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On Implication Bases in n-Lattices

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Abstract

Implication bases in n-lattices are not formally defined. We clarify the different types of implications we need to reconstruct a concept n-lattice and show they can be derived from the same set of implications. We use this to identify a particular type of implication base in n-contexts. Finally, we provide an algorithm for computing implicational closures with n-dimensional bases.

Keywords:
Implications, n-dimensional lattices

1. Introduction

Implications are important in fields that deal with lattices directly such as (obviously) lattice theory, formal concept analysis and boolean function theory or indirectly through an interest for data like relational databases theory, artificial intelligence and data mining. They are most often understood as rules of the form $A \rightarrow B$ that sum up the structure of the lattice and the regularities in the data. Their variant related to lattices and bi-dimensional, object-attribute, data has been extensively studied and, while some questions remain open, most of the algorithmic tools one would need are available. Notably, the literature is rich with implication bases - subsets of implications that sum up the rest - having different properties.

Much less is known about implications related to the multidimensional generalisations of lattices and object-attribute data: n-lattices and n-contexts. In this multidimensional case, no implication base has been defined yet. To the best of our knowledge, only one work has addressed the matter in 3-contexts by highlighting the various possible forms of implications and their semantic [3]. However, whether these implications allow for the reconstruction of the subjacent 3-lattice has yet to be shown. In order to remedy this lack of knowledge and kickstart a more widespread use of n-contexts and n-lattices, we propose here different implication bases for the n-dimensional case.

The remainder of this work is structured as follows: Section 2 presents the relevant definitions and properties concerning the 2-dimensional and n-dimensional cases, Section 3 makes explicit the implications one needs to be able to construct the elements of an n-lattice, Section 4 shows how to obtain them, Section 5 highlights a type of implication base and Section 6 contains an algorithm for computing implicational closures with those bases. Finally, Section 7 discusses the questions that are left unanswered.

2. Preliminary Definitions and Results

2.1. Lattices

A *lattice* $\mathcal{L}$ is a structure that is defined differently depending on the domain. In order theory, $\mathcal{L}$ is a partially ordered set $(E, \leq)$ in which every pair of elements $e_1, e_2 \in E$ has both a least upper bound and a greatest lower bound. In algebra, $\mathcal{L} = (E, \wedge, \vee)$ with $\wedge$ and $\vee$ being binary operations respecting the commutative, associative and absorption laws. In this work, we will prefer the order theoretic approach and consider our lattices to be complete - i.e. having a finite set of elements including a greatest and a least.

Definition 1. *A* formal context *is a triple* (S_1, S_2, R) *in which* S_1 *and* S_2 *are sets and* $R \subseteq S_1 \times S_2$ *is a binary relation between* S_1 *and* S_2*.*

A formal context can be represented by a crosstable. Hence, when $\mathcal{C} = (S_1, S_2, R)$, the elements of S_1 will be called the *rows* and those of S_2 the *columns* of $\mathcal{C}$.

	a	b	c	d	e
1	×	×			
2		×	×	×	
3		×		×	×
4			×		×
5				×	×

Figure 1: A formal context $\mathcal{C} = (S_1, S_2, R)$ with $S_1 = \{1, 2, 3, 4, 5\}$ and $S_2 = \{a, b, c, d, e\}$

For $X \subseteq S_1$ and $Y \subseteq S_2$, two *derivation operators* are defined:

$$X \mapsto X' : \{a \in S_2 \mid \forall x \in X, (x, a) \in R\}$$
$$Y \mapsto Y' : \{o \in S_1 \mid \forall y \in Y, (o, y) \in R\}$$

The composition of these two operators forms a Galois connection or, in other words, a closure operator.

Definition 2. *A* formal concept *of* (S_1, S_2, R) *is a pair* $(X_1, X_2) \in 2^{S_1} \times 2^{S_2}$ *such that* $X_1 \times X_2 \subseteq R$ *and there are no* $k_1 \in S_1 \setminus X_1$ *or* $k_2 \in S_2 \setminus X_2$ *such that* $(X_1 \cup \{k_1\}, X_2)$ *or* $(X_1, X_2 \cup \{k_2\})$ *respects this property.*

In other words, in the formal context, a formal concept is a maximal rectangle full of crosses (up to permutation of the rows and columns). The set of formal concepts existing in a formal context can be ordered by the inclusion relation on one of their components. The resulting partially ordered set forms a structure known as a *concept lattice*. It has been shown [4] that any complete lattice is isomorphic to the concept lattice of a formal context. Multiple formal contexts can correspond to the same lattice but the minimal one, called the *reduced context*, is such that no row (resp. column) is the intersection of other rows (resp. columns). In this reduced context, the rows and columns represent the $\vee$- and $\wedge$-irreducibles of the lattice with a cross meaning that the two elements are comparable.

Definition 3. *In a formal context, an* implication *is a rule $A \to B$ in which A and B are sets of columns.*

An implication $A \to B$ is said to *hold* in a context if every row that has crosses on the columns in A also has crosses on the columns in B. In the context presented in Figure 1, the implications $\{a\} \to \{ab\}$ and $\{bc\} \to \{bcd\}$ hold while $\{be\} \to \{abe\}$ does not. We will use $\mathcal{I}_{\mathcal{C}}$ to denote the set of all implications that hold in a context $\mathcal{C}$.

