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Ali Zemouche

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Observer Design for Nonlinear Systems by Using High-Gain and LPV/LMI-Based Technique

A. ZEMOUCHE[†‡]

Abstract— **This note deals with observer design for nonlinear systems. The main contribution of this work consists in providing a new high-gain observer design method with lower gain compared to the standard high-gain observer. This new observer, called HG/LMI observer is obtained by combining the standard high-gain methodology with the LPVLMI-based technique. We will show through analytical developments how the new observer provides a lower gain. A numerical example is given to illustrate the performance of the new HG/LMI observer.**

Index Terms— **Observers design, high-gain methodology, Lipschitz systems, LMI.**

I. INTRODUCTION

Nonlinear state observers have attracted a great attention from the automatic control community in the recent years [1], [2], [3], [4], [5], [6], [7]. This is due to the fact that in many real models, some variables are very expensive to measure, and in certain cases some variables are unmeasurable because they lost their physical sense through mathematical transformations. Therefore, designing state observers is a necessary step for diagnosis, control tracking, monitoring, and other control design problems. For instance, in the field of autonomous vehicles in a platoon, measurement of some variables, such as longitudinal distances, velocities and accelerations of other nearby vehicles, requires significant expense. Some of sensors, such as slip angle and roll angle, can be extremely expensive to measure, requiring sensors that cost thousands of dollars [8], [9]. In addition, several important tasks cannot be performed due to unavailability of sensors at any cost.

Because of the lack of a general design method for nonlinear systems like in the linear case, several methods have been developed in the literature, where each method corresponds to a specific class of nonlinear systems. we can quote the class of systems with Lipschitz nonlinearities [10], [11], [12], [13], [14], [15]. Specifically for this class of systems a lot of LMI techniques have been established in the literature. Each LMI technique aims to provide a better way to get less conservative LMI conditions as possible. Despite theoretical advances in this field and although some enhancements are proposed recently [16], [17], [18], the problem still remains open.

One of the most popular methods for state estimation of nonlinear systems is the well known high-gain observer. This later works for systems in triangular form or any system that can be transformed into a triangular structure. The advantage of the high-gain methodology is the fact that always it can guarantee the exponential convergence thanks to the tuning of only one parameter that should be large enough [3], [19]. Although the practicability of high-gain observer in output feedback control has been nicely demonstrated by Khalil's work [5], [20], it remains a major drawback to overcome. Indeed, high-gain observer is very sensitive to output measurement noises because of the value of the tuning parameter which may be very huge for higher dimensional systems having nonlinearities with high Lipschitz constants. To overcome this obstacle, many research works have been oriented to high-gain observers with time-varying parameter adaptation, and a lot of schemes have been proposed. For an overview of the literature, we refer the reader to [21], [22], [16], [23], [24], [25], [26], and the references therein.

Despite all these improvements the research activities in this direction still remains active and there are many problems to be solved to improve the performance of the high-gain observer with respect to measurement noises. A new and recent technique was proposed in [27] to solve this problem. Through elegant arguments, the authors have proposed a high-gain observer with limited gain power. Their observer structure is new and different from the standard high-gain structure. Indeed, for an n-dimensional system, instead of a Luenberger observer structure of dimension n, they designed an observer of dimension $2n-2$. Even if their gain power is limited to 2 instead of n with the standard high-gain, the higher dimension of the observer $(2n-2)$ may increase the tuning parameter. As shown in [27], overall, this new high-gain observer is better than the standard one from the performance point of view.

What we propose in this note is different from the approach in [27]. Our technique follows the standard high-gain methodology with the same state observer structure of dimension n. However, by exploiting the LPV/LMI technique developed in [17], we are able to decrease the gain power. We will introduce a so called "*compromise index*" j_0, with $0 \leq j_0 \leq n$ to decrease the high-gain tuning parameter.

II. PROBLEM FORMULATION

A. Preliminaries

We start by introducing some definitions and preliminaries which will be of crucial use in the developed LPV-approach for Lipschitz and not necessarily differentiable systems.

[†] University of Lorraine, CRAN CNRS UMR 7039, 54400 Cosnes et Romain, France (email: ali.zemouche@univ-lorraine.fr).

