^{-\delta d(\mathbf{o},\gamma\cdot\mathbf{o})} = c_j \Delta. \tag{1}$$

$\mathbf{H_3}$. *For any $p+1 \le j \le p+q$ and $\Delta > 0$,*

$$\lim_{R\to+\infty} \frac{R^\alpha}{L(R)} \sum_{\substack{\gamma\in\Gamma_j \\ R\le d(\mathbf{o},\gamma\cdot\mathbf{o})<R+\Delta}} e^{-\delta d(\mathbf{o},\gamma\cdot\mathbf{o})} = 0.$$

Then, there exists a constant $C_\Gamma > 0$ such that, as $R \to +\infty$,

$$\sharp\{\gamma \in \Gamma \mid d(\mathbf{o}, \gamma \cdot \mathbf{o}) \le R\} \quad \sim \quad C_\Gamma \frac{L(R)}{R^\alpha} e^{\delta R}.$$

The importance of the convergence hypothesis $\mathbf{H}_1$ in the previous theorem is illustrated by the following result, previous work of one of the authors.

Theorem 1.2 ([15], Theorem C) *Let Γ be a Schottky product of $p+q \ge 2$ elementary subgroups $\Gamma_1, \Gamma_2 \ldots, \Gamma_{p+q}$ of isometries of a pinched negatively curved manifold X. Assume that $p \ge 1$ and*
- *Γ is* divergent *and $\delta_\Gamma = \delta$,*
- *Hypotheses $\mathbf{H}_2$, $\mathbf{H}_3$ hold.*

Then, there exists $C_\Gamma > 0$ such that, as $R \to +\infty$,

$$\sharp\{\gamma \in \Gamma \mid d(\mathbf{o}, \gamma \cdot \mathbf{o}) \le R\} \quad \sim \quad C_\Gamma \frac{e^{\delta R}}{R^{2-\alpha} L(R)}.$$

The difference with the equivalent of Theorem 1.1 may surprise, since it is possible to vary smoothly the Riemannian metric $g_{\alpha,L}$ from a divergent to a convergent case, preserving hypotheses $\mathbf{H}_2$ and $\mathbf{H}_3$, cf [12]. Nevertheless, the proof of our Theorem 1.1 will illustrate the reasons of this difference. For groups $\Gamma = \Gamma_1 * ... * \Gamma_{p+q}$ satisfying $\mathbf{H}_2$ and $\mathbf{H}_3$, the counting estimate only depends on elements of the form $\gamma = a_1 \cdots a_k$, whith $a_i \in \Gamma_1 \cup \ldots \cup \Gamma_p$ and where a_i and $a_{i+1}, 1 \le i < k$, do not belong to the same Γ_j. In the divergent case (see the proof of Theorem C in [15]), the asymptotic of $\{\gamma \in \Gamma \mid d(\mathbf{o}, \gamma \cdot \mathbf{o}) \le R\}$ as $R \to +\infty$ only depends on the $\gamma = a_1 \cdots a_k$ with $k >> R$. On the opposite, in the convergent case, the dominant parabolic factors $\Gamma_1, \ldots, \Gamma_p$ are "heavy" and the asymptotic of the orbital function of Γ comes from the $\gamma = a_1 \cdots a_k$ with k bounded independently of R; the number of such isometries γ with $d(\mathbf{o}, \gamma \cdot \mathbf{o}) \le R$ is comparable to $\dfrac{L(R)}{R^\alpha} e^{\delta R}$. By a straightforward adaptation of Proposition 5.4, this last estimate remains valid in the divergent case; nevertheless, the fact that Γ is divergent implies that the contribution of these isometries is negligible.

Remark 1.3 *The condition $\alpha > 1$ assures that the parabolic groups $\Gamma_1, \ldots, \Gamma_p$ are convergent. The additive condition $\alpha < 2$ is used in Proposition 5.3 to obtain a uniform upper bound for the power $\widetilde{P}^k, k \ge 1$ of some operator $\widetilde{P}$ introduced in Section 5; the proof of this Proposition relies on a previous work of one of the authors [15] and is not valid for greater values of α. The analogous of our Theorem 1.1 when $\alpha \ge 2$ remains open.*

The article is organized as follows. In the next section, we recall some backgrounds on negatively curved manifolds, and we construct examples of metrics for which the hypotheses of Theorem 1.1 are satisfied. In section 3, we present Schottky groups and the coding which we use to express our geometric problem in terms of sub-shift of finite type on a countable alphabet. In section 4, we introduce the Ruelle operator for this sub-shift; this is the key analytical tool which is used. Eventually, section 5 is devoted to the proof of Theorem 1.1.

[5] A function $L(t)$ is said to be "slowly varying" it is positive, measurable and $L(\lambda t)/L(t) \to 1$ as $t \to +\infty$ for every $\lambda > 0$.

2 Geometry of negatively curved manifolds

2.1 Generalities

In the sequel, we fix $N \geq 2$ and consider a N-dimensional complete connected Riemannian manifold X with metric g whose sectional curvatures satisfy : $-B^2 \leq K_X \leq -A^2 < 0$ for fixed constants A and B; the metric g we consider in this paper be obtained by perturbation of a hyperbolic one and the curvature equal -1 on large subsets of X, thus we assume $0 < A \leq 1 \leq B$. We denote d the distance on X induced by the metric g.

Let ∂X be the boundary at infinity of X and let us fix an origin $\mathbf{o} \in X$. The family of functions $(\mathbf{y} \mapsto d(\mathbf{o}, \mathbf{x}) - d(\mathbf{x}, \mathbf{y}))_{\mathbf{x} \in X}$ converges uniformly on compact sets to the *Busemann function* $\mathcal{B}_x(\mathbf{o}, \cdot)$ for $\mathbf{x} \to x \in \partial X$. The *horoballs* $\mathcal{H}_x$ and the *horospheres* $\partial\mathcal{H}_x$ centered at x are respectively the sup-level sets and the level sets of the function $\mathcal{B}_x(\mathbf{o}, \cdot)$. For any $t \in \mathbb{R}$, we set $\mathcal{H}_x(t) := \{\mathbf{y}/\mathcal{B}_x(\mathbf{o}, \mathbf{y}) \geq t\}$ and $\partial\mathcal{H}_x(t) := \{\mathbf{y}/\mathcal{B}_x(\mathbf{o}, \mathbf{y}) = t\}$; the parameter $t = \mathcal{B}_x(\mathbf{o}, \mathbf{y}) - \mathcal{B}_x(\mathbf{o}, \mathbf{x})$ is the *height* of $\mathbf{y}$ with respect to x. When no confusion is possible, we omit the index $x \in \partial X$ denoting the center of the horoball. Recall that the Busemann function satisfies the fundamental cocycle relation: for any $x \in \partial X$ and any $\mathbf{x}, \mathbf{y}, \mathbf{z}$ in X

$$\mathcal{B}_x(\mathbf{x}, \mathbf{z}) = B_x(\mathbf{x}, \mathbf{y}) + B_x(\mathbf{y}, \mathbf{z}).$$

The Gromov product between $x, y \in \partial X \cong \partial X$, $x \neq y$, is defined as

$$(x|y)_{\mathbf{o}} = \frac{\mathcal{B}_x(\mathbf{o}, \mathbf{z}) + \mathcal{B}_y(\mathbf{o}, \mathbf{z})}{2}$$

where $\mathbf{z}$ is any point on the geodesic (x, y) joining x to y. By [3]), the expression

$$D(x, y) = e^{-A(x|y)_{\mathbf{o}}}$$

defines a distance on ∂X satisfying the following property: for any $\gamma \in \Gamma$

$$D(\gamma \cdot x, \gamma \cdot y) = e^{-\frac{A}{2}\mathcal{B}_x(\gamma^{-1}\cdot\mathbf{o}, \mathbf{o})} e^{-\frac{A}{2}\mathcal{B}_y(\gamma^{-1}\cdot\mathbf{o}, \mathbf{o})} D(x, y).$$

In other words, the isometry γ acts on $(\partial X, D)$ as a conformal transformation with coefficient of conformality $|\gamma'(x)|_{\mathbf{o}} = e^{-A\mathcal{B}_x(\gamma^{-1}\cdot\mathbf{o}, \mathbf{o})}$ at x and satisfies the following equality

$$D(\gamma \cdot x, \gamma \cdot y) = \sqrt{|\gamma'(x)|_{\mathbf{o}} |\gamma'(y)|_{\mathbf{o}}} D(x, y). \tag{2}$$

The function $x \mapsto \mathcal{B}_x(\gamma^{-1} \cdot \mathbf{o}, \mathbf{o})$ plays a central role to describe the action of the isometry γ on the boundary at infinity ∂X. From now on, we denote it $b(\gamma, \cdot)$ and notice that it satisfies the following "cocycle property": for any isometries γ_1, γ_2 of X and any $x \in \partial X$

$$b(\gamma_1\gamma_2, x) = b(\gamma_1, \gamma_2 \cdot x) + b(\gamma_2, x). \tag{3}$$

In order to describe the action on ∂X of the isometries of (X, g), it is useful to control precisely the behavior of the sequences $|(\gamma^n)'(x)|_{\mathbf{o}}$; the following fact provides a useful estimation of these quantities.

Fact 2.1 *(1) For any hyperbolic isometry h with repulsive and attractive fixed point $x_h^- = \lim_{n\to+\infty} h^{-n} \cdot \mathbf{o}$ and $x_h^+ = \lim_{n\to+\infty} h^n \cdot \mathbf{o}$ respectively, it holds*

$$b(h^{\pm n}, x) = d(o, h^{\pm n} \cdot o) - 2(x_h^{\pm}|x)_o + \epsilon_x(n)$$

with $\lim_{n\to+\infty} \epsilon_x(n) = 0$, the convergence being uniform on the compact sets of $\partial X \setminus \{x_h^{\mp}\}$.

(2) For any parabolic group $\mathcal{P}$ with fixed point $x_{\mathcal{P}} := \lim_{\substack{p\in\mathcal{P} \\ p\to+\infty}} p \cdot o$, it holds

$$b(p, x) = d(o, p \cdot o) - 2(x_{\mathcal{P}}|x)_o + \epsilon_x(p)$$

with $\lim_{\substack{p\in\mathcal{P} \\ p\to+\infty}} \epsilon_x(p) = 0$, the convergence being uniform on the compact sets of $\partial X \setminus \{x_{\mathcal{P}}\}$.

2.2 On the existence of convergent parabolic groups

In this section, we recall briefly the construction presented in [12] of convergent parabolic groups satisfying condition (1), up to a bounded term; we refer to [12] for the details.

