An LMI-Based $\mathcal{H}_\infty$ Discrete-Time Nonlinear State Observer Design for an Anaerobic Digestion Model

K. Chaib Draa * **H. Voos** * **M. Alma** ** **A. Zemouche** *** **M. Darouach** **

* *Interdisciplinary Centre for Security, Reliability and Trust (SnT) Université du Luxembourg. (e-mail: khadidja.chaibdraa/holger.voos@uni.lu)*
** *CRAN UMR CNRS Université de Lorraine, IUT de Longwy. (e-mail: marouane.alma/ mohamed.darouach@univ-lorraine.fr)*
*** *CRAN UMR CNRS Université de Lorraine, IUT de Longwy and EPI Inria DISCO, Laboratoire des Signaux et Systèmes, CNRS-CentraleSupélec, Gif-sur-Yvette, France.(e-mail: ali.zemouche@univ-lorraine.fr)*

Abstract: This paper is devoted to the design of an $\mathcal{H}_\infty$ discrete-time nonlinear state observer for an anaerobic digestion model. Indeed, the designed observer can be used for different class of systems, mainly linear systems, LPV systems with known and bounded parameters, and nonlinear Lipschitz systems. We use an LMI approach to guarantee the $\mathcal{H}_\infty$ asymptotic stability of the estimation error despite the disturbances affecting the system dynamics and measurements. The synthesised LMI is relaxed due to the inclusion of additional decision variables which enhance its feasibility. This was possible due to the use of a suitable reformulation of the Young's inequality. Simulation results are given to show the robustness of the designed observer.

Keywords: Anaerobic digestion, LMI approach, Discrete time observer design, Nonlinear systems, $\mathcal{H}_\infty$ Criterion.

1. INTRODUCTION AND PRELIMINARIES

1.1 Introduction

Anaerobic digestion (AD) is a promising process for green energy production. It occurs in an anaerobic digester where the organic matter, recovered from different types of waste, is converted to biogas. Inside the digester, several species of bacteria cohabit with each other and usually each species has different living conditions then the others and even sometimes compete for food. Thus, to ensure a healthy functioning of the process, continuous supervision and monitoring are needed.

However, unfortunately both the monitoring and supervision of the process are hampered by several hurdles. One of them is the poverty of knowledge about the bacteria behaviour and the microbial metabolic pathways, this problem has been widely investigated in the literature and as a result several models has been proposed (Bastin and Dochain, 1990), (K. Chaib Draa and M.Alma, 2015). Another hurdle, lies in the lack of reliable and cheap measurement devices. An alternative solution for such a problem is the design of software sensors (observers) to estimate the unmeasurable state variables. However, this does not seem to be an easy task due to the complexity of the biological models, the absence of key measurements and the sensitivity of estimations to the measurement noise (L. Yu and Chen, 2013). Therefore, many researchers have investigated the observation problem for the AD models. Among the designed observers, we can cite the asymptotic observer (Bastin and Dochain, 1990) which is quite simple and does not require the knowledge of some specific nonlinear functions. However, such an observer is very sensitive to model uncertainties and its convergence rate depends on the operational conditions. Therefore, it has been extended to interval observers (Bernard and Gouzé, 2004) which have the advantage of using reliable measurements, which are nonlinear functions of the state vector. The interval observers estimate the interval where the state is lying when the system has large uncertainties. However, generally the rate of convergence is partially tunable and it is not easy to exploit the intervals for control. In order to enhance the convergence rate of the observers the Kalman filter has been designed repeatedly in the literature (F. Haugen and Lie, 2014), (E. Rocha-Cozatl and Wouwer, 2015), which shows suitable results in different chemical applications, but unfortunately the convergence of estimation errors to zero is not guaranteed. The high gain observer (Gauthier et al., 1992), (M. Farza and Busawon, 1997) converges rapidely to the model state variables, however its synthesis is complex and it is very sensitive to noise (M. Lombardi and Laurent, 1999). Thus, it is relevant to design an observer which can overcome model disturbances and measurements corruption.

An other issue related to the design of observers for the AD models is that most of the observer designers assume that measurements are available on line and continuously. Whereas, this is not true in real applications. Therefore, in reality the observer operates in discrete time and is driven by discrete time measurements (sampled data) (Kravaris et al., 2013). Consequently, we design in the current paper a suitable $\mathcal{H}_\infty$ discrete time nonlinear observer for the AD process. The designed observer is simple to implement, robust as it can be seen further in the paper and its rate of convergence is tunable. Moreover, we provide a novel algorithm to ensure the stability of the observation error. Literally, we give a new and less conservative LMI to find the observer gains. Indeed, this was possible due to the use of a suitable reformulation of the Young's inequality (Zemouche et al., 2016). Actually, we include additional decision variables in the LMI to allow its feasibility.

The paper is organized as follows, in Section 2 we present the process modelling, then in Section 3 we state the problem of $\mathcal{H}_\infty$ observer design. Later, in Section 4 we give the new LMI synthesis conditions, then in Section 5 we simulate and apply the findings to the considered AD model and finally we conclude the paper in Section 6.

