Projective structures and neighborhoods of rational curves

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PROJECTIVE STRUCTURES AND NEIGHBORHOODS OF RATIONAL CURVES

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Abstract. We investigate the duality between local (complex analytic) projective structures on surfaces and two dimensional (complex analytic) neighborhoods of rational curves having self-intersection $+1$. We study the analytic classification, existence of normal forms, pencil/fibration decomposition, infinitesimal symmetries. Part of the results were announced in [13].

Contents

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1. Introduction

Duality between lines and points in $\mathbb{P}^2$ has a nice non linear generalization which goes back at least to the works of Élie Cartan[1]. The simplest (or more familiar) setting where this duality takes place is when considering the geodesics of a given Riemannian metric[2] on the neighborhood U of the origin in the plane. The space of geodesics is itself a surface U^* that can be constructed as follows. The projectivized tangent bundle $\mathbb{P}(TU)$ is naturally a contact manifold: given coordinates (x, y) on U, the open set of "non vertical" directions is parametrized by triples $(x, y, z) \in U \times \mathbb{C}$ where z represents the class of the vector field $\partial_x + z\partial_y$; the contact structure is therefore given by $dy - zdx = 0$. Each geodesic on U lifts uniquely as a Legendrian curve on $\mathbb{P}(TU)$, forming a foliation $\mathcal{G}$ on $\mathbb{P}(TU)$. A second Legendrian foliation $\mathcal{F}$ is defined by fibers of the canonical projection $\pi : \mathbb{P}(TU) \to U$. The two foliations $\mathcal{F}$ and $\mathcal{G}$ are transversal, spanning the contact distribution. Duality results from permuting of the role of these two foliations. The space of $\mathcal{F}$-leaves is the open set U; if U is small enough, then the space of $\mathcal{G}$-leaves is also a surface U^*. However, U^* is "semi-global" in the sense that it contains (projections of) $\mathbb{P}^1$-fibers of π. If U is a small ball, then it is convex, and we deduce that any two $\mathbb{P}^1$-fibers are connected by a unique geodesic ($\mathcal{G}$-leaf) on $\mathbb{P}(TU)$, i.e. intersect once on U^*. Finally, we get a 2-dimensional family (parametrized by U) of rational curves on U^* with normal bundle $\mathcal{O}_{\mathbb{P}^1}(1)$. Note that $\mathbb{P}(TU) \subset \mathbb{P}(TU^*)$ as contact 3-manifolds.

In fact, we do not need to have a metric for the construction, but only a collection of curves on U having the property that there is exactly one such curve passing through a given point with a given direction. This is what Cartan calls a *projective structure*[3]. In coordinates $(x, y) \in U$, such a family of curves is defined as the graph-solutions to a given differential equation of the form

$$y'' = A(x, y) + B(x, y)(y') + C(x, y)(y')^2 + D(x, y)(y')^3 \tag{1}$$

with A, B, C, D holomorphic on U. Then the geodesic foliation $\mathcal{G}$ is defined by the trajectories of the vector field

$$\partial_x + z\partial_y + \left[A + Bz + Cz^2 + Dz^3\right] \partial_z.$$

Since it is second order, we know by Cauchy-Kowalevski Theorem that there is a unique solution curve passing through each point and any non vertical direction. That the second-hand is cubic is exactly what we need to insure the existence and unicity for vertical directions. In a more intrinsec way, we can define a projective structure by an affine connection, i.e. a (linear) connection $\nabla : TU \to TU \otimes \Omega^1_U$ on the tangent bundle. Then, ∇-geodesics are parametrized curves $\gamma(t)$ on U such that, after lifting to TU as $(\gamma(t), \dot{\gamma}(t))$, they are in the kernel of ∇. All projective structures come from an affine connection, but there are many affine connections

[1] It even appears in Alfred Koppisch's thesis 1905.
[2] real analytic, or holomorphic
[3] not to be confused with the homonym notion of manifolds locally modelled on $\mathbb{P}^n$, see [11, 20]

giving rise to the same projective structure: the collection of curves is the same, but with different parametrizations. An example is the Levi-Civita connection associated to a Riemannian metric and this is the way to see the Riemannian case as a special case of projective structure. We note that a general projective connection does not come from a Riemannian metric, see [5].

A nice fact is that the duality construction can be reversed. Given a rational curve $\mathbb{P}^1 \simeq C_0 \subset S$ in a surface, having normal bundle $\mathcal{O}_{C_0}(1)$, then Kodaira Deformation Theory tells us that the curve C_0 can be locally deformed as a smooth 2-parameter family C_ϵ of curves, likely as a line in $\mathbb{P}^2$. We can lift it as a Legendrian foliation $\mathcal{F}$ defined on some tube $V \subset \mathbb{P}(TU^*)$ and take the quotient: we get a map $\pi : V \to U$ onto the parameter space of the family. Then fibers of $\pi^* : \mathbb{P}(TU^*) \to U^*$ project to the collection of geodesics for a projective structure on U. We thus get a *one-to-one correspondance between projectives structures at* $(\mathbb{C}^2, 0)$ *up to local holomorphic diffeomorphisms and germs of* $(+1)$*-neighborhoods* (U^*, C_0) *of* $C_0 \simeq \mathbb{P}^1$ *up to holomorphic isomorphisms* (see Le Brun's thesis [24]).

Section 2 recalls in more details this duality picture following Arnold's book [1], Le Brun's thesis [24] and Hitchin's paper [17]. In particular, the euclidean (or **trivial**) structure by lines, defined by the second order differential equation $y'' = 0$, corresponds to the **linear** neighborhood of the zero section C_0 in the total space of $\mathcal{O}_{C_0}(1)$, or equivalently of a line in $\mathbb{P}^2$. But as we shall see, the moduli space of projective structures up to local isomorphisms has infinite dimension.

We recall in section 2.5 some criteria of triviality/linearizability. The neighborhood of a rational curve C_0 in a projective surface S is always linear (see Proposition 2.12). As shown by Arnol'd, if the local deformations of C_0 are the geodesics of a projective structure on U^*, then we are again in the linear case. In fact, in the non linear case, it is shown in Proposition 2.9 that deformations C_ϵ of C_0 passing through a general point p of (U^*, C_0) are only defined for ϵ close to 0: there is no local **pencil** of smooth analytic curves through p that contains the germs C_ϵ at p. We show in Proposition 2.9 that, in the non linear case, there is at most one point p where we get such pencil.

Going back to real analytic metrics, the three geometries of Klein, considering metrics of constant curvature, give birth to the same (real) projective structure, namely the trivial one. Indeed, geodesics of the unit 2-sphere $\mathbb{S}^2 \subset \mathbb{R}^3$ are defined as intersections with planes passing through the origin: they project on lines, from the radial projection to a general affine plane. Similarly, for negative curvature, geodesics are lines in Klein model. It would be nice to understand *which* $(+1)$*-neighborhoods* (U^+, C_0) *come from the geosedics of a holomorphic metric.*

In section 3, we introduce the notion of **flat projective structure**, when the projective structure is defined by a flat affine connection ∇, i.e. satisfying $\nabla \cdot \nabla = 0$. Equivalently, the collection of geodesics decomposes as a pencil of geodesic foliations. On the dual picture (U^*, C_0), such a decomposition corresponds to an analytic **fibration** transversal to C_0, i.e. a holomorphic retraction $U^* \to C_0$. This dictionary appear in Kryński [19]. Our main result, announced in [13], is that non linear $(+1)$-neighborhoods (U^*, C_0) have 0, 1 or 2 transverse fibrations, no more (see Theorem 5.1). We show that each case occur with an infinite dimensional moduli space.

The main ingredient to study the existence and unicity of transverse fibrations is the classification of $(+1)$-neighborhoods which is due to Mishustin [27] (section 4).

It was known since the work of Grauert [16] that there are infinitely many obstructions to linearize such a neighborhood. Mishustin showed that any neighborhood can be described as the patching of two open sets of the linear neighborhood by a non linear cocycle, that can be reduced to an almost unique normal form (Theorem 4.2 and Proposition 4.5). The moduli space appears to be isomorphic to the space of convergent power series in two variables. Hurtubise and Kamran [18] provide explicit formulae linking the formal invariants of Mishustin (coefficients of the cocycle) with Cartan invariants for the equivalence problem for projective structures (or second order differential equations). They were not aware of Mishustin work and proved a formal version of the normal form; we also proved the normal form before S. Ivachkovitch informed us about Mishustin's paper. It is quite surprising that Mishustin's result has never been quoted although it answers a problem left opened since the celebrating works of Grauert and Kodaira. In Proposition 4.5, we get a more precise description of the freedom in the reduction to normal form which is necessary for our purpose, namely the action of a 4-dimensional linear group (see Corollary 4.6).

From Mishustin's cocycle (and its non unicity), we see in Proposition 4.9 that the first obstruction to the existence to a transverse fibration arise in 5-jet, i.e. in the 5th infinitesimal neighborhood of the rational curve, which was also surprising for us. Another surprising fact is the existence of many neighborhoods with two fibrations: we get a moduli space (Theorem 5.19) isomorphic to the space of power series in one variable. One remarquable example (see section 5.4) is given by the two-fold ramified covering $(U^*, C_0) \overset{2:1}{\to} (\mathbb{P}^1 \times \mathbb{P}^1, \Delta)$ of the diagonal $\Delta \subset \mathbb{P}^1 \times \mathbb{P}^1$ that ramifies along Δ: the two fibrations of $\mathbb{P}^1 \times \mathbb{P}^1$ lift as fibrations tangent all along C_0. This example is non linear, and in particular non algebraic (the covering can be only defined at the neighborhood of Δ for topological reasons). However, the field of meromorphic functions on (U^*, C_0) identifies with the field of rational functions on $\mathbb{P}^1 \times \mathbb{P}^1$ and has transcendance dimension 2. We expect that the general $(+1)$-neighborhood has no meromorphic function, but we have no proof, and no example. We are able to compute the differential equation defining the dual projective structure, namely $y'' = (xy' - y)^3$. This example is also remarquable because it has the largest symmetry group, namely $\mathrm{SL}_2(\mathbb{C})$, and this is an ingredient of the proof.

In the last section 6, we investigate the projective structures, or equivalently $(+1)$-neighborhoods with infinitesimal symmetries, or equivalently a positive dimensional Lie group of symmetries. From Lie's work, appart from the linear case $\mathrm{sl}_3(\mathbb{C})$ and the special case above $\mathrm{sl}_2(\mathbb{C})$, the group of symmetries is either 1-dimensional, or isomorphic to the affine group $\mathrm{aff}(\mathbb{C})$. For each group, we recall in Theorem 6.4 the classification mainly established by Bryant, Manno and Matveev in [6]. Then, in section 6.3, we end-up exploring the classification of flat projective structures with non trivial Lie symmetries (Theorem 6.7). The most remarquable fact is the generic affine case: the Lie algebra can be normalized to $\mathbb{C}\langle X, Y\rangle$ with

$$X = \partial_y \quad \text{and} \quad Y = \partial_x + y\partial_y, \quad [X, Y] = X,$$

and the projective structure is defined by (1) with coefficients

$$(A, B, C, D) = (\alpha e^x, \beta, 0, e^{-2x}), \quad \alpha, \beta \in \mathbb{C}.$$

These projective structures are two-by-two non isomorphic. The projective structure admits a flat structure if, and only if the parameters belong to the affine nodal

cubic curve

$$\Gamma = \{(\alpha, \beta)\ ;\ 27\alpha^2 + 4\beta^3 - 12\beta^2 + 9\beta - 2 = 0\};$$

more precisely, under the parametrization

$$\mathbb{C} \to \Gamma\ ;\ \gamma \mapsto (\gamma(2\gamma^2 - 1), 2 - 3\gamma^2),$$

the corresponding projective structure Π_γ is defined by the pencil of foliations

$$\mathcal{F}_z = \{\omega_z = 0\}, \quad z \in \mathbb{P}^1, \quad \text{with}$$

$$\omega_z = \left[e^x(\gamma y + (2\gamma^2 - 1)e^x)dx - (y + 2\gamma e^x)dy\right] + z\left[dy - \gamma e^x dx\right].$$

In other words, geodesics of Π_γ are the leaves of $\mathcal{F}_z$ while z runs over $\mathbb{P}^1$. The flat structure (i.e. decomposition as a pencil of foliations) is unique except in the two following cases:

$$(\alpha, \beta) = (0, \frac{1}{2}) \ \leftrightarrow\ \gamma = \pm\frac{1}{\sqrt{2}} \text{ (nodal point)}$$

$$(\alpha, \beta) = (0, 2) \ \leftrightarrow\ \gamma = 0$$

they respectively correspond to the two projective structures with larger symmetry Lie algebra $\mathrm{sl}_2(\mathbb{C})$ and $\mathrm{sl}_3(\mathbb{C})$. The former one admits exactly two flat structures: the two pencils of foliations are given by $\mathcal{F}_z$ as above, one for each value $\gamma = \pm\frac{1}{\sqrt{2}}$. The latter one is the linearizable case: a pencil of geodesic foliations corresponds in that case to the family of pencils of lines through all points of a given projective line.

As a conclusion, it would be nice to go back to the study of Painlevé equations from this point of view, as initiated in the paper [18] of Hurtubise and Kamran. We expect that the (+1)-neighborhood corresponding to the projective structure defined by the phase portrait of a Painlevé equation has no transverse fibration, no automorphism, and even no non constant meromorphic function, except possibly the very special Picard case (see [26, 10]).