We consider on $\mathbb{R}^{N-1}\times\mathbb{R}$ a Riemannian metric of the form $g = T^2(t)\mathrm{d}x^2 + \mathrm{d}t^2$ at point $\mathbf{x} = (x,t)$ where $\mathrm{d}x^2$ is a fixed euclidean metric on $\mathbb{R}^{N-1}$ and $T : \mathbb{R}\to\mathbb{R}^{*+}$ is a C^∞ non-increasing function. The group of isometries of g contains the isometries of $\mathbb{R}^{N-1}\times\mathbb{R}$ fixing the last coordinate. The sectional curvature at $\mathbf{x} = (x,t)$ equals $K_g(t) = -\dfrac{T''(t)}{T(t)}$ on any plane $\Big\langle \dfrac{\partial}{\partial X_i}, \dfrac{\partial}{\partial t}\Big\rangle, 1 \le i \le N-1$, and $-K_g^2(t)$ on any plane $\Big\langle \dfrac{\partial}{\partial X_i}, \dfrac{\partial}{\partial X_j}\Big\rangle, 1 \le i < j \le N-1$. Note that g has negative curvature if and only if T is convex; when $T(t) = e^{-t}$, one obtains a model of the hyperbolic space of constant curvature -1.

It is convenient to consider the non-decreasing function

$$u : \left\{ \begin{array}{lcl} \mathbb{R}^{*+} & \to & \mathbb{R} \\ s & \mapsto & T^{-1}(\frac{1}{s}) \end{array} \right. \tag{4}$$

which satisfies the following implicit equation $T(u(s)) = \dfrac{1}{s}$. The hyperbolic metric with constant curvature -1 correspond to the function $u(s) = \log s$. This function u is of interest since it gives precise estimates (up a *bounded* term) of the distance between points lying on the same horosphere $\mathcal{H}_t := \{(x,t) : x \in \mathbb{R}^{N-1}\}$ where $t \in \mathbb{R}$ is fixed. Namely, the distance between $\mathbf{x}_t := (x,t)$ and $\mathbf{y}_t := (y,t)$ for the metric $T^2(t)\mathrm{d}x^2$ induced by g on $\mathcal{H}_t$ is equal to $T(t)\|x-y\|$. Hence, this distance equals 1 when $t = u(\|x-y\|)$ and the union of the 3 segments $[\mathbf{x}_0,\mathbf{x}_t]$, $[\mathbf{x}_t,\mathbf{y}_t]$ and $[\mathbf{y}_t,\mathbf{y}_0]$ lies at a bounded distance of the hyperbolic geodesic joigning $\mathbf{x}_0$ and $\mathbf{y}_0$ (see [6], lemme 4) : this readily implies that $d(\mathbf{x}_0,\mathbf{y}_0) - 2u(\|x-y\|)$ is bounded.

The "curvature" function K_g may be expressed in term of u as follows:

$$K_g(u(s)) := -\frac{T''(u(s))}{T(u(s))} = -\frac{2u'(s) + su''(s)}{s^2(u'(s))^3}. \tag{5}$$

For any $\alpha \ge 0$, let us consider the non decreasing C^2-function $u = u_\alpha$ from $\mathbb{R}^{*+}$ to $\mathbb{R}$ such that

$$(i) \quad u_\alpha(s) = \log s \text{ if } 0 < s \le 1 \quad \text{and} \quad (ii) \quad u_\alpha(s) = \log s + \alpha\log\log s \quad \text{if} \quad s \ge s_\alpha$$

for some constant $s_\alpha > 1$ to be chosen in the following way. Using formula (5) and following Lemma 2.2 in [12], for any $A \in]0,1[$, one may choose $s_\alpha > 1$ in such a way the metric $g_\alpha = T^2_{u_\alpha}(t)\mathrm{d}x^2 + \mathrm{d}t^2$ on $\mathbb{R}^{N-1}\times\mathbb{R}$ has pinched negative curvature on X, bounded from above by $-A^2$. Let us emphasize that this metric coïncides with the hyperbolic one on the subset $\mathbb{R}^{N-1}\times\mathbb{R}^-$ and that we can enlarge this subset shifting the metric g_α along the axis $\{0\}\times\mathbb{R}$ as far as we want (see [12] § 2.2).

Now, let $\mathcal{P}$ be a discrete group of isometries of $\mathbb{R}^{N-1}$ spanned by k linearly independent translations $p_{\vec{\tau}_1}, \cdots, p_{\vec{\tau}_k}$ in $\mathbb{R}^{N-1}$. For any $\mathbf{n} = (n_1,\cdots,n_k) \in \mathbb{Z}^k$, we set $\vec{\mathbf{n}} = n_1\vec{\tau}_1 + \cdots + n_k\vec{\tau}_k$. The translations $p_{\vec{\mathbf{n}}}$ are also isometries of $(\mathbb{R}^N, g_\alpha)$ and the corresponding Poincaré series of $\mathcal{P}$ is given by, up to finitely many terms,

$$\begin{aligned} P_{\mathcal{P}}(s) = \sum_{\|\vec{\mathbf{n}}\| > s_{\alpha,\beta}} e^{-sd(\mathbf{o}, p_{\vec{\mathbf{n}}}\cdot\mathbf{o})} &= \sum_{\|\vec{\mathbf{n}}\| > s_{\alpha,\beta}} e^{-2su(\|\vec{\mathbf{n}}\|) - sO(1)} \\ &= \sum_{\|\vec{\mathbf{n}}\| > s_{\alpha,\beta}} \frac{e^{-sO(1)}}{\|\vec{\mathbf{n}}\|^{2s}\Big(\log\|\vec{\mathbf{n}}\|\Big)^{2s\alpha}}. \end{aligned}$$

Thus, the Poincaré exponent of $\mathcal{P}$ equals $k/2$ and $\mathcal{P}$ is convergent if and only if $\alpha > 1$.

Remark 2.2 *We can construct other similar metrics as follows. For $\alpha > 1$, $\beta > 0$, there exists $s_{\alpha,\beta} > 0$ and $u_{\alpha,\beta} : (0,+\infty) \to \mathbb{R}$ such that*

(i) $u_{\alpha,\beta}(s) = \log s$ *if* $0 < s \le 1$,

(ii) $u_{\alpha,\beta}(s) = \log s + \alpha\log\log s + \beta\log\log\log s$ *if* $s \ge s_{\alpha,\beta}$,

(iii) $K_g(u(s)) \leq -A$.
Hence, the Poincaré series of the parabolic subgroup $\mathcal{P}$ with respect to the metric $g_{\alpha,\beta} = T^2_{u_{\alpha,\beta}}(t)^2\mathrm{d}x^2 + \mathrm{d}t^2$ is given by, up to finitely many terms,

$$\begin{aligned}
P_{\mathcal{P}}(s) = \sum_{\|\vec{\mathbf{n}}\| > s_{\alpha,\beta}} e^{-sd(\mathbf{o}, p_{\vec{\mathbf{n}}}\cdot \mathbf{o})} &= \sum_{\|\vec{\mathbf{n}}\| > s_{\alpha,\beta}} e^{-2su(\|\vec{\mathbf{n}}\|) - sO(1)} \\
&= \sum_{\|\vec{\mathbf{n}}\| > s_{\alpha,\beta}} \frac{e^{-sO(1)}}{\|\vec{\mathbf{n}}\|^{2s}\Big(\log\|\vec{\mathbf{n}}\|\Big)^{2s\alpha}\Big(\log\log\|\vec{\mathbf{n}}\|\Big)^{2s\beta}}.
\end{aligned}$$

This implies that $\mathcal{P}$ converges as soon as $\alpha > 1$ but it is not enough to ensure that $\mathcal{P}$ satisfy hypothesis (1). In the next paragraph, we present new metrics g_α, close to those presented in the present section, for which it holds

$$d(o, p_{\vec{\mathbf{n}}} \cdot o) = 2\left(\log\|\vec{\mathbf{n}}\| + \alpha \log\log\|\vec{\mathbf{n}}\|\right) + C + \epsilon(n),$$

where $C \in \mathbb{R}$ is a constant and $\lim\limits_{n\to+\infty} \epsilon(n) = 0$.

2.3 On convergent parabolic groups satisfying condition (1)

Let us fix $N = 2, \alpha > 1$ and a slowly varying function $L : [0, +\infty[\to \mathbb{R}^{*+}$. We construct in this section a metric $g = g_{\alpha,L} = T^2(t)\mathrm{d}x^2 + \mathrm{d}t^2$ on $\mathbb{R}\times\mathbb{R}$ such that the group spanned by the translation $(x,t) \mapsto (x+1, t)$ satisfies our hypothesis (1). The generalization to higher dimension is immediate.

For any real t greater than some $\mathfrak{a} > 0$ to be chosen, let us set

$$T(t) = T_{\alpha,L}(t) = e^{-t}\frac{t^\alpha}{L(t)}.$$

Without loss of generalities, we assume that L is C^∞ on $\mathbb{R}^+$ and its derivates $L^{(k)}, k \geq 1$, satisfy $L^{(k)}(t) \longrightarrow 0$ and $\dfrac{L'(t)}{L(t)} \to 0$ as $t \to +\infty$ ([2], Section 1.3). Furthermore, for any $\theta > 0$, there exist $t_\theta \geq 0$ and $C_\theta \geq 1$ such that for any $t \geq t_\theta$

$$\frac{1}{C_\theta t^\theta} \leq L(t) \leq C_\theta t^\theta. \tag{6}$$

Notice that $-\dfrac{T''(t)}{T(t)} = -\left(1 - \dfrac{2\alpha}{t} + L'(t)\right)^2 + \left(\dfrac{\alpha}{t^2} + L''(t)\right) < 0$ for $t \geq \mathfrak{a}$.

We assume that $0 < A < 1 < B$ and, following Lemma 2.2 in [12], extend $T_{\alpha,L}$ on $\mathbb{R}$ as follows.

Lemma 2.3 *There exists $\mathfrak{a} = \mathfrak{a}_{\alpha,L} > 0$ such that the map $T = T_{\alpha,L} : \mathbb{R} \to (0, +\infty)$ defined by*

- $T(t) = e^{-t}$ *for* $t \leq 0$,
- $T(t) = e^{-t}\frac{t^\alpha}{L(t)}$ *for* $t \geq \mathfrak{a}_{\alpha,L}$,

admits a decreasing and 2-times continuously differentiable extension on $\mathbb{R}$ satisfying the following inequalities

$$-B \leq K(t) = -\frac{T''(t)}{T(t)} \leq -A < 0.$$

Notice that this property holds for any $t' \geq t_{\alpha,L}$; **for technical reasons** (see Lemma 2.7), we assume that $\mathfrak{a} > 4\alpha$. A direct computation yields the following estimate for the function $u = u_{\alpha,L}$ given by the implicit equation (4).

Lemma 2.4 *Let $u = u_{\alpha,L} : (0, +\infty) \to \mathbb{R}$ be such $T_{\alpha,L}(u(s)) = \dfrac{1}{s}$ for any $s > 0$. Then*

$$u(s) = \log s + \alpha \log\log s - \log L(\log s) + \epsilon(s)$$

with $\epsilon(s) \to 0$ as $s \to +\infty$.

We now consider the group $\mathcal{P}$ spanned by the translation p of vector $\vec{i} = (1,0)$ in $\mathbb{R}^2$; the map p is an isometry of $(\mathbb{R}^2, g_{\alpha,L})$ which fixes the point $x=\infty$. By Lemma 2.4, it holds

$$d(\mathbf{o}, p^n\cdot \mathbf{o}) = 2\Big(\log n + \alpha \log\log n - \log L(\log n)\Big)$$

up to a bounded term. Hence, the group $\mathcal{P}$ has critical exponent $\frac{1}{2}$; furthermore, it is convergent since $\alpha > 1$. [(6)] The following proposition ensures that $\mathcal{P}$ satisfies hypothesis (1); in other words, the "bounded term" mentioned above tends to 0 as $n \to +\infty$.

