Output feedback stabilization of switching discrete-time linear systems with parameter uncertainties

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Abstract

This paper deals with observer-based control design for a class of switched discrete-time linear systems with parameter uncertainties. The main contribution of the paper is to propose a convenient way based on Finsler's lemma to enhance the synthesis conditions, expressed in terms of Linear Matrix Inequalities (LMIs). Indeed, this judicious use of Finsler's lemma provides additional decision variables, which render the LMIs less conservative and more general than all those existing in the literature for the same class of systems. Two numerical examples followed by a Monte Carlo evaluation are proposed to show the superiority of the proposed design technique.

Keywords: Switched discrete-time systems; Output feedback control; Switched Lyapunov function (SLF); Finsler's lemma; LMI.

1. Introduction

Switching systems deserve to be investigated for theoretical motivations justified by their fascinating construction as well as practical reasons, due to several

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applications, such as networked control systems [46], air traffic control [35], servomechanism systems [45]. For an overview on stability analysis, we refer the reader to [16, 32, 33, 38], which summarize some contributions on the analysis and design of switching systems. New investigations on stabilization and control for both linear and nonlinear switched systems have been addressed in the monograph [47], see also [42].

Several methodologies have been developed in the literature for both continuous-time an discrete-time systems [2, 8, 12, 34, 42]. Among the existing methods, we have: dwell-time and average dwell-time approaches for stability analysis and stabilization problems [21, 43]; approaches based on a specific class of switching laws [4], [23] and under arbitrary switching sequences [3]; sliding mode technique [39]; algebraic approach [2]; Lyapunov-Metzler approach [14, 18]; input-output approach [31].

In this paper, we investigate the problem of robust observer-based stabilization for linear switched discrete-time systems in the presence of parameter uncertainties. The switching mode is assumed to be arbitrary, but its instantaneous values are available in real time.

Most of the existing control strategies of switched systems focus on full-state feedback; see, e.g., [44] and [37]. However, in practice, full measurement of the states of a switched system may be expensive or unavailable at any cost. For this reason, considerable efforts have been paid to state estimation of linear and nonlinear switched systems [4, 8, 19]. On the other hand, it is always more suitable to design a control system which is not only stable, but also guarantees an adequate level of performance. This is why control systems design in the presence of model uncertainties has been a challenging topic and received considerable attention [20, 28, 36, 42].

One of hot topics in switched systems is to find non conservative conditions to guarantee the stabilization of the systems under arbitrary switching rules. A breakthrough regarding this issue is the switched quadratic Lyapunov functions (SLF) introduced in [12]. Within LMI framework, control techniques by switching among different controllers have been applied extensively in recent years, see in particular [6, 7, 14, 15, 45]. Control synthesis techniques via static output feedback for switched systems under arbitrary switching rule have been first considered in [12]. Sufficient LMI conditions subject to an equality constraint that guarantees the asymptotic stability of the closed-loop system have been given. Using similar techniques, the issue has been reconsidered in [22], in the observer-based static output feedback context, in presence of parameter uncertainties. Relevant results and interesting improvements of [12] have been considered in [7]. As for the dynamic output feedback, it has been investigated in [14].

Finsler's lemma has been used previously in the control literature mainly in order to eliminate some unlike matrix terms, see e.g. [9]. Switched quadratic Lyapunov functions combined with Finsler's lemma have been used in [17] to get necessary and sufficient LMI conditions for the asymptotic stability issue. However, based on the pioneering work in [17], some attempts using Finsler's lemma-based approach have been presented in [29] and [36]. Unfortunately, the obtained LMI conditions still remain very conservative. Indeed, they are either subjected to strong equality constraints [36], or require particular choices of the decision variables [29]. From LMI point of view, the stabilization problem is far from being solved. Indeed, finding a systematic LMI technique for handling the bilinear matrix terms related to the controller gains and the Finsler inequality is a hard task. This is one of the main motivations of the work investigated in this paper.

The main objective of this paper consists in developing new and less conservative LMI synthesis conditions for the observer-based stabilization problem for switched discrete-time linear systems with parameter uncertainties. As mentioned previously, the addressed problem has been investigated in [29] and [36] in the LMI context by using switched Lyapunov functions combined with Finsler's lemma. However, the obtained LMI conditions are conservative because of the particular use of Finsler's lemma in its basic formulation. In addition, the switched Lyapunov matrices used in [29] are assumed to be diagonal, and the LMI synthesis conditions proposed in [36] require additional equality constraints, which turns out to be very conservative. To show the importance of the work we proposed in this paper, we summarize the contributions in the following items:

- A new linearization scheme is proposed, thanks to a convenient use of Finsler's lemma. This novel use of Finsler's lemma may open some new directions to solve more complicated control design problems.
- Analytical developments to show the superiority of the proposed design method compared to the existing techniques in the literature are proposed in Section 3.2.
- As compared to the Young inequality based approach introduced recently in [24] and [26], the proposed technique in this paper allows to eliminate some bilinear terms arising from the use of the Young relation-based approach, and then leads to less conservative LMI conditions.
- The proposed LMI method is more general than those established in the literature for the same stabilization problem. We essentially demonstrated that the way to select the matrices inferred from Finsler's lemma plays an important role in the feasibility of the obtained LMIs. Some scenarios are provided as comparisons to the existing results in the literature.

It is worth noticing that this paper considers only the stabilization problem in the LMI framework. Therefore, all the results provided in the paper are compared with existing LMI techniques only. The proposed method is completely different from, for instance, those in [1, 30, 39, 40, 41], which dealt with Fibonacci switched-capacitor (SC) DC-AC inverter, optimal switching approach, dwell-time approach, sliding mode approach, and model approximation problem for T-S fuzzy switched systems with stochastic disturbances, respectively.

The remainder of the paper is organized as follows. Section 2.1 is devoted to the problem formulation and some preliminary results. The main contribution of this paper is presented in Section 3. Some numerical design aspects and constructive comments are provided in Secton 4. Section 5 gives simulation examples and comparisons to show the superiority of the proposed design methodology. Finally, some conclusions and future works are reported in Section 6.

Notation. We provide some notations used throughout this paper. Given a symmetric matrix S, then the symbol $S > 0$ (< 0) means that the matrix S is positive (negative) definite. $\mathbb{R}^{m\times n}$ denotes the set of $m \times n$ real matrices; the symbol $\mathbb{S}_+^{n\times n}$ denotes the set of $n \times n$ real symmetric positive-definite matrices; the notation $(\star)$ is used for the blocks induced by symmetry; A^T denotes the transpose of A; $\mathrm{He}(A)$ denotes $A + A^T$, and G^{-T} denotes the transpose of G^{-1}.

Before formulating the problem, let us introduce the following lemma [13], which plays an important role and constitutes the main tool in this paper.

Lemma 1 (Finsler). *Let $x \in \mathbb{R}^n, P \in \mathbb{S}^{n\times n}$, and $H \in \mathbb{R}^{m\times n}$ such that* $\mathrm{rank}(H) = r < n$. *The following statements are equivalent:*

1. $x^T P x < 0$, $\forall Hx = 0, x \neq 0$;
2. $\exists X \in \mathbb{R}^{n\times m}$ *such that* $P + XH + H^T X^T < 0$.

2. Problem Formulation and Background Results

This section is devoted to the formulation of the problem and background results before stating the main results of the paper.

2.1. Formulation of the Problem

Consider a class of switched discrete-time linear systems described by the following equations:

$$\begin{cases} x_{t+1} = (A_\sigma + \Delta A_\sigma)x_t + (B_\sigma + \Delta B_\sigma)u_t \\ y_t = (C_\sigma + \Delta C_\sigma)\, x_t \end{cases} \tag{1}$$

where $x_t \in \mathbb{R}^n$ is the state vector, $y_t \in \mathbb{R}^m$ is the output measurement vector, $u_t \in \mathbb{R}^p$ is the control input vector, and $\sigma : \mathbb{N} \to \Lambda \triangleq \{1, 2, \ldots, N\}$, $t \mapsto \sigma_t$, is a switching rule. If there is no ambiguity, we simply write σ instead of σ_t. A_σ,

B_σ, and C_σ, $\sigma \in \Lambda$, are $n \times n$, $n \times m$ and $p \times n$ real matrices, respectively. Assume that

$$[\Delta A_\sigma, \Delta B_\sigma, \Delta C_\sigma] \triangleq M_\sigma D_\sigma [E_{\sigma 1}, E_{\sigma 2}, E_{\sigma 3}], \tag{2}$$

where, for each $\sigma \in \Lambda$, the uncertainty D_σ satisfies

$$D_\sigma^T D_\sigma \leq I. \tag{3}$$

$M_\sigma, E_{\sigma 1}, E_{\sigma 2}, E_{\sigma 3}$ are constant matrices characterizing the structure of the uncertainties. Note that such a model can be used to describe a large class of practical systems, such as cognitive radio networks [27], stepper motors [10], and control of an F-16 aircraft [35]. The references [32], [34], [38] provide a general and accurate modeling framework for many relevant real-world models and processes. In particular, a discrete-time version of the Lipschitz nonlinear switched system modeling the longitudinal dynamics of an F-18 aircraft [43] can be viewed as a switched linear discrete-time system with parameter uncertainties. Indeed, any Lipschitz system can be transformed to a linear system with structured and norm-bounded parameter uncertainties [25].

Throughout the paper, the following assumption is needed [29]:

Assumption 1. The switching function σ satisfies the two following items:

i) σ is unknown a priori, but it is available in real-time;

ii) the switching of the observer should coincide exactly with the switching of the system.

As speculated in [20], assuming an unknown switching rule σ can be very useful in many practical applications such as the case when σ is computed via complex algorithms by a higher level supervisor or when it is generated by a human operator (for instance the switch of gears in a car).

The observer-based controller we consider in this paper is under the form :

$$\begin{cases} \hat{x}_{t+1} = A_\sigma \hat{x}_t + L_\sigma (y_t - C_\sigma \hat{x}_t) + B_\sigma u_t \\ \hat{y}_t = C_\sigma \hat{x}_t \\ u_t = K_\sigma \hat{x}_t \end{cases} \tag{4}$$

where $\hat{x}_t \in \mathbb{R}^n$ is the estimate of x_t, $L_\sigma \in \mathbb{R}^{n\times p}$, $K_\sigma \in \mathbb{R}^{m\times n}$, $\sigma \in \Lambda$, are the observer-based controller gains. Consider the generalized state vector

$$\bar{x}_t = [\hat{x}_t^T \quad e_t^T]^T,$$

where $e_t = \hat{x}_t - x_t$ is the estimation error. Then the closed-loop system resulted from (1) and (4) can be written as:

$$\overline{x}_{t+1} = \overline{A}_\sigma \bar{x}_t \tag{5}$$

where

$$\overline{A}_\sigma = \begin{bmatrix} A_\sigma + B_\sigma K_\sigma + L_\sigma \Delta C_\sigma & -L_\sigma (C_\sigma + \Delta C_\sigma) \\ -(\Delta A_\sigma + \Delta B_\sigma K_\sigma - L_\sigma \Delta C_\sigma) & A_\sigma + \Delta A_\sigma - L_\sigma (C_\sigma + \Delta C_\sigma) \end{bmatrix}. \tag{6}$$

The objective is to design output feedback matrices K_σ and L_σ, $\sigma \in \Lambda$, so that the closed-loop system (5) is asymptotically stable. Let us define the indicator function

$$\xi(t) = [\xi_1(t), \xi_2(t), \ldots, \xi_N(t)]^T$$

as follows:

$$\xi_i(t) = \begin{cases} 1, & \sigma_t = i; \\ 0, & \text{otherwise.} \end{cases}$$

Therefore, system (5) can be rewritten in the unified form:

$$\bar{x}_{t+1} = \sum_{i=1}^{N} \xi_i(t) \overline{A}_i \bar{x}_t, \tag{7}$$

where $\overline{A}_i$ is defined in (6), when $\sigma_t = i$.