1.2 Notation and Preliminaries

Notation: The following notations will be used throughout this paper:

- $(\star)$ is used for the blocks induced by symmetry;
- A^T represents the transposed matrix of A;
- $\mathbb{I}_r$ represents the identity matrix of dimension r;
- for a square matrix S, $S > 0$ ($S < 0$) means that this matrix is positive definite (negative definite);
- the set $Co(x, y) = \{\lambda x + (1 - \lambda)y, 0 \leq \lambda \leq 1\}$ is the convex hull of $\{x, y\}$;
- $e_s(i) = \big(\underbrace{0, ..., 0, \overbrace{1}^{i\text{ th}}, 0, ..., 0}_{s\text{ components}}\big)^T \in \mathbb{R}^s, s \geq 1$ is a vector of the canonical basis of $\mathbb{R}^s$.

Preliminaries: The preliminaries provided herein are very useful in the design of the synthesis conditions to ensure the asymptotic convergence of the state observer that we will propose later.

Lemma 1. (*a variant of Lipschitz reformulation*). Let $\varphi : \mathbb{R}^n \to \mathbb{R}^q$ a differentiable function on $\mathbb{R}^n$. Then, the following items are equivalent (Zemouche et al., 2016):

- φ is a globally γ_φ-Lipschitz function;
- there exist finite and positive scalar constants a_{ij}, b_{ij} so that for all $x, y \in \mathbb{R}^n$ there exist $z_i \in Co(x, y), z_i \neq x, z_i \neq y$ and functions ψ_{ij}: $\mathbb{R}^n \to \mathbb{R}$ satisfying the following:

$$\varphi(x) - \varphi(y) = \sum_{i,j=1}^{q,n} \psi_{ij}(z_i)\mathcal{H}_{ij}\Big(x - y\Big) \tag{1}$$

$$a_{ij} \leq \psi_{ij}\Big(z_i\Big) \leq b_{ij}, \tag{2}$$

where

$$\psi_{ij}(z_i) = \frac{\partial \varphi_i}{\partial x_j}(z_i),\ \mathcal{H}_{ij} = e_q(i)e_n^T(j).$$

Notice that this lemma has been introduced in order to simplify the presentation of the proposed observer design method. Indeed, for our technique, we will exploit (1)-(2) instead of a direct use of Lipschitz property.

Lemma 2. (*(Zemouche et al., 2016)*). Let X and Y be two given matrices of appropriate dimensions. Then, for any symmetric positive definite matrix S of appropriate dimension, the following inequality holds:

$$X^T Y + Y^T X \leq \frac{1}{2}\Big[X + SY\Big]^T S^{-1}\Big[X + SY\Big]. \tag{3}$$

This lemma will be very useful for the main contributions of this paper. It allows providing less restrictive LMI conditions compared to the classical LMI techniques for the considered class of systems.

2. ANAEROBIC DIGESTION MODELING

The AD model proposed in O. Bernard and Steyer (2001) has been slightly modified by introducing two control inputs reflecting the addition of stimulating substrates (acids (S_{2ad}) and alkalinity (Z_{ad}). The mass balance model of the process is given by equations (4), where x_1 (g/l) represents the concentration of the organic matter to be digested and x_2 (g/l) the concentration of acidogenic bacteria, which degrades the organic matter. The volatile fatty acids concentration x_3 (mmol/l) is supposed to be pure acetate, x_4 (g/l) is the concentration of methanogenic bacteria, x_5 (mmol/l) represents the inorganic carbon concentration and x_6 (mmol/l) the alkalinity concentration. The other control inputs are $u_1 = \frac{F_{1in}}{v}$ ($1/day$) and $u_2 = \frac{F_{2in}}{v}$ ($1/day$), where F_{1in} is the input flow rate of waste to the digester and F_{2in} the input flow rate of the added alkalinity (Z_{ad}). Since the digester volume (v) is constant, the output flow rate $u_{out} = u_1 + u_2$, the rest of the used parameters in the model are defined in Table 1.

$$\begin{aligned}
\dot{x}_1 &= -k_1\mu_1(x_1)x_2 + u_1 S_{1in} - u_{out}x_1 \\
\dot{x}_2 &= (\mu_1(x_1) - \alpha)u_{out}x_2 \\
\dot{x}_3 &= k_2\mu_1(x_1)x_2 - k_3\mu_2(x_3)x_4 + u_1(S_{2in} + S_{2ad}) - \\
&\quad u_{out}x_3 \\
\dot{x}_4 &= (\mu_2(x_3) - \alpha)u_{out}x_4 \\
\dot{x}_5 &= k_4\mu_1(x_1)x_2 + k_5\mu_2(x_3)x_4 + u_1 C_{in} \\
&\quad -u_{out}x_5 - q_c(x) \\
\dot{x}_6 &= u_1 Z_{in} + u_2 Z_{ad} - u_{out}x_6
\end{aligned} \tag{4}$$