[‡] EPI Inria DISCO, Laboratoire des Signaux et Systèmes, CNRS-CentraleSupélec, 91192 Gif-sur-Yvette, France (email: ali.zemouche@inria.fr).

Definition 1 ([17]): Consider two vectors

$$X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n \quad \text{and} \quad Z = \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix} \in \mathbb{R}^n.$$

For all $i = 0, ..., n$, we define an auxiliary vector $X^{Z_i} \in \mathbb{R}^n$ corresponding to X and Z as follows:

$$\begin{cases} X^{Z_i} = \begin{pmatrix} z_1 \\ \vdots \\ z_i \\ x_{i+1} \\ \vdots \\ x_n \end{pmatrix} & \text{for } i = 1, ..., n \\ X^{Z_0} = X \end{cases} \tag{1}$$

Lemma 1 ([17]): Consider a continuous function $\Psi : \mathbb{R}^n \longrightarrow \mathbb{R}$. Then, for all

$$X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n \quad \text{and} \quad Z = \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix} \in \mathbb{R}^n$$

there exist functions $\psi_j : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$, $j = 1, ..., n$ so that

$$\Psi(X) - \Psi(Z) = \sum_{j=1}^{j=n} \psi_j\Big(X^{Z_{j-1}}, X^{Z_j}\Big)\mathrm{e}_n^\top(j)\Big(X - Z\Big) \tag{2}$$

where $\mathrm{e}_n(j)$ is the j^{th} vector of the canonical basis of $\mathbb{R}^n$.

Lemma 2 ([17]): Consider a function $\Psi : \mathbb{R}^n \longrightarrow \mathbb{R}^n$. Then, the two following items are equivalent:

- Ψ is γ_Ψ-Lipschitz with respect to its argument, i.e.:

$$\Big\|\Psi(X) - \Psi(Z)\Big\| \leq \gamma_\Psi \Big\|X - Y\Big\|, \quad \forall\ X, Z \in \mathbb{R}^n \tag{3}$$

- for all $i, j = 1, ..., n$, there exist functions

$$\psi_{ij} : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$$

and constants $\underline{\gamma}_{\psi_{ij}} \leq 0$, $\bar{\gamma}_{\psi_{ij}} \geq 0$, so that $\forall\ X, Z \in \mathbb{R}^n$,

$$\Psi(X) - \Psi(Z) = \sum_{i=1}^{i=n}\sum_{j=1}^{j=n} \psi_{ij} H_{ij}\Big(X - Z\Big) \tag{4}$$

and

$$-\gamma_\Psi \leq \underline{\gamma}_{\psi_{ij}} \leq \psi_{ij} \leq \bar{\gamma}_{\psi_{ij}} \leq \gamma_\Psi \tag{5}$$

where

$$\psi_{ij} \triangleq \psi_{ij}\Big(X^{Z_{j-1}}, X^{Z_j}\Big) \quad \text{and} \quad H_{ij} = \mathrm{e}_n(i)\mathrm{e}_n^\top(j)$$

B. System Description

Since this paper deals with high-gain observers, we will consider nonlinear systems in a triangular form. For simplicity of the presentation and to explain well what we propose in this note, we consider the following triangular form of nonlinear systems as in [3]:

$$\begin{cases} \dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ . \\ . \\ . \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} x_2 \\ x_3 \\ . \\ . \\ . \\ x_n \\ f(x) \end{bmatrix} \\ y = x_1 \end{cases} \tag{6}$$

with $f : \mathbb{R}^n \to \mathbb{R}$ satisfies the Lipschitz property formulated under the flowing form:

$$\Big|f(x_1 + \Delta_1, \ldots, x_n + \Delta_n) - f(x_1, \ldots, x_n)\Big| \leq \gamma_f \sum_{j=1}^{n} |\Delta_j|\ . \tag{7}$$

For the sake of compactness, we write system (6) under the form:

$$\begin{cases} \dot{x} = Ax + Bf(x) \\ y = Cx, \end{cases} \tag{8}$$

where

$$B = \begin{bmatrix} 0 & \ldots & 0 & 1 \end{bmatrix}^T, \quad C = \begin{bmatrix} 1 & 0 & \ldots & 0 \end{bmatrix}$$

and the state matrix A is defined by

$$(A)_{i,j} = \begin{cases} 1 & \text{if} \quad j = i + 1 \\ 0 & \text{if} \quad j \neq i + 1 \end{cases}.$$