To analyze stability of the closed-loop system (7), we use the switched Lyapunov function defined as:

$$\begin{aligned} V(\bar{x}_t, \xi(t)) &= \bar{x}_t^T \hat{P}(\xi(t)) \bar{x}_t \\ &= \sum_{i=1}^{N} \xi_i(t) \bar{x}_t^T \begin{bmatrix} \hat{P}_i^{11} & \hat{P}_i^{12} \\ (\star) & \hat{P}_i^{22} \end{bmatrix} \bar{x}_t. \end{aligned} \tag{8}$$

Notice that the Lyapunov function (8) is well known in the literature, (see for instance [11] and [5]). For shortness we use $\sigma_t = i$ and $\sigma_{t+1} = j$. This means that $\xi_i(t) = 1$ and $\xi_j(t+1) = 1$. Then we get

$$\begin{aligned}
\Delta V &\triangleq V(\bar{x}_{t+1}, \xi(t+1)) - V(\bar{x}_t, \xi(t)) \\
&= \bar{x}_{t+1}^T \left(\sum_{j=1}^{N} \xi_j(t+1)\hat{P}_j \right) \bar{x}_{t+1} - \bar{x}_t^T \left(\sum_{i=1}^{N} \xi_i(t)\hat{P}_i \right) \bar{x}_t \\
&= \sum_{j=1}^{N} \xi_j(t+1) \left(\sum_{i=1}^{N} \xi_i(t) \begin{bmatrix} \bar{x}_t \\ \bar{x}_{t+1} \end{bmatrix}^T \begin{bmatrix} -\hat{P}_i & 0 \\ 0 & \hat{P}_j \end{bmatrix} \begin{bmatrix} \bar{x}_t \\ \bar{x}_{t+1} \end{bmatrix} \right).
\end{aligned} \tag{9}$$

We have to show that, under suitable conditions, $\Delta V < 0$, which means that the closed-loop system (7) is asymptotically stable.

2.2. Background Results

This section is devoted to two LMI techniques reported in the literature, with which we will compare the proposed main contribution of this paper. On the other hand, it is worth mentioning that these two techniques can be considered as preliminary results because they are not available for the same class of systems. First, we will recall the standard Finsler lemma based approach in [29] that we will correct because of some erroneous mathematical decompositions in [29]. Second, we will generalize the Young inequality based approach introduced in [24, 26] to switched linear systems in the presence of parameter uncertainties.

2.2.1. Standard Finsler's lemma based approach [29]

Finsler's lemma has been frequently used in the literature for numerous control design problems. Especially in [29], this lemma has been used for the same class of systems in (1). Unfortunately, a mistake has significantly affected the LMI synthesis conditions, which renders the final result in [29] erroneous. Throughout this section, we will give a correct version of the result in [29] and we will provide some comments on the way Finsler's lemma has been used.

In [29], the Finsler lemma has been used with the following parameters

$$x = \begin{bmatrix} \overline{x}_t \\ \overline{x}_{t+1} \end{bmatrix},\ P_{ij} = \begin{bmatrix} -\hat{P}_i & 0 \\ 0 & \hat{P}_j \end{bmatrix},\ H_i = \begin{bmatrix} \overline{A}_i & -I \end{bmatrix},\ X_i = \begin{bmatrix} \hat{F}_i \\ \hat{G}_i^T \end{bmatrix}, \tag{10}$$

which lead to $\Delta V < 0$ if the following second Finsler inequality:

$$P_{ij} + X_i H_i + H_i^T X_i^T < 0 \tag{11}$$

holds for all $i, j \in \Lambda$. By substituting (10) in (11) we get the detailed inequality:

$$\begin{bmatrix} \hat{F}_i \overline{A}_i + \overline{A}_i^T \hat{F}_i^T - \hat{P}_i & -\hat{F}_i + \overline{A}_i^T \hat{G}_i \\ -\hat{F}_i^T + \hat{G}_i^T \overline{A}_i & \hat{P}_j - \hat{G}_i - \hat{G}_i^T \end{bmatrix} < 0, \tag{12}$$

instead of [29, Inequality (9)] which is erroneous.

Now, using the same matrices as in [29], defined as follows:

$$\hat{F}_i = \text{diag}(F_i, I), \hat{G}_i = \text{diag}(G_i, I),\ \hat{P}_i = \text{diag}(\hat{P}_i^{11}, \hat{P}_i^{22}),$$

with $F_i = G_i^T$.

To simplify the presentation and to understand more the corrected version of the result, we consider, as in [29], systems without uncertainties, i.e:

$$\Delta A_i = 0, \Delta B_i = 0, \Delta C_i = 0,\ \forall i \in \Lambda.$$

By substituting $\hat{F}_i$, $\hat{G}_i$, and $\hat{P}_i$ in (12), and after developing, we get the following detailed inequalities:

$$\begin{bmatrix} \Omega_{11}^i & -F_i L_i C_i & \Omega_{13}^i & 0 \\ (\star) & \Omega_{22}^i & -C_i^T L_i^T G_i & \Omega_{24}^i \\ (\star) & (\star) & \hat{P}_j^{11} - G_i^T - G_i & 0 \\ (\star) & (\star) & (\star) & \hat{P}_j^{22} - 2I \end{bmatrix} < 0,\ i, j \in \Lambda \tag{13}$$

where

$$\Omega_{11}^i = -\hat{P}_i^{11} + \text{He}\Big(F_i A_i + F_i B_i K_i\Big),$$
$$\Omega_{13}^i = -F_i + {A_i}^T G_i + K_i^T B_i^T G_i,$$
$$\Omega_{22}^i = -\hat{P}_i^{22} + \text{He}\Big(A_i - L_i C_i\Big),$$

$$\Omega_{24}^i = -I + A_i^T - C_i^T L_i^T.$$

Note that, for each $i, j \in \Lambda$, inequality (13) is a BMI, which cannot be linearized by choosing $F_i = G_i^T$ as in [29]. This difficulty is due to the presence of the coupling $F_i B_i K_i$ and $G_i^T B_i K_i$, which are vanished from [29, Inequality (9)] because of the mistake.

To linearize such a BMI, we use several steps. First, we pre- and post-multiply (13) by $\mathrm{diag}\Big(F_i^{-1}, I, G_i^{-T}, I\Big)$, we obtain:

$$\begin{bmatrix} \hat{\Omega}_{11}^i & -L_i C_i & \hat{\Omega}_{13}^i & 0 \\ (\star) & \Omega_{22}^i & -C_i^T L_i^T & \Omega_{24}^i \\ (\star) & (\star) & G_i^{-T} \hat{P}_j^{11} G_i^{-1} - G_i^{-T} - G_i^{-1} & 0 \\ (\star) & (\star) & (\star) & \hat{P}_j^{22} - 2I \end{bmatrix} < 0, \tag{14}$$

where

$$\hat{\Omega}_{11}^i = -F_i^{-1} \hat{P}_i^{11} F_i^{-T} + \mathrm{He}\Big(A_i F_i^{-T} + B_i K_i F_i^{-T}\Big),$$
$$\hat{\Omega}_{13}^i = -G_i^{-1} + F_i^{-1} A_i{}^T + F_i^{-1} K_i^T B_i^T.$$

There are still two bilinear terms in (14), namely $-F_i^{-1} \hat{P}_i^{11} F_i^{-T}$ and $G_i^{-T} \hat{P}_j^{11} G_i^{-1}$. To avoid these terms, we first introduce the following change of variables

$$F_i^{-1} = \tilde{F}_i, \; (\hat{P}_i^{11})^{-1} = \tilde{P}_i^{11}, \; G_i^{-1} = \tilde{G}_i, \;\; R_i = K_i F_i^{-T}.$$

Then, using the inequality

$$-F_i^{-1} \hat{P}_i^{11} F_i^{-T} \leq \tilde{P}_i^{11} - \tilde{F}_i - \tilde{F}_i^T,$$

it follows that (14) is fulfilled if the following inequality holds:

$$\begin{bmatrix} \Upsilon_{11}^i & -L_i C_i & \Upsilon_{13}^i & 0 \\ (\star) & \Omega_{22}^i & -C_i^T L_i^T & \Omega_{24}^i \\ (\star) & (\star) & \tilde{G}_i^T \hat{P}_j^{11} \tilde{G}_i - \tilde{G}_i^T - \tilde{G}_i & 0 \\ (\star) & (\star) & (\star) & \hat{P}_j^{22} - 2I \end{bmatrix} < 0, \tag{15}$$

where

$$\Upsilon_{11}^i = \tilde{P}_i^{11} + \mathrm{He}\Big(- \tilde{F}_i + A_i \tilde{F}_i^T + B_i R_i \Big),$$
$$\Upsilon_{13}^i = -\tilde{G}_i + \tilde{F}_i {A_i}^T + R_i^T B_i^T.$$

Finally, a simple application of Schur lemma [9] on (15) leads to the following theorem, which is a corrected version of [29, Theorem 1].

Theorem 1. *Assume that there exist matrices $\tilde{P}_i^{11}, \hat{P}_i^{22} \in \mathbb{S}_+^{n\times n}$, invertible matrices $\tilde{F}_i \in \mathbb{R}^{n\times n}$, $\tilde{G}_i \in \mathbb{R}^{n\times n}$, and arbitrary matrices $R_i \in \mathbb{R}^{m\times n}$, $L_i \in \mathbb{R}^{n\times p}$, $i \in \Lambda$, such that the following LMIs are fulfilled:*

$$\begin{bmatrix} \Upsilon_{11}^i & -L_i C_i & \Upsilon_{13}^i & 0 & 0 \\ (\star) & \Upsilon_{22}^i & -C_i^T L_i^T & -I + A_i^T - C_i^T L_i^T & 0 \\ (\star) & (\star) & -\tilde{G}_i - \tilde{G}_i^T & 0 & \tilde{G}_i^T \\ (\star) & (\star) & (\star) & \hat{P}_j^{22} - 2I & 0 \\ (\star) & (\star) & (\star) & (\star) & -\tilde{P}_j^{11} \end{bmatrix} < 0, \; i, j \in \Lambda \quad (16)$$

with

$$\Upsilon_{11}^i = \tilde{P}_i^{11} + \mathrm{He}\Big(- \tilde{F}_i + A_i \tilde{F}_i^T + B_i R_i \Big),$$
$$\Upsilon_{13}^i = -\tilde{G}_i + \tilde{F}_i {A_i}^T + R_i^T B_i^T,$$
$$\Upsilon_{22}^i = -\hat{P}_i^{22} + \mathrm{He}\Big(A_i - L_i C_i \Big).$$

Then, the closed-loop system (5) is asymptotically stable for the observer-based controller gains

$$K_i = R_i \tilde{F}_i^{-T}, \, i \in \Lambda \quad (17)$$

and L_i, $i \in \Lambda$, are free solutions of (16).