2. Projective structure, Geodesics and duality

2.1. **Second order differential equations and duality.** Let (x, y) be coordinates of $\mathbb{C}^2$. Given a 2nd order differential equation

$$y'' = f(x, y, y')$$

with $f(x, y, z)$ holomorphic at the neighborhood V of some point $(0, 0, z_0)$ say, local solutions $y(x)$ lift as Legendrian curves for the contact structure defined by

$$\alpha = 0 \quad \text{,where} \quad \alpha = dy - zdx.$$

We get two transversal Legendrian foliations on V. The first one $\mathcal{F}$ is defined by the fibers of the projection $V \to U; (x, y, z) \mapsto (x, y)$. The second one $\mathcal{G}$ is defined by solutions $x \mapsto (x, y(x), y'(x))$ or equivalently by trajectories of the vector field

$$v = \partial_x + z\partial_y + f(x, y, z)\partial_z.$$

More generally, given a germ of contact 3-manifold together with two transversal Legendrian foliations, the space of $\mathcal{F}$-leaves can be identified with an open set $U \subset \mathbb{C}^2$ with coordinates (x, y) and $\mathcal{G}$-leaves project on U as graphs of solutions of a 2nd order differential equation $y'' = f(x, y, y')$, see [1, Chapter 1, Section 6.F].

It is now clear that the role of $\mathcal{F}$ and $\mathcal{G}$ can be permuted: *on the space U^* of $\mathcal{G}$-leaves, $\mathcal{F}$-leaves project to solutions of a 2nd order differential equation $Y'' =$*

$g(X,Y,Y')$ (once we have choosen coordinates $(X,Y)\in U^*$). This is the duality introduced by Cartan (see also [1, Chapter 1, Sections 6.F, 6.G]). Points on U correspond to curves on U^* and vice-versa. We will call V the incidence variety by analogy with the case of lines in $\mathbb{P}^2$.

For instance, lines $y=ax+b$ are solutions of the differential equation $y''=0$. Using $(X,Y)=(a,b)\in\check{\mathbb{P}}^2$ for coordinates of dual points, we see that foliations $\mathcal{F}$ and $\mathcal{G}$ given before are liftings of lines on the projective and dual plane, thus the dual equation is also $Y''=0$.

If there is a diffeomorphism $\phi: U\to\tilde{U}$ sending solutions of the differential equation to the solutions of another one $y''=\tilde{f}(x,y,y')$ on $\tilde{U}$, then ϕ can be lifted to a diffeomorphism $\Phi: V\to\tilde{V}$ conjugating the pairs of Legendrian foliations: $\Phi_*\mathcal{F}=\tilde{\mathcal{F}}$ and $\Phi_*\mathcal{G}=\tilde{\mathcal{G}}$. We say that the two differential equations are *Cartan-equivalent* in this case.

2.2. **Projective structure and geodesics.** When the differential equation is cubic in y'

$$y''=A(x,y)+B(x,y)(y')+C(x,y)(y')^2+D(x,y)(y')^3$$

(with A,B,C,D holomorphic on U), then the foliation $\mathcal{G}$ is global on $V:=\mathbb{P}(TU)\simeq U\times\mathbb{P}^1_z$, $z=\frac{dy}{dx}$, and transversal to the fibration $\mathcal{F}$ everywhere. Precisely, setting $\tilde{z}=\frac{1}{z}=\frac{dx}{dy}$, then the foliation $\mathcal{G}$ is defined by the two vector field

$$v=\partial_x+z\partial_y+(A+Bz+Cz^2+Dz^3)\partial_z$$

for z finite, and

$$\tilde{v}=\tilde{z}\partial_x+\partial_y-(D+C\tilde{z}+B\tilde{z}^2+A\tilde{z}^3)\partial_{\tilde{z}}$$

near $z=\infty$.

Remark 2.1. For equations $y''=f(x,y,y')$ having right-hand-side $f(x,y,y')$ polynomial with respect to y', but higher than cubic degree, the foliation $\mathcal{G}$ globalizes on $U\times\mathbb{C}_z$ but transversality is violated at $z=\infty$. Indeed, the corresponding vector field

$$\tilde{v}=\tilde{z}\partial_x+\partial_y-\tilde{z}^3 f\left(x,y,\frac{1}{\tilde{z}}\right)\partial_{\tilde{z}}$$

becomes meromorphic; after multiplication by a convenient power of $\tilde{z}$, the vector field becomes holomorphic but tangent to $\mathcal{F}$ and leaves become singular after projection on U.

With the previous remark, it is easy to check that any foliation $\mathcal{G}$ on $\mathbb{P}(TU)$ which is

- Legendrian, i.e. tangent to the natural contact structure ($dy-zdx=0$),
- transversal to the projection $\mathbb{P}(TU)\to U$,

is locally defined by a vector field like above, cubic in z, i.e. by a second order differential equation with $y''=A+B(y')+C(y')^2+D(y')^3$. We call *projective structure* such a data. We call *geodesic* a curve on U obtained by projection of a $\mathcal{G}$-leaf on $\mathbb{P}(TU)$. The following is proved in [24, Section 1.3]

Proposition 2.2. *If U is a sufficiently small ball, then all geodesics are properly embedded discs and we have the following properties:*

- convexity: *through any two distinct points $p,q\in U$ passes a unique geodesic;*
- infinitesimal convexity: *through any point $p\in U$ and in any direction $l\in T_pU$ passes a unique geodesic.*

We say that U is geodesically convex in this case.

The second item just follows from Cauchy-Kowalevski Theorem for the differential equation defining the projective structure.

2.3. **Space of geodesics and duality.** It is proved in [24, Section 1.4] the following

Proposition 2.3. *If U is geodesically convex, then the space of geodesics, i.e. the quotient space*

$$U^* := \mathbb{P}(TU)/\mathcal{G}$$

is a smooth complex surface. Moreover, the projection map

$$\pi^* : \mathbb{P}(TU) \to U^*$$

restricts to fibers of $\pi : \mathbb{P}(TU) \to U$ as an embedding.

We thus get a two-parameter family (parametrized by U) of smooth rational curves covering the surface U^*: for each point $p \in U$, we get a curve $C_p \subset U^*$. The curve C_p parametrizes in U^* the set (pencil) of geodesics passing through p. Any two curves C_p and C_q, with $p \neq q$, intersect transversely through a single point in U^* representing the (unique) geodesic passing through p and q. The normal bundle of any such curve C_p is in fact $\mathcal{O}_{\mathbb{P}^1}(1)$ (after identification $C_p \simeq \mathbb{P}^1$).

One might think that rational curves define the geodesics of a projective structure on U^*, but it is almost never true: for instance, the set of rational curves (of the family C_p) through a given point of U^* cannot be completed as a pencil of curves (as it would be for geodesics of a projective structure), see [1, Chapter 1, Section 6-D]. In fact, we will prove that if such a pencil exists at two different points of U^*, then we are essentially in the standard linear case of lines in $\mathbb{P}^2$.

From a germ of projective structure at $p \in U$, we can deduce a germ of surface neighborhood of $C_p \simeq \mathbb{P}^1$. Conversely, it is proved in [24, Section 1.7] that we can reverse the construction. Indeed, given a rational curve $C \subset S$ in a surface (everything smooth holomorphic) having normal bundle $\mathcal{O}_{\mathbb{P}^1}(1)$, then C admits by Kodaira Deformation Theory a local 2-parameter family of deformation and the parameter space U is naturally equipped with a projective structure: geodesics on U are those rational curves passing to a common point in S.

In the sequel, we call $(+1)$-neighborhood of a rational curve C a germ (S, C) of a smooth complex surface S where C is embedded with normal bundle $NC \simeq \mathcal{O}_C(+1)$.

Theorem 2.4 (Le Brun)**.** *We have a one-to-one correspondance between germs of projective structures on $(\mathbb{C}^2, 0)$ up to diffeomorphism and germs of $(+1)$-neighborhood of $\mathbb{P}^1$ up to isomorphism.*

2.4. **Affine connections, metric.** Let S be a smooth complex surface. An *affine connection* on S is a (linear) holomorphic connection on the tangent bundle TS, i.e. a $\mathbb{C}$-linear morphism $\nabla : TS \to TS \otimes \Omega^1_S$ satisfying the Leibnitz rule

$$\nabla(f \cdot Z) = Z \otimes df + f \cdot \nabla(Z)$$

for any holomorphic function f and any vector field Z. Given a two vector fields Z, W, we denote as usual by $\nabla_W Z := i_W(\nabla Z)$ the contraction of ∇Z with W. By a *parametrized geodesic* for ∇, we mean a holomorphic curve $t \mapsto \gamma(t)$ on S such that $\nabla_{\dot\gamma(t)}\dot\gamma(t) = 0$ on the curve. The image of $\gamma(t)$ on S is simply called

a *(unparametrized) geodesic* and is characterized by $\nabla_{\dot\gamma(t)}\dot\gamma(t) = f(t)\dot\gamma(t)$ for any parametrization. Geodesics define a projective structure Π_∇ on S.

In coordinates $(x,y) \in U \subset \mathbb{C}^2$, a trivialization of TU is given by the basis (∂_x, ∂_y) and the affine connection is given by

$$\nabla(Z) = d(Z) + \Omega \cdot Z, \quad \Omega = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

where $Z = z_1\partial_x + z_2\partial_y$ and $\alpha, \beta, \gamma, \delta \in \Omega^1(U)$. On the projectivized bundle $\mathbb{P}(TU)$, with trivializing coordinate $z = z_2/z_1$, equation $\nabla = 0$ induces a *Riccati distribution*

$$dz = -\gamma + (\alpha - \delta)z + \beta z^2.$$

Intersection with the contact structure $dy = zdx$ gives the geodesic foliation $\mathcal{G}$ of the projective structure. Precisely, if we set

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \alpha_1 & \beta_1 \\ \gamma_1 & \delta_1 \end{pmatrix} dx + \begin{pmatrix} \alpha_2 & \beta_2 \\ \gamma_2 & \delta_2 \end{pmatrix} dy$$

(with $\alpha_i, \beta_i, \gamma_i, \delta_i \in \mathcal{O}(U)$) then the projective structure is given by

$$\frac{dz}{dx} = -\gamma_1 + (\alpha_1 - \delta_1 - \gamma_2)z + (\beta_1 + \alpha_2 - \delta_2)z^2 + \beta_2 z^3.$$

We say that two affine connections are (projectively) equivalent if they have the same family of geodesics, i.e. if they define the same projective structure. The following is straightforward

Lemma 2.5. *Two affine connections ∇ and ∇' on U, with matrices Ω and Ω' respectively, define the same projective structure if, and only if, there are $a, b, c, d \in \mathcal{O}(U)$ such that*

$$\Omega' = \Omega + a\begin{pmatrix} dx/2 & 0 \\ dy & -dx/2 \end{pmatrix} + b\begin{pmatrix} -dy/2 & dx \\ 0 & dy/2 \end{pmatrix} + c\begin{pmatrix} dx & 0 \\ 0 & dx \end{pmatrix} + d\begin{pmatrix} dy & 0 \\ 0 & dy \end{pmatrix}.$$

Remark 2.6. Any projective connection $\Pi : y'' = A + B(y') + C(y')^2 + D(y')^3$ can be defined by an affine connection: for instance, $\Phi = \Pi_\nabla$ with

$$\nabla = d + \begin{pmatrix} 0 & Cdx + Ddy \\ -Adx - Bdy & 0 \end{pmatrix}$$

or is equivalently defined by the Riccati distribution

$$dz + Adx + Bdy + z^2(Cdx + Ddy) = 0.$$

There exist also a unique affine connection defining Π which is trace-free and torsion-free (see [29, lemma 6.11]):

$$d + \begin{pmatrix} \frac{B}{3}dx + \frac{C}{3}dy & \frac{C}{3}dx + Ddy \\ -Adx - \frac{B}{3}dy & -\frac{B}{3}dx - \frac{C}{3}dy \end{pmatrix}$$

But mind that these two "special" representatives do not have intrinsec meaning (i.e. not preserved by change of coordinates).

One can also define a projective structure by a holomorphic Riemannian metric, by considering its geodesics defined by Levi-Civita (affine) connection. But it is not true that all projective structures come from a metric: in [5], it is proved that there are infinitely many obstructions, the first one arising at order 5.

Question 2.7. Can we characterize in a geometric way those projective structures arising from a holomorphic metric ? And what about the corresponding (+1)-neighborhood ?

2.5. **Some criteria of linearization.** A projective structure (U, Π) is said *linearizable* if it is Cartan-equivalent to the standard linear structure whose geodesics are lines: there is a diffeomorphism

$$\Phi : U \to V \subset \mathbb{P}^2$$

such that geodesics on U are pull-back of lines in $\mathbb{P}^2$. When U is geodesically convex, this is equivalent to say that (U^*, C_0) is the neighborhood of a line in $\mathbb{P}^2$. As we shall see later, there are many projective structures that are non linearizable (even locally). Here follow some criteria of local linearizability.

Proposition 2.8. *Let Π be a projective structure on a connected open set U. If Π is linearizable at the neighborhood of a point $p \in U$, then it is linearizable at the neighborhood of any otherpoint $q \in U$.*

The proof is postponed in section 5.3, using another criterium of linearization.

Proposition 2.9. *Let Π be a germ of projective structure at $(\mathbb{C}^2, 0)$ and let (U^*, C_0) be the corresponding $(+1)$-neighborhood. If for 2 distinct points $p_1, p_2 \in C_0$ the family of rational curves through p_i is contained in a pencil of curves based in p_i, then $(U^*, C_0) \simeq (\mathbb{P}^2, line)$ (and Π is linearizable).*

Proof. For $i = 1, 2$, let $F_i : U^* \dashrightarrow \mathbb{P}^1$ be the meromorphic map defining the pencil based at p_i: deformations of C_0 passing through p_i are (reduced) fibers of F_i. We can assume $C_0 = \{F_i = 0\}$ for $i = 1, 2$. Then, maybe shrinking U^*, the map

$$\Phi : U^* \to \mathbb{P}^2_{(z_0:z_1:z_2)} \ ; \ p \mapsto (1 : \frac{1}{F_1} : \frac{1}{F_2})$$

is an embedding of U^* onto a neighborhood of the line $z_0 = 0$. Indeed, Φ is well-defined and injective on $U^* \setminus C_0$; one can check that it extends holomorphically on C_0 and the extension does not contract this curve. □

Corollary 2.10 (Arnol'd)**.** *Let Π be a germ of projective structure at $(\mathbb{C}^2, 0)$ and let (U^*, C_0) be the corresponding $(+1)$-neighborhood. If deformations of C_0 are geodesics of a projective structure Π^* in a neighborhood of a point $p \in C_0$, then $(U^*, C_0) \simeq (\mathbb{P}^2, line)$ (and Π is linearizable).*

Remark 2.11. Arnol'd stated this result in [1, Chapter 1, Section 6.D] in terms of 2nd order differential equation: *"An equation $d^2y/dx^2 = \Phi(x, y, dy/dx)$ can be reduced to the form $d^2y/dx^2 = 0$ if and only if the right-hand side is a polynomial in dy/dx of order not greater than 3 both for the equation and for its dual".* Cartan had a similar discussion in [9].