Definition 4. *Let $\mathcal{I}$ be a set of implications and S a set of columns of $\mathcal{C}$. The* implicational closure *of $S0$, noted $\mathcal{I}(S)$, is the smallest set T such that $S \subseteq T$ and ($A \to B \in \mathcal{I}$ and $A \subseteq T$) implies $B \subseteq T$.*

We have the property that $\mathcal{I}_{\mathcal{C}}(S) = S''$. As such, the closure system induced by $\mathcal{I}_{\mathcal{C}}(\cdot)$ is the same as the one induced by $\cdot''$ and is isomorphic to the set of concepts. Thus, just as the context, the implications can be used to construct a structure that is isomorphic to the concept lattice.

Multiple implication sets can correspond to the same context/lattice. For example, if $\{a\} \to \{ab\}$ holds, then $\{ac\} \to \{abc\}$ obviously holds too and is redundant. An implication set that allows for the derivation of all the implications that hold in a context - and only them - through the application of Armstrong's axioms is called a *base*. Different such bases, with their own properties, have been studied, of which two are of particular interest: the Duquenne-Guigues [6] and the canonical direct [2] bases.

2.2. n-Lattices

The structures called n-lattices are multidimensional generalisations of lattices. An n-lattice $\mathcal{L}$ is, first of all, an n-ordered set $(E, \lesssim_1, \dots \lesssim_n)$ in which the $\lesssim_i$ are quasi-orders on E such that $\cap_{j \neq i} \lesssim_j \subseteq \gtrsim_i$. Additionally, it requires the existence of so-called $(j_{n-1}, \dots, j_1)$-*joins*, generalisations of the least upper bound and greatest lower bound taking into consideration $n-1$ quasi-orders [12].

Definition 5. *An* n-context *is an (n+1)-tuple* $\mathcal{C} = (S_1, ..., S_n, R)$ *in which the* S_i *are sets and* $R \subseteq \prod_{k \in \{1,...,n\}} S_k$ *is an* n*-ary relation between them.*

An n-context can be seen as an n-dimensional crosstable. The 3-context $\mathcal{C}$ shown in Figure 2 will be used throughout this work as a running example.

	1	2	3	1	2	3	1	2	3
a	×			×		×			×
b	×			×					
c	×	×			×			×	×
		α			β			γ	

Figure 2: An example of a 3-context with $S_{greek} = \{\alpha, \beta, \gamma\}$, $S_{latin} = \{\mathrm{a}, \mathrm{b}, \mathrm{c}\}$ and $S_{number} = \{1, 2, 3\}$.

It can be useful to view the n-context as a building in which one of the dimensions, d, serves as the front so that each element of S_d is a floor number. Each floor $s_d \in S_d$ is then an $(n-1)$-dimensional space that contains rooms corresponding to the coordinates $(s_1, ..., s_d, ..., s_n)$, $s_i \in S_i$, $\forall i \in \{1, \ldots, n\} \setminus \{d\}$.

Many k-contexts ($2 \leq k \leq n$) can be constructed from the n-context. Borrowing the notations from [12], let $\pi = (\pi_1, ..., \pi_k)$ be a partition of $\{1, ..., n\}$ into k sets. The k-context corresponding to π is $\mathcal{C}^\pi = (\prod_{i \in \pi_1} S_i, ..., \prod_{i \in \pi_k} S_i, R^\pi)$ with $(s^1, ..., s^k) \in R^\pi$ if and only if $(s_1, ..., s_n) \in R$ with $s_i \in s^j \Leftrightarrow i \in \pi_j$.

Figure 3 shows the 2-context $\mathcal{C}^{(latin, \{number, greek\})}$ resulting from the binary partition of our Figure 2 example.

	(1,α)	(1,β)	(1,γ)	(2,α)	(2,β)	(2,γ)	(3,α)	(3,β)	(3,γ)
a	×	×						×	×
b	×	×							
c	×			×	×	×			×

Figure 3: The 2-context $\mathcal{C}^{(latin, \{number, greek\})}$.

Let $D \subseteq \{1, ..., n\}$ be a set of dimensions and $\overline{D} = \{1, ..., n\} \setminus D$ be its complement. When D is a singleton, we will write d instead of $\{d\}$. Let S_D denote the collection $\langle S_d \mid d \in D \rangle$. Let $X_d \subseteq S_d$ be a set of elements in the dimension d, $X_D = \langle X_d \mid d \in D \rangle$ and $x_D \in \prod_{d \in D} S_d$. The $|\overline{D}|$-context associated to X_D is $\mathcal{C}_{X_D} = (S_{\overline{D}}, R_{X_D})$ such that, $\forall x_{\overline{D}} \in \prod_{d \in \overline{D}} S_d$, $x_{\overline{D}} \in R_{X_D}$ if and only if $x_{D \cup \overline{D}} \in R$.

Figure 4 shows two 2-contexts derived from our Figure 2 example by fixing subsets of the *greek* dimension.

	1	2	3
a	×		
b	×		
c	×	×	

	1	2	3
a			×
b			
c		×	

Figure 4: The 2-contexts $\mathcal{C}_{\langle\{\alpha\}\rangle}$ and $\mathcal{C}_{\langle\{\beta,\gamma\}\rangle}$.