Consider the following Luenberger observer:

$$\dot{\hat{x}} = A\hat{x} + Bf(\hat{x}) + L\Big(y - C\hat{x}\Big) \tag{9}$$

The dynamics of the estimation error $e = x - \hat{x}$ is then given by:

$$\dot{e} = \big(A - LC\big)e + B\big[f(x) - f(\hat{x})\big] \tag{10}$$

C. High-Gain Methodology

Here, we recall the basic high gain observer as in [19]. Basically, in the high-gain methodology, we write the observer gain L under the form:

$$L := \mathrm{T}(\theta)K, \ \theta \geq 1 \tag{11}$$

where

$$\mathrm{T}(\theta) := \mathrm{diag}\Big(\theta, \ldots, \theta^n\Big) \text{ and } K \in \mathbb{R}^{n \times p}.$$

In addition, the high-gain methodology is based on the transformed estimation error

$$\hat{e} := \mathrm{T}^{-1}(\theta)e \tag{12}$$

where $\mathrm{T}^{-1}(\theta)$ is the inverse of $\mathrm{T}(\theta)$ given by

$$\mathrm{T}^{-1}(\theta) = \mathrm{diag}\Big(\frac{1}{\theta}, \ldots, \frac{1}{\theta^n}\Big).$$

It is well-known that the dynamics of the error $\hat{e}$ is given by

$$\dot{\hat{e}} = \theta\big(A - KC\big)\hat{e} + \frac{1}{\theta^n} B\Delta f \tag{13}$$

with

$$\Delta f := f(x) - f(x - \mathrm{T}(\theta)\hat{e})$$

From the Lipschitz condition (7) and the fact that $\theta \geq 1$, we can show as in [25] that there always exists a positive scalar constant k_f, independent of θ, so that

$$\|\mathrm{T}^{-1}(\theta)B\Delta f\| \leq k_f\|\hat{e}\|. \tag{14}$$

Consequently, using the high-gain methodology we have the following theorem:

Theorem 2.1 ([19]): If there exist $P > 0$, $\lambda > 0$, Y, and $\theta \geq 1$ such that

$$A^T P + PA - C^T Y - Y^T C + \lambda I < 0 \tag{15}$$

$$\theta > \theta_0 = \frac{2k_f \lambda_{\max}(P)}{\lambda} \tag{16}$$

then the estimation error e is asymptotically stable with

$$K = P^{-1}Y^T,$$

where $\lambda_{\max}(P)$ is the maximum eigenvalue of the matrix P.

Proof: For more details about the proof of this theorem, we refer the reader to [19], [25], [26]. ■

D. Problem formulation and objectives

If the LPV/LMI based approach is the best LMI technique and avoids high-gain, this approach has a weakness from the complexity point of view. Indeed, to synthesize the observer gain, the LPV/LMI based approach needs to solve a high number of LMIs, $n_{\mathrm{LMI}} = 2^n$. In addition, this technique, as is the case for all LMI techniques, contrarily to high-gain method, provides sufficient LMI conditions for which we cannot guarantee convergence before solving the LMIs. On the other hand, it is true that before solving conditions (15)-(16), the high-gain methodology guarantees convergence, however the obtained gain is really high even for slightly high Lipschitz constants. This weakness affects strongly the performance of the high-gain observer, namely in case of systems with noise measurement.

To overcome the above drawbacks, we propose to combine the two designs. We will exploit the advantages of each method to get a new and improved observer design technique. Especially, the combined observer, that we will call "HG/LMI observer" will have smaller observer gain compared to the standard high-gain. On the other hand, the high number of LMIs n_{LMI} will be significantly decreased. Mainly we will reduce the value of the right hand side of the high-gain condition (16). To do this successfully, we will need to use the LPV/LMI based approach; then the new design method will reduce the number of LMIs related to the standard LPV/LMI technique. The next section is devoted to this issue.