Proof. Let $V \subset U^*$ the open set where the projective structure Π^* is defined. Then at any point $q \in C_0 \cap V$, deformations of C_0 through q are contained in a pencil. Choose two distinct points and apply the previous Proposition. □

Proposition 2.12 ([18, Proposition 4.7])**.** *Let S be a smooth projective surface with an embedded curve $C_0 \simeq \mathbb{P}^1$ with self-intersection $+1$. Then S is rational and $(S, C_0) \simeq (\mathbb{P}^2, L_0)$.*

Proof. As S contains a smooth rational curve with positive self-intersection, we deduce from [2, Proposition V.4.3] that S is rational. This implies $H^1(S,\mathcal{O}_S)\simeq H^{0,1}(S)\simeq H^{1,0}(S)\simeq H^0(S,\Omega^1_S)=0$ thus the Chern-class morphism $H^1(S,\mathcal{O}_S^*)\to H^2(S,\mathbb{Z})$ is injective. We can take deformations C_1, C_2 of C_0 such that $C_0\cap C_1\cap C_2=\emptyset$ and by the previous discussion the three curves determine the same element $\mathcal{O}_S(C)$ of $H^1(S,\mathcal{O}_S^*)$, then we have sections F_i of $\mathcal{O}_S(C)$ vanishing on C_i, $i=0,1,2$. We define

$$\sigma:=(F_0:F_1:F_2):S\dashrightarrow\mathbb{P}^2$$

which is in fact a morphism. Moreover, by the condition on the intersection of the curves we deduce that the generic topological degree of σ is 1. In particular σ is a sequence of blow-ups with no exceptional divisor intersecting C_0. □

Proposition 2.13. *There is a unique global projective structure on $\mathbb{P}^2$, namely the linear one.*

Proof. If $\mathcal{F}_\Pi$ is the associated regular foliation by curves defined in $M=\mathbb{P}(T\mathbb{P}^2)$ with cotangent bundle $\mathcal{O}_M(ah+b\check{h})$ and $\mathcal{V}$ stands for the foliation defined by the fibers, then $Tang(\mathcal{F}_\Pi,\mathcal{V})=(a+2)h+(b-1)\check{h}$ (see [12, Proposition 2.3]). So, the only second order differential equation totally transverse to $\mathcal{V}$ is the one given by $y''=0$. □

Finally, we end-up with an analytic criterium proved by Liouville in [23] (and later by Tresse and Cartan):

Proposition 2.14. *Given a projective structure Π defined by (1), then consider the two functions $L_i(x,y)$ defined by*

$$\begin{cases} L_1=2B_{xy}-C_{xx}-3A_{yy}-6AD_x-3A_xD+3(AC)_y+BC_x-2BB_y,\\ L_2=2C_{xy}-B_{yy}-3D_{xx}+6A_yD+3AD_y-3(BD)_x-B_yB+2CC_x.\end{cases}\tag{2}$$

Then, the form $\theta:=(L_1dx+L_2dy)\otimes(dx\wedge dy)$ is intrinsically defined by the projective structure Π, i.e. its definition does not depend on the choice of coordinates (x,y). Moreover, Π is linearizable if, and only if, $\theta\equiv 0$.

3. Flat structures vs transverse fibrations

A (non singular) foliation $\mathcal{F}$ on U, defined by say $y'=f(x,y)$, can be equivalently defined by its graph $S:=\{z=f(x,y)\}\subset\mathbb{P}(TU)$, a section of $\pi:\mathbb{P}(TU)\to U$. The foliation is geodesic iff the section S is invariant by $\mathcal{G}$; in this case, the section projects onto curve $D:=\pi^*(S)$ intersecting transversally the rational curve C_0 at a single point on U^*. We thus get a one-to-one correspondance between geodesic foliations on U and transversal curves on U^*.

3.1. **Pencil of foliations and Riccati foliation.** A (regular) *pencil of foliations* on U is a one-parameter family of foliations $\{\mathcal{F}_t\}_{t\in\mathbb{P}^1}$ defined by $\mathcal{F}_t=[\omega_t=0]$ for a pencil of 1-forms $\{\omega_t=\omega_0+t\omega_\infty\}_{t\in\mathbb{P}^1}$ with $\omega_0,\omega_\infty\in\Omega^1(U)$ and $\omega_0\wedge\omega_\infty\neq 0$ on U. The pencil of 1-forms defining $\{\mathcal{F}_t\}_{t\in\mathbb{P}^1}$ is unique up to multiplication by a non vanishing function: $\tilde{\omega}_t=f\omega_t$ for all $t\in\mathbb{P}^1$ and $f\in\mathcal{O}^*(U)$. In fact, the parametrization by $t\in\mathbb{P}^1$ is not intrinsec; we will say that $\{\mathcal{F}_t\}_{t\in\mathbb{P}^1}$ and $\{\mathcal{F}'_t\}_{t\in\mathbb{P}^1}$ define the same pencil on U if there is a Moebius transformation $\varphi\in\mathrm{Aut}(\mathbb{P}^1)$ such that $\mathcal{F}'_t=\mathcal{F}_{\varphi(t)}$ for all $t\in\mathbb{P}^1$.

There exists a unique projective structure Π whose geodesics are the leaves of the pencil. Indeed, the graphs S_t of foliations $\mathcal{F}_t$ are disjoint sections (since foliations are pairwise transversal) and form a codimension one foliation $\mathcal{H}$ on $\mathbb{P}(TU)$ transversal to the projection $\pi : \mathbb{P}(TU) \to U$. The foliation $\mathcal{H}$ is a Riccati foliation, i.e. a Frobenius integrable Riccati distribution:

$$\mathcal{H} : [\omega = 0], \quad \omega = dz + \alpha z^2 + \beta z + \gamma, \quad \omega \wedge d\omega = 0.$$

Intersecting $\mathcal{F}$ with the contact structure yields a Legendrian foliation $\mathcal{G}$ (also transversal to the $\mathbb{P}^1$-fibers) and thus a projective structure.

In local coordinates (x, y) such that $\mathcal{F}_0$ and $\mathcal{F}_\infty$ are respectively defined by $dx = 0$ and $dy = 0$, we can assume the pencil generated by $\omega_0 = dx$ and $\omega_\infty = u(x, y)dy$ (we have normalized ω_0) with $u(0, 0) \neq 0$. Then the graph of the foliation $\mathcal{F}_t$ is given by the section $S_t = \{z = -\frac{1}{tu(x,y)}\} \subset \mathbb{P}(TU)$. These sections are the leaves of the Riccati foliation $\mathcal{H} : [dz + \frac{du}{u} z = 0]$, and we can deduce the equation of the projective structure:

$$y'' + \frac{u_x}{u} y' + \frac{u_y}{u} (y')^2 = 0.$$

Note that the projective structure is also defined by the affine connection

$$\nabla = d + \Omega, \quad \Omega = \begin{pmatrix} \frac{1}{2}\frac{du}{u} & 0 \\ 0 & -\frac{1}{2}\frac{du}{u} \end{pmatrix}$$

which is flat (or integrable, curvature-free) $\Omega \wedge d\Omega = 0$, and trace-free $\mathrm{trace}(\Omega) = 0$.

Remark 3.1. A Riccati distribution $\mathcal{H} : [\omega = 0]$ on $\mathbb{P}(TU)$,

$$\omega = dz + \alpha z^2 + \beta z + \gamma, \quad \alpha, \beta, \gamma \in \Omega^1(U),$$

is the projectivization of a unique trace-free affine connection, namely

$$\nabla = d + \Omega, \quad \Omega = \begin{pmatrix} -\frac{\beta}{2} & -\alpha \\ \gamma & \frac{\beta}{2} \end{pmatrix}.$$

Are equivalent

- ∇ is flat: $\Omega \wedge \Omega + d\Omega = 0$;
- ω is Frobenius integrable: $\omega \wedge d\omega = 0$.

There are many other affine connections whose projectivization is ω which are not flat: in general, we only have the implication $[\nabla \text{ flat}] \Rightarrow [\mathcal{H} \text{ integrable}]$.

3.2. **Transverse fibrations on U^*.** If we have a Riccati foliation $\mathcal{H}$ on $\mathbb{P}(TU)$ which is $\mathcal{G}$-invariant, then it descends as a foliation $\underline{\mathcal{H}}$ on U^* transversal to C_0. Maybe shrinking U^*, we get a fibration by holomorphic discs transversal to C_0 that can be defined by a holomorphic submersion

$$H : U^* \to C_0$$

satisfying $H|_{C_0} = id|_{C_0}$ (a retraction). Indeed, we can define this map on $\mathbb{P}(TU)$ first (construct a first integral for $\mathcal{H}$) and, then descend it to U^*.

Conversely, if we have a holomorphic map $H : U^* \to \mathbb{P}^1$ which is a submersion in restriction to C_0, then fibers of $\tilde{H} := H \circ \pi^* : \mathbb{P}(TU) \to \mathbb{P}^1$ define the leaves of a Riccati foliation $\mathcal{H}$. Indeed, the restriction of $\tilde{H}$ to $\mathbb{P}^1$-fibers must be global diffeomorphisms, and in coordinates, $\tilde{H}$ take the form

$$\tilde{H}(x, y, z) = \frac{\alpha(x, y)z + \beta(x, y)}{\gamma(x, y)z + \delta(x, y)}$$

which, after derivation, give a Riccati distribution:

$$d\tilde{H} = 0 \quad \Leftrightarrow$$

$$(\alpha\delta - \beta\gamma)dz + (\gamma d\alpha - \alpha d\gamma)z^2 + (\gamma d\beta - \beta d\gamma + \delta d\alpha - \alpha d\delta)z + (\delta d\beta - \beta d\delta) = 0.$$

By construction, the Riccati foliation $\mathcal{H}$ is $\mathcal{G}$-invariant.

The following also appear in [19]

Proposition 3.2. *Let Π be a projective structure on $(U, 0)$ and (U^*, C_0) be the dual. The following data are equivalent:*

- *a pencil of geodesic foliations $\{\mathcal{F}_t\}_{t\in\mathbb{P}^1}$,*
- *a $\mathcal{G}$-invariant Riccati foliation $\mathcal{H}$ on $\mathbb{P}(TU)$,*
- *a fibration by discs transversal to C_0 on U^*.*

In this case, we say that the projective structure is *flat*[4].

Example 3.3. Let Π_0 be the trivial structure $y'' = 0$ with Riccati distribution $\omega_0 = dz$. In this case $\omega = \omega_0 + (F + zG)(dy - zdx)$ is integrable if and only if

$$F_x = F^2, \quad G_y = G^2, \quad G_x + F_y = 2FG. \tag{3}$$

On the other hand, since $C_0 \subseteq \check{\mathbb{P}}^2$, it is easy to see that every transverse fibration to C_0 extends to a pencil of lines passing through some point outside C_0 and in this case the foliation defined by ω corresponds to the pencil of foliations given dually in $\mathbb{P}^2$. We fix coordinates $(a, b) \in \check{\mathbb{P}}^2$ for the line $\{ax + by = 1\} \subseteq \mathbb{P}^2$ and observe that $\check{0}$ is the line of the infinity L_∞ in this coordinates. It is straightforward to see that the Riccati foliation associated to the pencil of lines through (a, b) is

$$\omega = dz + \left(\left(\frac{a}{1 - ax - by}\right) + z\left(\frac{b}{1 - ax - by}\right)\right)(dy - zdx).$$

We find in this way $F = \frac{a}{1-ax-by}$, $G = \frac{b}{1-ax-by}$ solutions of 3. Remark that the fibrations induced by ω_0 and ω have a common fiber, which is the fiber associated to the radial foliation with center at $\{ax + by = 1\} \cap L_\infty$.

3.3. **Webs and curvature.** We say that the projective structure Π is compatible with a regular web $\mathcal{W}$ if every leaf of $\mathcal{W}$ is a geodesic of Π. For 4-webs we have the following proposition.

Proposition 3.4 ([29, Proposition 6.1.6])**.** *If $\mathcal{W}$ is a regular 4-web on $(\mathbb{C}, 0)$ then there is a unique projective structure $\Pi_{\mathcal{W}}$ compatible with $\mathcal{W}$.*

Let $\mathcal{W} = \mathcal{F}_1 \boxtimes \mathcal{F}_2 \boxtimes \mathcal{F}_3 \boxtimes \mathcal{F}_4$ be a regular 4-web on $(\mathbb{C}, 0)$

$$\mathcal{F}_i = [X_i = \partial_x + e_i(x, y)\partial_y] = [\eta_i = e_i dx - dy], \quad i = 1, 2, 3, 4.$$

The cross-ratio

$$(\mathcal{F}_1, \mathcal{F}_2; \mathcal{F}_3, \mathcal{F}_4) := \frac{(e_1 - e_3)(e_2 - e_4)}{(e_2 - e_3)(e_1 - e_4)} \tag{4}$$

is a holomorphic function on $(\mathbb{C}, 0)$ intrinsically defined by $\mathcal{W}$. Then, we have:

[4]not to be confused with [20] where flat means locally linearizable

Proposition 3.5. *If $\mathcal{W} = \mathcal{F}_0 \boxtimes \mathcal{F}_1 \boxtimes \mathcal{F}_\infty$ is a regular 3-web on $(\mathbb{C}, 0)$, then there is a unique pencil $\{\mathcal{F}_t\}_{t\in\mathbb{P}^1}$ that contains $\mathcal{F}_0$, $\mathcal{F}_1$ and $\mathcal{F}_\infty$ as elements. Precisely, $\mathcal{F}_t$ is defined as the unique foliation such that*

$$(\mathcal{F}_0, \mathcal{F}_1; \mathcal{F}_t, \mathcal{F}_\infty) \equiv t.$$

We denote by $\Pi_{\mathcal{W}}$ the corresponding projective structure on $(\mathbb{C}^2, 0)$.