with

$$\mu_1(x_1) = \overline{\mu}_1 \frac{x_1}{x_1 + k_{s_1}}, \quad \mu_2(x_3) = \overline{\mu}_2 \frac{x_3}{x_3 + k_{s2} + \frac{x_3^2}{k_{i2}}}$$

$$co_2 = x_5 + x_3 - x_6, \quad \phi = co_2 + K_H P_T + \frac{k_6}{k_{La}\mu_2(x_3)x_4}$$

$$q_c(x) = k_{La}[co_2 - K_H P_C(x)], \quad q_m(x) = k_6\mu_2(x_3)x_4$$

$$P_C(x) = \frac{\phi - \sqrt{\phi^2 - 4K_H P_T co_2}}{2K_H}$$

where $q_m(x)$ and $q_c(x)$ represent the methane and co_2 gas flow rates, respectively.

We divide the system outputs into nonlinear (y_1) and linear (y_2) outputs

$$\begin{cases} y_1 = q_c(x) \\ y_2 = [x_1, x_3, x_6]^T \end{cases} \tag{5}$$

Table 1. Model Parameters (O. Bernard and Steyer, 2001)

Acronyms	Definition	Units	Value
k_1	Yield for substrate (x_1) degradation	$g/(g$ of $x_2)$	42.1
k_2	Yield for VFA (x_3) production	mmol/(g of x_2)	116.5
k_3	Yield for VFA consumption	mmol/(g of x_4)	268
k_4	Yield for co_2 production	mmol/g	50.6
k_5	Yield for co_2 production	mmol/g	343.6
k_6	Yield for ch_4 production	mmol/g	453
$\overline{\mu}_1$	Maximum acidogenic bacteria (x_2) growth rate	1/day	1.25
$\overline{\mu}_2$	Maximum methanogenic bacteria (x_4) growth rate	1/day	0.74
k_{s_1}	Half saturation constant associated with x_1	g/l	7.1
k_{s2}	Half saturation constant associated with x_3	mmol/l	9.28
k_{i2}	Inhibition constant associated with x_3	mmol/l	256
k_b	Acidity constant of bicarbonate	mol/l	$6.5\ 10^{-7}$
K_H	Henry's constant	mmol/($l.atm$)	27
P_T	Total preasure	atm	1.013
k_{La}	Liquid/gas transfer constant	1/day	19.8

3. PROBLEM FORMULATION OF THE $\mathcal{H}_\infty$ OBSERVER DESIGN

To render the findings general and suitable for other nonlinear models, we will present the results in a general way for a certain class of nonlinear models.

Motivated by the model of anaerobic digestion (4), we will investigate the general class of discrete-time models described by the following equations:

$$\begin{cases} x_{k+1} = A(\rho_k)x_k + B\gamma(x_k) + g(y_{1k}, u_k) + Ew_k \\ y_{2k} = Cx_k + Dw_k \end{cases} \tag{6}$$

where $x_k \in \mathbb{R}^n$ is the state vector, $y_{1k} \in \mathbb{R}^{p'}$ is the nonlinear output measurement and $y_{2k} \in \mathbb{R}^p$ is the linear output measurement, $u_k \in \mathbb{R}^q$ is an input vector, $w_k \in \mathbb{R}^z$ in the disturbance $\mathcal{L}_2$ bounded vector and $\rho_k \in \mathbb{R}^s$ is an $\mathcal{L}_\infty$ bounded and known parameter. The affine matrix $A(\rho_k)$ is expressed under the form

$$A(\rho_k) = A_0 + \sum_{j=1}^{s} \rho_k^j A_j$$

with

$$\rho^j_{\min} \le \rho^j_k \le \rho^j_{\max},$$

which means that the parameter ρ_k belongs to a bounded convex set for which the set of 2^s vertices can be defined by:

$$\mathbb{V}_\rho = \Big\{ \varrho \in \mathbb{R}^s : \varrho^j \in \{\rho^j_{\min}, \rho^j_{\max}\} \Big\}. \tag{7}$$

The matrices $A_i \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$, $E \in \mathbb{R}^{n\times z}$ and $D \in \mathbb{R}^{p\times z}$ are constant. The nonlinear function $\gamma :\ \mathbb{R}^n \longrightarrow\ \mathbb{R}^m$ is assumed to be globally Lipschitz. It is obvious that $\gamma(.)$ can always be written under the detailed form:

$$B\gamma(x_k) = \sum_{i=1}^{m} B_i \gamma_i(\overbrace{H_i x_k}^{\vartheta_i})$$

where $H_i \in \mathbb{R}^{n_i \times n}$ and B_i refers to the i^{th} column of the matrix B.

Remark 1. Notice that the fact we use the same disturbances vector w in the dynamics and the output measurements is not a restriction because the matrices E, D and the dimension of w are arbitrary. Indeed, if we assume that in the dynamics we have $E_1 w_1$, and in the measurement equation, we have $E_2 w_2$, then we can always write $E = [E_1\ \ 0]$, $D = [0\ \ E_2]$ and $w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$, which lead to the form (6).