Proof. The rest of the proof is omitted. It is based on the use of the Schur lemma on (15) to linearize the remaining bilinear term $G_i^{-T} \hat{P}_j^{11} G_i^{-1}$.

2.2.2. Young's inequality based approach

This section is dedicated for the application of Young's relation based approach introduced in [24, 26] to solve the problem of observer-based stabilization problem of linear uncertain systems. The Young inequality based approach in [24] corresponds to $\hat{F}_i = 0$ and $\hat{G}_i = \mathrm{diag}(G^{11}, G_i^{22})$, with $G_i^{22} = (G_i^{22})^T$. It follows that inequality (12) has the same structure than [24, Inequality (11)].

The crucial linearization problem lies in the presence of the isolated term $(C_i + \Delta C_i)^T L_i^T$, while the matrix L_i is elsewhere coupled with the matrix G_i^{22}. Then, to retrieve the term $L_i^T G_i^{22}$ and eliminate the isolated term related to L_i, in order the make a change of variables, a solution has been proposed in [24, 26], which provides straightforwardly the next theorem valid for linear switched systems (1).

Theorem 2. *Assume that for some fixed positive scalars ϵ_i, γ_i and μ_i, $i \in \Lambda$, there exist positive definite matrices $\mathcal{D}_i \triangleq \begin{bmatrix} \tilde{P}_i^{11} & \tilde{P}_i^{12} \\ (\star) & \hat{P}_i^{22} \end{bmatrix} \in \mathbb{R}^{2n\times 2n}$ and G_i^{22}, $\tilde{G}^{11} \in \mathbb{R}^{n\times n}$, $\tilde{K}_i \in \mathbb{R}^{m\times n}$, $\hat{L}_i \in \mathbb{R}^{n\times p}$, for $i \in \Lambda$, such that the LMI (18) holds for all $i, j \in \Lambda$*

$$\begin{bmatrix}
-\tilde{P}_i^{11} & -\tilde{P}_i^{12} & \mathbf{(1.3)} & 0 & 0 & 0 & \mathbf{(1.7)} & 0 & \mathbf{(1.9)} & 0 \\
(\star) & -\hat{P}_i^{22} & 0 & \mathbf{(2.4)} & -C_i^T \hat{L}_i^T & 0 & E_{i1}^T & 0 & -E_{i3}^T & 0 \\
(\star) & (\star) & \mathbf{(3.3)} & \tilde{P}_j^{12} & 0 & I & 0 & 0 & 0 & 0 \\
(\star) & (\star) & (\star) & \mathbf{(4.4)} & 0 & 0 & 0 & G_i^{22} M_i & 0 & \hat{L}_i M_i \\
(\star) & (\star) & (\star) & (\star) & -\epsilon_i G_i^{22} & 0 & 0 & 0 & 0 & \hat{L}_i M_i \\
(\star) & (\star) & (\star) & (\star) & (\star) & -\epsilon_i^{-1} G_i^{22} & 0 & 0 & 0 & 0 \\
(\star) & (\star) & (\star) & (\star) & (\star) & (\star) & -\gamma_i I & 0 & 0 & 0 \\
(\star) & (\star) & (\star) & (\star) & (\star) & (\star) & (\star) & -\gamma_i^{-1} I & 0 & 0 \\
(\star) & (\star) & (\star) & (\star) & (\star) & (\star) & (\star) & (\star) & -\mu_i I & 0 \\
(\star) & (\star) & (\star) & (\star) & (\star) & (\star) & (\star) & (\star) & (\star) & -\mu_i^{-1} I
\end{bmatrix} < 0 \tag{18}$$

$$\begin{aligned}
&\mathbf{(1.3)} = (\tilde{G}^{11})^T A_i^T + \tilde{K}_i^T B_i^T, && \mathbf{(1.7)} = -(\tilde{G}^{11})^T E_{i1}^T - \tilde{K}_i^T E_{i1}^T \\
&\mathbf{(1.9)} = (\tilde{G}^{11})^T E_{i3}^T, && \mathbf{(2.4)} = A_i^T G_i^{22} - C_i^T \hat{L}_i^T \\
&\mathbf{(3.3)} = \tilde{P}_j^{11} - \tilde{G}^{11} - (\tilde{G}^{11})^T, && \mathbf{(4.4)} = \hat{P}_j^{22} - 2G_i^{22}
\end{aligned} \tag{19}$$

Then the closed-loop system (5) is asymptotically stable with the observer-based controller gains:

$$K_i = \tilde{K}_i \tilde{G}^{11}, \; L_i = (G_i^{22})^{-1} \hat{L}_i. \tag{20}$$

Proof. The proof is omitted. It is straightforward and follows exactly the same steps than [24]. The matrix $\mathcal{D}_i$ comes from the change of variable:

$$\mathcal{D}_i \triangleq \begin{bmatrix} (\tilde{G}^{11})^T & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \hat{P}_i^{11} & \hat{P}_i^{12} \\ (\star) & \hat{P}_i^{22} \end{bmatrix} \begin{bmatrix} \tilde{G}^{11} & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} \tilde{P}_i^{11} & \tilde{P}_i^{12} \\ (\star) & \hat{P}_i^{22} \end{bmatrix}.$$

Although Theorem 1 and Theorem 2 provide solutions to the observer-based stabilization problem for switched linear systems, the obtained LMIs still remain conservative, and then there are some possibilities for improvements from LMI feasibility point of view. This is the objective of the next section, where new and enhanced LMI conditions will be proposed by exploiting the Finsler lemma in a non-standard way.

3. Main Results: Enhanced LMI Conditions

In this section, we introduce the main result of this paper, which consists in new LMI conditions to solve the problem of robust observer-based stabilization for switched systems. We will show that thanks to the use of convenient matrices in the Finsler lemma, we get more general and less conservative LMIs compared to the those presented in the previous section.

3.1. Introductory developments

We will analyze all the bilinear terms in (12) by considering the detailed structures of $\hat{F}_i, \hat{G}_i$, and $\hat{P}_i$ as follows:

$$\hat{F}_i \triangleq \begin{bmatrix} F_i^{11} & F_i^{12} \\ F_i^{21} & F_i^{22} \end{bmatrix}, \ \hat{G}_i \triangleq \begin{bmatrix} G_i^{11} & G_i^{12} \\ G_i^{21} & G_i^{22} \end{bmatrix}, \ \hat{P}_i \triangleq \begin{bmatrix} \hat{P}_i^{11} & \hat{P}_i^{12} \\ (\star) & \hat{P}_i^{22} \end{bmatrix}. \tag{21}$$

By substituting (21) in (12) and after developing, we get the new inequality:

$$\begin{bmatrix} \Omega_{11}^i & \Omega_{12}^i & \Omega_{13}^i & \Omega_{14}^i \\ (\star) & \Omega_{22}^i & \Omega_{23}^i & \Omega_{24}^i \\ (\star) & (\star) & \hat{P}_j^{11} - (G_i^{11})^T - G_i^{11} & \hat{P}_j^{12} - (G_i^{21})^T - G_i^{12} \\ (\star) & (\star) & (\star) & \hat{P}_j^{22} - G_i^{22} - (G_i^{22})^T \end{bmatrix} < 0, \tag{22}$$

where

$$
\begin{aligned}
\Omega_{11}^{i} &= -\hat{P}_i^{11} + \mathrm{He}\Big(F_i^{11}A_i + F_i^{11}B_iK_i + (F_i^{11} + F_i^{12})L_i\Delta C_i - F_i^{12}\Delta A_i - F_i^{12}\Delta B_iK_i\Big),\\
\Omega_{12}^{i} &= -\hat{P}_i^{12} + F_i^{12}(A_i + \Delta A_i) - (F_i^{11} + F_i^{12})L_i(C_i + \Delta C_i) + A_i^T(F_i^{21})^T,\\
&\quad + K_i^TB_i^T(F_i^{21})^T + \Delta C_i^TL_i^T(F_i^{21} + F_i^{22})^T - \Delta A_i^T(F_i^{22})^T - K_i^T\Delta B_i^T(F_i^{22})^T,\\
\Omega_{13}^{i} &= -F_i^{11} + A_i{}^TG_i^{11} + K_i^TB_i^TG_i^{11} - \Delta A_i^TG_i^{21} - K_i^T\Delta B_i^TG_i^{21} + \Delta C_i^TL_i^T(G_i^{11} + G_i^{21}),\\
\Omega_{14}^{i} &= -F_i^{12} + A_i^TG_i^{12} + K_i^TB_i^TG_i^{12} - \Delta A_i{}^TG_i^{22} - K_i^T\Delta B_i^TG_i^{22} + \Delta C_i^TL_i^T(G_i^{12} + G_i^{22}),\\
\Omega_{22}^{i} &= -\hat{P}_i^{22} + \mathrm{He}\Big(F_i^{22}(A_i + \Delta A_i) - (F_i^{21} + F_i^{22})L_i(C_i + \Delta C_i)\Big),\\
\Omega_{23}^{i} &= -F_i^{21} - (C_i + \Delta C_i)^TL_i^T(G_i^{11} + G_i^{21}) + A_i^TG_i^{21} + \Delta A_i^TG_i^{21},\\
\Omega_{24}^{i} &= -F_i^{22} + A_i^TG_i^{22} + \Delta A_i^TG_i^{22} - (C_i + \Delta C_i)^TL_i^T(G_i^{12} + G_i^{22}).
\end{aligned}
$$

As we can see, the linearization problem is a hard challenge due to the presence of twelve bilinear terms without counting the bilinearities related to the uncertainties. We cannot use change of variables because the matrices L_i, $i \in \Lambda$, are coupled with eight different matrices, namely G_i^{11}, G_i^{12}, G_i^{22}, G_i^{21}, F_i^{11}, F_i^{12}, F_i^{22}, and F_i^{21}. The strategy consists in exploiting the invertibility of the matrices G_i^{11}, and G_i^{22}, which is a consequence of (22). Then we use the congruence principle with convenient matrices. To do this, we first start by linearizing the bilinear terms related to the gains K_i. The linearization procedure is presented in the next section.

3.2. A new linearization procedure

To enhance the clarity of the contributions and to simplify the understanding of the main ideas, the proposed linearization strategy is shared into three steps.