Conversely, any flat projective structure comes from a 3-web: it suffices to choose 3 elements of a pencil. In particular, any 4 elements of a pencil $\{\mathcal{F}_t\}_{t\in\mathbb{P}^1}$ have constant cross-ratio.

We can define the curvature of a flat projective structure as follows. First of all, to a regular 3-web $\mathcal{W} = \mathcal{F}_0 \boxtimes \mathcal{F}_1 \boxtimes \mathcal{F}_\infty$, we can define the curvature $K_{\mathcal{W}}$ which is a 2-form. For instance, if $\mathcal{W}$ is in the normal form $\mathcal{W} = dx \boxtimes (dx + a(x, y)dy) \boxtimes dy$, with $a(0, 0) \neq 0$, then it is easy to see that the curvature is

$$K(\mathcal{W}) = \left[\frac{\partial}{\partial x}\frac{\partial}{\partial y}\log(a(x, y))\right] dx \wedge dy.$$

In particular, the curvature of the web is the same if we change the foliation $\mathcal{F}_1$ by any other member $\mathcal{F}_t : [dx + ta(x, y)dy = 0]$ of the pencil generated by $\mathcal{W}$. Consequently, the 2-form $K(\mathcal{W})$ does not depend of the 3-web inside the pencil.

On the other hand, let $\omega = dz + zd\log(a)$ be the Riccati 1-form given by the same pencil. The *Chern connection* associated to $\mathcal{W}$ is the unique torsion-free affine connection ∇_c associated to ω (see [28], [19]). A simple calculation shows that the connection matrix is

$$\Omega_c = \begin{pmatrix} -\frac{\partial}{\partial x}\log(a)dx & 0 \\ 0 & \frac{\partial}{\partial y}\log(a)dy \end{pmatrix}$$

and the curvature matrix of Ω_c is

$$d\Omega_c + \Omega_c \wedge \Omega_c = \begin{pmatrix} K(\mathcal{W}) & 0 \\ 0 & K(\mathcal{W}) \end{pmatrix}.$$

This implies again that the curvature on any 3-web in the pencil is always the same. In particular, the Chern connection associated to $\mathcal{W}$ has zero curvature if and only if $\Pi_{\mathcal{W}}$ is linearizable.

Example 3.6. In the case of the linear projective structure Π_0 by lines, all pencils have zero curvature $K_{\mathcal{W}} \equiv 0$ (i.e. hexagonal, see [3, section 6] or [29, Chap. 1, Sect. 2]) and can be defined by pencils of closed 1-forms. We can easily construct non linearizable projective structure by violating these properties. For instance, the projective structure generated by the pencil of 1-forms $\omega_t := dx + te^{xy}dy$ cannot be defined by a pencil of closed 1-forms; we have $K_{\mathcal{W}} \equiv 1$ in this case.

3.4. **About unicity of flat structure.** If the Riccati distribution $\omega = dz + \gamma + z(\delta - \alpha) - z^2\beta$ is integrable, then recall that any other Riccati distribution defining the same projective structure writes $\omega' = \omega + (F + zG)(dy - zdx)$. Then $\mathcal{H}' : [\omega' = 0]$ is Frobenius integrable if and only if

$$\begin{aligned}
\left(FG - \frac{F_y + G_x}{2}\right) dx \wedge dy + F(\beta \wedge dy) + G(dx \wedge \gamma) &= 0 \\
(G^2 - G_y)dx \wedge dy + 2G(\alpha \wedge dx) + (Fdx - Gdy) \wedge \beta &= 0 \\
(F_x - F^2)dx \wedge dy + 2F(dy \wedge \alpha) + \gamma \wedge (Fdx - Gdy) &= 0,
\end{aligned}$$

However, it seems impossible to see from these equations how many flat Riccati foliations are compatible with a given flat projective structure. We will give a complete answer by considering this question on the dual surface U^*.

4. Classification of neighborhoods of rational curves

Let $\mathbb{P}^1 \hookrightarrow S$ be an embedding of $\mathbb{P}^1$ into a smooth complex surface and let C be its image. The self-intersection of C is also the degree of the normal bundle of the curve $C \cdot C = \deg(N_C)$. When $C \cdot C < 0$, it follows from famous work of Grauert [16] that the germ of neighborhood (S, C) is linearizable, i.e. biholomorphically equivalent to $(N_C, 0)$ where 0 denotes the zero section. Such neighborhood is called rigid since there is no non trivial deformation. When $C \cdot C = k \geq 0$, it follows from Kodaira [21] that the deformation space of the curve C in its neighborhood is smooth of dimension $k+1$. In particular, for $C \cdot C = 0$, the neighborhood is a fibration by rational curves, which is thus trivial by Fisher-Grauert [15]: the neighborhood is again linearizable (see also [30]), thus rigid. However, in the positive case $C \cdot C > 0$, it is also well-known that we have huge moduli. The analytic classification (which is less known) is due to Mishustin[5] [27] and in this section, we recall the case $C \cdot C = 1$.

Let us first decompose $C = V_0 \cup V_\infty$ where $x_i : V_i \xrightarrow{\sim} \mathbb{C}$ are affine charts, $i = 0, \infty$, with $x_0 x_\infty = 1$ on $V_0 \cap V_\infty$. Then any germ of neighborhood (S, C) can be decomposed as the union $U_0 \cup U_\infty$ of two trivial neighborhoods $U_i \simeq V_i \times \mathbb{D}_\epsilon$ with coordinates (x_i, y_i) patched together by a holomorphic map

$$(x_\infty, y_\infty) = \Phi\left(\frac{1}{x_0} + \sum_{n\geq 1} a_n(x_0) y_0^n \ , \ \sum_{n \geq 1} b_n(x_0) y_0^n\right)$$

where a_n, b_n are holomorphic on $V_0 \cap V_\infty \simeq \mathbb{C}^*$. Moreover, b_1 does not vanish on $V_0 \cap V_\infty$ and, viewed as a cocycle $\{b_1\} \in H^1(\mathbb{P}^1, \mathcal{O}_C^*)$, defines the normal bundle N_C. Denote U_Φ the germ of neighborhood defined by such a gluing map. The gluing map Φ can also be viewed as a non linear cocycle encoding the biholomorphic class of the neighborhood, as illustrated by the following straightforward statement.

Proposition 4.1. *Given another map Φ', then the following data are equivalent:*

- *a germ of biholomorphism $\Psi : U_\Phi \xrightarrow{\sim} U_{\Phi'}$ inducing the identity on C,*
- *a pair of biholomorphism germs*

$$\Psi^i(x_i, y_i) = \left(x_i + \sum_{n\geq 1} a_n^i(x_i) y_i^n \ , \ \sum_{n\geq 1} b_n^i(x_i) y_i^n\right), \quad (i = 0, \infty)$$

(with b_1^0, b_1^∞ not vanishing) satisfying $\Phi' \circ \Psi^0 = \Psi^\infty \circ \Phi$:

$$\begin{array}{ccc} U_0 & \xrightarrow{\Phi} & U_\infty \\ {\scriptstyle \Psi^0}\downarrow & & \downarrow{\scriptstyle \Psi^\infty} \\ U_0 & \xrightarrow{\Phi'} & U_\infty \end{array}$$

We will say that the two "cocycles" Φ and Φ' are equivalent in this case.

Since $H^1(\mathbb{P}^1, \mathcal{O}_C^*) = \mathbb{Z}$ there exist $b^i \in \mathcal{O}^*(V_i)$, $i = 0, \infty$, such that $b^\infty b_1 = x_0^k b^0$. Thus, the pair $\Psi^i(x_i, y_i) = (x_i, b^i y_i)$, $i = 0, \infty$, provides us with an equivalent

[5] we thank S. Ivachkovitch for the reference

cocycle such that $b_1(x_0) = x_0^k$. Now, this exactly means that $C \cdot C = -k$. As conclusion, $(+1)$-neighborhoods can be defined by a cocycle of the form

$$\Phi(x_0, y_0) = \left(\frac{1}{x_0} + \sum_{n\geq 1} a_n(x_0) y_0^n \ , \ \frac{y_0}{x_0} + \sum_{n\geq 2} b_n(x_0) y_0^n \right) =: \left(\frac{1}{x_0} + a \ , \ \frac{y_0}{x_0} + b \right).$$

4.1. **Normal form.** Using the equivalence defined in Proposition 4.1 above, we can reduce the cocycle Φ into an almost unique normal form:

Theorem 4.2 (Mishustin)**.** *Any germ (S, C) of $(+1)$-neighborhood is biholomorphically equivalent to a germ U_Φ for a cocycle Φ of the following "normal form"*

$$\Phi = \left(\frac{1}{x} + \sum_{n\geq 4} \left(\sum_{m=3}^{n-1} \frac{a_{m,n}}{x^m} \right) y^n, \frac{y}{x} + \sum_{n\geq 3} \left(\sum_{m=2}^{n-1} \frac{b_{m,n}}{x^m} \right) y^n \right). \tag{5}$$

Moreover, when the neighborhood (S, C) admits a fibration transverse to C, then one can choose all $a_{m,n} = 0$ so that the fibration is given by $x_0 = \frac{1}{x_\infty} : S \to C$.

As we shall see in the next section, this normal form is unique up to a 4-dimensional group action.

In order to give the proof of the theorem, let us introduce the following notation. For a subset $E \subset \mathbb{Z}^2$, denote by $\sum_E$ the sum over indices belonging to E. For instance, setting

$$V(k, l) := \{(m + k, n + l) \in \mathbb{Z}^2 \ ; \ -n \leq m \leq 0\}, \quad (k, l) \in \mathbb{Z}^2$$

then normal form of Theorem 4.2 writes

$$\Phi = \left(\frac{1}{x} + \sum_{V(-3,4)} a_{m,n} x^m y^n \ , \ \frac{y}{x} + \sum_{V(-2,3)} b_{m,n} x^m y^n \right).$$

In view of this normal form, a huge step can be done by a simple geometrical argument using blow-up and rigidity in the case of zero self-intersection.

Lemma 4.3 (Prenormal Form)**.** *Any germ (S, C) of $(+1)$-neighborhood is biholomorphically equivalent to a germ U_Φ for a cocycle $\Phi = (a(x, y), b(x, y))$ of the following "prenormal form"*

$$\Phi = \left(\sum_{V(-1,0)} a_{m,n} x^m y^n \ , \ \sum_{V(-1,1)} b_{m,n} x^m y^n \right)$$

(with $a_{-1,0} = 1$). Moreover, an equivalent cocycle $\Phi' = \Psi^\infty \circ \Phi \circ \Psi^0$ is also in prenormal form (with possibly different coefficients) if, and only if

$$\Psi^i(x, y) = (\alpha^i(y)x + \beta^i(y), \varphi^i(y)), \quad i = 0, \infty$$

with $\alpha^i, \beta^i, \varphi^i \in \mathbb{C}\{y\}$, $\alpha^i(0) = 1$, $\beta^i(0) = 0$, $\varphi^{i\prime}(0) \neq 0$. When the neighborhood (S, C) admits a fibration transverse to C, then one can choose $a \equiv 0$ in the prenormal form so that the fibration is given by $x_0 = \frac{1}{x_\infty} : S \to C$.

Proof. Since a coordinate $x_0 = \frac{1}{x_\infty}$ has been fixed on C, we can consider the following two points $p_i := \{x_i = \infty\}$, $i = 0, \infty$. Consider for each $i = 0, \infty$ the blow-up $\pi^i : S^i \to S$ of the surface at the point p_i, and denote by D_i the exceptional

divisor. The strict transform $C^i := (\pi^i)^*C$ of the rational curve has now zero self-intersection. Following [21] and [15] (see also [30]), one can find a neighborhood U^i of C^i in S^i which is trivial: there are coordinates $(x_i, y_i) : U^i \to \mathbb{P}^1 \times \mathbb{C}$ such that $C^i = \{y_i = 0\}$ and $D_i = \{x_i = \infty\}$ extending the coordinate x_i initially defined on $V_i \subset C$. This system of coordinates is clearly unique up to the freedom settled in the statement. By abuse of notation, we still denote U^i (the open part of) its image by π^i in U.

We now have to check that the cocycle Φ given by these systems of coordinates satisfy precisely the conditionof the statement. First of all, note that after blowing-up p_i, both coordinates x_0 and x_∞ are well defined at the intersection point $C^i \cap D_i$, and have opposite divisor. Therefore, the function

$$x_0 \cdot x_\infty = 1 + \sum_{m,n} a_{m,n} x_0^{m+1} y_0^n = 1 + \sum_{m,n} a_{m,n} x_0^{m+n+1} \left(\frac{y_0}{x_0}\right)^n$$

must be a holomorphic (and non vanishing) function of

- (x_0^{-1}, y_0) at $C^0 \cap D_0$ implying $m + 1 \leq 0$ in the support of a-component,
- $(x_0, \frac{y_0}{x_0})$ at $C^\infty \cap D_\infty$ implying $m + n + 1 \geq 0$ in the same support.