Proposition 2.5 *The parabolic group $\mathcal{P} = \langle p\rangle$ on $(\mathbb{R}^2, g_{\alpha,L})$ satisfies the following property: for any $n\in\mathbb{N}$,*

$$d(\mathbf{o}, p^n\cdot \mathbf{o}) = 2\Big(\log n + \ \alpha \log\log n - \log L(\log n)\Big) + \epsilon(n)$$

with $\lim\limits_{n\to+\infty}\epsilon(n) = 0$. In particular, if $\alpha>1$, then $\mathcal{P}$ is convergent with respect to g_α.

Let $\mathcal{H} = \mathbb{R}\times[0,+\infty)$ be the upper half plane $\{(x,t)\mid t\geq 0\}$ and $\mathcal{H}/\mathcal{P}$ the quotient cylinder endowed with the metric $g_{\alpha,L} = T_{\alpha,L}(t)^2\mathrm{d}x^2 + \mathrm{d}t^2$. We do not estimate directly the distances $d(\mathbf{o}, p^n\cdot\mathbf{o})$, since the metric $g_{\alpha,L}$ is not known explicitely for $t\in[0,\mathfrak{a}]$. Let us introduce the point $\mathbf{a} = (0,\mathfrak{a})\in\mathbb{R}^2$. The union of the three geodesic segments $[\mathbf{o},\mathbf{a}]$, $[\mathbf{a}, p^n\cdot\mathbf{a})]$ and $[p^n\cdot\mathbf{a}, p^n\cdot\mathbf{o}]$ is a quasi-geodesic; more precisely, since $d(\mathbf{o},\mathbf{a}) = d(p^n\cdot\mathbf{o}, p^n\cdot\mathbf{a})$ is fixed and $d(\mathbf{a}, p^n\cdot\mathbf{a})\to+\infty$, the following statement holds.

Lemma 2.6 *Under the previous notations,*

$$\lim_{n\to+\infty} d(\mathbf{o}, p^n\cdot\mathbf{o}) - d(\mathbf{a}, p^n\cdot\mathbf{a}) = 2\mathfrak{a}.$$

Proposition 2.5 follows from the following lemma.

Lemma 2.7 *Assume that $\mathfrak{a}\geq 4\alpha$. Then*

$$d(\mathbf{a}, p^n\cdot\mathbf{a}) = 2(\log n + \ \alpha\log\log n - \log L(\log n) - \mathfrak{a}) + \epsilon(n)$$

with $\lim\limits_{n\to+\infty}\epsilon(n) = 0$.

Proof. Throughout this proof, we work on the upper half-plane $\mathbb{R}\times[\mathfrak{a},+\infty[$ whose points are denoted $(x,\mathfrak{a}+t)$ with $x\in\mathbb{R}$ and $t\geq 0$; we set

$$\mathcal{T}(t) = T_\alpha(t+\mathfrak{a}) = e^{-\mathfrak{a}-t}\frac{(t+\mathfrak{a})^\alpha}{L(t+\mathfrak{a})}.$$

In these coordinates, the quotient cylinder $\mathbb{R}\times[\mathfrak{a},+\infty[/\mathcal{P}$ is a surface of revolution endowed with the metric $\mathcal{T}(t)^2\mathrm{d}x^2+\mathrm{d}t^2$. For any $n\in\mathbb{Z}$, denote h_n the maximal height at which the geodesic segment $\sigma_n = [\mathbf{a}, p^n\cdot\mathbf{a}]$ penetrates inside the upper half-plane $\mathbb{R}\times[\mathfrak{a},+\infty[$; it tends to $+\infty$ as $n\to\pm\infty$. The relation between n, h_n and $d_n := d(\mathbf{a}, p^n\cdot\mathbf{a})$ may be deduced from the Clairaut's relation ([5], section 4.4, Example 5) :

$$\frac{n}{2} = \mathcal{T}(h_n)\int_0^{\mathfrak{h}_n}\frac{\mathrm{d}t}{\mathcal{T}(t)\sqrt{\mathcal{T}^2(t)-\mathcal{T}^2(h_n)}} \quad\text{and}\quad d_n = 2\int_0^{h_n}\frac{\mathcal{T}(t)\mathrm{d}t}{\sqrt{\mathcal{T}^2(t)-\mathcal{T}^2(h_n)}}.$$

These identities may be rewritten as

$$\frac{n}{2} = \frac{1}{\mathcal{T}(h_n)}\int_0^{\mathfrak{h}_n}\frac{f_n^2(s)\mathrm{d}s}{\sqrt{1-f_n^2(s)}} \qquad\text{and}\qquad d_n = 2h_n + 2\int_0^{h_n}\Big(\frac{1}{\sqrt{1-f_n^2(s)}}-1\Big)\mathrm{d}s$$

where $f_n(s) := \dfrac{\mathcal{T}(h_n)}{\mathcal{T}(h_n-s)}1_{[0,h_n]}(s)$.

First, for any $s\geq 0$, the quantity $\dfrac{f_n^2(s)}{\sqrt{1-f_n^2(s)}}$ converges towards $\dfrac{e^{-2s}}{\sqrt{1-e^{-2s}}}$ as $n\to+\infty$. In order to use the dominated convergence theorem, we need the following property.

[6] Notice that the group $\mathcal{P}$ also converges when $\alpha=1$ and $\sum\limits_{n\geq 1}\dfrac{L(n)}{n} < +\infty$; this situation is not explore here.

Fact 2.8 *There exists $n_0 > 0$ such that for any $n \geq n_0$ and any $s \geq 0$,*

$$0 \leq f_n(s) \leq f(s) := e^{-s/2}$$

Proof. Assume first $h_n/2 \leq s \leq h_n$; taking $\theta = \alpha/2$ in (6) yields

$$\begin{aligned} 0 \leq f_n(s) &= \left(\frac{\mathfrak{a}+h_n}{\mathfrak{a}+h_n-s}\right)^{\alpha} \frac{L(\mathfrak{a}+h_n-s)}{L(\mathfrak{a}+h_n)} e^{-s} \\ &\leq C_{\alpha/2}^2 \frac{(\mathfrak{a}+h_n)^{3\alpha/2}}{(\mathfrak{a}+h_n-s)^{\alpha/2}} e^{-s} \\ &\leq \frac{C_{\alpha/2}^2}{\mathfrak{a}^{\alpha/2}} (\mathfrak{a}+h_n)^{3\alpha/2} e^{-s} \\ &\leq \frac{C_{\alpha/2}^2}{\mathfrak{a}^{\alpha/2}} (\mathfrak{a}+h_n)^{3\alpha/2} e^{-\frac{h_n}{4}} e^{-\frac{s}{2}} \leq e^{-\frac{s}{2}} \end{aligned}$$

where the last inequality holds if h_n is great enough, only depending on $\mathfrak{a}$ and α.

Assume now $0 \leq s \leq h_n/2$; it holds $\dfrac{1}{2} \leq \dfrac{\mathfrak{a}+h_n-s}{\mathfrak{a}+h_n} \leq 1$ and $0 \leq \frac{s}{\mathfrak{a}+h_n} \leq \min(\frac{1}{2}, \frac{s}{\mathfrak{a}})$. Recall that $L'(t)/L(t) \to 0$ as $t \to +\infty$ and $0 \leq \frac{1}{1-v} \leq e^{2v}$ for $0 \leq v \leq \frac{1}{2}$; hence, for any $\varepsilon > 0$ and n great enough (say $n \geq n_\varepsilon$), there exists $s_n \in (0, s)$ such that

$$\begin{aligned} 0 \leq f_n(s) &= \frac{L(\mathfrak{a}+h_n-s)}{L(\mathfrak{a}+h_n)} \left(\frac{1}{1-\frac{s}{\mathfrak{a}+h_n}}\right)^{\alpha} e^{-s} \\ &\leq \left(1 - s\frac{L'(a+h_n-s_n)}{L(a+h_n)}\right) e^{-(1-\frac{2\alpha}{\mathfrak{a}})s} \\ &\leq (1+\epsilon s) e^{-(1-\frac{2\alpha}{\mathfrak{a}})s} \\ &\leq e^{-(1-\varepsilon-\frac{2\alpha}{\mathfrak{a}})s}. \end{aligned}$$

Consequently, fixing $\epsilon > 0$ in such a way $2\dfrac{\alpha}{\mathfrak{a}} + \epsilon \leq \dfrac{1}{2}$, it yields $0 \leq f_n(s) \leq e^{-s/2}$ for n great enough. □

Therefore,

$$0 \leq \frac{f_n^2(s)}{\sqrt{1-f_n^2(s)}} \leq F(s) := \frac{f^2(s)}{\sqrt{1-f^2(s)}}$$

where the function F is integrable on $\mathbb{R}^+$. By the dominated convergence theorem, it yields

$$\frac{n}{2} = \frac{1+\epsilon(n)}{\mathcal{T}(h_n)} \int_0^{+\infty} \frac{e^{-2s}}{\sqrt{1-e^{-2s}}} \mathrm{d}s = \frac{1+\epsilon(n)}{\mathcal{T}(h_n)}.$$

Consequently $h_n = \log n + \alpha \log\log n - \log L(\log n) - \log 2 - \mathfrak{a} + \epsilon(n)$.

Similarly $\displaystyle\lim_{n\to+\infty} \int_0^{h_n} \Big(\frac{1}{\sqrt{1-f_n^2(s)}} - 1\Big) \mathrm{d}s = \int_0^{+\infty} \Big(\frac{1}{\sqrt{1-e^{-2s}}} - 1\Big) \mathrm{d}s = \log 2$, which yields

$$d_n = 2(\log n + \alpha \log\log n - \log L(\log n) - \mathfrak{a}) + \epsilon(n).$$

□

The Poincaré exponent of $\mathcal{P}$ equals 1/2 and, as $R \to +\infty$,

$$\sharp\{p \in \mathcal{P} \mid 0 \leq d(\mathbf{o}, p\cdot\mathbf{o}) < R\} \sim e^{R/2} \frac{L(R)}{(R/2)^{\alpha}}.$$

Hence, for any $\Delta > 0$,

$$\sharp\{p \in \mathcal{P} \mid R \leq d(\mathbf{o}, p\cdot\mathbf{o}) < R+\Delta\} \sim \frac{1}{2} \int_R^{R+\Delta} e^{t/2} \frac{L(t)}{(t/2)^{\alpha}} \mathrm{d}t \quad \text{as} \quad R \to +\infty$$

and

$$\lim_{R\to+\infty}\frac{R^\alpha}{L(R)}\sum_{\substack{p\in\mathcal{P}\\ R\le d(\mathbf{o},p\cdot\mathbf{o})<R+\Delta}} e^{-\frac{1}{2}d(\mathbf{o},p\cdot\mathbf{o})}=2^{\alpha-1}\Delta$$

which is precisely Hypothesis 1.