Less formally, the contexts $\mathcal{C}^{(\pi)}$ are obtained from $\mathcal{C}$ by merging dimensions using Cartesian products while the contexts $\mathcal{C}_{X_D}$ are obtained by choosing subsets of some dimensions D and keeping only what they have in common. For example, let us go back to our building analogy. Let d be the "front" dimension. The floors use the other dimensions or, in other words, $\overline{d}$. Let us arbitrarily fix $X_d = \{1\}$. Thus, the context $\mathcal{C}_{\langle\{1\}\rangle}$ is the first floor. If we now choose $X_d = \{1, 2\}$, $\mathcal{C}_{\langle\{1,2\}\rangle}$ is the context resulting from the intersection of the first and second floors. That is, this context has a cross in a room iff both the first and second floors have one. If we want $D = \{d_1, d_2\}$ to have more than one dimension, we can start by considering only one, obtaining $\mathcal{C}_{X_{d_1}}$. Viewing this intersection of floors as a new $(n-1)$-context/building, we can then fix the other dimension by considering it as the new "front", obtaining $(\mathcal{C}_{X_{d_1}})_{X_{d_2}} = \mathcal{C}_{X_D}$.

We will use $\mathcal{C}^{\pi}_{X_D}$ as a shortcut to denote the context $(\mathcal{C}_{X_D})^{\pi}$.

Binary partitions $\pi = (\pi_1, \pi_2)$ give rise to the derivation operators $A \mapsto A^{\pi}$ based on the derivation operators defined on the 2-contexts $\mathcal{C}^{\pi}$. Similarly, 2-elements sets $\overline{D} = \{d_1, d_2\}$ and sets $X_d \subseteq S_d$, $d \in D$ give rise to the derivation operators $A \mapsto A^{(d_1,d_2,X_D)}$ based on the derivation operators defined on the 2-contexts $\mathcal{C}^{(d_1,d_2)}_{X_D}$.

Definition 6. *An n-*concept *of* $(S_1, ..., S_n, R)$ *is an n-tuple* $(X_1, ..., X_n) \in \prod_{i\in\{1,...,n\}} 2^{S_i}$ *such that* $\prod_{i\in\{1,...,n\}} X_i \subseteq R$ *and there are no* $d \in \{1, ..., n\}$ *and* $k \in S_d \setminus X_d$ *such that* $(X_1, ..., X_d \cup \{k\}, ..., X_n)$ *respects this property.*

In other words, an n-concept is a maximal n-dimensional box full of crosses in $(S_1, ..., S_n, R)$ (up to permutations inside dimensions).

Our Figure 2 example contains the following 3-concepts (brackets are left out for the sake of legibility): (abc,1,α), (c,12,α), (ab,1,$\alpha\beta$), (a,13,β), (c,2,$\alpha\beta\gamma$), (a,3,$\beta\gamma$), (c,23,γ), (ac,3,γ), ($\emptyset$,123,$\alpha\beta\gamma$), (abc,$\emptyset$,$\alpha\beta\gamma$) and (abc,123,$\emptyset$).

Proposition 1. *(From [12]) Let* $\{j_1, ..., j_n\} = \{1, ..., n\}$ *and* $X_i \subseteq S_i$, $i \neq j_n$. *Define*

$$A_{j_n} = X_{j_{n-1}}^{(j_n, j_{n-1}, X_{\overline{\{j_n, j_{n-1}\}}})} \tag{1}$$

$$A_{j_{n-1}} = A_{j_n}^{(j_n, j_{n-1}, X_{\overline{\{j_n, j_{n-1}\}}})} \tag{2}$$

$$A_{j_{n-2}} = A_{\{j_n,j_{n-1}\}}^{(\{j_n,j_{n-1}\},j_{n-2},X_{\overline{\{j_n,j_{n-1},j_{n-2}\}}})} \tag{3}$$

$$\vdots$$

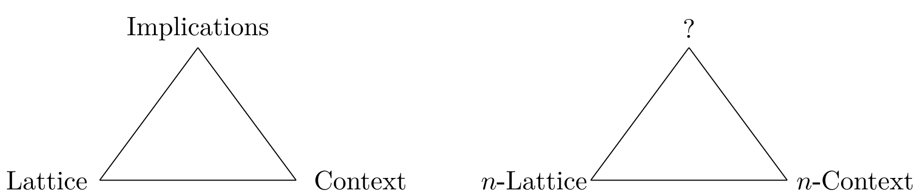
$$A_{j_k} = A_{\{j_n,j_{n-1},...,j_{k+1}\}}^{(\{j_n,j_{n-1},...,j_{k+1}\},j_k,X_{\{j_1,...,j_{k-1}\}})} \tag{4}$$

$$\vdots$$

$$A_{j_1} = A_{\{j_n,...,j_2\}}^{(\{j_n,...,j_2\},j_2)} \tag{5}$$

Then $(A_1, ..., A_n)$ *is the* n*-concept* $\mathfrak{b}_{j_{n-1},...,j_1}(X_{\overline{j_n}})$ *with the property that it has the largest* j_2*-component among all* n*-concepts* $(B_1, ..., B_n)$ *with the largest* j_3*-component among those with the largest* j_4*-component, ..., among all those with the largest* j_n*-component, satisfying* $X_i \subseteq B_i$*,* $i \neq j_n$*. Thus, if* $(C_1, ..., C_n)$ *is an* n*-concept, then* $\mathfrak{b}_{j_{n-1},...,j_1}(C_{\overline{j_n}}) = (C_1, ..., C_n)$*.*