III. Main Results

A. Introduction and motivating example

The fact that k_f in inequality (14) is independent of θ is not necessary an advantage. Indeed, this depends on how θ would be involved in k_f. Also, the fact that k_f is independent of θ does not come only from the condition $\theta \geq 1$, but essentially from the presence of the last component of x in f. Because of this last component, the parameter θ vanishes from the term $\frac{1}{\theta^n}\Delta f$ for $\theta \geq 1$. This can be shown easily by using the Lipschitz property (7). To illustrate this point and to motivate our study, let us consider a simple three dimensional system. If we take a nonlinear function

$$f(x) = \gamma_f \sin(x_3)$$

then we get from (7)

$$\frac{1}{\theta^3}\|\Delta f\| \leq \frac{\gamma_f}{\theta^3} \times \mid \theta^3 \hat{e}_3 \mid = \gamma_f \mid \hat{e}_3 \mid \leq k_f\|\hat{e}\|$$

where $k_f = \gamma_f$ in this case. However, if we take

$$f(x) = \gamma_f \sin(x_2)$$

then we get

$$\frac{1}{\theta^3}\|\Delta f\| \leq \frac{\gamma_f}{\theta^3} \times \mid \theta^2 \hat{e}_2 \mid = \frac{\gamma_f}{\theta} \mid \hat{e}_2 \mid \leq \frac{k_f}{\theta}\|\hat{e}\|.$$

Hence, by replacing in (16) k_f by $\frac{k_f}{\theta}$, θ_0 will be reduced to $\sqrt{\theta_0}$, which will reduce significantly the values of the observer gain.

The main result of this paper is based on the above idea. Thanks to the LPV/LMI technique combined to the standard high-gain methodology, we will be able to obtain a high-gain observer with a lower gain.

B. More general case: preliminary Result

This section is devoted to the preliminary key idea of this paper. The high gain methodology exploits the fact that k_f in (14) is independent of θ. Our key idea lies in this inequality. Indeed, under a simple assumption, we will show that we can obtain a lower high-gain. That is, the value of θ_0 in (16) will be reduced thanks to this assumption.

Assumption 3.1: There exists $j_0 \geq 0$ so that

$$\frac{\partial f}{\partial x_j}(x) \equiv 0, \forall\ j > n - j_0. \tag{17}$$

This assumption means that the nonlinear function f does not depend on the j_0 last components of the state vector x. Notice that we consider that Assumption 3.1 is not fulfilled if $j_0 = 0$.

Under this assumption, inequality (14) becomes

$$\|\mathrm{T}^{-1}(\theta)B\Delta f\| \leq \frac{k_{j_0}}{\theta^{j_0}}\|\hat{e}\| \tag{18}$$

where k_{j_0} is independent of θ and $k_{j_0} \leq k_f$, where k_f is the same than that in (14). It is clear that with inequality (18), we reduce significantly the value of θ_0. Therefore, we get the following theorem providing our preliminary result, which is the key idea of this paper.

Theorem 3.2: Under the Assumption 3.1, if there exist $P > 0$, $\lambda > 0$, Y, and $\theta \geq 1$ such that

$$A^T P + PA - C^T Y - Y^T C + \lambda I < 0 \tag{19}$$

$$\theta^{1+j_0} > \theta_{j_0} = \frac{2k_{j_0}\lambda_{\max}(P)}{\lambda} \tag{20}$$

then the estimation error e is asymptotically stable with

$$K = P^{-1}Y^T.$$

As can be shown in (20), the value θ_0 is decreased to $\theta_{j_0}^{\frac{1}{1+j_0}}$, which is a very significant attenuation of the standard high-gain.

C. HG/LMI Observer

This section is devoted to the main contribution of this note. We will exploit the LPV/LMI based technique to extend the previous preliminary result to systems which do not satisfy Assumption 3.1.

Using the LPV/LMI method in [17], Δf in (13) can be rewritten under the following form:

$$\Delta f = \underbrace{\overbrace{\sum_{j=1}^{n-j_0} \theta^j \psi_j \hat{e}_j}^{\text{for HG}}}_{\Delta f_1} + \overbrace{\sum_{j=1}^{j_0} \theta^{k(j)} \psi_{k(j)} \hat{e}_{k(j)}}^{\text{for LPV/LMI}} \tag{21}$$

where

$$k(j) = n - (j_0 - j),$$
$$0 \leq j_0 \leq n.$$

Hence, the error dynamics (13) is rewritten as follows:

$$\dot{\hat{e}} = \theta\Big(\mathcal{A}(\Psi^\theta) - KC\Big)\hat{e} + \frac{1}{\theta^n} B\Delta f_1 \tag{22}$$