On the other hand

$$y_\infty = \sum_{m,n} b_{m,n} x_0^{m+n} \left(\frac{y_0}{x_0}\right)^n$$

must be a holomorphic function of $(x_0, \frac{y_0}{x_0})$ at $C^\infty \cap D_\infty$ implying $m+n \geq 0$ in the support of the b-component. Also,

$$x_0 y_\infty = (x_0 x_\infty) \cdot \left(\frac{y_\infty}{x_\infty}\right) = \sum_{m,n} b_{m,n} x_0^{m+1} y_0^n$$

must be a holomorphic function of (x_0^{-1}, y_0) at $C^0 \cap D_0$ implying $m+1 \leq 0$ for the same support.

If the neighborhood admits a transverse fibration, then we can preliminarily extend each coordinate $x_i : V_i \to \mathbb{P}^1$ as a submersion $\tilde{x}_i : U \to \mathbb{C}$ defining the fibration on the neighborhood U of C. After blowing-up p_i, the exceptional divisor D_i is clearly defined by $\tilde{x}_i = \infty$ so that we can write $\tilde{x}_i = \alpha^i(y_i)x_i + \beta^i(y_i)$. Finally note that, by construction, $\tilde{x}_\infty = 1/\tilde{x}_0$ so that these new coordinates provide a cocycle $\tilde{\Phi}$ satisfying $a \equiv 0$. □

Proof of Theorem 4.2. We now use the freedom in pre-normal forms of Proposition 4.3 to sharpen the support of coefficients. First decompose

$$(x_\infty, y_\infty) = \Phi(x_0, y_0) =$$

$$\left(\frac{f(y_0)}{x_0} + \frac{g(y_0)}{x_0^2} + \sum_{V(-3,2)} a_{m,n} x_0^m y_0^n \ , \ \frac{h(y_0)}{x_0} + \sum_{V(-2,2)} b_{m,n} x_0^m y_0^n\right)$$

with $f, g, h \in \mathbb{C}\{y\}$, $f(0) = 1$, $h(y)$ invertible. We want first to normalize coefficients f, g, h by conveniently changing coordinates (x_0, y_0). More precisely, setting

$$(x_0, y_0) = (\alpha(\tilde{y}_0)\tilde{x}_0 + \beta(\tilde{y}_0), \varphi(\tilde{y}_0)),$$

then the first component x_∞ of Φ is given by

$$\frac{f\circ\varphi(\tilde y_0)}{\alpha(\tilde y_0)\tilde x_0+\beta(\tilde y_0)}+\frac{g\circ\varphi(\tilde y_0)}{(\alpha(\tilde y_0)\tilde x_0+\beta(\tilde y_0))^2}+\sum_{V(-3,2)}a_{m,n}(\alpha(\tilde y_0)\tilde x_0+\beta(\tilde y_0))^m\varphi(\tilde y_0)^n.$$

We note that, if $(m,n)\in V(p,q)$, then the support of

$$x_0^m y_0^n=(\alpha(\tilde y_0)\tilde x_0+\beta(\tilde y_0))^m\varphi(\tilde y_0)^n=\tilde x_0^m\alpha(\tilde y_0)^m\varphi(\tilde y_0)^n(1+\frac{\beta(\tilde y_0)}{\alpha(\tilde y_0)\tilde x_0})^m$$

is still contained in $V(p,q)$, as a power series in $(\tilde x_0,\tilde y_0)$. Therefore, we can rewrite

$$x_\infty=\frac{f\circ\varphi(\tilde y_0)}{\alpha(\tilde y_0)\tilde x_0}+\frac{g\circ\varphi(\tilde y_0)-\beta(\tilde y_0)f\circ\varphi(\tilde y_0)}{\alpha(\tilde y_0)^2\tilde x_0^2}+\sum_{V(-3,2)}\tilde a_{m,n}\tilde x_0^m\tilde y_0^n$$

(with new coefficients $\tilde a_{m,n}$). In a similar way, we have

$$y_\infty=\frac{h\circ\varphi(\tilde y_0)}{\alpha(\tilde y_0)\tilde x_0}+\sum_{V(-2,2)}\tilde b_{m,n}\tilde x_0^m\tilde y_0^n$$

Therefore, we want

$$\frac{f\circ\varphi(\tilde y_0)}{\alpha(\tilde y_0)}=1,\quad g\circ\varphi(\tilde y_0)-\beta(\tilde y_0)f\circ\varphi(\tilde y_0)=0\quad\text{and}\quad\frac{h\circ\varphi(\tilde y_0)}{\alpha(\tilde y_0)}=\tilde y_0$$

which rewrites

$$\alpha=f\circ h^{-1},\quad\beta=\frac{g\circ h^{-1}}{f\circ h^{-1}}\quad\text{and}\quad\varphi=h^{-1}.$$

Reversing now $(\tilde x_0,\tilde y_0)=\Psi^0(x_0,y_0)$, we get

$$\Psi^0(x_0,y_0)=\left(\frac{x_0-\beta\circ\varphi^{-1}(y_0)}{\alpha\circ\varphi^{-1}(y_0)},\varphi^{-1}(y_0)\right)=\left(\frac{x_0}{f(y_0)}-\frac{g(y_0)}{f(y_0)^2},\frac{h(y_0)}{f(y_0)}\right)\tag{6}$$

we get $\Phi=\tilde\Phi\circ\Psi^0$ with $\tilde\Phi$ half-normalized

$$\tilde\Phi(\tilde x_0,\tilde y_0)=\left(\frac{1}{\tilde x_0}+\sum_{V(-3,2)}\tilde a_{m,n}\tilde x_0^m\tilde y_0^n\ ,\ \frac{\tilde y_0}{\tilde x_0}+\sum_{V(-2,2)}\tilde b_{m,n}\tilde x_0^m\tilde y_0^n\right).$$

In a similar way, write

$$(x_\infty,y_\infty)=\tilde\Phi(\tilde x_0,\tilde y_0)=$$

$$\left(\frac{f}{\tilde x_0}+g+\sum_{V(-3,4)}\tilde a_{m,n}\tilde x_0^m\tilde y_0^n\ ,\ h+\underbrace{\tilde x_0k+\sum_{V(-2,4)}\tilde b_{m,n}\tilde x_0^m\tilde y_0^n}_{V(-2,3)}\right)$$

with $f,g,h,k\in\mathbb{C}\{\frac{\tilde y_0}{\tilde x_0}\}$. Then, setting $(\tilde x_\infty,\tilde y_\infty)=(\alpha(y_\infty)x_\infty+\beta(y_\infty),\varphi(y_\infty))$, and using Taylor expansion

$$\alpha\left(h+\tilde x_0k\right)=\alpha\left(h\right)+\tilde x_0k\cdot\alpha'(h)+\underbrace{(\tilde x_0k)^2\cdot\frac{\alpha''(h)}{2}+\cdots}_{V(-2,4)}$$

$$\text{we get:}\quad\tilde x_\infty=\alpha\left(h+\tilde x_0k\right)\left(\frac{f}{\tilde x_0}+g\right)+\beta(h)+\sum_{V(-3,4)}*\ \tilde x_0^m\tilde y_0^n$$

$$= \frac{f\cdot\alpha\circ h}{\tilde{x}_0} + (g\cdot\alpha\circ h + f\cdot k\cdot\alpha'\circ h + \beta\circ h) + \sum_{V(-3,4)} * \tilde{x}_0^m\tilde{y}_0^n$$

$$\text{and}\quad \tilde{y}_\infty = \varphi\circ h + \sum_{V(-2,3)} * \tilde{x}_0^m\tilde{y}_0^n.$$

Finally, we want

$$f\cdot\alpha\circ h = 1,\quad g\cdot\alpha\circ h + f\cdot k\cdot\alpha'\circ h + \beta\circ h = 0\quad\text{and}\quad \varphi\circ h = \mathrm{id}.$$

Deriving the first equality gives

$$f'\cdot\alpha\circ h + f\cdot\alpha'\circ h\cdot h' = 0\quad\text{i.e. } \alpha'\circ h = -\frac{f'\cdot\alpha\circ h}{f\cdot h'} = -\frac{f'}{f^2\cdot h'}$$

so that we can fix α,β,φ, and after inversion we get

$$(x_\infty, y_\infty) = \Psi^\infty(\tilde{x}_\infty,\tilde{y}_\infty) = \left(f(\tilde{y}_\infty)\tilde{x}_\infty + g(\tilde{y}_\infty) - \frac{f'(\tilde{y}_\infty)k(\tilde{y}_\infty)}{h'(\tilde{y}_\infty)}, h(\tilde{y}_\infty)\right). \tag{7}$$

By construction, $\Phi = \Psi^\infty\circ\Phi'\circ\Psi^0$ where $(\tilde{x}_\infty,\tilde{y}_\infty) = \Phi'(\tilde{x}_0,\tilde{y}_0)$ is a cocycle in normal form (5). □

Remark 4.4. We have the following geometric interpretation of the normal form (5). We go back to the geometric construction of prenormal forms in the proof of Lemma 4.3. For $i = 0,\infty$, the coordinate y_i in normal form is such that after blowing-up the point p_i, it is linear in restriction to the exceptional divisor D_i. On the other hand, near the divisor D_i, the two fibrations given by x_0 and x_∞ are well defined and have D_i as a common fiber. The coordinates x_0 and x_∞ in normal form are such that the two fibrations have contact of order 3 along D_i, for $i = 0,\infty$. For instance, blowing-up the point $x_0 = 0$ in prenormal form gives

$$x_0\cdot x_\infty = f(t) + x_0 g(t) + x_0^2\sum_{V(-1,2)} a_{m,n}x_0^{m+n-1}t^n$$

where $t = \frac{y_0}{x_0}$, $f(0) = 1$ and $g(0) = 0$. The contact between the two fibrations is defined by the vanishing divisor of

$$dx_0\wedge d\left(\frac{1}{x_\infty}\right);$$

since $x_0x_\infty \neq 0$, it is equivalently defined by

$$-(x_0x_\infty)^2 dx_0\wedge d\left(\frac{1}{x_\infty}\right) = x_0^2 dx_0\wedge dx_\infty$$

$$= x_0 dx_0\wedge d(x_0x_\infty) = \left(x_0f'(t) + x_0^2g'(t) + x_0^3\sum_{V(-1,2)} a_{m,n}x_0^{m+n-1}t^n\right)dx_0\wedge dx_\infty.$$

The multiplicity of $\{x_0 = 0\}$ is 3 precisely when $f(t) = 1$ and $g(t) = 0$ like in the normal form (5).

4.2. **Isotropy group for normal forms.** During the proof of Theorem 4.2, we have had the possibility to normalize coefficients

$$a_{-1,1} = a_{-2,1} = a_{-2,2} = 0 \quad \text{and} \quad b_{-1,1} = 1$$

by either using Ψ^0 or Ψ^∞. This underline a 4-parameter freedom in the choice of normalizing coordinates systems (x_0, y_0) and (x_∞, y_∞). For instance, if $\Phi_0 = (\frac{1}{x}, \frac{y}{x})$ is the linear neighborhood, then we know that it admits the following family of automorphisms:

$$\left(\frac{x_\infty + \alpha y_\infty}{1+\beta y_\infty}, \frac{\theta y_\infty}{1+\beta y_\infty}\right) \circ \Phi_0 = \Phi_0 \circ \left(\frac{x_0 + \beta y_0}{1+\alpha y_0}, \frac{\theta y_0}{1+\alpha y_0}\right).$$

We will see that this group acts on the set of normal forms (5), having Φ_0 as fixed point. On the other hand, we can easily check that if Φ is in normal form, then

$$(x_\infty + \gamma y_\infty^2, y_\infty) \circ \Phi = \Phi' \circ (x_0 - \gamma y_0^2, y_0)$$

gives another a new normal form. The 4-parameter of freedom is a combination of those two actions.

Proposition 4.5. *Consider a cocycle in normal form*

$$\Phi = \left(\frac{1}{x} + \sum_{V(-3,4)} a_{m,n} x^m y^n \ , \ \frac{y}{x} + \sum_{V(-2,3)} b_{m,n} x^m y^n\right).$$

Then an equivalent cocycle $\Psi^\infty \circ \Phi = \Phi' \circ \Psi^0$ is also in normal form if, and only if, there are constants $\alpha, \beta, \gamma \in \mathbb{C}$ and $\theta \in \mathbb{C}^*$ such that

$$\begin{array}{rcl} \Psi^\infty & = & \left(\frac{x_\infty+\alpha y_\infty}{1+\beta y_\infty} + \gamma\left(\frac{y_\infty}{1+\beta y_\infty}\right)^2 + \frac{\beta k^\infty(y_\infty)}{(1+\beta y_\infty)^2} \ , \ \frac{\theta y_\infty}{1+\beta y_\infty}\right) \\ \Psi^0 & = & \left(\frac{x_0+\beta y_0}{1+\alpha y_0} - \gamma\left(\frac{y_0}{1+\alpha y_0}\right)^2 - \frac{\alpha k^0(y_0)}{(1+\alpha y_0)^2} \ , \ \frac{\theta y_0}{1+\alpha y_0}\right) \end{array} \tag{8}$$

where $k^0(y_0) = \sum_{n\geq 3} b_{-2,n} y_0^n$ and $k^\infty(y_\infty) = \sum_{n\geq 3} b_{-n+1,n} y_\infty^n$.