Implication bases in the n-dimensional case have yet to be formally defined, which is the motivation behind this work. Ganter and Obiedkov brought up the topic in the 3-dimensional case [3] and defined different forms of implications that correspond to $\mathcal{I}_{\mathcal{C}^{(d_1,d_3)}_{X_{d_2}}}$, $\mathcal{I}_{\mathcal{C}^{(d_1,d_2)}_{X_{d_3}}}$ and $\mathcal{I}_{\mathcal{C}^{(d_1,\{d_2,d_3\})}}$. However, to the best of our knowledge, it has yet to be proven that they form a base that allows for the construction of the elements of the 3-lattice. Despite this, the topic is growing in popularity [7, 5, 9] and we believe that it calls for more stable foundations.



3. Computing n-Concepts with Implications

An *implication base* of an n-context $\mathcal{C}$ must be a set of rules of the form $A \to B$ that provides enough information for the construction of a set of objects isomorphic to the set of n-concepts of $\mathcal{C}$. From Proposition 1, we know that computing the n-concepts can be done with the knowledge of the derivation operators induced by the various 2-contexts that can be constructed from $\mathcal{C}$. In the 2-dimensional case, the composition of the derivation operators and the implicational closure form the same closure operator. However, in Proposition

1, the computation of n-concepts is done using single derivation operators and not their composition. Hence, we have to reformulate this proposition.

Proposition 2. *Let* $\mathcal{C} = (S_1, ..., S_n, R)$, $\{j_1, ..., j_n\} = \{1, ..., n\}$ *and* $X_i \subseteq S_i$, $i \neq j_n$. *Define*

$$A_{j_n} = X_{j_{n-1}}^{(j_n, j_{n-1}, X_{\overline{\{j_n, j_{n-1}\}}})} \tag{6}$$

$$A_{j_{n-1}} = \mathcal{I}_{\mathcal{C}^{(j_n, j_{n-1})}_{X_{\{j_1, \ldots, j_{n-2}\}}}}(X_{j_{n-1}}) \tag{7}$$

$$A_{j_{n-2}} = \mathcal{I}_{\mathcal{C}^{(j_n, j_{n-2})}_{A_{j_{n-1}} \cup X_{\{j_1, \ldots, j_{n-3}\}}}}(X_{j_{n-2}}) \tag{8}$$

$$\vdots$$

$$A_{j_k} = \mathcal{I}_{\mathcal{C}^{(j_n, j_k)}_{A_{\{j_{k+1}, \ldots, j_{n-1}\}} \cup X_{\{j_1, \ldots, j_{k-1}\}}}}(X_{j_k}) \tag{9}$$

$$\vdots$$

$$A_{j_1} = \mathcal{I}_{\mathcal{C}^{(j_n, j_1)}_{A_{\{j_2, \ldots, j_{n-1}\}}}}(X_{j_1}) \tag{10}$$

Then $(A_1, ..., A_n)$ *is the* n*-concept* $\mathfrak{b}_{j_{n-1}, \ldots, j_1}(X_{\overline{j_n}})$ *with the property that it has the largest* j_2*-component among all* n*-concepts* $(B_1, ..., B_n)$ *with the largest* j_3*-component among those with the largest* j_4*-component, ..., among all those with the largest* j_n*-component, satisfying* $X_i \subseteq B_i$, $i \neq j_n$. *Thus, if* $(C_1, ..., C_n)$ *is an* n*-concept, then* $\mathfrak{b}_{j_{n-1}, \ldots, j_1}(C_{\overline{j_n}}) = (C_1, ..., C_n)$.

Proof We know that $(s_{j_n}, x_{j_k}) \in R^{(j_n, j_k)}_{A_{\{j_{k+1}, \ldots, j_{n-1}\}} \cup X_{\{j_1, \ldots, j_{k-1}\}}}$ if and only if

$$\forall (a_{j_{k+1}}, ..., a_{j_{n-1}}) \in \prod_{i \in \{k+1, \ldots, n-1\}} A_{j_i},$$

$$(x_{j_k}, a_{j_{k+1}}, ..., a_{j_{n-1}}, s_{j_n}) \in R_{X_{\{j_1, \ldots, j_{k-1}\}}}.$$