3.2.1. First step: Linearization with respect to K_i

Since the matrices G_i^{11} and G_i^{22} are necessarily invertible, then using a congruence transformation on (22) by pre- and post-multiplying by

$\mathrm{diag}\Big((G_i^{11})^{-T}, I, (G_i^{11})^{-T}, I\Big)$, we get

$$\begin{bmatrix} \hat{\Omega}_{11}^i & \hat{\Omega}_{12}^i & \hat{\Omega}_{13}^i & \hat{\Omega}_{14}^i \\ (\star) & -\hat{\Omega}_{22}^i & \hat{\Omega}_{23}^i & \hat{\Omega}_{24}^i \\ (\star) & (\star) & \hat{\Omega}_{33}^{ij} & \hat{\Omega}_{34}^{ij} \\ (\star) & (\star) & (\star) & \hat{P}_j^{22} - G_i^{22} - (G_i^{22})^T \end{bmatrix} < 0, \tag{23}$$

where

$$\begin{aligned}
\hat{\Omega}_{11}^i =& - (G_i^{11})^{-T}\hat{P}_i^{11}(G_i^{11})^{-1} + \mathrm{He}\Big((G_i^{11})^{-T}F_i^{11}A_i(G_i^{11})^{-1} \\
&+ (G_i^{11})^{-T}F_i^{11}B_iK_i(G_i^{11})^{-1} + ((G_i^{11})^{-T}F_i^{11} + (G_i^{11})^{-T}F_i^{12})L_i\Delta C_i(G_i^{11})^{-1} \\
&- (G_i^{11})^{-T}F_i^{12}\Delta A_i(G_i^{11})^{-1} - (G_i^{11})^{-T}F_i^{12}\Delta B_iK_i(G_i^{11})^{-1}\Big), \\
\hat{\Omega}_{12}^i =& - (G_i^{11})^{-T}\hat{P}_i^{12} + (G_i^{11})^{-T}F_i^{12}(A_i + \Delta A_i) \\
&- ((G_i^{11})^{-T}F_i^{11} + (G_i^{11})^{-T}F_i^{12})L_i(C_i + \Delta C_i) + (G_i^{11})^{-T}A_i^T(F_i^{21})^T \\
&+ (G_i^{11})^{-T}K_i^TB_i^T(F_i^{21})^T + (G_i^{11})^{-T}\Delta C_i^TL_i^T(F_i^{21} + F_i^{22})^T \\
&- (G_i^{11})^{-T}\Delta A_i^T(F_i^{22})^T - (G_i^{11})^{-T}K_i^T\Delta B_i^T(F_i^{22})^T, \\
\hat{\Omega}_{13}^i =& - (G_i^{11})^{-T}F_i^{11}(G_i^{11})^{-1} + (G_i^{11})^{-T}A_i{}^T + (G_i^{11})^{-T}K_i^TB_i^T \\
&- (G_i^{11})^{-T}\Delta A_i^TG_i^{21}(G_i^{11})^{-1} - (G_i^{11})^{-T}K_i^T\Delta B_i^TG_i^{21}(G_i^{11})^{-1} \\
&+ (G_i^{11})^{-T}\Delta C_i^TL_i^T(I + G_i^{21}(G_i^{11})^{-1}), \\
\hat{\Omega}_{14}^i =& - (G_i^{11})^{-T}F_i^{12} + (G_i^{11})^{-T}A_i^TG_i^{12} + (G_i^{11})^{-T}K_i^TB_i^TG_i^{12} \\
&- (G_i^{11})^{-T}\Delta A_i{}^TG_i^{22} - (G_i^{11})^{-T}K_i^T\Delta B_i^TG_i^{22} \\
&+ (G_i^{11})^{-T}\Delta C_i^TL_i^T(G_i^{12} + G_i^{22}), \\
\hat{\Omega}_{22}^i =& \Omega_{22}^i, \\
\hat{\Omega}_{23}^i =& - F_i^{21}(G_i^{11})^{-1} - (C_i + \Delta C_i)^TL_i^T(I + G_i^{21}(G_i^{11})^{-1}) + A_i^TG_i^{21}(G_i^{11})^{-1} \\
&+ \Delta A_i^TG_i^{21}(G_i^{11})^{-1}, \\
\hat{\Omega}_{24}^i =& \Omega_{24}^i, \\
\hat{\Omega}_{33}^{ij} =& (G_i^{11})^{-T}\hat{P}_j^{11}(G_i^{11})^{-1} - (G_i^{11})^{-1} - (G_i^{11})^{-T} \\
\hat{\Omega}_{34}^{ij} =& (G_i^{11})^{-T}\hat{P}_j^{12} - (G_i^{11})^{-T}G_i^{12} - (G_i^{11})^{-T}(G_i^{21})^T.
\end{aligned}$$

Then, we can be see, there are two "similar" bilinear terms in inequality (23), namely, $(G_i^{11})^{-T}\hat{P}_i^{11}(G_i^{11})^{-1}$ and $(G_i^{11})^{-T}\hat{P}_j^{11}(G_i^{11})^{-T}$, in the expressions of $\hat{\Omega}_{11}^{i}$, and $\hat{\Omega}_{33}^{ij}$, respectively.

By choosing $G_i^{11} = G^{11}$ for all i, then we can introduce a suitable change of variables. On the other hand, in order to avoid some bilinear terms containing K_i, we focus on the case where $F_i^{11} = 0$. To sum up, we introduce the convenient change of variables:

$$(G^{11})^{-1} \triangleq \tilde{G}^{11}, \quad \tilde{K}_i \triangleq K_i\tilde{G}^{11}, \; (\tilde{G}^{11})^T\hat{P}_i^{11}\tilde{G}^{11} \triangleq \tilde{P}_i^{11}, \; (\tilde{G}^{11})^T\hat{P}_i^{12} \triangleq \tilde{P}_i^{12}.$$

Therefore, inequality (23) becomes:

$$\begin{bmatrix} \tilde{\Omega}_{11}^{i} & \tilde{\Omega}_{12}^{i} & \tilde{\Omega}_{13}^{i} & \tilde{\Omega}_{14}^{i} \\ (\star) & \tilde{\Omega}_{22}^{i} & \tilde{\Omega}_{23}^{i} & \tilde{\Omega}_{24}^{i} \\ (\star) & (\star) & \tilde{P}_j^{11} - \tilde{G}^{11} - (\tilde{G}^{11})^T & \tilde{P}_j^{12} - \tilde{G}^{11}G_i^{12} - \tilde{G}^{11}(G_i^{21})^T \\ (\star) & (\star) & (\star) & \hat{P}_j^{22} - G_i^{22} - (G_i^{22})^T \end{bmatrix} < 0, \quad (24)$$

where

$$\begin{aligned}
\tilde{\Omega}_{11}^{i} =& -\tilde{P}_i^{11} + \mathrm{He}\Big((\tilde{G}^{11})^T F_i^{12} L_i \Delta C_i \tilde{G}^{11} - (\tilde{G}^{11})^T F_i^{12}\Delta A_i(\tilde{G}^{11}) - (\tilde{G}^{11})^T F_i^{12}\Delta B_i \tilde{K}_i\Big),\\
\tilde{\Omega}_{12}^{i} =& -\tilde{P}_i^{12} + (\tilde{G}^{11})^T F_i^{12}(A_i + \Delta A_i) - (\tilde{G}^{11})^T F_i^{12} L_i (C_i + \Delta C_i)\\
&+ (\tilde{G}^{11})^T A_i^T (F_i^{21})^T + \tilde{K}_i^T B_i^T (F_i^{21})^T + (\tilde{G}^{11})^T \Delta C_i^T L_i^T (F_i^{21} + F_i^{22})^T\\
&- (\tilde{G}^{11})^T \Delta A_i^T (F_i^{22})^T - \tilde{K}_i^T \Delta B_i^T (F_i^{22})^T,\\
\tilde{\Omega}_{13}^{i} =& (\tilde{G}^{11})^T A_i^T + \tilde{K}_i^T B_i^T - (\tilde{G}^{11})^T \Delta A_i^T G_i^{21}\tilde{G}^{11} - \tilde{K}_i^T \Delta B_i^T G_i^{21}\tilde{G}^{11}\\
&+ (\tilde{G}^{11})^T \Delta C_i^T L_i^T (I + G_i^{21}\tilde{G}^{11}),\\
\tilde{\Omega}_{14}^{i} =& -(\tilde{G}^{11})^T F_i^{12} + (\tilde{G}^{11})^T A_i^T G_i^{12} + \tilde{K}_i^T B_i^T G_i^{12} - (\tilde{G}^{11})^T \Delta A_i{}^T G_i^{22}\\
&- \tilde{K}_i^T \Delta B_i^T G_i^{22} + (\tilde{G}^{11})^T \Delta C_i^T L_i^T (G_i^{12} + G_i^{22}),\\
\tilde{\Omega}_{22}^{i} =& -\hat{P}_i^{22} + \mathrm{He}\Big(F_i^{22}(A_i + \Delta A_i) - (F_i^{21} + F_i^{22}) L_i (C_i + \Delta C_i)\Big),\\
\tilde{\Omega}_{23}^{i} =& -F_i^{21}\tilde{G}^{11} - (C_i + \Delta C_i)^T L_i^T (I + G_i^{21}\tilde{G}^{11}) + A_i^T G_i^{21}\tilde{G}^{11} + \Delta A_i^T G_i^{21}\tilde{G}^{11},\\
\tilde{\Omega}_{24}^{i} =& -F_i^{22} + A_i^T G_i^{22} + \Delta A_i^T G_i^{22} - (C_i + \Delta C_i)^T L_i^T (G_i^{12} + G_i^{22}).
\end{aligned}$$

Inequality (24) is still a BMI with respect to $\tilde{K}_i$, even in the uncertainty free case. This is due to their coupling with the matrices F_i^{21} and G_i^{12}. Then, these bilinear terms vanish if $G_i^{12} = F_i^{21} = 0$. Then, in such a case, inequality (24) is equivalent to the following one:

$$\begin{bmatrix} \Theta_{11}^i & \Theta_{12}^i & \Theta_{13}^i & \Theta_{14}^i \\ (\star) & \Theta_{22}^i & \Theta_{23}^i & \Theta_{24}^i \\ (\star) & (\star) & \tilde{P}_j^{11} - \tilde{G}^{11} - (\tilde{G}^{11})^T & \tilde{P}_j^{12} - \tilde{G}^{11}(G_i^{21})^T \\ (\star) & (\star) & (\star) & \hat{P}_j^{22} - G_i^{22} - (G_i^{22})^T \end{bmatrix} < 0, \qquad (25)$$

where

$$\begin{aligned}
\Theta_{11}^i =& -\tilde{P}_i^{11} + \mathrm{He}\Big((\tilde{G}^{11})^T F_i^{12} L_i \Delta C_i \tilde{G}^{11} - (\tilde{G}^{11})^T F_i^{12} \Delta A_i (\tilde{G}^{11}) - (\tilde{G}^{11})^T F_i^{12} \Delta B_i \tilde{K}_i\Big), \\
\Theta_{12}^i =& -\tilde{P}_i^{12} + (\tilde{G}^{11})^T F_i^{12}(A_i + \Delta A_i) - (\tilde{G}^{11})^T F_i^{12} L_i (C_i + \Delta C_i) \\
& + (\tilde{G}^{11})^T \Delta C_i^T L_i^T (F_i^{22})^T - (\tilde{G}^{11})^T \Delta A_i^T (F_i^{22})^T - \tilde{K}_i^T \Delta B_i^T (F_i^{22})^T, \\
\Theta_{13}^i =& (\tilde{G}^{11})^T A_i^T + \tilde{K}_i^T B_i^T - (\tilde{G}^{11})^T \Delta A_i^T G_i^{21} \tilde{G}^{11} - \tilde{K}_i^T \Delta B_i^T G_i^{21} \tilde{G}^{11} \\
& + (\tilde{G}^{11})^T \Delta C_i^T L_i^T (I + G_i^{21} \tilde{G}^{11}), \\
\Theta_{14}^i =& -(\tilde{G}^{11})^T F_i^{12} - (\tilde{G}^{11})^T \Delta {A_i}^T G_i^{22} - \tilde{K}_i^T \Delta B_i^T G_i^{22} + (\tilde{G}^{11})^T \Delta C_i^T L_i^T G_i^{22}, \\
\Theta_{22}^i =& -\hat{P}_i^{22} + \mathrm{He}\Big(F_i^{22}(A_i + \Delta A_i) - F_i^{22} L_i (C_i + \Delta C_i)\Big), \\
\Theta_{23}^i =& -(C_i + \Delta C_i)^T L_i^T (I + G_i^{21} \tilde{G}^{11}) + A_i^T G_i^{21} \tilde{G}^{11} + \Delta A_i^T G_i^{21} \tilde{G}^{11}, \\
\Theta_{24}^i =& -F_i^{22} + A_i^T G_i^{22} + \Delta A_i^T G_i^{22} - (C_i + \Delta C_i)^T L_i^T G_i^{22}.
\end{aligned}$$

Now that the BMI (22) is linearized with respect to the controller matrices $\tilde{K}_i$, we will proceed to the linearization with respect to the observer gains L_i. This is the aim of the next linearization step.