2.4 On the existence of non elementary exotic groups

Explicit constructions of exotic groups, i.e. non-elementary groups Γ containing a parabolic $\mathcal{P}$ whose Poinacré exponent equals δ_Γ, have been detailed in several papers; first in [6], then in [12], [7] and [15]. Let us describe them in the context of the metrics $g=g_{\alpha;L}$ presented above.

For any $a>0$ and $t\in\mathbb{R}$, we write

$$T_{\alpha,L,a}=\begin{cases} e^{-t} & \text{if} \quad t\le a\\ e^{-a}T_{\alpha,L}(t-a) & \text{if} \quad t\ge a\end{cases},$$

where $T_{\alpha,L}$ is defined in the previous paragraph. As in [12], we consider the metric on $\mathbb{R}^2$ given by $g_{\alpha,L,a}=T^2_{\alpha,L,a}(t)\mathrm{d}x^2+\mathrm{d}t^2$. It is a complete smooth metric, with pinched negative curvature, and which equals the hyperbolic one on $\mathbb{R}\times(-\infty,a)$. Note that $g_{\alpha,L,0}=g_{\alpha,L}$ and $g_{\alpha,L,+\infty}$ is the hyperbolic metric on $\mathbb{H}^2$. Note the previous subsection, for any $a\in(0,+\infty)$ and any $\tau\in\mathbb{R}^*$, a parabolic group of the form $\mathcal{P}=<(x,t)\mapsto(x+\tau,r)>$ is convergent. This allows to reproduce the construction of a non-elementary group given in [6] and [12].

Let h be a hyperbolic isometry of $\mathbb{H}^2$ and p be a parabolic isometry in Schottky position with h (cf next section for a precise definition). They generate a free group $\Gamma=<h,p>$ which acts discretely without fixed point on $\mathbb{H}^2$. Up to a global conjugacy, we can suppose that p is $(x,t)\mapsto(x+\tau,t)$ for some $\tau\in\mathbb{R}^*$. The surface $S=\mathbb{H}^2/\Gamma$ has a cusp, isometric to $\mathbb{R}/\tau\mathbb{Z}\times(a_0,+\infty)$ for some $a_0>0$. Therefore, we can replace in the cusp the hyperbolic metric by $g_{\alpha,L,a}$ for any $a\ge a_0$; we also denote $g_{\alpha,L,a}$ the lift of $g_{\alpha,L,a}$ to $\mathbb{R}^2$.

For any $n\in\mathbb{Z}^*$, the group $\Gamma_n=<h^n,p>$ acts freely by isometries on $(\mathbb{R}^2,g_{\alpha,L,a})$. It is shown in [6] that, for $n>0$ great enough, the group Γ_n also converges. This provides a family of examples for Theorem 1.1. By [12], if Γ_n is convergent for some $a_0>0$, then there exists $a^*>a_0$ such that for any $a\in[a_0,a^*)$, the group Γ_n acting on $(\mathbb{R}^2,g_{\alpha,L,a})$ is convergent, whereas for $a>a^*$, it has finite Bowen-Margulis measure and hence diverges. In some sense, the case $a=a^*$ is "critical"; it is proved in [12] that Γ also diverges in this case. With additive hypotheses on the tail of the Poincaré series associated to the factors $\Gamma_j,1\le j\le p$ of Γ, P. Vidotto has obtained a precise estimate of the orbital function of Γ in the case when its Bowen-Margulis measure is infinite [15] ; this is the analogous of Theorem 1.1, under slightly more general assumptions.

In [7], the authors propose another approach based on a "strong" perturbation of the metric inside the cusp. Starting from a N-dimensional finite volume hyperbolic manifold with cuspidal ends, they modify the metric far inside one end in such a way the corresponding parabolic group is convergent with Poincaré exponent >1 and turns the fundamental group of the manifold into a convergent group; in this construction, the sectional curvature of the new metric along certain planes is <-4 far inside the modified cusp.

3 Schottky groups: generalities and coding

From now on, we fix two integers $p\ge1$ and $q\ge0$ such that $\ell:=p+q\ge2$ and consider a Schottky group Γ generated by ℓ elementary groups $\Gamma_1,\dots,\Gamma_\ell$ of isometries of X. These elementary groups are in Schottky position, i.e. there exist disjoint closed sets F_j in ∂X such that, for any $1\le j\le\ell$

$$\Gamma_j^*(\partial X\setminus F_j)\subset F_j.$$

The group Γ spanned by the $\Gamma_j,1\le j\le\ell$, is called the Schottky product of the Γ_j's and denoted $\Gamma=\Gamma_1\star\Gamma_2\star\cdots\star\Gamma_\ell$.

In this section, we present general properties of Γ. In particular, we do not require that conditions **H1**, **H2** and **H3** hold; these hypotheses are only needed in the last section of this paper.

By the Klein's tennis table criteria, Γ is the free product of the groups Γ_i; any element in Γ can be uniquely written as the product

$$\gamma = a_1 \dots a_k$$

for some $a_j \in \cup \Gamma_j^*$ with the property that no two consecutive elements a_j belong to the same group. The set $\mathcal{A} = \cup \Gamma_j^*$ is called the *alphabet* of Γ, and $a_1, \dots, a_k$ the *letters* of γ. The number k of letters is the *symbolic length* of γ; let us denote $\Gamma(k)$ the set of elements of Γ with symbolic length k. The last letter of γ plays a special role, and the index of the group it belongs to be denoted by l_γ. Applying Fact 2 one gets

Property 3.1 *There exists a constant $C > 0$ such that*

$$d(\mathbf{o}, \gamma.\mathbf{o}) - C \leq B_x(\gamma^{-1}.\mathbf{o}, \mathbf{o}) \leq d(\mathbf{o}, \gamma.\mathbf{o})$$

for any $\gamma \in \Gamma = \star_i \, \Gamma_i$ and any $x \in \cup_{i \neq l_\gamma} F_i$.

This fact implies in particular the following crucial contraction property [1].

Proposition 3.2 *There exist a real number $r \in]0, 1[$ and $C > 0$ such that for any γ with symbolic length $n \geq 1$ and any x belonging to the closed set $\cup_{i \neq i(\gamma)} F_i$ one has*

$$|\gamma'(x)| \leq C r^n.$$

The following statement, proved in [1], provides a coding of the limit set $\Lambda(\Gamma)$ but the Γ-orbits of the fixed points of the generators.

Proposition 3.3 *Denote by Σ^+ the set of sequences $(a_n)_{n \geq 1}$ for which each letter a_n belongs to the alphabet $\mathcal{A} = \cup \Gamma_i^*$ and such that no two consecutive letters belong to the same group (these sequences are called admissible). Fix a point x_0 in $\partial X \setminus F$. Then*

- *(a) For any $\mathbf{a} = (a_n)_{n \geq 1} \in \Sigma^+$, the sequence $(a_1 \dots a_n \cdot x_0)_{n \geq 1}$ converges to a point $\pi(\mathbf{a})$ in the limit set of Γ, independent on the choice of x_0.*
- *(b) The map $\pi : \Sigma^+ \to \Lambda(\Gamma)$ is one-to-one and $\pi(\Sigma^+)$ is contained in the radial limit set of Γ.*
- *(c) The complement of $\pi(\Sigma^+)$ in the limit set of Γ equals the Γ-orbit of the union of the limits sets $\Lambda(\Gamma_i)$*

From now on, we consider a Schottky product group Γ. Thus, up to a denumerable set of points, the limit set of Γ coincides with $\pi(\Sigma^+)$. For any $1 \leq i \leq \ell$, let $\Lambda_i = \Lambda \cap F_i$ be the closure of the set of those limit points with first letter in Γ_i (not to be confused with the limit set of Γ_i). The following description of $\Lambda = \Lambda(\Gamma)$ be useful:

a) Λ is the finite union of the sets Λ_i,

b) the closes sets $\Lambda_i, 1 \leq i \leq \ell$, are pairwise disjoints,

c) each of these sets is partitioned into a countable number of subsets with disjoint closures :

$$\Lambda_i = \cup_{a \in \Gamma_i^*} \cup_{j \neq i} \; a.\Lambda_j \; .$$

Now, we enlarge the set Λ in order to take into account the finite admissible words. We fix a point $x_0 \notin \cup_j F_j$. There exists a one-to-one correspondence between $\Gamma \cdot x_0$ and Γ; furthermore, the point $\gamma \cdot x_0 \in F_j$ for any $\gamma \in \Gamma^*$ with first letter in Γ_j. We set $\widetilde{\Sigma}_+ = \Sigma^+ \cup \Gamma$ and notice that, by the previous Proposition, the natural map $\pi : \widetilde{\Sigma}_+ \to \Lambda(\Gamma) \cup \Gamma \cdot x_0$ is one-to-one with image $\pi(\Sigma^+) \cup \Gamma \cdot x_0$. Thus we introduce the following notations

a) $\tilde{\Lambda} = \Lambda \cup \Gamma \cdot x_0$;

b) $\tilde{\Lambda}_i = \tilde{\Lambda} \cap F_i$ for any $1 \leq i \leq \ell$.

The set $\tilde{\Lambda}$ is the disjoint union of $\{x_0\}$ and the sets $\tilde{\Lambda}_i, 1 \leq i \leq \ell$; furthermore, each $\tilde{\Lambda}_i$ is partitioned into a countable number of subsets with disjoint closures:

$$\tilde{\Lambda}_i = \cup_{a \in \Gamma_i^*} \cup_{j \neq i} \ a \cdot \tilde{\Lambda}_j \ .$$

The cocycle b defined in (3) play a central role in the sequel. In order to calculate the distance between two points of the orbit $\Gamma \cdot \mathbf{o}$, we consider an extension $\tilde{b}$ of this cocycle defined as follow on $\tilde{\Lambda}$: for any $\gamma \in \Gamma$ and $x \in \tilde{\Lambda}$,

$$\tilde{b}(\gamma, x) := \begin{cases} b(\gamma, x) = \mathcal{B}_x(\gamma^{-1}\mathbf{o}, \mathbf{o}) & \text{if} \quad x \in \Lambda; \\ d(\gamma^{-1} \cdot \mathbf{o}, g \cdot \mathbf{o}) - d(\mathbf{o}, g \cdot \mathbf{o}) & \text{if} \quad x = g \cdot x_0 \quad \text{for some} \quad g \in \Gamma. \end{cases}$$

The cocycle equality (3) is still valid for the function $\tilde{b}$; furthermore, if $\gamma \in \Gamma$ decomposes as $\gamma = a_1 \cdots a_k$, then

$$d(\mathbf{o}, \gamma \cdot \mathbf{o}) = b(a_1, \gamma_2 \cdot x_0) + b(a_2, \gamma_3 \cdot x_0) + \cdots + b(a_k, x_0),$$

where $\gamma_l = a_l \cdots a_k$ for $2 \leq l \leq k$.