For instance, starting with the linear neighborhood $\Phi_0 = (\frac{1}{x}, \frac{y}{x})$, then we obtain the following equivalent cocycles in normal form (with $c = \frac{\gamma}{\theta^2} \in \mathbb{C}$ arbitrary)

$$\Phi = \left(\frac{1}{x}\frac{(1+2c\frac{y^2}{x})}{(1+c\frac{y^2}{x})^2} \ , \ \frac{y}{x}\frac{1}{(1+c\frac{y^2}{x})}\right) \quad \sim \quad \Phi_0.$$

We promptly deduce from Proposition 4.5 that any change of normalization

$$\Phi' = \Psi^\infty \circ \Phi \circ (\Psi^0)^{-1}$$

of a given cocycle in normal form Φ is determined by the quadratic part of Ψ^0:

$$\Psi^0 = (x + (\beta - \alpha x)y + (\alpha^2 x - (\alpha\beta + \gamma))y^2 + \cdots \ , \ \theta y - \theta\alpha y^2 + \cdots). \tag{9}$$

Conversely, for any $\vartheta = (\alpha, \beta, \gamma, \theta) \in \mathbb{C}^3 \times \mathbb{C}^*$, the above quadratic part can be extended as a new normalization $(\Psi^0_{\vartheta,\Phi}, \Psi^\infty_{\vartheta,\Phi})$ for each cocycle Φ in normal form. We thus get an action of $\mathbb{C}^3 \times \mathbb{C}^*$ on the set of normal forms

$$(\vartheta, \Phi) \ \mapsto \ \vartheta \cdot \Phi := \Psi^\infty_{\vartheta,\Phi} \circ \Phi \circ (\Psi^0_{\vartheta,\Phi})^{-1}$$

with the group law given by

$$\begin{aligned}\vartheta_1\cdot(\vartheta_2\cdot\Phi) &= \Psi^\infty_{\vartheta_1,\Phi'}\circ\left(\underbrace{\Psi^\infty_{\vartheta_2,\Phi}\circ\Phi\circ(\Psi^0_{\vartheta_2,\Phi})^{-1}}_{\Phi'}\right)\circ(\Psi^0_{\vartheta_1,\Phi'})^{-1}\\ &= \left(\Psi^\infty_{\vartheta_1,\Phi'}\circ\Psi^\infty_{\vartheta_2,\Phi}\right)\circ\Phi\circ\left(\Psi^0_{\vartheta_1,\Phi'}\circ\Psi^0_{\vartheta_2,\Phi}\right)^{-1}\\ &= \Psi^\infty_{\vartheta_3,\Phi}\circ\Phi\circ(\Psi^0_{\vartheta_3,\Phi})^{-1} \qquad = \quad \vartheta_3\cdot\Phi\end{aligned}$$

This group law can be easily computed by composing the quadratic parts (9) of $\Psi^0_{\vartheta_1}$ and $\Psi^0_{\vartheta_2}$, and we get

$$\vartheta_3=(\alpha_2+\theta_2\alpha_1,\beta_2+\theta_2\beta_1,\gamma_2+\theta_2^2\gamma_1,\theta_1\theta_2).$$

In other words, the group law on parameters $\vartheta=(\alpha,\beta,\gamma,\theta)$ is equivalent to the matrix group law

$$\Gamma:=\left\{\begin{pmatrix}1&\alpha&\beta&\gamma\\0&\theta&0&0\\0&0&\theta&0\\0&0&0&\theta^2\end{pmatrix}\ ,\ (\alpha,\beta,\gamma,\theta)\in\mathbb{C}^3\times\mathbb{C}^*\right\}\quad\subset\ \mathrm{GL}_4(\mathbb{C}).$$

We deduce:

Corollary 4.6. *The* 4*-dimensional matrix group* Γ *acts on the set of normal forms (5) as defined in Proposition 4.5 and the set of equivalence classes is in one-to-one correspondance with the set of isomorphisms classes of germs of* $(+1)$*-neighborhoods* (S,C) *of the rational curve* $C\simeq\mathbb{P}^1_x$ *(with fixed coordinate* x*).*

Let us describe this action on the first coefficients of the cocycle $\Phi'=\vartheta\cdot\Phi$:

$$\begin{cases}a'_{-3,4}=&\frac{a_{-3,4}-\gamma^2+2\gamma b_{-2,3}-\beta b_{-2,4}+\alpha b_{-3,4}}{\theta^4}\\ a'_{-3,5}=&\frac{a_{-3,5}+\alpha a_{-3,4}+(2\gamma-\alpha\beta)b_{-2,4}+\alpha^2 b_{-3,4}-\beta b_{-2,5}+\alpha b_{-3,5}}{\theta^5}\\ a'_{-4,5}=&\frac{a_{-4,5}+2\beta a_{-3,4}+3\beta b^2_{-2,3}-\beta^2 b_{-2,4}+(\alpha\beta+2\gamma)b_{-3,4}-\beta b_{-3,5}+\alpha b_{-4,5}}{\theta^5}\\ &\dots\end{cases}\tag{10}$$

$$\begin{cases}b'_{-2,3}=\frac{b_{-2,3}-\gamma}{\theta^2}\\ b'_{-2,4}=\frac{b_{-2,4}}{\theta^3} & b'_{-3,4}=\frac{b_{-3,4}}{\theta^3}\\ b'_{-2,5}=\frac{b_{-2,5}+\alpha b_{-2,4}}{\theta^4} & b'_{-3,5}=\frac{b_{-3,5}+\alpha b_{-3,4}-2\gamma b_{-2,3}+\gamma^2}{\theta^4} & b'_{-4,5}=\frac{b_{-4,5}+\beta b_{-3,4}}{\theta^4}\\ \quad\dots\end{cases}$$

Proof of Proposition 4.5. The existence of a 4-parameter group acting on normal forms is clear from the proof of Theorem 4.2: we have 4 degrees of freedom in the construction of (Ψ^0,Ψ^∞) as mentionned at the beginning of the section. So it just remains to check that formula (8) in the statement is indeed a normalizing pair. It is enough (and easier) to show it for elements of the decomposition

$$\vartheta=(\alpha,\beta,\gamma,\theta)=(0,0,0,\theta)\cdot(\alpha,0,0,1)\cdot(0,\beta,0,1)\cdot(0,0,\gamma,1).$$

We easily check that the following pairs preserve normal forms:

$$\begin{array}{ccc}\Psi^0=(x_0,\theta y_0) & \text{and} & \Psi^\infty=(x_\infty,\theta y_\infty)\\ \Psi^0=(x_0-\gamma y_0^2,y_0) & \text{and} & \Psi^\infty=(x_\infty+\gamma y_\infty^2,y_\infty)\end{array}$$

Now let us set $\Psi^\infty := (x_\infty + \alpha y_\infty, y_\infty)$ and compute $\tilde{\Phi} := \Psi^\infty \circ \Phi$:

$$\tilde{\Phi} = \Psi^\infty \circ \left(\frac{1}{x} + \sum_{V(-3,4)} a_{m,n} x^m y^n \ , \ \frac{y}{x} + \underbrace{\frac{k^0(y)}{x^2} + \sum_{V(-3,4)} b_{m,n} x^m y^n}_{\sum_{V(-2,3)} b_{m,n} x^m y^n} \right)$$

$$= \left(\frac{1+\alpha y}{x} + \frac{\alpha k^0(y)}{x^2} + \sum_{V(-3,4)} (a_{m,n} + \alpha b_{m,n}) x^m y^n \ , \ \frac{y}{x} + \sum_{V(-2,3)} b_{m,n} x^m y^n \right).$$

Looking at the proof of Theorem 4.2, formula (6) gives $\tilde{\Phi} = \Phi' \circ \Psi^0$ with

$$\Psi^0 = \left(\frac{x_0}{1+\alpha y_0} - \frac{\alpha k^0(y_0)}{(1+\alpha y_0)^2} \ , \ \frac{y_0}{1+\alpha y_0} \right)$$

and Φ' $(= \Psi^\infty \circ \Phi \circ (\Psi^0)^{-1})$ is in normal form.

In a similar way, if we set $\Psi^0 = (x_0 + \beta y_0, y_0)$ and compute $\tilde{\Phi} := \Phi \circ (\Psi^0)^{-1}$:

$$\tilde{\Phi} = \left(\frac{1}{x} + \sum_{V(-3,4)} a_{m,n} x^m y^n \ , \ \frac{y}{x} + \underbrace{x k^\infty(\frac{y}{x}) + \sum_{V(-2,4)} b_{m,n} x^m y^n}_{\sum_{V(-2,3)} b_{m,n} x^m y^n} \right) \circ (\Psi^0)^{-1}$$

$$= \left(\frac{1}{x-\beta y} + \sum_{V(-3,4)} \tilde{a}_{m,n} x^m y^n \ , \ \frac{y}{x-\beta y} + (x-\beta y) k^\infty(\frac{y}{x-\beta y}) + \sum_{V(-2,4)} \tilde{b}_{m,n} x^m y^n \right)$$

$$= \left(\frac{f(\frac{y}{x})}{x} + \sum_{V(-3,4)} \tilde{a}_{m,n} x^m y^n \ , \ h(\frac{y}{x}) + x\tilde{k}(\frac{y}{x}) + \sum_{V(-2,4)} \tilde{b}_{m,n} x^m y^n \right)$$

where $f(y_\infty) = \dfrac{1}{1-\beta y_\infty}$, $h(y_\infty) = \dfrac{y_\infty}{1-\beta y_\infty}$ and $\tilde{k}(y_\infty) = (1+\beta y_\infty) k^\infty(y_\infty)$.

Formula (7) in the proof of Theorem 4.2 gives a normal form $\Phi' = \Psi^\infty \circ \tilde{\Phi}$ with

$$\Psi^\infty = \left(\frac{x_\infty}{1+\beta y_\infty} + \frac{\beta k^\infty(y_\infty)}{(1+\beta y_\infty)^2} \ , \ \frac{y_\infty}{1+\beta y_\infty} \right).$$

□

Remark 4.7. In this classification, we have fixed a coordinate $x : C \to \mathbb{P}^1$. One could consider the action of Moebius transformations on C and therefore on x. For instance, the action of homotheties $x \mapsto \lambda x$ on normal forms is easy:

$$a'_{m,n} = \lambda^{m-1} a_{m,n} \quad \text{and} \quad b'_{m,n} = \lambda^{m-1} b_{m,n}.$$

If we add this action to the 4-parameter group Γ, then orbits correspond to the analytic class of $(S, C \supset \{p_0, p_\infty\})$ where we have fixed two points $p_i = \{x = i\}$ without fixing the coordinate on C. If we blow-up p_0 and p_∞, and then contract the strict transform of C, then we get a germ $(\tilde{S}, C_0 \cup C_\infty)$ of neighborhood of the union of two rational curves C_0 and C_∞ (exceptional divisors) with self-intersection

$C_i \cdot C_i = 0$, that intersect transversally at a single point $p = C_0 \cap C_\infty$ (the contraction of C). In fact, we can reverse this construction and have a one-to-one corespondance

$$(S, C \supset \{p_0, p_\infty\}) \ \leftrightarrow \ (\tilde{S}, C_0 \cup C_\infty)$$

so that analytic classifications are the same. We note that the action of other Moebius transformations on normal forms Φ are much more difficult to compute.

4.3. **Existence of transverse fibration.** We go back to the notion of transversal fibration by discs on (S, C) considered in 3.2. If we have such a fibration, it can be defined by a submersion $H : S \to C$ inducing the identity on C; equivalently, after composition with $x : C \to \mathbb{P}^1$ we get an extension of the coordinate x on the neighborhood S. In this case, recall (see Theorem 4.2) that one can choose a normal form with zero a-part:

$$\Phi(x, y) = \left(\frac{1}{x}, \frac{y}{x} + \sum_{V(-2,3)} b_{m,n} x^m y^n \right)$$

compatible with the fibration in the sense that $x \circ H = x_0 = \frac{1}{x_\infty}$.

Proposition 4.8. *A $(+1)$-neighborhood with normal form Φ admits a transversal fibration if, and only if, there is a $\vartheta = (\alpha, \beta, \gamma, \theta) \in \Gamma$ such that the a-part of $\vartheta \cdot \Phi$ is trivial. Moreover, the set of transversal fibrations is in one-to-one correspondance with the set of those $(\alpha, \beta, \gamma) \in \mathbb{C}^3$ for which the a-part of $(\alpha, \beta, \gamma, 1).\Phi$ is zero.*

Proof. Just observe that, once we get a normal form with trivial a-part, the fibration given by $x_0 = 1/x_\infty$ is only preserved by the action of $(0, 0, 0, \theta)$. We thus have to divide the group action by this normal subgroup to get a bijection with the set of fibrations. □

It is clear that we cannot kill the a-part in general by means of the above 3-dimensional group action and therefore that we have infinitely many obstructions to have a transversal fibration.

Proposition 4.9. *Any $(+1)$-neighborhood admits a normal form with a-part vanishing up to the order 4 in y-variable, i.e. with $a_{-3,4} = 0$. In other words, the 4th infinitesimal neighborhood always admit a transverse fibration; the first obstruction to extend it arrives at order 5.*

Example 4.10. The neighborhood U_Φ given by the cocycle in normal form

$$\Phi = \left(\frac{1}{x} + \frac{y^5}{x^3} + \sum_{V(-3,4) \cap \{n>5\}} a_{m,n} x^m y^n, \frac{y}{x} + \sum_{V(-2,3) \cap \{n>5\}} b_{m,n} x^m y^n \right)$$

does not admit transversal fibration.

Proof of Proposition 4.9 and Example 4.10. Looking back at the explicit action (10) of Γ on the a-coefficients, we see that whatever is $a_{-3,4}$, we can assume $a'_{-3,4} = 0$ by setting $\alpha = \beta = 0$, $\theta = 1$ and $\gamma^2 = a_{-3,4}$. On the other hand, the coefficient $a_{-3,5}$ cannot be killed in general, and in particular in the example. Indeed, since all other coefficients $a_{m,n}, b_{m,n}$ occuring in $a'_{-3,5}$ - formula (10) - are zero, we see that $a'_{-3,5} = \theta^{-5} \neq 0$ whatever are $(\alpha, \beta, \gamma) \in \mathbb{C}^3$. □

Example 4.11. The neighborhood U_Φ given by the cocycle in normal form

$$\Phi = \left(\frac{1}{x}, \frac{y}{x} + \sum_{V(-2,3)\cap\{n\geq 5\}} b_{m,n} x^m y^n \right) \quad \text{with} \quad b_{-2,5}b_{-4,5} \neq b_{-3,5}$$

admits no other transversal fibration than $dx = 0$. Indeed, following Proposition 4.8, another fibration would correspond to a triple $(\alpha, \beta, \gamma) \neq (0, 0, 0)$ such that the a-part of $(\alpha, \beta, \gamma, 1).\Phi$ is zero. However, formula (10) gives

$$a'_{-3,4} = -\gamma^2, \quad a'_{-3,5} = \alpha b_{-3,5} - \beta b_{-2,5} \quad \text{and} \quad a'_{-4,5} = \alpha b_{-4,5} - \beta b_{-3,5}$$

which shows that we must have $\alpha = \beta = \gamma = 0$.