It is easy to show that the model of anaerobic digestion (4) can be written under the form (6) with particular parameters that will be provided in a detailed description in Section 4. It is also obvious to see that the assumptions on the LPV parameter ρ_k and the global Lipschitz property of $\gamma(.)$ corresponding to (4) are preserved.

In order to estimate the unmeasurable variables of the model (4), we use the following observer structure:

$$\hat{x}_{k+1} = A(\rho_k)\hat{x}_k + \sum_{i=1}^{m} B_i \gamma_i(\hat{\vartheta}_i) + g(y_{1k}, u_k) + L(\rho_k)\Big(y_{2k} - C\hat{x}_k\Big) \tag{8}$$

$$\hat{\vartheta}_i = H_i \hat{x}_k + K_i(\rho_k)\Big(y_k - C\hat{x}_k\Big) \tag{9}$$

$$L(\rho_k) = L_0 + \sum_{j=1}^{s} \rho_k^j L_j, \quad K_i(\rho_k) = K_i^0 + \sum_{j=1}^{s} \rho_k^j K_i^j \tag{10}$$

where $\hat{x}_k$ is the estimate of x_k. The matrices $L_i \in \mathbb{R}^{n\times p}$ and $K_i^j \in \mathbb{R}^{n_i \times p}$ are the observer parameters to be determined so that the estimation error $e_k = x_k - \hat{x}_k$, be $\mathcal{H}_\infty$ asymptotically stable.

Since $\gamma(.)$ is globally Lipschitz, then from Lemma 1 there exist $z_i \in Co(\vartheta_i, \hat{\vartheta}_i)$, functions

$$\phi_{ij}\ :\ \mathbb{R}^{n_i} \longrightarrow\ \mathbb{R}$$

and constants a_{ij}, b_{ij}, such that

$$\gamma(x) - \gamma(\hat{x}) = \sum_{i,j=1}^{m,n_i} \phi_{ij}(z_i)\mathcal{H}_{ij}\Big(\vartheta_i - \hat{\vartheta}_i\Big) \tag{11}$$

and

$$a_{ij} \le \phi_{ij}\Big(z_i\Big) \le b_{ij}, \tag{12}$$

where

$$\phi_{ij}(z_i) = \frac{\partial \gamma_i}{\partial \vartheta_i^j}(z_i),\ \mathcal{H}_{ij} = B_i e_{n_i}(j).$$

For shortness, we set $\phi_{ij} \triangleq \phi_{ij}\Big(z_i\Big)$. Without loss of generality, we assume that $a_{ij} = 0$ for all $i = 1, \dots, m$ and $j = 1, \dots, n_i$. For more details about this, we refer the reader to (Arcak and Kokotovic, 2001).

Since $\vartheta_i - \hat{\vartheta}_i = \Big(H_i - K_i(\rho_k)C\Big)e_k - K_i(\rho_k)Dw_k$, then we have

$$\begin{aligned}\gamma(x_k) - \gamma(\hat{x}_k) = &\left[\sum_{i,j=1}^{m,n_i} \phi_{ij}\mathcal{H}_{ij}\Big(H_i - K_i(\rho_k)C\Big)\right] e_k \\ &- \left[\sum_{i,j=1}^{m,n_i} \phi_{ij}\mathcal{H}_{ij}K_i(\rho_k)D\right] w_k \end{aligned} \quad (13)$$

Therefore, the dynamic equation of the estimation error can be obtained as

$$\begin{aligned} e_{k+1} = &\left(\mathbb{A}_L(\rho_k) + \sum_{i,j=1}^{m,n_i}\Big[\phi_{ij}\mathcal{H}_{ij}\mathbb{H}_{K_i}\Big]\right) e_k \\ &+ \left(\mathbb{E}_L(\rho_k) + \sum_{i,j=1}^{m,n_i}\Big[\phi_{ij}\mathcal{H}_{ij}\mathbb{D}_{K_i}\Big]\right) w_k \end{aligned} \quad (14)$$

with

$$\mathbb{A}_L = A(\rho_k) - L(\rho_k)C,\ \mathbb{H}_{K_i} = H_i - K_i(\rho_k)C. \quad (15)$$
$$\mathbb{E}_L = E - L(\rho_k)D,\ \mathbb{D}_{K_i} = K_i(\rho_k)D.$$

As already mentioned, our target is to make the estimation error dynamics (14) $\mathcal{H}_\infty$ asymptotically stable. In other words, we want to find the observer gains such that the following $\mathcal{H}_\infty$ criterion is satisfied:

$$\|e\|_{\mathcal{L}_2^n} \leq \sqrt{\mu\|w\|^2_{\mathcal{L}_2^z} + \nu\|e_0\|^2} \quad (16)$$

Usually, a quadratic Lyapunov function is used to analyse the $\mathcal{H}_\infty$ stability of the estimation error

$$V(e_k) = e_k^T\mathbb{P}e_k,\quad \mathbb{P} = \mathbb{P}^T > 0.$$

Consequently the $\mathcal{H}_\infty$ criterion is satisfied if the following holds

$$\mathcal{W} \triangleq \Delta V + \|e\|^2 - \mu\|w\|^2 \leq 0. \quad (17)$$

where $\Delta V = V(e_{k+1}) - V(e_k)$.