4 On the Ruelle operators $\mathcal{L}_s, s \in \mathbb{R}$

In this section, we describe the main properties of the transfer operators $\mathcal{L}_s, s \in \mathbb{R}$, defined formally by: for any function $\phi : \tilde{\Lambda} \to \mathbb{C}$ and $x \in \tilde{\Lambda}$,

$$\mathcal{L}_s\phi(x) = \sum_{\gamma \in \Gamma(1)} \mathbf{1}_{x \notin \tilde{\Lambda}_{l_\gamma}} e^{-s\tilde{b}(\gamma, x)} \phi(\gamma \cdot x) = \sum_{j=1}^{\ell} \sum_{\gamma \in \Gamma_j^*} \mathbf{1}_{x \notin \tilde{\Lambda}_j} e^{-s\tilde{b}(\gamma, x)} \phi(\gamma \cdot x).$$

For any $1 \leq j \leq \ell$, the sequence $(\gamma \cdot \mathbf{o})_{\gamma \in \Gamma_j}$ accumulates on the fixed point(s) of Γ_j. So for any $x \notin \tilde{\Lambda}_j$, the sequence $\left(\tilde{b}(\gamma, x) - d(\mathbf{o}, \gamma.\mathbf{o})\right)_{\gamma \in \Gamma_j}$ is bounded uniformly in $x \notin \tilde{\Lambda}_j$. Therefore the quantity $\mathcal{L}_s 1(x)$ is well defined as soon as $s \geq \delta := \max\{\delta_{\Gamma_j} \mid 1 \leq j \leq \ell\}$. The powers of $\mathcal{L}_s, s \geq \delta$, are formally given by: for any $k \geq 1$, any function $\phi : \tilde{\Lambda} \to \mathbb{C}$ and any $x \in \tilde{\Lambda}$,

$$\mathcal{L}_s^k \phi(x) = \sum_{\gamma \in \Gamma(k)} \mathbf{1}_{x \notin \tilde{\Lambda}_j} e^{-s\tilde{b}(\gamma, x)} \phi(\gamma \cdot x).$$

It is easy to check that the operator $\mathcal{L}_s, s \geq \delta$, act on $(C(\tilde{\Lambda}), |\cdot|_\infty)$; we denote $\rho_s(\infty)$ it spectral radius on this space.

4.1 Poincaré series versus Ruelle operators

By the "ping-pong dynamic" between the subgroups $\Gamma_j, 1 \leq j \leq \ell$, and Property 3.1, we easily check that the difference $\tilde{b}(\gamma, x) - d(\mathbf{o}, \gamma \cdot \mathbf{o})$ is bounded uniformly in $k \geq 0, \gamma \in \Gamma(k)$ and $x \notin \tilde{\Lambda}_{l_\gamma}$. Consequently, there exists a constant $C > 0$ such that, for any $x \in \tilde{\Lambda}$, any $k \geq 1$ and any $s \geq \delta$,

$$\mathcal{L}_s^k 1(x) \stackrel{c}{\asymp} \sum_{\gamma \in \Gamma(k)} e^{-sd(\mathbf{o}, \gamma \cdot \mathbf{o})}$$

where $A \stackrel{c}{\asymp} B$ means $\frac{A}{c} \leq B \leq cA$. Hence,

$$P_\Gamma(s) := \sum_{\gamma \in \Gamma} e^{-sd(\mathbf{o}, \gamma \cdot \mathbf{o})} = +\infty \quad \Longleftrightarrow \quad \sum_{k \geq 0} \mathcal{L}_s^k 1(x) = +\infty. \tag{7}$$

In particular

$$\delta_\Gamma = \sup\{s \geq \delta \mid \rho_s(\infty) \geq 1\} = \inf\{s \geq \delta \mid \rho_s(\infty) \leq 1\}. \tag{8}$$

It is proved in the next paragraph that Γ is convergent if and only if $\rho_\delta(\infty) < 1$.

4.2 On the spectrum of the operators $\mathcal{L}_s, s \geq \delta$

In order to control the spectral radius (and the spectrum) of the transfer operators $\mathcal{L}_s$, we study their restriction to the space $\mathbf{Lip}(\tilde\Lambda)$ of Lipschitz functions from $\tilde\Lambda$ to $\mathbb{C}$ defined by

$$\mathbf{Lip}(\tilde\Lambda) = \{\phi \in C(\tilde\Lambda);\ \|\phi\| = |\phi|_\infty + [\phi] < +\infty\}$$

where $[\phi] = \sup_{0\leq i\leq p} \sup_{\substack{x,y\in\tilde\Lambda_j\\ x\neq y}} \dfrac{|\phi(x)-\phi(y)|}{D(x,y)}$ is the Lipschitz coefficient of ϕ on $(\partial X, D)$.

The space $(\mathbf{Lip}(\tilde\Lambda), \|.\|)$ is a Banach space and the identity map from $(\mathbf{Lip}(\tilde\Lambda), \|.\|)$ into $(C(\tilde\Lambda), |.|_\infty)$ is compact. It is proved in [1] that the operators $\mathcal{L}_s, s \geq \delta$, act both on $(C(\Lambda), |\cdot|_\infty)$ and $(\mathbf{Lip}(\Lambda), \|\cdot\|)$; P. Vidotto has extended in [15] this property to the Banach spaces $(C(\tilde\Lambda), |\cdot|_\infty)$ and $(\mathbf{Lip}(\tilde\Lambda), \|\cdot\|)$. We denote ρ_s the spectral radius of $\mathcal{L}_s$ on $\mathbf{Lip}(\tilde\Lambda)$; in the following proposition, we state the spectral properties of the $\mathcal{L}_s$ we need in the present paper.

Proposition 4.1 *We assume $\ell = p+q \geq 3$ [7]. For any $s \geq \delta$,*

1. *$\rho_s = \rho_s(\infty)$;*
2. *ρ_s is a simple eigenvalue of $\mathcal{L}_s$ acting on $\mathbf{Lip}(\tilde\Lambda)$ and the associated eigenfunction h_s is non negative on $\tilde\Lambda$;*
3. *there exists $0 \leq r < 1$ such that the rest of the spectrum of $\mathcal{L}_s$ on $\mathbf{Lip}(\tilde\Lambda)$ is included in a disc of radius $\leq r\rho_s$.*

Sketch of the proof. We refer to [1] and [15] for the details. For any $s \geq 0$ and γ in Γ^*, let $w_s(\gamma, .)$ be the *weight function* defined on $\Lambda(\Gamma)$ by: for any $s \geq \delta$ and $\gamma \in \Gamma$

$$w_s(\gamma, x) := \begin{cases} e^{-s\tilde b(\gamma,x)} & \text{if} \qquad x \in \tilde\Lambda_j, j \neq l_\gamma, \\ 0 & \text{otherwise.} \end{cases}$$

Observe that these functions satisfy the following cocycle relation : if $\gamma_1, \gamma_2 \in \mathcal{A}$ do not belong to the same group Γ_j, then

$$w_s(\gamma_1\gamma_2, x) = w_s(\gamma_1, \gamma_2\cdot x)w_s(\gamma_2, x).$$

Due to this cocycle property, we may write, for any $k \geq 1$, any bounded function $\varphi : \tilde\Lambda \to \mathbb{R}$ and any $x \in \tilde\Lambda$

$$\mathcal{L}_s^k\varphi(x) = \sum_{\gamma\in\Gamma(k)} w_s(\gamma, x)\varphi(\gamma\cdot x).$$

In [1], it is proved that the restriction of the functions $w_s(\gamma, .)$ to the set Λ belong to $\mathbf{Lip}(\Lambda)$ and that for any $s \geq \delta$ there exists $C = C(s) > 0$ such that, for any γ in Γ^*

$$\|w_s(\gamma, .)\| \leq Ce^{-sd(\mathbf{o},\gamma.\mathbf{o})}.$$

In [15], Proposition 8.5, P. Vidotto has proved that the same inequality holds for the functions $w_s(\gamma, .)$ on $\tilde\Lambda$. Thus, the operator $\mathcal{L}_s$ is bounded on $\mathbf{Lip}(\tilde\Lambda)$ when $s \geq \delta$.

In order to describe its spectrum on $\mathbf{Lip}(\tilde\Lambda)$, we first write a "contraction property" for the iterated operators $\mathcal{L}_s^k$; indeed,

$$|\mathcal{L}_s^k\varphi(x) - \mathcal{L}_s^k\varphi(y)| \leq \sum_{\gamma\in\Gamma(k)} |w_s(\gamma, x)|\ |\varphi(\gamma\cdot x) - \varphi(\gamma\cdot y)| + \sum_{\gamma\in\Gamma(k)} [w_s(\gamma, .)]\ |\varphi|_\infty D(x, y).$$

By Proposition 3.2 and the mean value relation (2), there exist $C > 0$ and $0 \leq r < 1$ such that $D(\gamma\cdot x, \gamma\cdot y) \leq Cr^kD(x, y)$ whenever $x, y \in \tilde\Lambda_j$, $j \neq l_\gamma$. This leads to the following inequality

$$[\mathcal{L}_s^k\varphi] \leq r_k[\varphi] + R_k|\varphi|_\infty \tag{9}$$

[7] Recall that $\ell \geq 2$ since Γ is non-elementary. When $\ell = 2$, the real $-\rho_s$ is also a a simple eigenvalue of $\mathcal{L}_s$; a similar statement to Proposition 4.1 holds for the restriction of $\mathcal{L}_s$ to each space $\mathbf{Lip}(\tilde\Lambda_i), i = 1, 2$ [1].

where $r_k = \left(Cr^k\right) |\mathcal{L}_s^k 1|_\infty$ and $R_k = \sum_{\gamma\in\Gamma(k)}[w_s(\gamma, .)]$. Observe that

$$\limsup_k r_k^{1/k} = r \limsup_k |\mathcal{L}_s^k 1|_\infty^{1/k} = r\rho_s(\infty)$$

where $\rho_s(\infty)$ is the spectral radius of the positive operator $\mathcal{L}_s$ on $C(\tilde{\Lambda}(\Gamma))$. Inequality (9) is crucial in the Ionescu-Tulcea-Marinescu theorem for quasi-compact operators. By Hennion's work [9], it implies that the essential spectral radius of $\mathcal{L}_s$ on $\mathbf{Lip}(\tilde{\Lambda})$ is less than $r\rho_s(\infty)$; in other words, any spectral value of $\mathcal{L}_s$ with modulus strictly larger than $r\rho_s(\infty)$ is an eigenvalue with finite multiplicity and is isolated in the spectrum of $\mathcal{L}_s$.

This implies in particular $\rho_s = \rho_s(\infty)$. Indeed, the inequality $\rho_s \geq \rho_s(\infty)$ is obvious since the function 1 belongs to $\mathbf{Lip}(\tilde{\Lambda})$. Conversely, the strict inequality would imply the existence of a function $\phi \in \mathbf{Lip}(\tilde{\Lambda})$ such that $\mathcal{L}_s\phi = \lambda\phi$ for some $\lambda \in \mathbb{C}$ of modulus $> \rho_s(\infty)$; this yields $|\lambda||\phi| \leq \mathcal{L}_s|\phi|$ so that $|\lambda| \leq \rho_s(\infty)$. Contradiction.