where

$$\mathcal{A}(\Psi^\theta) = A + B\sum_{j=1}^{j_0} \psi_j^\theta \mathrm{e}_n^\top(k(j)) \tag{23}$$

$$\Psi^\theta = \begin{pmatrix} \psi_1^\theta \\ \vdots \\ \psi_{j_0}^\theta \end{pmatrix} \in \mathbb{R}^{j_0} \tag{24}$$

$$\psi_j^\theta = \frac{\psi_{k(j)}}{\theta^{1+(j_0-j)}}. \tag{25}$$

Now define the convex bounded set

$$\mathcal{H}_{j_0}^{\sigma} = \left\{ \Phi \in \mathbb{R}^{j_0} \;:\; \frac{\underline{\gamma}_{\gamma_{k(j)}}}{\sigma^{1+(j_0-j)}} \leq \Phi_j \leq \frac{\bar{\gamma}_{\gamma_{k(j)}}}{\sigma^{1+(j_0-j)}} \right\} \tag{26}$$

for which the set of vertices is defined by

$$\mathcal{V}_{\mathcal{H}_{j_0}^{\sigma}} = \left\{ \Phi \in \mathbb{R}^{j_0} \;:\; \Phi_j \in \left\{ \frac{\underline{\gamma}_{\gamma_{k(j)}}}{\sigma^{1+(j_0-j)}}, \frac{\bar{\gamma}_{\gamma_{k(j)}}}{\sigma^{1+(j_0-j)}} \right\} \right\}. \tag{27}$$

Since $\bar{\gamma}_{\gamma_{k(j)}} \geq 0$ and $\underline{\gamma}_{\gamma_{k(j)}} \leq 0$, then it is obvious that for two positive scalars σ_1, σ_2, we have the following implication:

$$\sigma_1 < \sigma_2 \implies \mathcal{H}_{j_0}^{\sigma_1} \supset \mathcal{H}_{j_0}^{\sigma_2}. \tag{28}$$

It follows that

$$\lim_{\sigma\to+\infty} \left(\mathcal{H}_{j_0}^{\sigma}\right) = \left\{0_{\mathbb{R}^{j_0}}\right\}. \tag{29}$$

On the other hand, we can show that there exists a positive real number $k_{j_0} \leq k_f$ so that Δf_1 satisfies

$$\|\mathrm{T}^{-1}(\theta)B\Delta f_1\| \leq \frac{k_{j_0}}{\theta^{j_0}}\|\hat{e}\|. \tag{30}$$

Consequently, by analogy to Theorem 3.2 and by exploiting the LPV/LMI method in [17], we get the following more general theorem.

Theorem 3.3: If there exist $P > 0$, $\lambda > 0$, Y, and $\sigma > 0$ such that

$$\mathcal{A}(\Psi^\sigma)^T P + P\mathcal{A}(\Psi^\sigma) - C^T Y - Y^T C + \lambda I < 0, \forall \Psi^\sigma \in \mathcal{V}_{\mathcal{H}_{j_0}^{\sigma}} \tag{31}$$

$$\theta^{1+j_0} > \theta_{j_0} = \frac{2k_{j_0}\lambda_{\max}(P)}{\lambda} \tag{32}$$

then the estimation error e is asymptotically stable with

$$\boxed{L = \mathrm{T}(\theta)\overbrace{P^{-1}Y^T}^{K},\;\; \theta \geq \max\left(\sigma, \theta_{j_0}^{\frac{1}{1+j_0}}\right).}$$

Proof: A direct application of Theorem 3.2 leads to $\dot{V}(\hat{e}) < 0$, for all $\hat{e} \neq 0$, with $V(\hat{e}) = \hat{e}^\top P\hat{e}$, if

$$\mathcal{A}(\Psi^\theta)^T P + P\mathcal{A}(\Psi^\theta) - C^T Y - Y^T C + \lambda I < 0, \forall \Psi^\theta \in \mathcal{H}_{j_0}^{\theta} \tag{33}$$

and

$$\theta^{1+j_0} > \theta_{j_0} = \frac{2k_{j_0}\lambda_{\max}(P)}{\lambda}. \tag{34}$$