4. NEW LMI SYNTHESIS CONDITIONS

The main results related to the convergence analysis of the estimation error is summarized in the following theorem, which provides new LMI conditions.

Theorem 3. If there exist symmetric positive definite matrices $\mathbb{P} \in \mathbb{R}^{n\times n}$, $\mathcal{S}_i \in \mathbb{R}^{n_i\times n_i}$, matrices $\mathcal{X}_0, \mathcal{X}_l \in \mathbb{R}^{p\times n}$, and $\mathcal{Y}_0, \mathcal{Y}_i^l \in \mathbb{R}^{p\times n_i}$, $i = 1, \dots, m; l = 1, \dots, s$, of appropriate dimensions so that the convex optimization problem $\min(\mu)$ subject to the constraint (18) is solvable:

$$\begin{bmatrix} \mathbb{M}(\varrho) & \big[\Pi_1^T\ \dots\ \Pi_m^T\big] \\ (\star) & -\Lambda\mathbb{S} \end{bmatrix} < 0,\ \forall \varrho \in \mathbb{V}_\rho \quad (18)$$

with

$$\mathbb{M}(\varrho) = \begin{bmatrix} -\mathbb{P} & 0 & \mathbb{M}_{13}(\varrho) \\ 0 & -\mu\mathbb{I}_z & \mathbb{M}_{23}(\varrho) \\ \mathbb{M}_{13}^\top(\varrho) & \mathbb{M}_{23}^\top(\varrho) & -\mathbb{P} \end{bmatrix} \quad (19)$$

$$\mathbb{M}_{13}(\varrho) = \big(A_0^T\mathbb{P} - C^T\mathcal{X}_0\big) + \sum_{l=1}^{s}\varrho^l\big(A_l^T\mathbb{P} - C^T\mathcal{X}_l\big) \quad (20)$$

$$\mathbb{M}_{23}(\varrho) = E^\top\mathbb{P} - D^\top\left(\mathcal{X}_0 + \sum_{l=1}^{s}\varrho^l\mathcal{X}_l\right) \quad (21)$$

$$\Pi_i = \Big[\Pi_{i1}^T \dots \Pi_{in_i}^T\Big]^T,\quad \Pi_{ij}^T = \begin{bmatrix} \mathbb{H}\Big(\mathcal{S}_i, \mathcal{Y}_i^l\Big) \\ \mathbb{D}\Big(\mathcal{S}_i, \mathcal{Y}_i^l\Big) \\ \mathbb{P}\mathcal{H}_{ij} \end{bmatrix} \quad (22)$$

$$\mathbb{H}\Big(\mathcal{S}_i, \mathcal{Y}_{i,l}\Big) = H_i^T\mathcal{S}_i - C^T\left(\mathcal{Y}_i^0 + \sum_{l=1}^{s}\varrho^l\mathcal{Y}_i^l\right) \quad (23)$$

$$\mathbb{D}\Big(\mathcal{S}_i, \mathcal{Y}_{i,l}\Big) = D^T\left(\mathcal{Y}_i^0 + \sum_{l=1}^{s}\varrho^l\mathcal{Y}_i^l\right) \quad (24)$$

$$\Lambda = \text{block-diag}\Big(\Lambda_1, ..., \Lambda_m\Big) \quad (25)$$

$$\Lambda_i = \text{block-diag}\left(\frac{2}{b_{i1}}\mathbb{I}_{n_i}, \dots, \frac{2}{b_{in_i}}\mathbb{I}_{n_i}\right) \quad (26)$$

$$\mathbb{S} = \text{block-diag}\Big(\mathbb{S}_1, \dots, \mathbb{S}_m\Big) \quad (27)$$

$$\mathbb{S}_i = \text{block-diag}\Big(\overbrace{\mathcal{S}_i, \dots, \mathcal{S}_i}^{n_i\ \text{times}}\Big) \quad (28)$$

then, the estimation error satisfies the $\mathcal{H}_\infty$ criterion (16) with $\nu = \lambda_{\max}(\mathbb{P})$. Consequently, the observer parameters L_l and K_i^l are to be computed as follows:

$$L_l = \mathbb{P}^{-1}\mathcal{X}_l^T,\quad K_i^l = \mathcal{S}_i^{-1}(\mathcal{Y}_i^l)^T.$$

Proof.