3.2.2. Second step: Linearization of (25) with respect to L_i

Throughout this step, we aim to linearize all the bilinear terms related to the observer gains L_i, namely the terms $(\tilde{G}^{11})^T F_i^{12} L_i C_i$, $(I + G_i^{21}\tilde{G}^{11})^T L_i C_i$, $F_i^{22} L_i C_i$, and $(G_i^{22})^T L_i C_i$. The other terms containing the uncertainties will be handled in the third linearization step. To avoid all the previous bilinear terms, the strategy consists in taking

$$(\tilde{G}^{11})^T F_i^{12} = I + G_i^{21}\tilde{G}^{11} = (G_i^{22})^T \text{ and } F_i^{22} = 0.$$

This identities lead to

$$F_i^{12} = (G^{11})^T (G_i^{22})^T,\ G_i^{21} = G_i^{22} G^{11} - G^{11} \ \text{ and } \ F_i^{22} = 0,$$

which means that the matrices $\hat{F}_i$ and $\hat{G}_i$ have the following structures:

$$\hat{F}_i = \begin{bmatrix} 0 & (G^{11})^T (G_i^{22})^T \\ 0 & 0 \end{bmatrix}, \ \hat{G}_i = \begin{bmatrix} G^{11} & 0 \\ G_i^{22} G^{11} - G^{11} & G_i^{22} \end{bmatrix}. \tag{26}$$

It follows that the following change of variable

$$\hat{L}_i = (G_i^{22})^T L_i$$

is possible.

By substituting (26) in (25) we get the new inequality:

$$\begin{bmatrix} \hat{\Theta}_{11}^i & \hat{\Theta}_{12}^i & \hat{\Theta}_{13}^i & \hat{\Theta}_{14}^i \\ (\star) & -\hat{P}_i^{22} & \hat{\Theta}_{23}^i & \hat{\Theta}_{24}^i \\ (\star) & (\star) & \tilde{P}_j^{11} - \tilde{G}^{11} - (\tilde{G}^{11})^T & \tilde{P}_j^{12} + I - (G_i^{22})^T \\ (\star) & (\star) & (\star) & \hat{P}_j^{22} - G_i^{22} - (G_i^{22})^T \end{bmatrix} < 0, \tag{27}$$

where

$$\begin{aligned}
\hat{\Theta}_{11}^i =& - \tilde{P}_i^{11} + \mathrm{He}\Big(\hat{L}_i \Delta C_i \tilde{G}^{11} - (G_i^{22})^T \Delta A_i \tilde{G}_i^{11} - (G_i^{22})^T \Delta B_i \tilde{K}_i\Big), \\
\hat{\Theta}_{12}^i =& - \tilde{P}_i^{12} + (G_i^{22})^T (A_i + \Delta A_i) - \hat{L}_i (C_i + \Delta C_i), \\
\hat{\Theta}_{13}^i =& (\tilde{G}^{11})^T {A_i}^T + \tilde{K}_i^T B_i^T - (\tilde{G}^{11})^T \Delta {A_i}^T (G_i^{22} - I) \\
& - \tilde{K}_i^T \Delta B_i^T (G_i^{22} - I) + (\tilde{G}^{11})^T \Delta C_i^T \hat{L}_i^T, \\
\hat{\Theta}_{14}^i =& - (G_i^{22})^T - (\tilde{G}^{11})^T \Delta {A_i}^T G_i^{22} - \tilde{K}_i^T \Delta B_i^T G_i^{22} + (\tilde{G}^{11})^T \Delta C_i^T \hat{L}_i^T, \\
\hat{\Theta}_{23}^i =& - (C_i + \Delta C_i)^T \hat{L}_i^T + (A_i^T + \Delta A_i^T)(G_i^{22} - I), \\
\hat{\Theta}_{24}^i =& A_i^T G_i^{22} + \Delta A_i^T G_i^{22} - (C_i + \Delta C_i)^T \hat{L}_i^T.
\end{aligned} \tag{28}$$

All the bilinear terms, except those related to ΔA_i, ΔB_i, and ΔC_i, are avoided. These terms will be handled in the next and last linearization step.

3.2.3. *Third step: Full linearization*

This step is classic and well-known in the literature, see in particular [24]. By developing ΔA_i, ΔB_i, and ΔC_i, we can rewrite (27) in the following convenient from:

$$\Xi^{ij} + \mathrm{He}\Big(Z_{i1}^T D_i^T Z_{i2} + Z_{i3}^T D_i^T Z_{i4} \Big) < 0, \tag{29}$$

where

$$Z_{i1} = \begin{bmatrix} (\Theta_{15}^i)^T & E_{i1} & 0 & 0 \end{bmatrix}^T, \; Z_{i3} = \begin{bmatrix} E_{i3}\tilde{G}^{11} & -E_{i3} & 0 & 0 \end{bmatrix}^T,$$

$$Z_{i2} = \begin{bmatrix} M_i^T G_i^{22} & 0 & M_i^T (G_i^{22} - I) & M_i^T G_i^{22} \end{bmatrix},$$

$$Z_{i4} = \begin{bmatrix} M_i^T \hat{L}_i^T & 0 & M_i^T \hat{L}_i^T & M_i^T \hat{L}_i^T \end{bmatrix},$$

$$\Xi^{ij} = \begin{bmatrix} -\tilde{P}_i^{11} & \tilde{\Theta}_{12}^i & \tilde{\Theta}_{13}^i & -(G_i^{22})^T \\ (\star) & -\hat{P}_i^{22} & \tilde{\Theta}_{23}^i & \tilde{\Theta}_{24}^i \\ (\star) & (\star) & \tilde{P}_j^{11} - \tilde{G}^{11} - (\tilde{G}^{11})^T & \tilde{P}_j^{12} + I - (G_i^{22})^T \\ (\star) & (\star) & (\star) & \hat{P}_j^{22} - G_i^{22} - (G_i^{22})^T \end{bmatrix},$$

$$\begin{aligned} \tilde{\Theta}_{12}^i &= -\tilde{P}_i^{12} + (G_i^{22})^T A_i - \hat{L}_i C_i, \\ \tilde{\Theta}_{13}^i &= (\tilde{G}^{11})^T A_i^T + \tilde{K}_i^T B_i^T, \\ \tilde{\Theta}_{15}^i &= -(\tilde{G}^{11})^T E_{i1}^T - \tilde{K}_i^T E_{i2}^T, \\ \tilde{\Theta}_{23}^i &= -C_i^T \hat{L}_i^T + A_i^T (G_i^{22} - I), \\ \tilde{\Theta}_{24}^i &= A_i^T G_i^{22} - C_i^T \hat{L}_i^T, \\ \Theta_{15}^i &= -(\tilde{G}^{11})^T E_{i1}^T - \tilde{K}_i^T E_{i2}^T. \end{aligned}$$

Using the Young inequality [9] and the fact that $D_\sigma^T D_\sigma \le I$, we deduce that inequality (29) is fulfilled if the following one holds:

$$\Xi^{ij} + \alpha_i^{-1} Z_{i1}^T Z_{i1} + \alpha_i Z_{i2}^T Z_{i2} + \lambda_i^{-1} Z_{i3}^T Z_{i3} + \lambda_i Z_{i4}^T Z_{i4} < 0, \tag{30}$$

where α_i and λ_i are some positive scalars. Now, it remains to use Schur lemma on the right hand side of (30) to get an LMI. This LMI is stated in the next theorem.

Theorem 3. *Assume that there exist positive definite matrices*

$$\mathcal{D}_i \triangleq \begin{bmatrix} \tilde{P}_i^{11} & \tilde{P}_i^{12} \\ (\star) & \hat{P}_i^{22} \end{bmatrix} \in \mathbb{R}^{2n \times 2n},$$

invertible matrices G_i^{22} *and* $\tilde{G}^{11} \in \mathbb{R}^{n\times n}$*, and matrices* $\tilde{K}_i \in \mathbb{R}^{m\times n}, \hat{L}_i \in \mathbb{R}^{n\times p}$*, for* $i \in \Lambda$*, such that the LMI (31) holds for some positive constants* α_i *and* λ_i*, for all* $i, j \in \Lambda$*.*

$$
\begin{bmatrix}
-\tilde{P}_i^{11} & \mathbf{(1.2)} & \mathbf{(1.3)} & -(G_i^{22})^T & \mathbf{(1.5)} & (G_i^{22})^T M_i & (\tilde{G}^{11})^T E_{i3}^T & \hat{L}_i M_i \\
(\star) & -\hat{P}_i^{22} & \mathbf{(2.3)} & \mathbf{(2.4)} & E_{i1}^T & 0 & -E_{i3}^T & 0 \\
(\star) & (\star) & \mathbf{(3.3)} & \mathbf{(3.4)} & 0 & \mathbf{(3.6)} & 0 & \hat{L}_i M_i \\
(\star) & (\star) & (\star) & \mathbf{(4.4)} & 0 & (G_i^{22})^T M_i & 0 & \hat{L}_i M_i \\
(\star) & (\star) & (\star) & (\star) & -\alpha_i I & 0 & 0 & 0 \\
(\star) & (\star) & (\star) & (\star) & (\star) & -\alpha_i^{-1} I & 0 & 0 \\
(\star) & (\star) & (\star) & (\star) & (\star) & (\star) & -\lambda_i I & 0 \\
(\star) & (\star) & (\star) & (\star) & (\star) & (\star) & (\star) & -\lambda_i^{-1} I
\end{bmatrix} < 0 \tag{31}
$$

$$
\begin{aligned}
&\mathbf{(1.2)} = -\tilde{P}_i^{12} + (G_i^{22})^T A_i - \hat{L}_i C_i, && \mathbf{(1.3)} = (\tilde{G}^{11})^T A_i^T + \tilde{K}_i^T B_i^T \\
&\mathbf{(1.5)} = -(\tilde{G}^{11})^T E_{i1}^T - \tilde{K}_i^T E_{i2}^T, && \mathbf{(2.3)} = -C_i^T \hat{L}_i^T + A_i^T (G_i^{22} - I) \\
&\mathbf{(3.3)} = \tilde{P}_j^{11} - \tilde{G}^{11} - (\tilde{G}^{11})^T, && \mathbf{(2.4)} = A_i^T G_i^{22} - C_i^T \hat{L}_i^T \\
&\mathbf{(3.4)} = \tilde{P}_j^{12} + I - (G_i^{22})^T, && \mathbf{(4.4)} = \hat{P}_j^{22} - G_i^{22} - (G_i^{22})^T \\
&\mathbf{(3.6)} = (G_i^{22} - I)^T M_i, &&
\end{aligned}
$$