It remains to control the value ρ_s in the spectrum of $\mathcal{L}_s$. By the above, we know that ρ_s is an eigenvalue of $\mathcal{L}_s$ with (at least) one associated eigenfunction $h_s \geq 0$. This function is strictly positive on $\tilde{\Lambda}$: otherwise, there exist $1 \leq j \leq p+q$ and a point $y_0 \in \tilde{\Lambda}_j$ such that $h_s(y_0) = 0$. The equality $\mathcal{L}_s h_s(y_0) = \rho_s h_s(y_0)$ implies $h_s(\gamma \cdot y_0) = 0$ for any $\gamma \in \Gamma$ with last letter $\neq j$. The minimality of the action of Γ on Λ and the fact that $\Gamma \cdot x_0$ accumulates on Λ implies $h_s = 0$ on $\tilde{\Lambda}$. Contradiction.

In order to prove that ρ_s is a simple eigenvalue of $\mathcal{L}_s$ on $\mathbf{Lip}(\tilde{\Lambda})$, we use a classical argument in probability theory related to the "Doob transform" of a sub-markovian transition operator. For any $s \geq \delta$, we denote P_s the operator defined formally by: for any bounded Borel function $\phi : \tilde{\Lambda} \to \mathbb{C}$ and $x \in \tilde{\Lambda}$,

$$P_s\phi(x) = \frac{1}{\rho h_s(x)}\mathcal{L}(h_s\phi)(x) = \frac{1}{\rho h_s(x)} \sum_{\gamma\in\Gamma(1)} e^{-\delta\tilde{b}(\gamma,x)} h(\gamma \cdot x)\phi(\gamma \cdot x).$$

The iterates of P_s are given by: $P_s^0 = \mathrm{Id}$ and for $k \geq 1$

$$P_s^k\phi(x) = \int_X \phi(y)P_s^k(x, dy) = \frac{1}{\rho_s^k h_s(x)} \sum_{\gamma\in\Gamma(k)} e^{-\delta b(\gamma,x)} h(\gamma \cdot x)\phi(\gamma \cdot x).w \tag{10}$$

The operator P_s acts on $\mathbf{Lip}(\tilde{\Lambda})$ as a Markov operator, i.e. $P_s\phi \geq 0$ if $\phi \geq 0$ and $P_s\mathbf{1} = \mathbf{1}$. It inherits the spectral properties of $\mathcal{L}_s$ and is in particular quasi-compact with essential spectral radius < 1. The spectral value 1 is an eigenvalue and it remains to prove that the associated eigenspace is $\mathbb{C} \cdot 1$. Let $f \in \mathbf{Lip}(\tilde{\Lambda})$ such that $P_s f = f$ and $1 \leq j \leq p+q$ and $y_0 \in \tilde{\Lambda}_j$ such that $|f(y_0)| = |f|_\infty$. An argument of convexity applied to the inequality $P|f| \leq |f|$ readily implies $|f(y_0)| = |f(\gamma \cdot y_0)|$ for any $\gamma \in \Gamma$ with last letter $\neq j$; by minimality of the action of Γ on $\tilde{\Lambda}$, it follows that the modulus of f is constant on $\tilde{\Lambda}$. Applying again an argument of convexity, the minimality of the action of Γ on $\tilde{\Lambda}$ and the fact that $\Gamma \cdot x_0$ accumulates on Λ, one proves that f is in fact constant on $\tilde{\Lambda}$. Finally, the eigenspace of $\mathcal{L}_s$ associated with ρ_s equals $\mathbb{C} \cdot 1$.

Similarly, using the fact that $\ell \geq 3$, we may prove that the peripherical spectrum of $\mathcal{L}_s$, i.e. the eigenvalues λ with $|\lambda| = \rho_s$, is reduced to ρ_s; we refer the reader to Proposition III.4 of [1] and Proposition 8.6 of [15]. □

Expression (10) yields to the following.

Notations 4.2 *For any $s \geq \delta$, any $x \in \tilde{\Lambda}$, any $k \geq 0$ and any $\gamma \in \Gamma(k)$, set*

$$p_s(\gamma, x) \quad := \quad \frac{1}{\rho_s^k}\frac{h_s(\gamma \cdot x)}{h_s(x)} w_s(\gamma, x). \tag{11}$$

As for the $w_s(\gamma, \cdot)$, these "weight functions" are positive and satisfy the cocycle property

$$p_s(\gamma_1\gamma_2, x) = p_s(\gamma_1, \gamma_2 \cdot x) \cdot p_s(\gamma_2, x)$$

for any $s \geq \delta, x \in \tilde{\Lambda}$ and $\gamma_1, \gamma_2 \in \Gamma$. Let us emphasize that $\sum_{\gamma\in\Gamma(k)} p_s(\gamma, x) = 1$; in other words, the operator P_s is markovian.

□

Corollary 4.3 *The group Γ is convergent if and only if $\rho_\delta < 1$.*

Proof. If $\rho_\delta = \rho_\delta(\infty) < 1$ then $\rho_s < 1$ for any $s \geq \delta$, since $s \mapsto \rho_s(\infty) = \rho_s$ is decreasing on $[\delta, +\infty[$. Equality (8) implies $\delta_\Gamma \leq \delta$ and so $\delta_\Gamma = \delta$; by (7), it follows that Γ is convergent.

Assume now $\rho_\delta \geq 1$. When Γ is non exotic, it is divergent by [6]. Otherwise, $\delta_\Gamma = \delta$ and since the eigenfunction h_δ is non negative on $\tilde{\Lambda}$, we have, for any $k \geq 1$ and $x \in \tilde{\Lambda}$

$$\mathcal{L}_\delta^k 1(x) \asymp \mathcal{L}_\delta^k h_\delta(x) = \rho_\delta^k h_\delta(x) \asymp \rho_\delta^k.$$

Consequently $\sum_{k\geq 0} \mathcal{L}_\delta^k 1(x) = +\infty$ and the group Γ is divergent, by (7).

□

5 Counting for convergent groups

Throughout this section we assume that Γ is convergent on (X, g); by Corollary 4.3 it is equivalent to the fact that $\rho_\delta < 1$.

For any $\phi \in \mathbf{Lip}(\tilde{\Lambda})$, any $x \in \tilde{\Lambda}$ and $R > 0$, let us denote by $M(R, \phi \times \cdot\ )(x)$ the measure on $\mathbb{R}$ defined by:

$$M(R, \phi \otimes u)(x) := \sum_{\gamma\in\Gamma} e^{-\delta\tilde{b}(\gamma,x)}\phi(\gamma\cdot x)u(-R + \tilde{b}(\gamma, x)).$$

It holds $0 \leq M(R, \phi \otimes u)(x) < +\infty$ when u has a compact support in $\mathbb{R}$ since the group Γ is discrete.

The orbital function of Γ may be decomposed as

$$N_\Gamma(R) = e^{\delta R} \sum_{n\geq 0} M(R, \mathbf{1} \otimes e_n)(x_0)$$

with $e_n(t) := e^{\delta t}\mathbf{1}_{]-(n+1),-n]}(t)$. Hence, Theorem 1.1 is a direct consequence of the following statement.

Proposition 5.1 *For any positive function $\phi \in \mathbf{Lip}(\tilde{\Lambda})$ and any $x \in \tilde{\Lambda}$, there exists $C_\phi(x) > 0$ such that for any continuous function $u : \mathbb{R} \to \mathbb{R}$ with compact support,*

$$\lim_{R\to+\infty} \frac{R^\alpha}{L(R)} M(R, \phi \otimes u)(x) = C_\phi(x) \int_{\mathbb{R}} u(t)\mathrm{d}t.$$

This section is devoted to the proof of Proposition 5.1. From now on, we fix a positive function $\phi \in \mathbf{Lip}(\tilde{\Lambda})$ and a continuous function $u : \mathbb{R} \to \mathbb{R}^+$ with compact support. Let us decompose $M(R, \phi \otimes u)(x)$ as

$$M(R, \phi \otimes u)(x) = \sum_{k\geq 0} M_k(R, \phi \otimes u)(x)$$

with

$$M_k(R, \phi \otimes u)(x) := \sum_{\gamma\in\Gamma(k)} e^{-\delta\tilde{b}(\gamma,x)}\phi(\gamma\cdot x)u(-R + \tilde{b}(\gamma, x)).$$

Thus, it is natural to associate to $P_s, s \geq \delta$, a new transition operator $\widetilde{P}_s$ on $\tilde{\Lambda} \times \mathbb{R}$, setting: for any $\phi \in \mathbf{Lip}(\tilde{\Lambda})$, any Borel function $v : \mathbb{R} \to \mathbb{R}$ and any $(x, s) \in \tilde{\Lambda}) \times \mathbb{R}$,

$$\begin{aligned}\widetilde{P}_s(\phi \otimes v)(x, t) &= \frac{1}{\rho h_s(x)} \sum_{\gamma\in\Gamma(1)} e^{-s\tilde{b}(\gamma,x)}h_s(\gamma\cdot x)\phi(\gamma\cdot x)u(t + \tilde{b}(\gamma, x)) \\ &= \sum_{\gamma\in\Gamma(1)} p_s(\gamma, x)\phi(\gamma\cdot x)u(t + \tilde{b}(\gamma, x))\end{aligned}$$

Notice that $\widetilde{P}_s$ is a also a Markov operator on $\tilde{\Lambda} \times \mathbb{R}$; it commutes with the action of translations on $\mathbb{R}$ and one usually says that it defines a semi-markovian random walk on $\tilde{\Lambda} \times \mathbb{R}$. Its iterates are given by: $\widetilde{P}_s^0 = \mathrm{Id}$ and, for any $k \geq 1$,

$$\widetilde{P}_s^k(\phi \otimes v)(x, s) = \sum_{\gamma\in\Gamma(k)} p_s\gamma, x)\phi(\gamma\cdot x)u(s + \tilde{b}(\gamma, x)).$$

From now on, to lighten notations we write $P = P_\delta$, $\tilde{P} = \tilde{P}_\delta$, $h = h_\delta$, $p = p_\delta$ and $\rho = \rho_\delta < 1$. We rewrite the quantity $M_k(R, \phi \otimes u)(x)$ as

$$M_k(R, \phi \otimes u)(x) = \rho^k h(x) \widetilde{P}^k \left(\frac{\phi}{h} \otimes u \right)(x, -R),$$

so that,

$$M(R, \phi \otimes u)(x) = h(x) \sum_{k \geq 0} \rho^k \widetilde{P}^k \left(\frac{\phi}{h} \otimes u \right)(x, -R). \tag{12}$$

We first control the behavior as $R \to +\infty$ of the quantity $M_1(R, \phi \otimes u)(x)$.