At this stage, inequality (33) is not exploitable because it depends on θ. However, from the inclusion implication (28), we get $\Psi^\theta \in \mathcal{H}_{j_0}^{\sigma}$ for all $\theta \geq \sigma$. Hence from the convexity principle [28], inequality (33) holds if (31) is satisfied. Therefore, the observer gain

$$L = \mathrm{T}(\theta)P^{-1}Y^T \tag{35}$$

ensures the exponential convergence of the estimation error towards zero for all θ such that

$$\theta \geq \max\left(\sigma, \theta_{j_0}^{\frac{1}{1+j_0}}\right). \tag{36}$$

This ends the proof. ■

It is worth interesting that there always exists $\sigma > 0$ so that the LMIs (31) admit solutions. Indeed, from (29), we have

$$\lim_{\sigma\to+\infty}\Big(\mathcal{A}(\Psi^\sigma)\Big) = \mathcal{A}\Big(\lim_{\sigma\to+\infty}(\Psi^\sigma)\Big) = A. \tag{37}$$

This proves that the LPV/LMI technique (corresponding to $j_0 = 0$) for this class of systems with the high-gain structure can always provide solutions for the set of sufficient LMIs (31) for σ large enough. In this case, the high-gain inequality (32) vanishes.

The parameter σ is generally not large, as can be shown through the comparisons in [11]. The simulations in [11] provide small gain K with $\sigma = 1$. We could take $\sigma = 1$ in (31). However, the parameter σ is introduced to guarantee the existence of solutions for (31). In addition, since the high-gain constraint (32) depends on P and (31) depends on σ and P, then even if the LMI (31) is feasible for $\sigma = 1$, it is always possible to find better and lower solutions for $\sigma > 1$. On the other hand, from homogeneity of (31)-(32), the decision variable λ can be fixed to $\lambda = 1$. As for the selection of σ, the best solution we found efficient for the numerical procedure is the use of the gridding method. For this, we introduce a bijective change of variable $\tau = \frac{\sigma}{1+\sigma}$ ($\sigma = \frac{\tau}{1-\tau}$). Hence when $\sigma \in [1 \ +\infty[$, the new variable $\tau \in [\frac{1}{2} \ 1[$. Then we can use the gridding method on τ. The following algorithm summarizes the numerical design procedure we proposed to get a lower observer gain.

- **Algorithm**
  (i) Choose a small $\epsilon > 0$ for the gridding, take $\tau = \frac{1}{2}$, a high value $v_{gain} > 0$ and go to step (ii);
  (ii) Solve LMIs (31). If (31) is found feasible, then go to step (iv). Else go to step (iii);
  (iii) While $\tau + \epsilon < 1$, take $\tau := \tau + \epsilon$ and return to step (ii);
  (iv) Take $\theta = \max\left(\frac{\tau}{1-\tau}, \theta_{j_0}^{\frac{1}{1+j_0}}\right)$ and compute L as in (35). If $v_{gain} > \|L\|$, then put $v_{gain} := \|L\|$ and go to step (ii);

This algorithm will be used in the next section to show the performance of the new HG/LMI observer.

IV. Numerical Example

We consider the case of a five dimensional system with a nonlinearity

$$f(x) = \frac{k_f}{5}\sum_{i=1}^{5}\sin(x_i).$$

This nonlinearity satisfies (7) and (14) with $\gamma_f = k_f$. We can show easily that

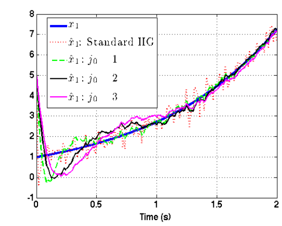
$$k_{j_0} = \frac{k_f(5 - j_0)}{5}.$$

We will provide some comparisons between the standard high-gain method and the new HG/LMI technique. The advantage of the "*compromise index*" j_0 will be shown for different values of k_f. Table I illustrates how the values of the proposed HG/LMI observer gain are smaller than those of the standard high-gain observer.