Then the closed-loop system (5) is asymptotically stable with the observer-based controller gains:

$$K_i = \tilde{K}_i \tilde{G}^{11},\ L_i = (G_i^{22})^{-T} \hat{L}_i,\ i \in \Lambda. \tag{32}$$

Proof. The proof is done in the three previous linearization steps. It remains to apply the Schur lemma on the right hand side of (30) to get the LMI (31). The matrices $\mathcal{D}_i$, for $i \in \Lambda$, come from the change of variable:

$$\mathcal{D}_i \triangleq \begin{bmatrix} \tilde{P}_i^{11} & \tilde{P}_i^{12} \\ (\star) & \hat{P}_i^{22} \end{bmatrix} = \begin{bmatrix} (\tilde{G}^{11})^T & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \hat{P}_i^{11} & \hat{P}_i^{12} \\ (\star) & \hat{P}_i^{22} \end{bmatrix} \begin{bmatrix} \tilde{G}^{11} & 0 \\ 0 & I \end{bmatrix}, \quad \forall i \in \Lambda.$$

4. Numerical Design Aspects and Some Comments

4.1. On the Optimization of the uncertainty bounds

Notice that the scalars α_i and λ_i are to be fixed *a priori* to render linear the condition (31). Moreover, in order to overcome this drawback and to maximize the uncertainty bounds tolerated by (31), the uncertainties are replaced by the more general form:

$$\Delta A_i = M_A^i D_i(t) E_{i1},\, \Delta B_i = M_B^i F_i(t) E_{i2},\, \Delta C_i = M_C^i H_i(t) E_{i1}. \tag{33}$$

The uncertain matrices, containing the uncertainty bounds, are replaced by:

$$D_i^T(t) D_i(t) \leq \delta_i^2 I,\, F_i^T(t) F_i(t) \leq \beta_i^2 I,\, H_i^T(t) H_i(t) \leq \gamma_i^2 I \tag{34}$$

instead of (3). This formulation is often used in decentralized stabilization problem of interconnected systems. The objective consists in maximizing the bounds δ_i, β_i, and γ_i. Such a strategy leads to an LMI without *a priori* choice of the scalars α_i and λ_i.

Under these new considerations, inequality (29) becomes

$$\Xi^{ij} + \mathrm{He}\Big(Z_{i1}^T\, D_i^T\, Z_{i2} + Z_{i3}^T\, F_i^T\, Z_{i4} + Z_{i5}^T\, H_i^T\, Z_{i6} \Big) < 0, \tag{35}$$

where

$$\begin{aligned} Z_{i1} &= \begin{bmatrix} -E_{i1}\tilde{G}^{11} & E_{i1} & 0 & 0 \end{bmatrix}^T,\, Z_{i2} = {M_A^i}^T \begin{bmatrix} G_i^{22} & 0 & (G_i^{22} - I) & G_i^{22} \end{bmatrix} \\ Z_{i3} &= \begin{bmatrix} -E_{i2}\tilde{K}_i & 0 & 0 & 0 \end{bmatrix}^T,\, Z_{i4} = {M_B^i}^T \begin{bmatrix} G_i^{22} & 0 & (G_i^{22} - I) & G_i^{22} \end{bmatrix} \\ Z_{i5} &= \begin{bmatrix} E_{i3}\tilde{G}^{11} & -E_{i3} & 0 & 0 \end{bmatrix}^T,\, Z_{i6} = \begin{bmatrix} {M_C^i}^T \hat{L}_i^T & 0 & {M_C^i}^T \hat{L}_i^T & {M_C^i}^T \hat{L}_i^T \end{bmatrix}. \end{aligned}$$

Using the classical Young's relation and taking into account (34), we deduce that (35) holds if the following one is fulfilled:

$$\begin{aligned} \Xi^{ij} + \Big(a_i \delta_i^2 Z_{i1}^T Z_{i1} + a_i^{-1} Z_{i2}^T Z_{i2} + b_i \beta_i^2 Z_{i3}^T Z_{i3} \\ + b_i^{-1} Z_{i4}^T Z_{i4} + c_i \gamma_i^2 Z_{i5}^T Z_{i5} + c_i^{-1} Z_{i6}^T Z_{i6} \Big) < 0. \end{aligned}$$

Finally, with the change of variables

$$\xi_i = \frac{1}{a_i\delta_i^2},\ \nu_i = \frac{1}{b_i\beta_i^2},\ \kappa_i = \frac{1}{c_i\gamma_i^2}$$

and by using the Schur lemma, we get the following enhanced version of Theorem 3.

Theorem 4. *Assume that there exist positive definite matrices* $\mathcal{D}_i \in \mathbb{R}^{2n\times 2n}$ *and* $G_i^{22},\ \tilde{G}^{11} \in \mathbb{R}^{n\times n}, \tilde{K}_i \in \mathbb{R}^{m\times n}, \hat{L}_i \in \mathbb{R}^{n\times p}$*, for* $i \in \Lambda$*, such that the following convex optimization problem holds:*

$$\min \operatorname{Trace}(\Gamma_i) \quad \text{subject to}$$

$$\begin{bmatrix} \Xi^{ij} & \begin{bmatrix} Z_{i1}^T & Z_{i2} & Z_{i3} & Z_{i4} & Z_{i5} & Z_{i6} \end{bmatrix} \\ (\star) & -\Gamma_i \end{bmatrix} < 0,\ i, j \in \Lambda \tag{36}$$

$$\Gamma_i = \operatorname{diag}\Big\{\xi_i I,\ a_i I,\ \nu_i I,\ b_i I,\ \kappa_i I,\ c_i I\Big\}.$$

Then the closed-loop system (5) is asymptotically stable with the observer-based controller gains:

$$K_i = \tilde{K}_i\tilde{G}^{11},\ \ L_i = (G_i^{22})^{-T}\hat{L}_i, \tag{37}$$

for all $\delta_i,\ \beta_i,\ \gamma_i,\ i \in \Lambda$*, satisfying*

$$\delta_i \leq \frac{1}{\sqrt{a_i\xi_i}},\ \beta_i \leq \frac{1}{\sqrt{b_i\nu_i}},\ \gamma_i \leq \frac{1}{\sqrt{c_i\kappa_i}}.$$

4.2. Some comments

This section is dedicated to some constructive remarks, which may be helpful and useful for any application of the proposed enhanced LMI methodology.

4.2.1. On the a priori choice of some scalar variables

Conditions (31) and (18) are LMIs if the positive scalars $\alpha_i,\ \lambda_i, \epsilon_i, \gamma_i$ and μ_i are fixed a priori. Then to get LMIs we need to use some techniques providing these a priori choices of the scalar variables. One of the famous techniques can be found in [24, Remark 3], namely the gridding method. It is worth noticing that this alternative solution will be used in case where the bounds of the uncertainties are fixed and not to be maximized. Indeed, this latter may be handled by using Theorem 4.

4.2.2. Comparison with the Young relation based approach

As compared to the Young inequality based approach, the judicious choice of the slack variable in (21) coming from Finsler's lemma (especially the triangular structure of $\hat{G}_i$), has eliminated the isolated term $(C_i + \Delta C_i)^T L_i^T$ arising from the diagonal structure of $\hat{G}_i$ used in the Young inequality based approach. Hence, the proposed enhanced LMI design methodology based on a convenient use of Finsler's inequality allows to avoid all these bilinear terms without using Young's inequality several times, which leads to conservative LMI conditions like in the Young relation based approach [24].

4.2.3. Handling the uncertainties to get full linearization

It should be mentioned that Young's relation is almost unavoidable when dealing with uncertainties satisfying equations (2)-(3). This is due mainly to their structure and the condition (3), namely $D_\sigma^T(t) D_\sigma(t) \leq I$. This technique is standard and well known in the literature. We can proceed otherwise if we are dealing with other uncertainties, such as LPV uncertainties. These latter can be handled more easily, thanks to the use of the convexity principle. This LPV reformulation of the uncertainties is not suitable in the context of the paper dealing with switched systems. Indeed, in case of switched systems with large number of subsystems we have a large number of LMIs to solve. Then the LPV reformulation of the uncertainties leads to a higher number of LMIs, which may causes numerical problems from computational point of view.

On the other hand, the Young inequality based approach may be used even in the uncertainty free case to handle the BMI coming from some coupling between decision variables. Young's inequality based approach is more conservative than the new proposed enhanced LMI methodology based on the novel and non standard use of the Finsler lemma. As we have mentioned above, the source of the conservatism is the diagonal structure of the variables G_i. The conservatism comes also crucially from the manner to handle the term $(C_i + \Delta C_i)^T L_i^T$.

4.3. On the numerical complexity of the proposed LMI techniques

The numerical complexity associated with the proposed LMI conditions can be computed in terms of the number of scalar variables and number of LMI to be solved. As for the relaxed algorithm proposed in Theorem 4, the computational complexity can easily be evaluated. Indeed, we must solve N^2 LMIs conditions to get $12N+1$ decision variables, or $N(3n^2+n(m+p+1)+6)+n^2$ scalar variables. As compared with the other conditions presented in this paper, following the comment in subsection 4.2.3, if we use the gridding method, we must solve conditions (18) by scaling the parameters ϵ_i, γ_i, μ_i via the change of variables $s_i := \epsilon_i/(1+\epsilon_i)$, $t_i := \gamma_i/(1+\gamma_i)$, $\kappa_i := \mu_i/(1+\mu_i)$, with $s_i, t_i, \kappa_i \in (0,1)$. Thus, for each mode i, we have to make a (uniform) mesh of the interval $(0,1)$ with length equal to $\Delta_i^{s_i}$, $\Delta_i^{t_i}$ and $\Delta_i^{\kappa_i}$, respectively. If conditions (18) are found feasible for $(s_i^*, t_i^*, \kappa_i^*)$, then this means that we solved, for each $i, j \in \Lambda$, the following number of LMI conditions:

$$\left[\frac{s_i^*}{\Delta_i^{s_i}}\right]\left[\frac{t_i^*}{\Delta_i^{t_i}}\right]\left[\frac{\kappa_i^*}{\Delta_i^{\kappa_i}}\right]$$

where $[x]$ denotes the integer part of a real number x. This amounts to solving a number of LMI equal to

$$\sum_{j=1}^{N}\sum_{i=1}^{N}\left[\frac{s_i^*}{\Delta_i^{s_i}}\right]\left[\frac{t_i^*}{\Delta_i^{t_i}}\right]\left[\frac{\kappa_i^*}{\Delta_i^{\kappa_i}}\right].$$

It should be noted that this number is greater than or equal to N^2. We have N^2 LMIs to solve only if the LMI (18) is found feasible at the first step when the gridding method is applied, i.e: $\left[\frac{s_i^*}{\Delta_i^{s_i}}\right] = \left[\frac{t_i^*}{\Delta_i^{t_i}}\right] = \left[\frac{\kappa_i^*}{\Delta_i^{\kappa_i}}\right] = 1$. The computational complexity of the algorithm given by (31) can be evaluated similarly. Table 1 shows the number, $\sharp$SV, of the scalar variables, the number, $\sharp$DV, of decision variables, and the number, $\sharp$LMI, of LMI conditions to be solved for the three tests presented here. From a numerical complexity point of view, the superiority of (36) is quite clear.