By calculating $\mathcal{W}$ (17) along the trajectories of (14), we obtain:

$$
\begin{aligned}
\mathcal{W} = e_k^T & \Bigg[\Bigg(\mathbb{A}_L(\rho_k) + \sum_{i,j=1}^{m,n_i}\Big[\phi_{ij}\mathcal{H}_{ij}\mathbb{H}_{K_i}\Big]\Bigg)^T \mathbb{P} \\
& \times \Bigg(\mathbb{A}_L(\rho_k) + \sum_{i,j=1}^{m,n_i}\Big[\phi_{ij}\mathcal{H}_{ij}\mathbb{H}_{K_i}\Big]\Bigg) - \mathbb{P} + \mathbb{I}_n\Bigg] e_k \\
& + w_k^T \Bigg[\Bigg(\mathbb{E}_L(\rho_k) + \sum_{i,j=1}^{m,n_i}\Big[\phi_{ij}\mathcal{H}_{ij}\mathbb{D}_{K_i}\Big]\Bigg)^T \mathbb{P} \\
& \times \Bigg(\mathbb{E}_L(\rho_k) + \sum_{i,j=1}^{m,n_i}\Big[\phi_{ij}\mathcal{H}_{ij}\mathbb{D}_{K_i}\Big]\Bigg) - \mu\mathbb{I}_z\Bigg] w_k \\
& + e_k^T \Bigg[\Bigg(\mathbb{A}_L(\rho_k) + \sum_{i,j=1}^{m,n_i}\Big[\phi_{ij}\mathcal{H}_{ij}\mathbb{H}_{K_i}\Big]\Bigg)^T \mathbb{P} \\
& \times \Bigg(\mathbb{E}_L(\rho_k) + \sum_{i,j=1}^{m,n_i}\Big[\phi_{ij}\mathcal{H}_{ij}\mathbb{D}_{K_i}\Big]\Bigg)\Bigg] w_k \\
& + w_k^T \Bigg[\Bigg(\mathbb{E}_L(\rho_k) + \sum_{i,j=1}^{m,n_i}\Big[\phi_{ij}\mathcal{H}_{ij}\mathbb{D}_{K_i}\Big]\Bigg)^T \mathbb{P} \\
& \times \Bigg(\mathbb{A}_L(\rho_k) + \sum_{i,j=1}^{m,n_i}\Big[\phi_{ij}\mathcal{H}_{ij}\mathbb{H}_{K_i}\Big]\Bigg)\Bigg] e_k \qquad (29)
\end{aligned}
$$

Using the Schur lemma we deduce that $\mathcal{W} < 0$ if the subsequent matrix inequality holds:

$$
\begin{bmatrix}
\begin{bmatrix} -\mathbb{P} + \mathbb{I}_n & 0 \\ 0 & -\mu\mathbb{I}_z \end{bmatrix} & \begin{bmatrix} \Big(\mathbb{A}_L(\rho_k) + \sum_{i,j=1}^{m,n_i}\Big[\phi_{ij}\mathcal{H}_{ij}\mathbb{H}_{K_i}\Big]\Big)^T \mathbb{P} \\ \Big(\mathbb{E}_L(\rho_k) + \sum_{i,j=1}^{m,n_i}\Big[\phi_{ij}\mathcal{H}_{ij}\mathbb{D}_{K_i}\Big]\Big)^T \mathbb{P} \end{bmatrix} \\
(\star) & -\mathbb{P}
\end{bmatrix} < 0 \qquad (30)
$$

which can be rewritten under the following form:

$$
\overbrace{\begin{bmatrix}
\begin{bmatrix} -\mathbb{P} + \mathbb{I}_n & 0 \\ 0 & -\mu\mathbb{I}_z \end{bmatrix} & \begin{bmatrix} \mathbb{A}_L^\top(\rho_k)\mathbb{P} \\ \mathbb{E}_L^\top(\rho_k)\mathbb{P} \end{bmatrix} \\
(\star) & -\mathbb{P}
\end{bmatrix}}^{\mathbb{M}(\varrho)} +
\sum_{i,j=1}^{m,n_i} \phi_{ij} \left(\overbrace{\begin{bmatrix} 0 \\ 0 \\ \mathbb{P}\mathcal{H}_{ij} \end{bmatrix}}^{\mathbb{X}_{ij}^T} \overbrace{[\mathbb{H}_{K_i} \;\; \mathbb{D}_{K_i} \;\; 0]}^{\mathbb{Y}_i} + \mathbb{Y}_i^T \mathbb{X}_{ij} \right) < 0. \qquad (31)
$$