Proposition 5.2 *For any continuous function $u : \mathbb{R} \to \mathbb{R}$ with compact support, there exists a constant $C_u > 0$ such that, for any $\varphi \in \mathbf{Lip}(\tilde{\Lambda})$, any $x \in \tilde{\Lambda}$ and $R \geq 1$,*

$$\left| \widetilde{P}(\varphi \otimes u)(x, -R) \right| \leq C_u \|\varphi\|_\infty \times \frac{L(R)}{R^\alpha}. \tag{13}$$

Furthermore,

$$\lim_{R \to +\infty} \frac{R^\alpha}{L(R)} \widetilde{P}(\varphi \otimes u)(x, -R) = \sum_{j=1}^{p} C_j(x) \varphi(x_j) \int_{\mathbb{R}} u(t) \mathrm{d}t, \tag{14}$$

where C_j is defined by: for $1 \leq j \leq p$,

$$C_j(x) := c_j \frac{h(x_j)}{\rho h(x)} \times \begin{cases} e^{2\delta(x_j|x)_{\mathbf{o}}} & \text{when} \quad x \in \Lambda \backslash \tilde{\Lambda}_j; \\ e^{\mathcal{B}_{x_j}(o, g \cdot o) + d(o, g \cdot o)} & \text{when} \quad x = g \cdot x_0 \notin \tilde{\Lambda}_j; \\ \\ 0 & \text{otherwise.} \end{cases} \tag{15}$$

Proof. Let $x \in \tilde{\Lambda}$ be fixed and assume that the support of u is included in the interval $[a, b]$. For any $R \geq -a$, it holds

$$\widetilde{P}(\varphi \otimes u)(x, -R) = \frac{1}{\rho h(x)} \sum_{j=1}^{p+q} \sum_{\gamma \in \Gamma_j} e^{-\delta \tilde{b}(\gamma x)} \mathbf{1}_{x \notin \tilde{\Lambda}_j} h(\gamma \cdot x) \varphi(\gamma \cdot x) u(-R + \tilde{b}(\gamma, x)).$$

It follows from hypotheses $\mathbf{H_2}$ and $\mathbf{H_3}$ and Fact 2.1 that for any $j = 1, ..., p+q$, there exists a constant $K_j > 0$ such that for any $R \geq 1$,

$$\sum_{\substack{\gamma \in \Gamma_j \\ R + a \leq \tilde{b}(\gamma, x) \leq R + b}} e^{-\delta \tilde{b}(\gamma, x)} \leq K_j (b - a) \frac{L(R)}{R^\alpha}.$$

Together with the fact that L has slow variation, this implies (13).

Now, in order to establish (14), it is sufficient to prove that for any $j = 1, ..., p+q$,

$$\lim_{R \to +\infty} \frac{R^\alpha}{L(R)} \sum_{\gamma \in \Gamma_j} p(\gamma, x) \varphi(\gamma \cdot x) u(-R + \tilde{b}(\gamma, x)) = C_j(x) \varphi(x_j) \int_{\mathbb{R}} u(t) \mathrm{d}t, \tag{16}$$

where $C_j(x)$ is given by (15) for $1 \leq j \leq p$ and $C_j(x) = 0$ for $j = p+1, ..., q$. By a classical approximation argument, we may assume that u is the characteristic function of the interval $[a, b]$; it yields

$$\sum_{\gamma \in \Gamma_j} p(\gamma, x) \varphi(\gamma \cdot x) u(-R + \tilde{b}(\gamma, x)) = \frac{1}{h(x)} \sum_{\substack{\gamma \in \Gamma_j \\ R + a \leq \tilde{b}(\gamma, x) \leq R + b}} e^{-\delta \tilde{b}(\gamma x)} \mathbf{1}_{x \notin \tilde{\Lambda}_j} h(\gamma \cdot x) \varphi(\gamma \cdot x).$$

First, assume that $x = g \cdot x_0$ belongs to $\Gamma \cdot x_0$. For any $j = 1, ..., p$ and any $\gamma \neq Id$ in Γ_j, the sequence $(\gamma^n \cdot o)_{n \geq 1}$ tends to x_j as $n \to \pm\infty$; it yields

$$\begin{array}{rcl} \tilde{b}(\gamma^n, x) - d(o, \gamma^n \cdot o) & = & d(\gamma^{-n} \cdot o, g \cdot o) - d(\gamma^{-n} \cdot o, o) - d(o, g \cdot o) \\ & \overset{n \to \pm\infty}{\longrightarrow} & -\mathcal{B}_{x_j}(o, g \cdot o) - d(o, g \cdot o). \end{array}$$

When $x \in \Lambda$, Fact 2.1 yields

$$\lim_{n \to \pm\infty} \tilde{b}(\gamma^n, x) - d(o, \gamma^n \cdot o) = -2(x_j \mid x).$$

Eventually, by hypotheses $\mathbf{H_2}$ and $\mathbf{H_3}$, for any $1 \leq j \leq p + q$,

$$\lim_{R \to +\infty} \frac{R^\alpha}{L(R)} \sum_{\substack{\gamma \in \Gamma_j \\ R + a \leq d(o, \gamma \cdot o) \leq R + b}} \tilde{p}(\gamma, x) = C_j(x)|b - a|.$$

Hence,

$$\lim_{R \to +\infty} \frac{R^\alpha}{L(R)} \sum_{\gamma \in \Gamma_j} \tilde{p}(\gamma, x) \varphi(\gamma \cdot x) u(-R + \tilde{b}(\gamma, x)) = C_j(x) \varphi(x_j)|b - a|.$$

□

Now, we extend (13) and (14) to the powers $\widetilde{P}^k, k \geq 1$, of the Markov operator $\widetilde{P}$.

Proposition 5.3 *For any continuous function $u : \mathbb{R} \to \mathbb{R}^+$ with compact support, there exists a constant $C_u > 0$ such that, for any $\varphi \in \mathbf{Lip}(\tilde{\Lambda})$, any $x \in \tilde{\Lambda}$, any $k \geq 1$ and any $R \geq 1$,*

$$\left| \widetilde{P}^k \left(\varphi \otimes u \right) (x, -R) \right| \leq C_u \ k^2 \ \|\varphi\|_\infty \times \frac{L(R)}{R^\alpha}. \tag{17}$$

Proposition 5.4 *For any continuous function $u : \mathbb{R} \to \mathbb{R}^+$ with compact support, any $\varphi \in \mathbf{Lip}(\tilde{\Lambda})$, any $x \in \tilde{\Lambda}$ and any $k \geq 1$,*

$$\lim_{R \to +\infty} \frac{R^\alpha}{L(R)} \widetilde{P}^k \left(\varphi \otimes u \right) (x, -R) = \sum_{j=1}^{p} \left(\sum_{l=0}^{k-1} P^l C_j(x) P^{k-1-l} \varphi(x_j) \right) \int_{\mathbb{R}} u(t) \mathrm{d}t \tag{18}$$

where, for any $1 \leq j \leq p$, the Lipschitz functions is $C_j : \tilde{\Lambda} \to \mathbb{R}$ is given by (15).

Proposition 5.1 follows immediately from these statements and (12). Indeed, Propositions 5.3 and 5.4 and the dominated convergence theorem yield

$$\lim_{R \to +\infty} \frac{R^\alpha}{L(R)} M(R, \phi \otimes u)(x) = \left(h(x) \sum_{k \geq 1} \rho^k \sum_{j=1}^{p} \left(\sum_{l=0}^{k-1} P^l C_j(x) P^{k-1-l} \left(\frac{\phi}{h} \right) (x_j) \right) \right) \times \int_{\mathbb{R}} u(t) \mathrm{d}t.$$

□

Let us now prove Propositions 5.3 and 5.4. For the convenience of the reader, we assume that all subgroups $\Gamma_j, 1 \leq j \leq p + q$, are parabolic. Hence, they have a unique fixed point at infinity x_j and for any $x \in \tilde{\Lambda}$, it holds

$$\lim_{\substack{\gamma \in \Gamma_j \\ d(o, \gamma \cdot o) \to +\infty}} \gamma \cdot x = x_j.$$

Namely, if one of the non-influent elementary group $\Gamma_j, p + 1 \leq j \leq p + q$, was generated by some hyperbolic isometry h_j, we would have in the next proofs to distinguish between positive and negative power of h_j and this would only overcharge our notations without interest.

Proof of Proposition 5.3. We apply here overestimations given in [15], whose proofs follow the approach developed in [8]. We set $\alpha = 1 + \beta$ with $0 < \beta < 1$; this restriction on the values of the parameter β is of major importance to get the following estimations. Following [15], we introduce the non negative sequence $(a_k)_{k\geq 1}$ defined implicitely by $\dfrac{a_k^\beta}{L(a_k)} = k$ for any $k\geq 1$. By Propositions A.1 and A.2 in [15], there exists a constant $C_1 = C_1(u) > 0$ such that, for any $\varphi \in \mathbf{Lip}(\tilde\Lambda)$, any $x\in\tilde\Lambda$, any $k\geq 1$ and any $R\geq 1$,

- if $1\leq R\leq 2a_k$ then $\left|\widetilde{P}^k(\varphi\otimes u)(x,-R)\right| \leq C_1\|\varphi\|_\infty \times \dfrac{1}{a_k}$;
- if $R\geq 2a_k$ then $\left|\widetilde{P}^k(\varphi\otimes u)(x,-R)\right| \leq C_1 k\|\varphi\|_\infty \times \dfrac{L(R)}{R^{1+\beta}}$.

The definition of the a_k yields, for $1\leq R\leq 2a_k$,

$$\frac{1}{a_k} = k\frac{L(a_k)}{a_k^{1+\beta}} \leq \frac{k}{2^{1+\beta}} \times \frac{L(R)}{R^{1+\beta}} \times \frac{L(a_k)}{L(R)}.$$

By Potter's lemma (see [15], lemma 3.4), it exists $C_2>0$ such that $\dfrac{1}{a_k}\leq C_2 k^2 \times \dfrac{L(R)}{R^{1+\beta}}$ for $R\geq 1$ great enough. We set $C=\max(C_1,C_2)$.