Algorithm	$\sharp$SV	$\sharp$DV	$\sharp$LMIs
LMI (36)	$N\Big(3n^2+n(m+p+1)+6\Big)+n^2$	$12N+1$	N^2
LMI (31)	$N\Big(3n^2+n(m+p+1)+2\Big)+n^2$	$8N+1$	$N\sum_{i=1}^{N}\left[\frac{\tau_i^*}{\Delta_i^{\tau_i}}\right]\left[\frac{\upsilon_i^*}{\Delta_i^{\upsilon_i}}\right]$
LMI (18)	$N\Big(3n^2+n(m+p+1)+3\Big)+n^2$	$9N+1$	$N\sum_{i=1}^{N}\left[\frac{s_i^*}{\Delta_i^{s_i}}\right]\left[\frac{t_i^*}{\Delta_i^{t_i}}\right]\left[\frac{\kappa_i^*}{\Delta_i^{\kappa_i}}\right]$

Table 1: Numerical complexity associated with the proposed algorithms

5. Numerical examples and comparisons

In this section, we present numerical examples to show the validity and effectiveness of the proposed design methodology. For a comparison reason, we reconsider the examples given in [24] and [29]. We will also provide a Monte Carlo simulation to evaluate the superiority of the enhanced LMI conditions (31) in the uncertainty free case.

5.1. Example 1

Here we consider the example proposed in [29]. First, we take exactly the same example (given without uncertainties). That is $\Delta A_i = \Delta B_i = \Delta C_i = 0$ and the other parameters are described as follows:

$$A_1 = \begin{bmatrix} 1.5 & 1 \\ 0 & 2 \end{bmatrix},\ B_1 = \begin{bmatrix} 1 & 2 \\ 4 & 0 \end{bmatrix},\ C_1 = \begin{bmatrix} 3 & 2 \\ 0 & -1 \end{bmatrix}, \tag{38a}$$

$$A_2 = \begin{bmatrix} 1.7 & 1 \\ 0.5 & 2 \end{bmatrix},\ B_2 = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix},\ C_2 = \begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix}. \tag{38b}$$

It is clear that all the matrices A_1 and A_2 are unstable. It should be noticed that the LMI conditions proposed in [29] are not feasible for this example, contrarily to what has been speculated in [29]. Indeed, first, the LMI conditions given in [29] are false because the authors made a mistake in [29, Inequality (9)]. This mistake removes many bilinear terms and conducted the authors to very simple LMIs. On the other hand, despite this error, the LMIs in [29] are not feasible for this example because of the particular choice $\hat{G}_i = \hat{F}_i^T$ and a conservative way of using Finsler's inequality. The same goes to the approach presented in [36], which is found infeasible due to a strong equality constraint. However, using Matlab LMI toolbox, we get that both LMI (31) and LMI (18) are feasible. Note that the solvability of (18) is performed via the gridding technique with respect to $epsilon_i$. Indeed, by scaling ϵ_i, $i \in \{1, 2\}$, by defining $s_i = \epsilon_i/(1 + \epsilon_i)$, with $s_i \in [0.1, 0.9]$, then with a uniform subdivision of the interval $[0.1, 0.9]$ of length equal to $\Delta_i^{s_i} = 0.1$, $\forall i \in \{1, 2\}$, we get LMI (18) feasible for $s_1 = 0.7$, $s_2 = 0.8$, i.e., $\epsilon_1 = 2.3333$, $\epsilon_2 = 4$. The observer-based controller gains are given in Table 2. Notice that the symbol (!) means that the corresponding LMI condition is found infeasible.

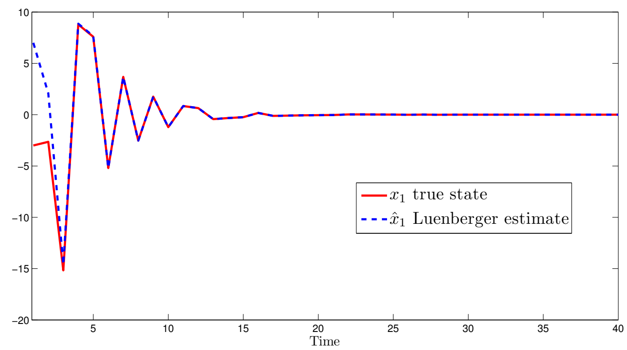
	LMI (16)	**LMI [36]**	**LMI (31)**	**LMI (18)**
				$\epsilon_1 = 2.3333$, $\epsilon_2 = 4$
K_1	(!)	(!)	$\begin{bmatrix} -0.1737 & -0.6394 \\ -0.2222 & -0.0530 \end{bmatrix}$	$\begin{bmatrix} -0.1373 & -0.5938 \\ -0.2100 & -0.0146 \end{bmatrix}$
K_2	(!)	(!)	$\begin{bmatrix} -2.3124 & -4.2526 \\ -0.1131 & 1.4825 \end{bmatrix}$	$\begin{bmatrix} -2.0309 & -3.6428 \\ -0.1534 & 1.3106 \end{bmatrix}$
L_1	(!)	(!)	$\begin{bmatrix} +0.3877 & -0.3601 \\ 0.0103 & -1.9046 \end{bmatrix}$	$\begin{bmatrix} +0.3403 & -0.5823 \\ 0.0435 & -1.7723 \end{bmatrix}$
L_2	(!)	(!)	$\begin{bmatrix} 0.5731 & -1.0100 \\ -1.0873 & -1.7078 \end{bmatrix}$	$\begin{bmatrix} -0.7444 & -1.7542 \\ -0.9839 & -1.6725 \end{bmatrix}$

Table 2: Observer-based controllers for the proposed LMI design applied to system (38)

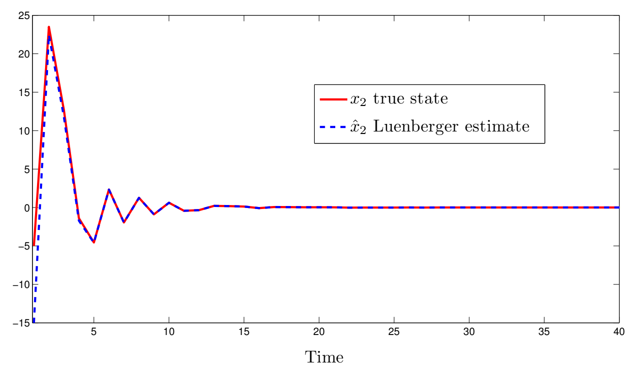
The simulation results corresponding to these observer-based controller gains obtained by solving LMIs (31) are given in Figure 1. These simulations are done for an horizon $T = 40s$, with $x_0 = \begin{bmatrix} -3 & -5 \end{bmatrix}^T$ and $\hat{x}_0 = \begin{bmatrix} 7 & -15 \end{bmatrix}^T$. The switching rule is taken in this form:

$$\sigma_t = 1 + \text{round}\,(\omega_t) \tag{39}$$

for $t = 1$ to T, where ω_t is an uniformly distributed random variable on the interval $[0, 1]$, and round(x) is the nearest integer function of real number x. Then, the switching signal can be realized by Matlab and a possible case is shown in Figure 1(d). Note that the switching instants in Figure 1(d) are arbitrary.



(a) Example 1: Time-behaviors of x_1 and $\hat{x}_1$ in the uncertainty free case.

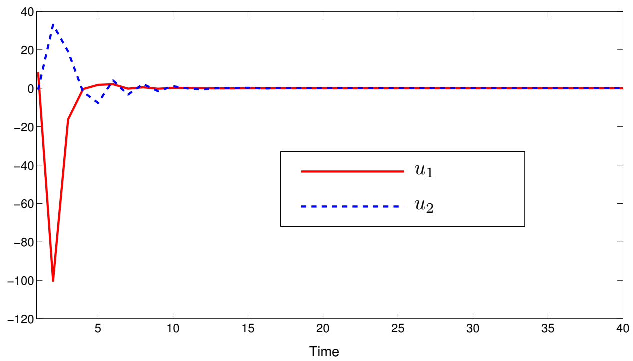


(b) Example 1: Time-behaviors of x_2 and $\hat{x}_2$ in the uncertainty free case.

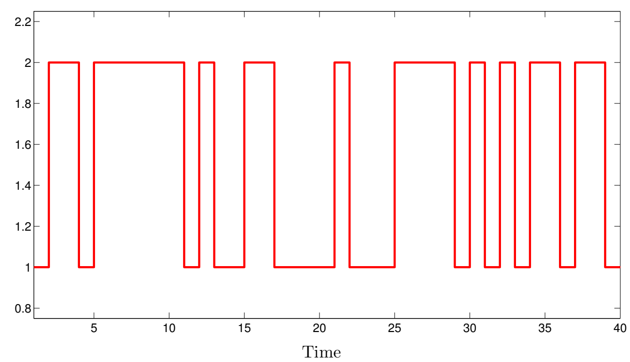
In order to boost comparisons between the proposed LMI conditions (31), (18) and (36), we add to the previous example parameter uncertainties as follows:

$$M_1 = \begin{bmatrix} 0.35 & 0.2 \\ 0.3 & 0.15 \end{bmatrix},\ M_2 = \begin{bmatrix} 0.3 & -0.1 \\ 0.3 & 0.2 \end{bmatrix}, \tag{40a}$$

$$E_{11} = \begin{bmatrix} 0.22 & 0.22 \\ 0.2 & 0.25 \end{bmatrix},\ E_{12} = \begin{bmatrix} 0.4 & 0.5 \\ 0.6 & 0.5 \end{bmatrix},\ E_{13} = \begin{bmatrix} 0.2 & 0.21 \\ 0.15 & 0.25 \end{bmatrix}, \tag{40b}$$



(c) Example 1: Time-behaviors of u_1 and u_2 in the uncertainty free case.