According to Proposition 1, $(A_{j_{k+1}}, ..., A_{j_n})$ is an $(n-k)$-concept of $\mathcal{C}_{X_{\{j_1, \ldots, j_k\}}}$. This means that $\prod_{i \in \{k+1, \ldots, n\}} A_{j_i} \subseteq R_{X_{\{j_1, \ldots, j_{k-1}, j_k\}}}$. From this, we deduce that $X_{j_k} \times \prod_{i \in \{k+1, \ldots, n\}} A_{j_i} \subseteq R_{X_{\{j_1, \ldots, j_{k-1}\}}}$. Consequently, $\forall (a_{j_n}, x_{j_k}) \in A_{j_n} \times X_{j_k}$, we have that

$$\forall (a_{j_{k+1}}, ..., a_{j_{n-1}}) \in \prod_{i \in \{k+1, \ldots, n-1\}} A_{j_i},$$

$$(x_{j_k}, a_{j_{k+1}}, ..., a_{j_{n-1}}, a_{j_n}) \in R_{X_{\{j_1, \ldots, j_{k-1}\}}}.$$

This tells us that $A_{j_n} \times X_{j_k} \subseteq R^{(j_n,j_k)}_{A_{\{j_{k+1},\ldots,j_{n-1}\}} \cup X_{\{j_1,\ldots,j_{k-1}\}}}$.
If $X_{j_k} \to \{x\} \in \mathcal{I}_{\mathcal{C}^{(j_n,j_k)}_{A_{\{j_{k+1},\ldots,j_{n-1}\}} \cup X_{\{j_1,\ldots,j_{k-1}\}}}}$, then, by definition,

$$\forall s_{j_n} \in S_{j_n},\ (\forall x_{j_k} \in X_{j_k}, (s_{j_n}, x_{j_k}) \in R^{(j_n,j_k)}_{A_{\{j_{k+1},\ldots,j_{n-1}\}} \cup X_{\{j_1,\ldots,j_{k-1}\}}})$$

$$\Downarrow$$

$$(s_{j_n}, x) \in R^{(j_n,j_k)}_{A_{\{j_{k+1},\ldots,j_{n-1}\}} \cup X_{\{j_1,\ldots,j_{k-1}\}}}.$$

Hence, from the previous paragraph, we deduce that the fact that

$$X_{j_k} \to \{x\} \in \mathcal{I}_{\mathcal{C}^{(j_n,j_k)}_{A_{\{j_{k+1},\ldots,j_{n-1}\}} \cup X_{\{j_1,\ldots,j_{k-1}\}}}}$$

implies that $\forall a_{j_n} \in A_{j_n}$, $(a_{j_n}, x) \in R^{(j_n,j_k)}_{A_{\{j_{k+1},\ldots,j_{n-1}\}} \cup X_{\{j_1,\ldots,j_{k-1}\}}}$ and, thus, that

$$\forall (a_{j_{k+1}}, ..., a_{j_{n-1}}, a_{j_n}) \in \prod_{i \in \{k+1,\ldots,n\}} A_{j_i},$$

$$(x, a_{j_{k+1}}, ..., a_{j_{n-1}}, a_{j_n}) \in R_{X_{\{j_1,\ldots,j_{k-1}\}}}.$$

Consequently, $(X_{j_k} \cup \{x\}) \times \prod_{i \in \{k+1,\ldots,n\}} A_{j_i} \subseteq R_{X_{\{j_1,\ldots,j_{k-1}\}}}$. The tuple $(X_{j_k} \cup \{x\}, A_{j_{k+1}}, \ldots, A_{j_n})$ is thus an $(n-k+1)$-dimensional box full of crosses in $\mathcal{C}_{X_{\{j_1,\ldots,j_{k-1}\}}}$. Additionally, the fact that $(A_{j_{k+1}}, ..., A_{j_n})$ is an $(n-k)$-concept of $\mathcal{C}_{X_{\{j_1,\ldots,j_k\}}}$ ensures that $(X_{j_k} \cup \{x\}, A_{j_{k+1}}, \ldots, A_{j_n})$ cannot be extended on dimensions $k+1, \ldots, n$.

From all of this, we can see that

$$(\mathcal{I}_{\mathcal{C}^{(j_n,j_k)}_{A_{\{j_{k+1},\ldots,j_{n-1}\}} \cup X_{\{j_1,\ldots,j_{k-1}\}}}}(X_{j_k}), A_{j_{k+1}}, \ldots, A_{j_n})$$

is the $(n-k+1)$-concept of $\mathcal{C}_{X_{\{j_1,\ldots,j_{k-1}\}}}$ with the property that it has the largest j_{k+1}-component among all $(n-k+1)$-concepts $(B_{j_k}, ..., B_n)$ with the largest j_{k+2}-component among those with the largest j_{k+3}-component, ..., among all those with the largest j_n-component, satisfying $X_i \subseteq B_i$, $i \in \{k, ..., n-1\}$. $\square$

As n-concepts are totally defined by $n-1$ of their components, the set of $(n-1)$-tuples of the form $(A_{j_{n-1}}, ..., A_{j_1})$ is isomorphic to the set of n-concepts. Since this set can be constructed using the implications bases of the different $\mathcal{C}^{(j_n,j_k)}_{A_{\{j_{k+1},\ldots,j_{n-1}\}} \cup X_{\{j_1,\ldots,j_{k-1}\}}}$, an implication base of $\mathcal{C}$ must, at least, allow for the computation of the implication bases of all such 2-contexts.

4. Deriving the Implications of Derived Contexts

As we have seen, only two operations are needed to derive subcontexts from $\mathcal{C}$:

- partitioning the dimensions ($\mathcal{C}^{\pi}$)
- "fixing" a subset X_d of a dimension d ($\mathcal{C}_{X_d}$).