From previous examples, we understand that neighborhoods with exactly one transversal fibration have infinite dimension and codimension in the moduli of all (+1)-neighborhoods.

4.4. **The case of general positive self-intersection $C \cdot C > 1$.** Mishustin gave in [27] a normal form for (k)-neighborhoods for arbitrary $k \in \mathbb{Z}_{>0}$ and the story is similar to the case $k = 1$. More geometrically, we can link the general case to the case $k = 1$ as follows. Let $k > 1$ and (S, C) a (k)-neighborhood. Then, maybe shrinking S, the topology of (S, C) is the same than the topology of $(N_C, 0)$, in particular:

$$\pi_1(S \setminus C) \simeq \mathbb{Z}/ < k >$$

i.e. the fundamental group of the complement of C is cyclic of order k. We can consider the corresponding ramified cover

$$(\tilde{S}, \tilde{C}) \overset{k:1}{\to} (S, C)$$

totally ramifying at order k over C, and inducing the cyclic cover of order k over the complement $S \setminus C$. If we do this with S being the total space of $\mathcal{O}_{\mathbb{P}^1}(k)$, then $\tilde{S}$ will be the total space of $\mathcal{O}_{\mathbb{P}^1}(1)$, the neighborhood of a line in $\mathbb{P}^2$. Likely as in the linear case, the lifted curve $\tilde{C}$ will have self-intersection $\tilde{C} \cdot \tilde{C} = 1$ and we are back to the case $k = 1$. Moreover, $\tilde{S}$ is equipped with the Galois transformation, of order k, which has $\tilde{C}$ as a fixed point curve.

Proposition 4.12. *Isomorphism classes (S, C) of germs of (k)-neighborhoods of the (parametrized) rational curve $x : C \to \mathbb{P}^1$ are in one-to-one correspondance with isomorphism classes of germs of $(+1)$-neighborhoods $(\tilde{S}, C)$ equipped with a cyclic automorphism of order k fixing C point-wise.*

5. Neighborhoods with several transverse fibrations

In this section, we study (+1)-neighborhoods having several fibrations. The following was recently announced and partly proved in [13]; Paulo Sad and the first author gave another proof in [14].

Theorem 5.1 ([13, 14])**.** *If a germ (S, C) of $(+1)$-neighborhood admits at least 3 distinct fibrations $\mathcal{H}$, $\mathcal{H}'$ and $\mathcal{H}''$ transversal to C, then (S, C) is equivalent to $(\mathbb{P}^2, L)$, where L is a line in $\mathbb{P}^2$.*

In this case, recall (see example 3.3) that there is a 2-parameter family of transverse fibrations (each of them is a pencil of lines through a point) and any two have a common fiber. Before proving this theorem is full details in section 5.6, we need first to classify pairs of fibrations on (+1)-neighborhoods.

5.1. **Tangency between two fibrations.** Given two (possibly singular) distinct foliations $\mathcal{H}$ and $\mathcal{H}'$ on a complex surface X, define the tangency divisor $\mathrm{Tang}(\mathcal{H},\mathcal{H}')$ as follows. Locally, we can define the two foliations respectively by $\omega=0$ and $\omega'=0$ for holomorphic 1-forms ω,ω' without zero (or with isolated zero in the singular case); then the divisor $\mathrm{Tang}(\mathcal{H},\mathcal{H}')$ is locally defined as the (zero) divisor of $\omega\wedge\omega'$.

Proposition 5.2. *If a germ (S,C) of $(+1)$-neighborhood admits 2 distinct fibrations $\mathcal{H}$ and $\mathcal{H}'$, then*

- *either* $\mathrm{Tang}(\mathcal{H},\mathcal{H}')=[C]$ *(without multiplicity),*
- *or* $\mathrm{Tang}(\mathcal{H},\mathcal{H}')\cdot[C]$ *is a single point (without multiplicity).*

The former case is rigid and will be described in section 5.4. In the latter case, the tangency divisor is reduced and transversal to C (equivalently the restriction $\mathrm{Tang}(\mathcal{H},\mathcal{H}')|_C$ has degree one); moreover, the support T of the divisor is

- either a common fiber of $\mathcal{H}$ and $\mathcal{H}'$,
- or is generically transversal to $\mathcal{H}$ and $\mathcal{H}'$ (but might be tangent at some point).

Proof. If C is contained into the support T of $\mathrm{Tang}(\mathcal{H},\mathcal{H}')$, then we just have to check that $\mathrm{Tang}(\mathcal{H},\mathcal{H}')\cdot[C']$ is a single point (without multiplicity) for any small deformation C' of C. Note that the multiplicity of $\mathrm{Tang}(\mathcal{H},\mathcal{H}')\cdot[C']$ (or equivalently the degree of the restricted divisor on C') is invariant by deformation of C'. We can thus assume without loss of generality that the support T intersects C tranversely, outside say $p_0=\{x=0\}$. After blowing-up p_0, we get a new surface $\pi_0:\tilde{S}\to S$ with exceptional divisor D_0 and lifted foliations $\tilde{\mathcal{H}}$ and $\tilde{\mathcal{H}}'$ with tangency divisor

$$\mathrm{Tang}(\tilde{\mathcal{H}},\tilde{\mathcal{H}}')=\pi_0^*\mathrm{Tang}(\mathcal{H},\mathcal{H}')+[D_0].$$

By assumption, D_0 does not intersect the support π_0^*T of $\pi_0^*\mathrm{Tang}(\mathcal{H},\mathcal{H}')$. The strict transform $\tilde{C}$ of C has self-intersection $\tilde{C}\cdot\tilde{C}=0$. We can therefore trivialize (see beginning of section 4) its neighborhood with coordinates $(x,y)\in\mathbb{P}^1\times\mathbb{C}$ such that $\tilde{C}=\{y=0\}$ and $\tilde{\mathcal{H}}=\{dx=0\}$, extending the original coordinate x on $C\simeq\tilde{C}$. Note that $\tilde{\mathcal{H}}$ and $\tilde{\mathcal{H}}'$ are (smooth) transversal to $\tilde{C}$, having the exceptional divisor D_0 as a common leaf near $\tilde{C}$, so that $D_0=\{x=0\}$. The other foliation $\tilde{\mathcal{H}}'$, being transversal to y-fibration, must be defined by a Riccati equation

$$\frac{dx}{dy}=a(y)x^2+b(y)x+c(y),\quad a,b,c\in\mathbb{C}\{y\}.$$

Since D_0 is a leaf, we have $c(y)\equiv 0$; in this chart, the tangency locus is given by

$$\mathrm{Tang}(\tilde{\mathcal{H}},\tilde{\mathcal{H}}')=[a(y)x^2+b(y)x=0]=\underbrace{[a(y)x+b(y)=0]}_{\pi_0^*\mathrm{Tang}(\mathcal{H},\mathcal{H}')}+[D_0]$$

and by assumption the two components do not intersect: $b(0)\neq 0$. Now, replacing x by its inverse $x_\infty=1/x$, we get

$$\mathcal{H}\ :\ dx_\infty=0\quad\text{and}\quad\mathcal{H}'\ :\ dx_\infty+(b(y)x_\infty+a(y))dy=0.$$

We note that π_0 is biregular in restriction to this chart $(x\infty,y)\in\mathbb{C}\times(\mathbb{C},0)$ and the support of $\pi_0^*\mathrm{Tang}(\mathcal{H},\mathcal{H}')$ is totally contained in this chart, coinciding with $\mathrm{Tang}(\tilde{\mathcal{H}},\tilde{\mathcal{H}}')$ there. It is given by

$$\pi_0^*\mathrm{Tang}(\mathcal{H},\mathcal{H}')=[b(y)x_\infty+a(y)=0],\quad\text{with } b(0)\neq 0$$

and restricts to $\tilde{C}:\{y=0\}$ as a degree 1 divisor. □

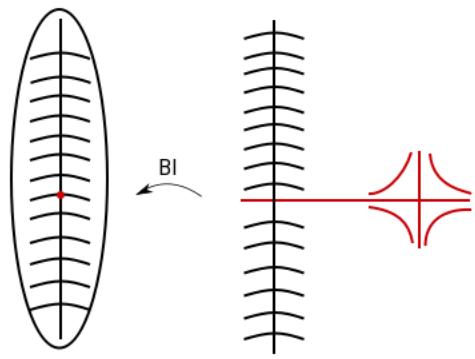

FIGURE 1. Blowing up

5.2. **Two fibrations having a common leaf.** Let us start with the simplest case.

Theorem 5.3. *Let (S, C) be a $(+1)$-neighborhood that admits two transversal fibrations $\mathcal{H}$ and $\mathcal{H}'$ with a common leaf T. Then (S, C) is linearizable.*

Proof. By Proposition 5.2, $\mathcal{H}$ and $\mathcal{H}'$ have contact of order 1 along T and are transversal outside. Let $H, H' : S \to \mathbb{P}^1$ be the two submersions defining these foliations and coinciding with $x : C \to \mathbb{P}^1$ in restriction to C. For simplicity, assume $T = \{H = \infty\} = \{H' = \infty\}$. We can use H and H' as a system of coordinates to embed (S, C) into $\mathbb{P}^2$/ Precisely, consider the map given in homogeneous coordinate $(u : v : w) \in \mathbb{P}^2$ by

$$\Phi := (H : H' : 1) : S \to \mathbb{P}^2.$$

The complement $S \setminus T$ is clearly embedded by Φ as a neighborhood of the diagonal $\Delta = \{u = v\}$ in the chart $w = 1$. Moreover, the two fibrations are send to fibrations $du = 0$ and $dv = 0$. We just have to check that this map is (well-defined and) still a local diffeomorphism at $T \cap C$. In local convenient coordinates $(x_\infty, y_\infty) \sim (0, 0)$ at the source, we have

$$\frac{1}{H} = x_\infty \quad \text{and} \quad \frac{1}{H'} = x_\infty(1 + y_\infty \cdot f(x_\infty, y_\infty)), \quad f(0, 0) \neq 0$$

(we used that $1/H$ and $1/H'$ coincide along $y_\infty = 0$, vanish at $x_\infty = 0$ and have simple tangency there). Coordinates at the target are given by

$$(X, Y) = \left(\frac{1}{u}, \frac{u}{v} - 1\right).$$

Therefore, our map is given by

$$\Phi \ : \ (X, Y) = (x_\infty, y_\infty \cdot f(x_\infty, y_\infty))$$

which is clearly a local diffeomorphism. □

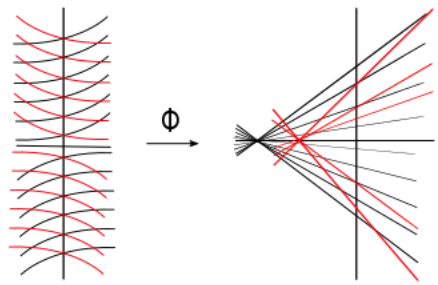

FIGURE 2. Linearization

5.3. **Application: cross-ratio and analytic continuation.** Let Π be a projective structure on a geodesically convex open set U (see definition in Proposition 2.2). For each point $p \in U$, we can consider the singular foliation $\mathcal{F}_p$ whose leaves are geodesics through p. Given 4 distinct points $p_1, p_2, p_3, p_4 \in U$, we can define by formula (4) the cross-ratio function:

$$(\mathcal{F}_{p_1}, \mathcal{F}_{p_2}; \mathcal{F}_{p_3}, \mathcal{F}_{p_4}) : U \setminus \cup_{i \neq j} \gamma_{i,j} \to \mathbb{C}$$

(where $\gamma_{i,j}$ is the geodesic passing through p_i and p_j, for each $i, j \in \{1, 2, 3, 4\}$); it is clearly holomorphic outside of the six geodesics $\gamma_{i,j}$.

Example 5.4. In $U = \mathbb{P}^2$ equipped with the standard structure Π_0, the fibers of the cross-ratio function $(\mathcal{F}_{p_1}, \mathcal{F}_{p_2}; \mathcal{F}_{p_3}, \mathcal{F}_{p_4})$ are those conics passing through p_1, p_2, p_3, p_4 provided that not three of them lie on the same line. On the other hand, when all four points are on the same line, then the cross-ratio is constant.