□

Proof of Proposition 5.4. We work by induction. By Proposition 5.2, convergence (18) holds for $k=1$. Now, we assume that it holds for some $k\geq 1$. Let $R>0$ and $r\in[0,R/2]$ be fixed. Recall that

$$\begin{aligned}\widetilde{P}^{k+1}(\varphi\otimes u)(x,-R) &= \sum_{\gamma\in\Gamma(k+1)} p(\gamma,x)\varphi(\gamma\cdot x)u(-R+\tilde b(\gamma,x))\\ &= \sum_{\gamma\in\Gamma(k)}\sum_{\beta\in\Gamma(1)} p(\gamma,\beta\cdot x)p(\beta,x)\varphi(\gamma\beta\cdot x)u\Big(-R+\tilde b(\gamma,\beta\cdot x)+\tilde b(\beta,x)\Big).\end{aligned}$$

We decompose $\widetilde{P}^{k+1}(\varphi\otimes u)(x,-R)$ as $A_k(x,r,R)+B_k(x,r,R)+C_k(x,r,R)$ where

$$\begin{aligned}A_k(x,r,R) &:= \sum_{\gamma\in\Gamma(k)}\sum_{\substack{\beta\in\Gamma(1)\\ d(\mathbf{o},\beta\cdot\mathbf{o})\leq r}} p(\gamma,\beta\cdot x)p(\beta,x)\varphi(\gamma\beta\cdot x)u\Big(-R+\tilde b(\gamma,\beta\cdot x)+\tilde b(\beta,x)\Big),\\ B_k(x,r,R) &:= \sum_{\substack{\gamma\in\Gamma(k)\\ d(\mathbf{o},\gamma\cdot\mathbf{o})\leq r}}\sum_{\substack{\beta\in\Gamma(1)\\ d(\mathbf{o},\beta\cdot\mathbf{o})> r}} p(\gamma,\beta\cdot x)p(\beta,x)\varphi(\gamma\beta\cdot x)u\Big(-R+\tilde b(\gamma,\beta\cdot x)+\tilde b(\beta,x)\Big)\\ \text{and}\quad C_k(x,r,R) &:= \sum_{\substack{\gamma\in\Gamma(k)\\ d(\mathbf{o},\gamma\cdot\mathbf{o})> r}}\sum_{\substack{\beta\in\Gamma(1)\\ d(\mathbf{o},\beta\cdot\mathbf{o})> r}} p(\gamma,\beta\cdot x)p(\beta,x)\varphi(\gamma\beta\cdot x)u\Big(-R+\tilde b(\gamma,\beta\cdot x)+\tilde b(\beta,x)\Big).\end{aligned}$$

Step 1. Let us first prove that

$$\lim_{R\to+\infty}\frac{R^\alpha}{L(R)}A_k(x,r,R) = \sum_{\substack{\beta\in\Gamma(1)\\ d(\mathbf{o},\beta\cdot\mathbf{o})\leq r}} p(\beta,x)\times\lim_{R\to+\infty}\frac{R^\alpha}{L(R)}\tilde P^k(\varphi\otimes u)(\beta\cdot x,-R). \tag{19}$$

Indeed, the set of $\beta\in\Gamma(1)$ such that $d(\mathbf{o},\beta\cdot\mathbf{o})\leq r$ is finite and $\tilde b(\beta,x)\leq r$ for such an isometry β; furthermore, if $p(\beta,x)\neq 0$ then $\frac{R}{2}\leq R-\tilde b(\gamma\beta\cdot x)\leq R+C$ where $C>0$ is the constant which appears in Property 3.1. Using the induction hypothesis, it yields, for any $\beta\in\Gamma(1)$ such that $d(\mathbf{o},\beta\cdot\mathbf{o})\leq r$,

$$\begin{aligned}&\lim_{R\to+\infty}\frac{R^\alpha}{L(R)}p(\beta,x)\sum_{\gamma\in\Gamma(k)} p(\gamma,\beta\cdot x)\varphi(\gamma\beta\cdot x)u\Big(-R+\tilde b(\beta,x)+\tilde b(\gamma,\beta\cdot x)\Big)\\ &\qquad\qquad = p(\beta,x)\times\lim_{R\to+\infty}\frac{R^\alpha}{L(R)}\tilde P^k(\varphi\otimes u)(\beta\cdot x,R).\end{aligned}$$

Convergence (19) follows, summing over β. It yields

$$\lim_{r\to+\infty}\lim_{R\to+\infty}\frac{R^\alpha}{L(R)}A_k(x,r,R) = \sum_{j=1}^{p}\left(\sum_{l=1}^{k}P^lC_j(x)P^{k-l}\varphi(x_j)\right)\times\int_{\mathbb{R}}u(t)\mathrm{d}t. \tag{20}$$

Step 2. We prove that there exists $\epsilon(r) > 0$, with $\lim_{r\to+\infty} \epsilon(r) = 0$, such that, for any $k \geq 1$,

$$\begin{aligned}\liminf_{R\to+\infty} \frac{R^\alpha}{L(R)} B_k(x,r,R) &\overset{\epsilon(r)}{\simeq} \limsup_{R\to+\infty} \frac{R^\alpha}{L(R)} B_k(x,r,R)\\ &\overset{\epsilon(r)}{\simeq} \sum_{j=1}^{p} \sum_{\substack{\gamma\in\Gamma(k)\\ d(o,\gamma\cdot o)\leq r}} p(\gamma,x_j)\varphi(\gamma\cdot x_j)C_j(x)\int_{\mathbb{R}} u(t)\mathrm{d}t, \qquad (21)\end{aligned}$$

where we write $a \overset{\epsilon}{\simeq} b$ if $1-\epsilon \leq \dfrac{a}{b} \leq 1+\epsilon$. Since each Γ_j has a unique fixed point, there exists a map $\epsilon : (0,+\infty) \to (0,+\infty)$ which tends to 0 as $r \to +\infty$, such that

$$\frac{p(\gamma,\beta\cdot x)}{p(\gamma,x_j)} \overset{\epsilon(r)}{\simeq} 1$$

for any $j = 1, ..., p+q$, any $\beta \in \Gamma_j$ with $d(o,\beta\cdot o) \geq r$, any $x \in \tilde{\Lambda}$ and any $\gamma \in \Gamma$ with $l_\gamma \neq j$.

The set of $\gamma \in \Gamma(k)$ such that $d(\mathbf{o},\gamma\cdot\mathbf{o}) \leq r$ is a finite subset of $\Gamma(k)$; furthermore, for such γ and any $\beta \in \Gamma(1)$, it holds $\frac{R}{2} \leq R - \tilde{b}(\gamma,\beta\cdot x) \leq R + C$, as above. Therefore,

$$\begin{aligned}&\sum_{\substack{\gamma\in\Gamma(k)\\ d(\mathbf{o},\gamma\cdot\mathbf{o})\leq r}} \sum_{\substack{\beta\in\Gamma(1)\\ d(\mathbf{o},\beta\cdot\mathbf{o})>r}} p(\gamma,\beta\cdot x)p(\beta,x)\varphi(\gamma\beta\cdot x)u\Big(-R+\tilde{b}(\gamma,\beta\cdot x)+\tilde{b}(\beta,x)\Big)\\ &\qquad\overset{\epsilon(r)}{\simeq} \sum_{j=1}^{p+q} \sum_{\substack{\gamma\in\Gamma(k)\\ d(\mathbf{o},\gamma\cdot\mathbf{o})\leq r}} p(\gamma,x_j)\varphi(\gamma\cdot x_j) \sum_{\substack{\beta\in\Gamma_j\\ d(\mathbf{o},\beta\cdot\mathbf{o})>r}} p(\beta,x)u\Big(-R+\tilde{b}(\gamma,\beta\cdot x)+\tilde{b}(\beta,x)\Big)\end{aligned}$$

Convergence (21) follows, using (16). In particular, letting $r \to +\infty$, it holds

$$\begin{aligned}\lim_{r\to+\infty}\liminf_{R\to+\infty} \frac{R^\alpha}{L(R)} B_k(x,r,R) &= \lim_{r\to+\infty}\limsup_{R\to+\infty} \frac{R^\alpha}{L(R)} B_k(x,r,R)\\ &= \sum_{j=1}^{p} P^k\varphi(x_j)C_j(x)\int_{\mathbb{R}} u(t)\mathrm{d}t. \qquad (22)\end{aligned}$$

Step 3. We prove that there exists a constant $C > 0$ such that, for any $R \geq 2r \geq 1$,

$$C_k(x,r,R) \leq Ck^2\|\varphi\|_\infty \frac{L(R)}{R^\alpha} \sum_{n=[r]}^{+\infty} \frac{L(n)}{n^\alpha}. \qquad (23)$$

By property 3.1, the condition $u\Big(-R+\tilde{b}(\gamma\beta\cdot x)+\tilde{b}(\beta,x)\Big) \neq 0$ implies

$$d(\mathbf{o},\gamma\cdot\mathbf{o}) + d(\mathbf{o},\beta\cdot\mathbf{o}) = R \pm c \quad \text{and} \quad \tilde{b}(\gamma\beta\cdot x)+\tilde{b}(\beta,x) = R \pm c \quad [8]$$

for some constant $c > 0$ which depends on u.

We decompose $C_k(x,r,R)$ into $C_k(x,r,R) = C_{k,1}(x,r,R) + C_{k,2}(x,r,R)$ with

$$C_{k,1}(x,r,R) := \sum_{\substack{\gamma\in\Gamma(k)\\ r<d(\mathbf{o},\gamma\cdot\mathbf{o})\leq R/2}} \sum_{\substack{\beta\in\Gamma(1)\\ d(\mathbf{o},\beta\cdot\mathbf{o})>r}} p(\gamma,\beta\cdot x)p(\beta,x)\varphi(\gamma\beta\cdot x)u\Big(-R+\tilde{b}(\gamma,\beta\cdot x)+\tilde{b}(\beta,x)\Big).$$

and

$$C_{k,2}(x,r,R) := \sum_{\substack{\gamma\in\Gamma(k)\\ d(\mathbf{o},\gamma\cdot\mathbf{o})\geq R/2}} \sum_{\substack{\beta\in\Gamma(k)\\ d(\mathbf{o},\beta\cdot\mathbf{o})>r}} p(\gamma,\beta\cdot x)p(\beta,x)\varphi(\gamma\beta\cdot x)u\Big(-R+\tilde{b}(\gamma,\beta\cdot x)+\tilde{b}(\beta,x)\Big).$$

[8] the notation $A = B \pm c$ means $|A-B| \leq c$.

We control the term $C_{k,1}(x,r,R)$. Assuming $c \geq 1$, one may write

$$\begin{aligned}
C_{k,1}(x,r,R) &\leq \|\varphi\|_\infty \|u\|_\infty \sum_{n=[r]}^{[R/2]} \sum_{\substack{\gamma\in\Gamma(k)\\ d(\mathbf{o},\gamma\cdot\mathbf{o})=n\pm c}} \sum_{\substack{\beta\in\Gamma(k)\\ d(\mathbf{o},\beta\cdot\mathbf{o})=R-n\pm c}} p(\gamma,\beta\cdot x)p(\beta,x) \\
&\leq \|\varphi\|_\infty \|u\|_\infty \sum_{n=[r]}^{[R/2]} \sum_{\substack{\beta\in\Gamma(1)\\ d(\mathbf{o},\beta\cdot\mathbf{o})=R-n\pm c}} p(\beta,x) \left(\sum_{\substack{\gamma\in\Gamma(k)\\ d(\mathbf{o},\gamma\cdot\mathbf{o})=n\pm c}} p(\gamma,\beta\cdot x) \right).
\end{aligned}$$

Using (17), this yields, for some constant $C > 0$,

$$\begin{aligned}
C_{k,1}(x,r,R) &\leq Ck^2 \|\varphi\|_\infty \|u\|_\infty \sum_{n=[r]}^{[R/2]} \frac{L(R-n)}{(R-n)^\alpha} \frac{L(n)}{n^\alpha} \\
&\leq C\ k^2 \|\varphi\|_\infty \|u\|_\infty \frac{L(R)}{R^\alpha} \sum_{n=[r]}^{+\infty} \frac{L(n)}{n^\alpha},
\end{aligned}$$

where the last inequality is based on the facts that $R - n \geq R/2 - 1$ and L is slowly varying. The same inequality holds for $C_{k,2}(x,r,R)$, by reversing in the previous argument the role of γ and β. Hence,

$$\lim_{r\to+\infty} \limsup_{R\to+\infty} \frac{R^\alpha}{L(R)} C_k(x,r,R) = 0. \tag{24}$$

Proposition 5.4 follows, combining (20), (22) and (24).

□

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