Every 2-context $\mathcal{K}^{(\pi_1,\pi_2)}$ used in Proposition 2 is the binary partition of an n-context $\mathcal{K} = \mathcal{C}_{X_{\overline{\pi_1 \cup \pi_2}}}$. We will start by showing how to compute the implication base of $\mathcal{C}_{X_D}^{(k,\overline{D}\setminus\{k\})}$ from the one of $\mathcal{C}_{X_{D\setminus\{d\}}}^{(k,(\overline{D}\cup\{d\})\setminus\{k\})}$.

Let $\{j_1,...,j_k\} \subset \{1,...,n\}$, $A = \{a_1,...,a_m\}$ with $a_l = (a_{l,j_1},...,a_{l,j_k}) \in \prod_{i\in\{1,...,k\}} S_{j_i}$ be a set of k-tuples and $x \in S_p$ with $p \in \{1,...,n\} \setminus \{j_1,...,j_k\}$. Let us define the notation

$$A * \{x\} = \{(a_{l,j_1},...,x,...,a_{l,j_k}) \mid (a_{l,j_1},...,a_{l,j_k}) \in A\}$$

which simply means adding to every k-tuple in A the element x at the right position.

Proposition 3. *$\mathcal{I}_{\mathcal{C}_{X_D}^{(k,\overline{D}\setminus\{k\})}} = \{A \to B \mid \exists d \in D$ such that $\bigcup_{x_d\in X_d} A * \{x_d\} \to \bigcup_{x_d\in X_d} B * \{x_d\} \in \mathcal{I}_{\mathcal{C}_{X_{D\setminus\{d\}}}^{(k,(\overline{D}\cup\{d\})\setminus\{k\})}}\}$*

Proof $\Leftarrow$. Let us suppose that

$$\bigcup_{x_d\in X_d} A * \{x_d\} \to \bigcup_{x_d\in X_d} B * \{x_d\} \in \mathcal{I}_{\mathcal{C}_{X_{D\setminus\{d\}}}^{(k,(\overline{D}\cup\{d\})\setminus\{k\})}}$$

for some $d \in D$. This means that

$$\forall x_k \in S_k, (\bigcup_{x_d\in X_d} A * \{x_d\} \subseteq R_{X_{D\setminus\{d\}}\cup\{x_k\}}) \Rightarrow (\bigcup_{x_d\in X_d} B * \{x_d\} \subseteq R_{X_{D\setminus\{d\}}\cup\{x_k\}})$$

Consequently,

$$\forall x_k \in S_k, x_d \in X_d, (A \subseteq R_{X_{D\setminus\{d\}}\cup\{x_k\}\cup\{x_d\}}) \Rightarrow (B \subseteq R_{X_{D\setminus\{d\}}\cup\{x_k\}\cup\{x_d\}})$$

So,

$$\forall x_k \in S_k, (A \subseteq R_{X_D\cup\{x_k\}}) \Rightarrow (B \subseteq R_{X_D\cup\{x_k\}})$$

Which finally means that

$$A \to B \in \mathcal{I}_{\mathcal{C}_{X_D}^{(k,\overline{D}\setminus\{k\})}}$$

$\Rightarrow$. Let us now suppose that $A \to B \in \mathcal{I}_{\mathcal{C}_{X_D}^{(k,\overline{D}\setminus\{k\})}}$. This means that

$$\forall x_k \in S_k, (A \subseteq R_{X_D\cup\{x_k\}}) \Rightarrow (B \subseteq R_{X_D\cup\{x_k\}})$$

However,

$$a \in R_{X_D \cup \{x_k\}} \Leftrightarrow \{a\} * \{x_d\} \in R_{X_{D\setminus\{d\}} \cup \{x_k\}}, \forall x_d \in X_d$$

Consequently,

$$\bigcup_{x_d \in X_d} A * \{x_d\} \to \bigcup_{x_d \in X_d} B * \{x_d\} \in \mathcal{I}_{\mathcal{C}^{(k,(\overline{D}\cup\{d\})\setminus\{k\})}_{X_{D\setminus\{d\}}}}$$

□

This proposition concerns only binary partitions (π_1, π_2) in which π_1 is a singleton. These are the only ones used in Proposition 2, and thus the only ones which implication bases are necessary for computing n-concepts. However, completely understanding the n-context and the various subcontexts that can be derived from it requires knowing the derivation operators of other binary partitions. For this reason, we would like to make sure that we can also derive the implication bases of every binary partition.

Proposition 4. $\forall d \in \pi_1, \mathcal{I}_{\mathcal{C}^{(\pi_1,\pi_2)}} = \bigcap_{s_d \in S_d} \mathcal{I}_{\mathcal{C}^{(\pi_1\setminus\{d\},\pi_2)}_{\{\{s_d\}\}}}$

Proof By looking at Figure 4, it is easy to see that an implication has to hold in every $\mathcal{C}^{(\pi_1\setminus\{d\},\pi_2)}_{\{\{s_d\}\}}$ to hold in $\mathcal{C}^{(\pi_1,\pi_2)}$. □

5. A Type of Implication Base

Now that we know how to derive the implication bases of all the possible binary partitions of subcontexts of $\mathcal{C}$, we can identify implication bases for $\mathcal{C}$.