Now, by applying Lemma 2 we have

$$
\mathbb{X}_{ij}^T\mathbb{Y}_i + \mathbb{Y}_i^T\mathbb{X}_{ij} \le \frac{1}{2}\Big(\mathbb{X}_{ij} + \mathbb{S}_{ij}\mathbb{Y}_i\Big)^T \mathbb{S}_{ij}^{-1} \overbrace{\Big(\mathbb{X}_{ij} + \mathbb{S}_{ij}\mathbb{Y}_i\Big)}^{\Pi_{ij}}
$$

for any symmetric positive definite matrices $\mathbb{S}_{ij}$. Since the matrix block $\mathbb{Y}_i$ does not depend on the index j and depends on the same $K_i(\rho_k)$, then to obtain an LMI, we need to put $\mathbb{S}_{ij} = \mathcal{S}_i$, $\forall (i,j)$ as in (27)-(28), with $\mathcal{S}_i \in \mathbb{R}^{n_i \times n_i}$. Consequently, from (12) and the fact that $a_{ij} = 0$, inequality (31) holds if

$$
\mathbb{M}(\varrho) - \sum_{i,j=1}^{m,n_i}\left(\Pi_{ij}^T\Big(-\frac{2}{b_{ij}}\mathcal{S}_i\Big)^{-1}\Pi_{ij}\right) < 0. \qquad (32)
$$

consequently, by Schur lemma, inequality (32) is equivalent to (18), and to solve the LMI we use the change of variable $\mathcal{X}_i = L_i^T\mathbb{P}$ and $\mathcal{Y}_i^l = (K_i^l)^T\mathcal{S}_i$. This ends the proof.

5. SIMULATION RESULTS

In this section, we apply the designed observer in Section 3 to the AD model (4) in order to estimate the main state variables of the model in presence of disturbance in the model dynamics and measurements. Therefore, we first write the model (4) in the form (6). This is done by using the first order Euler discretization method with sampling time T_s. After discretization, the model (4) is written under the form (6) with the following parameters:

$$A(\rho_k) = \mathbb{I}_6 - T_s u_{out}(k)A_1, \; A_1 = \text{block-diag}(1, \alpha, 1, \alpha, 1, 1)$$

$$B \quad = \quad T_s \begin{bmatrix} -k_1 & 1 & k_2 & 0 & k_4 & 0 \\ 0 & 0 & -k_3 & 1 & k_5 & 0 \end{bmatrix}^T$$

$$
g(y_{1k}, u_k) = T_s \begin{bmatrix} u_1(k)S_{1in} \\ 0 \\ u_1(k)(S_{2in} + S_{2ad}(k)) \\ 0 \\ u_1(k)C_{in} - q_c(k) \\ u_1(k)Z_{in} + u_2(k)Z_{ad}(k) \end{bmatrix}
$$

$$
\gamma(x_k) = \begin{bmatrix} \mu_1(x_{1k})x_{2k} \\ \mu_2(x_{3k})x_{4k} \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}
$$

For a demonstrative simulation, we suppose that the dynamics and measurements are disturbed by different signals w_1 and $w2$, respectively. They hold over a finite interval of time as depicted in Figure 1. Thus, we choose the matrices E_1 and E_2 , defined in Remark 1, as the following:

$$E_1 = [\,0.1,\ 0.2,\ 1,\ 0.1,\ 0.3,\ 0.2\,]^T, \quad E_2 = [\,0.1,\ 0.5,\ 1\,]^T$$

For the considered model we have $\rho_k = u_{out}(k)$ and for the observer design we have, $m = 2$, $s = 1$, $n_i = 2$, $\gamma_1(x_k) = \mu_1(x_{1k})x_{2k}$, $\gamma_2(x_k) = \mu_2(x_{3k})x_{4k}$,

$$
H_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad H_2 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}
$$

The simulation have been run for $T_s = 0.001$ day, $\rho_{min} = 0.1$ day^{-1}, $\rho_{max} = 0.9$ day^{-1}, $S_{1in} = 16$ g/l, $S_{2in} = 170$ mmol/l, $C_{in} = 76.15$ mmol/l, $Z_{in} = 200$ mmol/l, $Z_{ad} = 700$ mmol/l, $S_{2ad} = 0$ mmol/l, $u_1 = 0.6$ day^{-1}, $u_2 = 0.02$ day^{-1} and the parameter values given in Table 1. Moreover, the system and the observer were initialized, respectively, by $x(0) = [2, 0.5, 12, 0.7, 53.48, 55]^T$ and $\hat{x}(0) = [2, 1, 12, 0.4, 28.5, 55]^T$.

Figures 2, 3, 4 represent the system states and their estimates, obtained by using the proposed observer in this paper and another observer of the same structure (8)-(10) but without including the $\mathcal{H}_\infty$ criterion (16). It is clear that the proposed $\mathcal{H}_\infty$ observer gives the better results in presence of disturbances.

Indeed, as it can be seen from the same Figures 2, 3, 4, despite the initial estimation error, the corrupted measurements and the disturbed dynamics, the designed $\mathcal{H}_\infty$ observer shows satisfactory results and good performances. Actually the figures give a clear idea on the level of the attenuation obtained with the $\mathcal{H}_\infty$ criterion. This demonstrates the efficiency of the proposed approach.

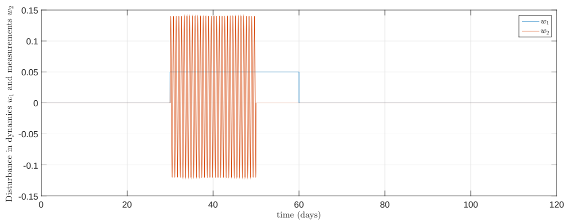

Fig. 1. Disturbance in dynamics (w_1) and measurements (w_2).