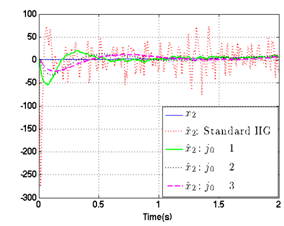
The simulations are done using an additive noise measurement, which is a Gaussian distributed random signal with mean zero and standard deviation 0.1. The initial conditions used for simulations are

$$x_0 = \begin{bmatrix}1\\1\\1\\1\\1\end{bmatrix}, \quad \hat{x}_0 = \begin{bmatrix}5\\5\\5\\5\\5\end{bmatrix}.$$

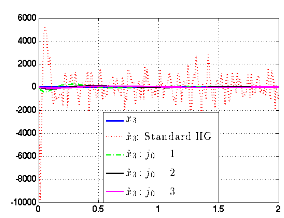
The simulation results for $k_f = 1$ are shown in Figure 1. The superiority of the proposed new HG/LMI observer is quite clear. Three values of j_0 are tested. With all these values the proposed new HG/LMI observer provides lower gains compared with the standard high-gain. Notice that the LPV/LMI technique, which corresponds to $j_0 = 5$, provides lower observer gains, but we need to solve $2^5 = 32$ LMIs. However, this high number of LMIs would complicate the numerical solving of these LMIs for higher dimensional systems. This can lead to infeasible LMIs. Hence the importance of the Proposed HG/LMI method. For instance, it suffices to solve 2 LMIs instead of only one to reduce significantly the value of θ from $\theta = 31.72$ to $\theta = 5.25$ for $k_f = 1$ and from $\theta = 273.03$ to $\theta = 17.63$ for $k_f = 10$. We can reduce more the observer gain, but we have to solve more LMIs, as can be shown in Table I (4 LMIs for $j_0 = 2$ and 8 LMIs for $j_0 = 3$). This is the reason why the index j_0 is called the "*compromise index*".



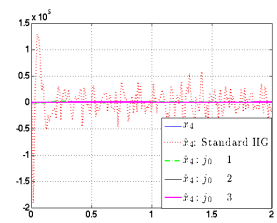
(a) Estimation of x_1.



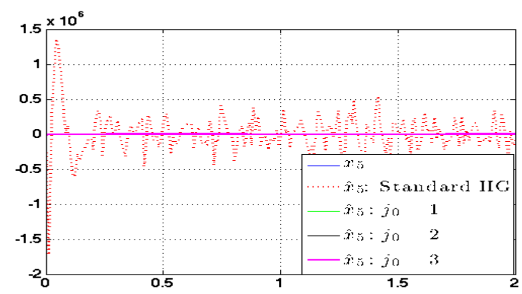
(b) Estimation of x_2.



(c) Estimation of x_3.



(d) Estimation of x_4.



(e) Estimation of x_5.

Fig. 1. Simulation results for $k_f = 1$ and different values of j_0.

V. Conclusions

In this note we presented a new state observer design for a class of triangular systems with Lipschitz nonlinearities. This new observer, called HG/LMI observer, has the advantage to provide lower gain compared to the standard high-gain observer. The key idea behind this observer is based on

		Standard HG						HG/LMI technique							
j_0	k_f	θ	K					n_{LMI}	σ	θ	K				
1	1	31.72	5.66	13.12	16.49	11.12	3.42	2	5.2	5.25	5.38	12.32	15.54	10.71	3.51
	10	273.03	12.07	33.47	45.54	31.58	9.92		17.5	17.63	5.46	12.77	16.61	12.10	4.51
	100	2594.20	32.51	97.87	137.15	96.01	30.39		62	62	11.43	34.53	53.42	46.94	24.56
2	1	SAME RESULT: INDEPENDENT OF j_0						4	2.5	2.88	5.11	11.57	14.74	10.65	4.08
	10								7	7.06	5.34	12.86	17.89	15.08	7.92
	100								23	24.32	6.22	18.01	32.19	40.23	39.86
3	1							8	2	2.06	4.73	10.34	13.03	9.67	4.19
	10								4.4	4.47	5.22	12.83	19.04	18.62	13.31
	100								14	15.12	6.92	23.02	50.57	86.80	133.18

TABLE I

COMPARISONS FOR DIFFERENT VALUES OF k_f AND j_0

the use of the LPV/LMI technique to modify the high-gain constraint, which reduces significantly the Lipschitz constant and leads to smaller observer gains compared to the classical high-gain. A design algorithm was provided and a numerical example has shown the effectiveness and performances of the HG/LMI technique.

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