Theorem 1. *Let $\mathcal{C} = (S_1, ..., S_n, R)$ be an n-context and $k \in \{1, ..., n\}$ be a dimension. An implication base of $\mathcal{C}^{(k,\overline{k})}$ is an implication base of $\mathcal{C}$.*

Proof From Proposition 2, we know that a set of $(n-1)$-tuples that is isomorphic to the set of n-concepts of $\mathcal{C}$ can be computed from the implications in binary contexts of the form $\mathcal{C}^{(k,d)}_{X_{\overline{\{k,d\}}}}$ where $d \in \{1, ..., n\} \setminus \{k\}$. From Proposition 3, we know that $\mathcal{I}_{\mathcal{C}^{(k,\overline{D}\setminus\{k\})}_{X_D}}$ can be derived from $\mathcal{I}_{\mathcal{C}^{(k,(\overline{D}\cup\{d\})\setminus\{k\})}_{X_{D\setminus\{d\}}}}$. Hence, $\forall d \in \{1, ..., n\} \setminus \{k\}$, $\mathcal{I}_{\mathcal{C}^{(k,d)}_{X_{\overline{\{k,d\}}}}}$ can be derived from $\mathcal{I}_{\mathcal{C}^{(k,\overline{k})}}$. Since an implication base of $\mathcal{C}^{(k,\overline{k})}$, by definition, allows for the derivation of $\mathcal{I}_{\mathcal{C}^{(k,\overline{k})}}$, it also allows for the derivation of $\mathcal{I}_{\mathcal{C}^{(k,d)}_{X_{\overline{\{k,d\}}}}}$, $\forall d \in \{1, ..., n\} \setminus \{k\}$. Thus an implication base of $\mathcal{C}^{(k,\overline{k})}$ contains enough information for the computation of a set of $(n-1)$-tuples that is isomorphic to the set of n-concepts of $\mathcal{C}$. □

$\mathcal{C}^{(k,\overline{k})}$ being a simple 2-context, results on the implication bases of 2-contexts apply to those of n-contexts. Most importantly, implication bases of $\mathcal{C}^{(k,\overline{k})}$ can be computed using known algorithms [11, 1].

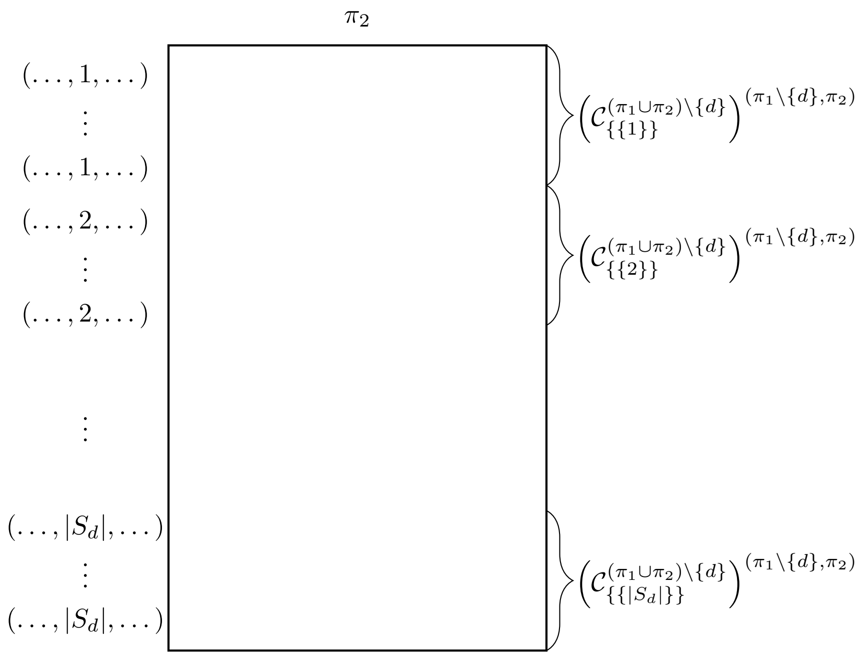


Figure 5: Illustration of the proof for Proposition 4

6. Computing Implicational Closures

Now, let us suppose that we have $X_{\overline{j_n}}$ and want to compute the n-concept $\mathfrak{b}_{j_{n-1},\dots,j_1}(X_{\overline{j_n}})$ as in Proposition 2. Explicitly computing the implication bases of the 2-contexts of the form $\mathcal{C}^{(j_n,j_d)}_{X_{\overline{\{j_d,j_n\}}}}$ would be inefficient as many implications can be irrelevant and/or found in multiple contexts. In this section, we propose an algorithm for computing the implicational closure $\mathcal{I}_{\mathcal{C}^{(k,j)}_{X_{\overline{\{k,j\}}}}}(X)$ from an implication base of $\mathcal{C}$.