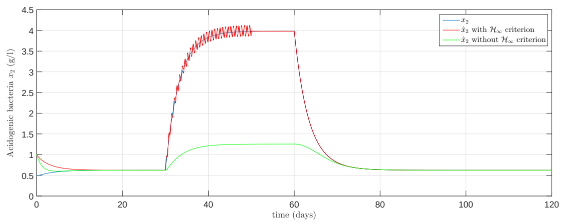

Fig. 2. Acidogenic bacteria x_2 and its estimate $\hat{x}_2(g/l)$.

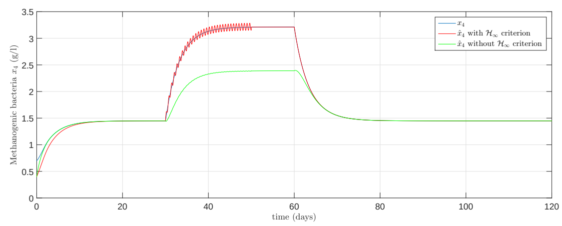

Fig. 3. Mathenogenic bacteria x_4 and its estimate $\hat{x}_4(g/l)$.

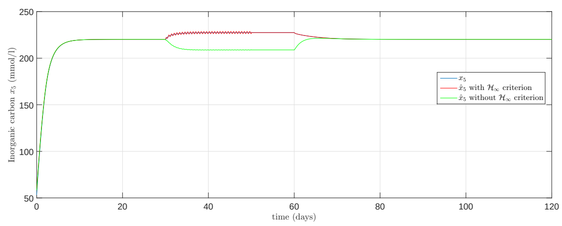

Fig. 4. Inorganic carbon x_5 and its estimate $\hat{x}_5$(mmol/l).

6. CONCLUSION

In this paper we have designed an $\mathcal{H}_\infty$ discrete time nonlinear observer for a disturbed AD model, and in presence of corrupted measurements data. The advantage of the findings is the robustness and the fastness of the designed observer against the model and measurement disturbances. This opens up new prospects to synthesize an optimal control for the biogas plants based on the designed observer.

REFERENCES

Arcak, M. and Kokotovic, P. (2001). Observer-based control of systems with slope-restricted nonlinearities. *IEEE Transactions on Automatic Control*, 46(7), 1146–1150.

Bastin, G. and Dochain, D. (1990). *On-line Estimation and Adaptive Control of Bioreactors*. Process Measurement and Control. Elsevier.

Bernard, O. and Gouzé, J. (2004). Closed loop observers bundle for uncertain biotechnological models. *Journal of Process Control*, 14(7), 765–774.

E. Rocha-Cozatl, M. Sbarciog, L.D.J.M.J. and Wouwer, A.V. (2015). State and input estimation of an anaerobic digestion reactor using a continuous-discrete unknown input observer. *IFAC-PapersOnLine*, 48(8), 129–134.

F. Haugen, R.B. and Lie, B. (2014). State estimation and model-based control of a pilot anaerobic digestion reactor. *Journal of Control Science and Engineering*, 2014, 3.

Gauthier, J., Hammouri, H., and Othman, S. (1992). A simple observer for nonlinear systems applications to bioreactors. *IEEE Transactions on Automatic Control*, 37(6), 875–880.

K. Chaib Draa, H. Voos, M.D. and M.Alma (2015). A formal modeling framework for anaerobic digestion systems. In *Proceedings of the* 2015 17^{th} *UKSIM-AMSS International Conference on Modelling and Simulation*, 426 – 431. IEEE Computer Society.

Kravaris, C., Hahn, J., and Chu, Y. (2013). Advances and selected recent ddevelopments in state and parameter estimation. *Computers and Chemical Engineering*, 51, 111 – 123.

L. Yu, P.C Wensel, J.M. and Chen, S. (2013). Mathematical modeling in anaerobic digestion (ad). *Journal of Bioremediation & Biodegradation*, 2014.

M. Farza, H. Hammouri, S.O. and Busawon, K. (1997). Nonlinear observers for parameter estimation in bioprocesses. *Chemical Engineering Science*, 52(23), 4251–4267.

M. Lombardi, K.F. and Laurent, P. (1999). Implementation of observer for on-line estimation of concentration in continuous-stirred membrane bioreactor: Application to the fermentation of lactose. *Chemical Engineering Science*, 54(13), 2689–2696.

O. Bernard, Z. Hadj-Sadok, D.D.A.G. and Steyer, J. (2001). Dynamical model development and parameter identification for an anaerobic wastewater treatment process. *Biotechnology and bioengineering*, 75(4), 424–438.

Zemouche, A., Rajamani, R., Boulkroune, B., Rafaralahy, H., and Zasadzinski, M. (2016). Convex optimization based dual gain observer design for Lipschitz nonlinear systems. In *IEEE American Control Conference*. Boston, MA, USA.