(d) Example 1: Switching mode

Figure 1: Example 1: Controlled states and their estimates in the uncertainty free case.

$$E_{21} = \begin{bmatrix} 0.25 & 0.15 \\ 0.31 & 0.25 \end{bmatrix}, E_{22} = \begin{bmatrix} 0.15 & 0.25 \\ 0.15 & 0.2 \end{bmatrix}, E_{23} = \begin{bmatrix} 0.15 & 0.2 \\ 0.2 & 0.2 \end{bmatrix}. \quad (40c)$$

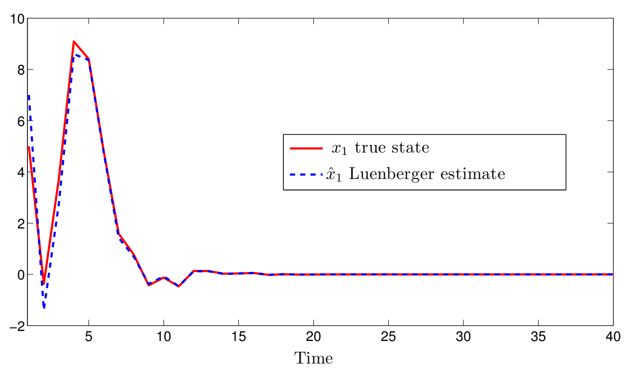
Then, the proposed LMIs (31) and (36) work successfully. LMI (31) is found feasible for $\tau_1^* = \tau_2^* = 0.5$, $\upsilon_1^* = 0.7$, $\upsilon_2^* = 0.6$, and $\Delta_i^{\tau_i} = \Delta_i^{\upsilon_i} = 0.1$, for all i. We obtain then, $\alpha_1 = \alpha_2 = \frac{\tau_1^*}{1-\tau_1^*} = 1$ and $\lambda_1 = \frac{\upsilon_1^*}{1-\upsilon_1^*} = 2.3333$, $\lambda_2 = \frac{\upsilon_2^*}{1-\upsilon_2^*} = 1.5$. However, the Young inequality-based approach is found infeasible for the same values of ϵ_i given in Table 2, and for the same step of discretization. The results are summarized in Table 3.

The simulation results corresponding to these observer-based controller gains

	LMI (31)	**LMI (36)**	**LMI (18)**
	$\Delta_i^{\tau_i} = \Delta_i^{\upsilon_i} = 0.1, \alpha_1 = \alpha_2 = 1$ $\lambda_1 = 2.3333, \lambda_2 = 1.5$		$\Delta_i^{t_i} = \Delta_i^{\kappa_i} = 0.1$ $\epsilon_1 = 2.3333$, $\epsilon_2 = 4$
K_1	$\begin{bmatrix} -0.2532 & -0.6666 \\ -0.1582 & 0.1107 \end{bmatrix}$	$\begin{bmatrix} -0.4175 & -0.9023 \\ 0.3077 & 0.7274 \end{bmatrix}$	(!)
K_2	$\begin{bmatrix} -2.9813 & -5.7095 \\ 0.5477 & 2.6315 \end{bmatrix}$	$\begin{bmatrix} -3.6037 & -6.2027 \\ 1.2277 & 3.0381 \end{bmatrix}$	(!)
L_1	$\begin{bmatrix} +0.4065 & -0.2117 \\ 0.2954 & -0.8230 \end{bmatrix}$	$\begin{bmatrix} +0.3752 & -0.2373 \\ 0.1966 & -0.9884 \end{bmatrix}$	(!)
L_2	$\begin{bmatrix} +1.1317 & -0.7526 \\ 0.1649 & -1.3045 \end{bmatrix}$	$\begin{bmatrix} 0.6219 & -0.8185 \\ -0.5084 & -1.3906 \end{bmatrix}$	(!)

Table 3: Observer-based controllers for the proposed LMI design applied to system (38)-(40)

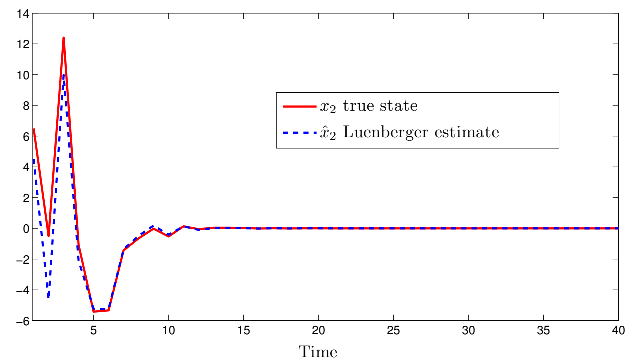
returned by LMIs (31) are shown in Figure 2. These simulations shown in Figure 2 are done over an horizon of length $T = 40s$ with $x_0 = \begin{bmatrix} 5 & 6.5 \end{bmatrix}^T$, $\hat{x}_0 = \begin{bmatrix} 7 & 4.5 \end{bmatrix}^T$. The switching rule is generated randomly as in (39).



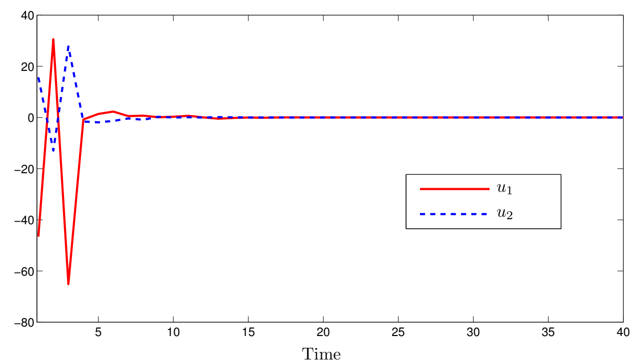
(a) Example 1: Time-behaviors of x_1 and $\hat{x}_1$ in the presence of uncertainties

5.2. Example 2 (Evaluation of maximum admissible uncertainty)

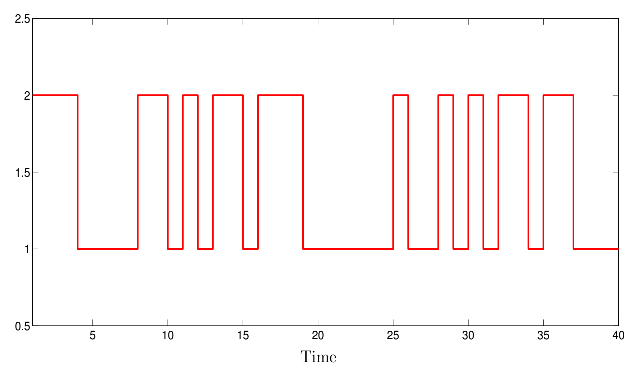
Through this example, we will show that the proposed LMI conditions are less conservative than those provided in [24]. We reconsider the same system



(b) Example 1: Time-behaviors of x_2 and $\hat{x}_2$ in the presence of uncertainties



(c) Example 1: Time-behaviors of u_1 and u_2 in the presence of uncertainties



(d) Example 1: Switching mode

Figure 2: Example 1: Simulation results in the presence of uncertainties.

as in [24, Example 1]. Obviously, this example can be viewed as a switching system under the form (1) with only one mode (there is no switching). The system is described by the following matrices:

$$A = \begin{bmatrix} 1 & 0.1 & 0.4 \\ 1 & 1 & 0.5 \\ -0.3 & 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 0.1 & 0.3 \\ -0.4 & 0.5 \\ 0.6 & 0.4 \end{bmatrix}, C = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix},$$

$$M_A = \begin{bmatrix} 0 & 0 & 0 \\ 0.1 & 0.3 & 0.1 \\ 0 & 0.2 & 0 \end{bmatrix}, N_A = \begin{bmatrix} 0 & 0 & 0 \\ 0.2 & 0 & 0.4 \\ 0 & 0.1 & 0 \end{bmatrix},$$

$$M_C = \begin{bmatrix} 0 & 0 & 0.3 \\ 0 & 0 & 0.8 \end{bmatrix}, N_C = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0.2 \end{bmatrix}.$$

The proposed design methodology works successfully. Solving the LMI (31) of Theorem 3 with $\alpha_1 = 1$ and $\lambda_1 = 0.5$, we get the following gains:

$$K = \begin{bmatrix} 1.2322 & 0.8710 & -0.8064 \\ -1.9374 & -1.1776 & -2.0770 \end{bmatrix}, L = \begin{bmatrix} 0.6909 & -0.2591 \\ 0.9596 & -0.3599 \\ 0.6405 & -0.2402 \end{bmatrix}.$$

To show the superiority of the proposed design methodology as compared with [24], we considered uncertain matrices, scaled by the parameters γ_1 and γ_2, as follows:

$$M_A = \gamma_1 \begin{bmatrix} 0 & 0 & 0 \\ 0.1 & 0.3 & 0.1 \\ 0 & 0.2 & 0 \end{bmatrix}, \quad M_C = \gamma_2 \begin{bmatrix} 0 & 0 & 0.3 \\ 0 & 0 & 0.8 \end{bmatrix}.$$

We look for the maximum values of γ_1 and γ_2 that satisfy LMI (16) and LMI (31). The results summarized in Table 4, reflect the superiority of the proposed methodology as compared to the Young inequality based approach [24] and the approach in [29].

5.3. *Numerical evaluation by Monte Carlo in the uncertainty free case*

Here we investigate the uncertainty-free case. The aim consists in evaluating numerically the necessary conditions required by each method. For this, we

	LMI (16) [29]	**LMI (9) in [24]**	**LMI (31)**
		$\epsilon_1 = 2.33$ $\epsilon_3 = 1.42$ $\epsilon_4 = 0.08$	$\alpha = 6$ $\lambda = 51.594$
$\max \gamma_1$	(!)	4.64	5.4
$\max \gamma_2$	(!)	10^{13}	10^{15}

Table 4: Comparison between different LMI design methods

generate randomly 1000 stabilizable and detectable systems of dimension $n = 3; p = 2$ and ranging from 1 to n (with switching rule $\sigma_t \in \{1, 2\}$). The results are summarized in Table 5, which gives the percentage of systems for which the different methods addressed in this note succeeded for each value of m.

Method	**LMI (16)**	**LMI (31)**	**LMI (18)**	**LMI (60) in [36]**
			with $\epsilon_i = 10$	
$m = 1$	0 %	31.5 %	28.6 %	0.5%
$m = 2$	0 %	100 %	84.1%	1.5%
$m = 3$	0 %	90.8 %	78.7 %	2 %

Table 5: Superiority of the proposed LMI methodology

6. Conclusions and Future Work

This paper developed new LMI conditions for the problem of stabilization of discrete-time uncertain switched linear systems. First, we revisited and corrected the approach proposed in [29] that combines Finsler's lemma and the switched Lyapunov function approach. A general theoretical method was proposed, which leads to less conservative LMI conditions. This is due to the use of Finsler's inequality in a new and convenient way. Illustrative examples are presented to demonstrate the effectiveness and superiority of the proposed design methodology.

There are several important issues which should be considered in the future. First, an extension to the problem of $\mathcal{H}_\infty$ analysis for Linear Parameter Varying (LPV) systems with uncertain parameters seems natural. Indeed, the stability analysis of LPV systems with inexact parameters can be performed following the almost the same arguments. Second, the stabilizability conditions (31) should be relaxed more by relaxing the independence of the matrix G^{11} from the mode i in (26).

Finally, one of the most important problem is to consider switched systems with arbitrary switching without any real-time information on the switching signal. To the best of authors knowledge, there are few results for this class of systems, and the available methods still remain conservative.

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