From Proposition 3, we know that, for some $d \in D$,

$$\mathcal{I}_{\mathcal{C}^{(k,\overline{D}\setminus\{k\})}_{X_D}} = \{A \to B \mid \bigcup_{x_d \in X_d} A * \{x_d\} \to \bigcup_{x_d \in X_d} B * \{x_d\} \in \mathcal{I}_{\mathcal{C}^{(k,(\overline{D}\cup\{d\})\setminus\{k\})}_{X_{D\setminus\{d\}}}}\}$$

This gives us that, for a set $A \subseteq S_j$ and a dimension $d \in \overline{\{k,j\}}$,

$$b \in \mathcal{I}_{\mathcal{C}^{(k,j)}_{X_{\overline{\{k,j\}}}}}(A) \Leftrightarrow P \subseteq \mathcal{I}_{\mathcal{C}^{(k,\{j,d\})}_{X_{\overline{\{k,j,d\}}}}}(Q)$$

with $P = \bigcup_{x_d \in X_d}(b) * \{x_d\}$ and $Q = \bigcup_{a \in A}\bigcup_{x_d \in X_d}(a) * \{x_d\}$. This naturally means that

$$b \in \mathcal{I}_{\mathcal{C}^{(k,j)}_{X_{\overline{\{k,j\}}}}}(A) \Leftrightarrow L \subseteq \mathcal{I}_{\mathcal{C}^{(k,\overline{k})}}(M)$$

with $L = \{(x_1, ..., x_{k-1}, x_{k+1}, ..., x_{j-1}, b, x_{j+1}, ..., x_n) \mid x_i \in X_i, i \in \overline{\{k,j\}}\}$ and $M = \bigcup_{a\in A}\{(x_1, ..., x_{k-1}, x_{k+1}, ..., x_{j-1}, a, x_{j+1}, ..., x_n) \mid x_i \in X_i, i \in \overline{\{k,j\}}\}$. Or, in other words,

$$M \rightarrow L \in \mathcal{I}_{\mathcal{C}^{(k,\overline{k})}}$$

A naive algorithm would be, given a basis $\mathcal{B} \subseteq \mathcal{I}_{\mathcal{C}^{(k,\overline{k})}}$, to compute $\mathcal{B}(M)$ and check whether L is in it. However, $\mathcal{B}(M)$ can contain many unnecessary elements.

Let $\mathcal{B}$ be an implication base of $\mathcal{C}^{(k,\overline{k})}$, $\mathcal{C}^{(k,j)}_{X_{\overline{\{k,j\}}}}$ be the 2-context in which we want to compute the implicational closure and $A \subseteq S_j$ be a set of columns of $\mathcal{C}^{(k,j)}_{X_{\overline{\{k,j\}}}}$. We propose the following algorithm to compute $\mathcal{I}_{\mathcal{C}^{(k,j)}_{X_{\overline{\{k,j\}}}}}(A)$ from $\mathcal{B}$, $X_{\overline{\{k,j\}}}$ and A. Let us start by creating a list associating, to each element $x \in \prod_{d\in\overline{k}} X_d$, the set of premises of implications in $\mathcal{B}$ that have x in their conclusions. We then want to check whether there is an implication which premise contains A in the list corresponding to each of the n-tuples $\prod_{d\in\overline{k,j}} X_d * \{b\}$ with $b \in S_j \setminus A$. If this is the case, b is added to the output.

Algorithm 1 MDLClosure($\mathcal{B}$,$X_{\overline{\{k,j\}}}$,A)

$R \leftarrow A$
Create the lists of implications for each column of $\mathcal{C}^{(k,\overline{k})}$
for each $b \in S_j \setminus A$ **do**
  $Add \leftarrow true$
  for each $B \in \prod_{d\in\overline{\{k,j\}}} X_d$ **do**
    if there is no implication containing $A \times \prod_{d\in\overline{\{k,j\}}} X_d$ in its premise and $B * \{b\}$ in its conclusion **then**
      $Add \leftarrow false$
    end if
  end for
  if Add **then**
    $R \leftarrow R \cup \{b\}$
  end if
end for
return R

The algorithm runs in $O(|\prod_{d\in\overline{\{k,j\}}} X_d| \times |S_j| \times |\mathcal{B}| \times K)$ where K is the complexity of checking whether $A \times \prod_{d\in\overline{\{k,j\}}} X_d$ is in a premise.

7. Discussion

While the proposed implication base of an n-context is apparently a simple implication base of a 2-context, the higher dimensionality opens new questions and challenges some previous results. First of all, it is not certain that the properties of the bases still hold. The Duquenne-Guigues base of $\mathcal{C}^{(k,\overline{k})}$ is the smallest base of this context but not necessarily the smallest of $\mathcal{C}$ as other bases could also be considered - including the union of the bases of the various 2-contexts used in Proposition 2. Such results should be carefully reexamined.

The problem of actually computing the n-lattice - whole, partially, with or without the quasi-orders - from implication bases should also be the subject of in-depth studies. This should include improving the algorithm proposed in Section 6 and comparing its runtime against the previously mentioned naive approach in various scenarios.

Finally, Proposition 4 should be elaborated upon in the context of data mining and partial implications. Indeed, while contexts of the form $\mathcal{C}^{(\pi_1,\pi_2)}$ with $|\pi_1| > 1$ are not needed in this work, they are important for computing frequent n-dimensional association rules [10, 9]. As was the case for the Luxenburger [8] base in the 2-dimensional case, bases for partial implications should be defined for n-lattices.

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