We say that Π satisfies the *cross-ratio condition* on U if, for any geodesic curve γ, and every four points in $p_1, p_2, p_3, p_4 \in \gamma$, then the cross-ratio function $(\mathcal{F}_{p_1}, \mathcal{F}_{p_2}; \mathcal{F}_{p_3}, \mathcal{F}_{p_4})$ is constant. Since this property is invariant by change of coordinates, any linearizable structure locally satisfies cross-ratio condition; conversely:

Lemma 5.5. *If the projective structure Π satisfies the cross-ratio condition on U, then it is linearizable at every point of U.*

Proof. Let $p \in U$ be an arbitrary point. We claim that Π is flat in a neighborhood of p. In fact, take a geodesic γ not passing through p and choose three distinct points p_1, p_2, $p_3 \in \gamma$. Consider the flat structure Π' defined by the 3-web $\mathcal{F}_{p_1} \boxtimes \mathcal{F}_{p_2} \boxtimes \mathcal{F}_{p_3}$ around p (see definition in Proposition 3.5). For any fourth point $p_4 \in \gamma$, the foliation $\mathcal{F}_{p_4}$ of Π-geodesics by p_4 has constant cross-ratio with $\mathcal{F}_1$, $\mathcal{F}_2$ and $\mathcal{F}_3$, then $\mathcal{F}_4$ is Π'-geodesic. We infer from Proposition 3.4 that $\Pi = \Pi_\gamma$. By taking another geodesic γ' intersecting γ at some point q, we obtain a second flat structure $\Pi_{\gamma'}$ equal to Π around p and having $\mathcal{F}_q$ as common foliation with Π_γ. Passing to the dual picture (U^*, C_0) of the germ of Π at (U, p), the two flat structures give two fibrations on U^* transversal to C_0 that share a common fiber given by $\mathcal{F}_q$ (see Proposition 3.2). We finally apply Theorem 5.3 to conclude that Π is linearizable in a neighborhood of p. $\square$

Corollary 5.6. *If U is geodesically convex and Π is linearizable at $p \in U$, then Π is linearizable at every point $q \in U$.*

Proof. For every geodesic curve $\gamma \subseteq U$ and every 4-tuple $P = (p_1, p_2, p_3, p_4)$ of different points of γ, the cross-ratio function $f_P = (\mathcal{F}_{p_1}, \mathcal{F}_{p_2}; \mathcal{F}_{p_3}, \mathcal{F}_{p_4})$ is holomorphic on $U \setminus \gamma$. Since U is convex, the set of such 4-tuples is a smooth connected manifold $\mathcal{P}$ of dimension 6: once we have chosen γ in the dual 2-manifold U^*, we have to choose distinct p_i's on γ, which is a disc. Moreover, the cross-ratio function f_P depends holomorphically on $P \in \mathcal{P}$ (where it makes sense, i.e. locally outside γ).

Now, take an open subset V containing p, where Π is linearizable. In particular, Π satisfies the cross-ratio condition on V: for those P such that all $p_i \in V$, the differential 1-form df_P is identically vanishing. By analytic continuation, we have $df_P \equiv 0$ for all $P \in \mathcal{P}$. Therefore, the cross-ratio condition propagates on the whole of U, and the previous lemma finishes the proof. □

Corollary 5.7. *If U is connected and Π is linearizable at some point $p \in U$, then Π is linearizable at every point of U.*

Proof. Given any other point $q \in U$, take a path $[0,1] \to U$ starting at p and ending at q, cover it by a finite number of geodesically convex open sets, and use the previous Corollary successively of these sets. □

5.4. **Two fibrations that are tangent along a rational curve.** The goal of this section is to describe a very special neighborhood. As we shall see later, it is the only one having a large group of symetries (i.e. of dimension > 2) but not linearizable.

Let us consider the diagonal $\Delta \subset \mathbb{P}^1 \times \mathbb{P}^1$. The self-intersection is $\Delta \cdot \Delta = 2$ and its neighborhood is a $(+2)$-neighborhood. Therefore, we can take (see section 4.4) the 2-fold ramified cover, ramifying over Δ:

$$\pi : (S, C) \overset{2:1}{\to} (\mathbb{P}^1 \times \mathbb{P}^1, \Delta)$$

and we get a $(+1)$-neighborhood. Moreover, the two fibrations on $\mathbb{P}^1 \times \mathbb{P}^1$ defined by projections on the two factors lift as fibrations $\mathcal{H}_1$ and $\mathcal{H}_2$ on S transversal to C whose tangent locus is $\mathrm{Tang}(\mathcal{H}_1, \mathcal{H}_2) = C$.

Remark 5.8. The two fibers passing through a given point $p \in S$ close enough to C intersect twice: the Galois involution $\iota : (S, C) \to (S, C)$ permutes these two points.

Proposition 5.9. *The germ (S, C) is not linearizable.*

Proof. Assume by contradiction that (S, C) is equivalent to the neighborhood of a line $L \subset \mathbb{P}^2$. Then each foliation $\mathcal{H}_i$ extends as a global singular foliation on $\mathbb{P}^2$. Since $\mathcal{H}_i$ is totally tranversal to L, it must be a foliation of degree 0, i.e. a pencil of lines. But if $\mathcal{H}_1$ and $\mathcal{H}_2$ are pencil of lines, their tangency must be invariant (the line through the to base points), contradiction. □

Corollary 5.10. *The germ (S, C) is not algebrizable but the field $\mathcal{M}(S, C)$ of meromorphic functions has transcendance degree 2 over $\mathbb{C}$.*

Proof. It immediately follows from Proposition 2.12 that (S, C) is not algebrizable; the field $\mathcal{M}(S, C)$ contains the field or rational functions on $\mathbb{P}^1 \times \mathbb{P}^1$ which has indeed, as an algebraic surface, transcendance degree 2 over $\mathbb{C}$. □

Remark 5.11. The fundamental group $\pi_1(\mathbb{P}^1\times\mathbb{P}^1\setminus\Delta)$ is trivial, and this is a reason why we cannot extend the ramified cover to the whole of the algebraic surface.

The cocycle Φ defining the germ (S, C_0) can be constructed as follows. We can give local coordinates (x_0, y_0), (x_∞, y_∞) on S and (u_0, v_0), (u_∞, v_∞) on $V \supset \Delta$ such that

$$\begin{cases} u_0 &= x_0 \\ v_0 &= x_0 - y_0^2 \end{cases}, \quad \begin{cases} u_\infty &= x_\infty \\ v_\infty &= x_\infty + y_\infty^2 \end{cases}, \quad (u_\infty, v_\infty) = \left(\frac{1}{u_0}, \frac{1}{v_0}\right) \tag{11}$$

where $\Delta = \{v_0 = u_0\} = \{v_\infty = u_\infty\}$. So the cocycle is explicitely given by

$$\Phi(x_0, y_0) = \left(\frac{1}{x_0}, \frac{y_0}{x_0}\left(1 - \frac{y_0^2}{x_0}\right)^{-1/2}\right) = \left(\frac{1}{x_0}, \frac{y_0}{x_0} + \frac{y^3}{2x^2} + \frac{3y^5}{8x^3} + \dots\right), \tag{12}$$

which is already in normal form. The fibrations are given by

$$h_1 = x_0 = \frac{1}{x_\infty} \quad \text{and} \quad h_2 = x_0 - y_0^2 = \frac{1}{x_\infty + y_\infty^2}.$$

Proposition 5.12. *The fibrations $\mathcal{H}_1$ and $\mathcal{H}_2$ are the only one fibrations on S transverse to C_0.*

Proof. Recall (see Proposition 4.8) that transverse fibrations are in one-to-one correspondance with $\vartheta = (\alpha, \beta, \gamma, \theta) \in \mathbb{C}^3 \times \{1\}$ such that the a-part of the equivalent cocycle $\vartheta \cdot \Phi$ is zero. Substituting the explicit cocycle above in formula (10), we get

$$a'_{-3,4} = \gamma(1-\gamma), \quad a'_{-3,5} = \frac{3}{8}\alpha \quad \text{and} \quad a'_{-4,5} = \frac{3}{8}\beta$$

so that the only two possibilities are $(\alpha, \beta, \gamma, \theta) = (0, 0, 0, 1)$ or $(0, 0, 1, 1)$ which respectively correspond to the two fibrations $\mathcal{H}_1$ and $\mathcal{H}_2$. □

Remark 5.13. Another proof runs as follows. Let $\mathcal{H}$ be a transversal fibration on (S, C). Assume first that $\mathcal{H}$ is invariant by the involution ι. One easily check that its projection on the germ of neighborhood $(\mathbb{P}^1 \times \mathbb{P}^1, \Delta)$ is a foliation $\mathcal{F}$ transversal to Δ. Since Δ is ample, $\mathcal{F}$ extends as a singular foliation on the whole of $\mathbb{P}^1 \times \mathbb{P}^1$. The tangent bundle writes $T\mathcal{F} = \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(m[F_1] + n[F_2])$ where F_i is a fiber for the projection on the ith factor and $m, n \in \mathbb{Z}$. But for any curve $C \subset \mathbb{P}^1 \times \mathbb{P}^1$ which is not $\mathcal{F}$-invariant, we have $T\mathcal{F} \cdot C = C \cdot C - \mathrm{Tang}(\mathcal{F}, C)$ where $\mathrm{Tang}(\mathcal{F}, C) \geq 0$ is the number (counted with multiplicities) of tangencies between $\mathcal{F}$ and C (see [7, Chapter 2, Section 2, Proposition 2]). For the two fibrations we have $(m, n) = (2, 0)$ or $(0, 2)$, and for other foliations, the non negativity of $\mathrm{Tang}(\mathcal{F}, F_i)$ implies $m, n \leq 0$. Finally, for $C = \Delta$, we obtain $\mathrm{Tang}(\mathcal{F}, \Delta) = 2 - m - n$ which is only possible for the two fibrations. Assuming now that $\mathcal{H}$ is not invariant by the involution, we get, after projection and extension, a singular 2-web $\mathcal{W}$ on $\mathbb{P}^1 \times \mathbb{P}^1$ given by $\omega \in H^0(\mathbb{P}^1 \times \mathbb{P}^1, \mathrm{Sym}^2\Omega^1_{\mathbb{P}1\times\mathbb{P}^1} \otimes N_{\mathcal{W}})$, where $N_W = a[F_1] + b[F_2]$ stands by its normal bundle. We see locally that in some neighborhood U of Δ the web is irreducible and has discriminant $\Delta(\mathcal{W}) \cap U = \Delta$. Pulling-back ω by the inclusion $i : C \hookrightarrow \mathbb{P}^1 \times \mathbb{P}^1$ of a not invariant curve we are able to count tangencies and obtain $\mathrm{Tang}(\mathcal{W}, C) = N_{\mathcal{W}} \cdot C - 2\chi(C)$. Taking in particular horizontal and vertical lines we see $a, b \geq 4$. On the other hand [29, Proposition 1.3.3] gives $\Delta(\mathcal{W}) = (2a-4)[F_1] + (2b-4)[F_2]$, thus $\Delta(\mathcal{W})$ must have some component distinct from Δ intersecting Δ, a contradiction.

Theorem 5.14. *Let (S, C_0) be a $(+1)$-neighborhood and suppose that there are two fibrations $\mathcal{H}_1$ and $\mathcal{H}_2$ transverse to C such that* $\mathrm{Tang}(\mathcal{H}_1, \mathcal{H}_2) = C_0$. *Then (S, C_0) is the previous example.*

Proof. Let $x : C_0 \to \mathbb{P}^1$ be a global coordinate on C_0 and consider $h_1, h_2 : S \to \mathbb{P}^1$ first integrals of $\mathcal{H}_1$ and $\mathcal{H}_2$ such that $h_1|_{C_0} = h_2|_{C_0} = x$. Consider the map

$$\Phi : S \to \mathbb{P}^1 \times \mathbb{P}^1 \ ; \ \Phi(p) = (h_1(p), h_2(p)).$$

Clearly, $\Phi|_{C_0} : C_0 \xrightarrow{\simeq} \Delta$ is an isomorphism, and we claim that Φ is a 2-fold covering of a neighborhood V of Δ, ramifying over Δ. In order to prove this claim, it suffices to check it near $p_0 : \{x = 0\}$ since x is well-defined up to a Moebius transform. Fix local coordinates (x, y) on (S, p_0) such that $C_0 = \{y = 0\}$ and $h_1(x, y) = x$. Therefore, $h_2(x, y) = x - y^2 f(x, y)$ with $f(0, 0) \neq 0$; here we have used that h_1 and h_2 coincide on C_0 and are tangent there, without multiplicity. We can change coordinate $(X, Y) = (x, y\sqrt{f(x, y)})$ so that

$$h_1(X, Y) = X \quad \text{and} \quad h_2(X, Y) = X - Y^2.$$

From this it is clear that Φ is a 2-fold cover ramifying over $\Delta \subset \mathbb{P}^1_u \times \mathbb{P}^1_v$ since in coordinates $(U, V) = (u, u - v)$ we have $\Delta = \{V = 0\}$ and $\Phi : (X, Y) \mapsto (U, V) = (X, Y^2)$. By construction, $\mathcal{H}_1$ and $\mathcal{H}_2$ are sent to $dX = 0$ and $dY = 0$. $\square$

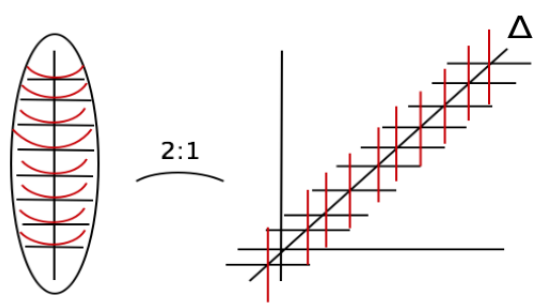

FIGURE 3. Covering

Now we want to write explicitly the differential equation associated to this example. In order to do that, we consider the automorphism group

$$\mathrm{Aut}(\mathbb{P}^1_u \times \mathbb{P}^1_v, \Delta) = \mathrm{PSL}_2(\mathbb{C}) \times \mathbb{Z}/2\mathbb{Z}$$

where the $\mathrm{PSL}_2(\mathbb{C})$-action is diagonal

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \ : \ (u, v) \mapsto \left(\frac{au + b}{cu + d}, \frac{av + b}{cv + d} \right)$$

and the $\mathbb{Z}/2\mathbb{Z}$-action is generated by the involution $(u, v) \mapsto (v, u)$. After 2-fold cover $(S, C) \to (\mathbb{P}^1_u \times \mathbb{P}^1_v, \Delta)$, we get an action of

$$\Gamma \simeq \{M \in \mathrm{GL}_2(\mathbb{C}) \ ; \ \det(M) = \pm 1\}$$

where $-I$ is the Galois involution of the covering, and we have

$$\mathrm{Aut}(\mathbb{P}^1_u \times \mathbb{P}^1_v, \Delta) = \Gamma / \{\pm I\}.$$

Indeed, the $\mathrm{PGL}_2(\mathbb{C})$-action is given by $\mathrm{SL}_2(\mathbb{C})/\{\pm I\}$ and $\langle\sqrt{-1}I\rangle/\{\pm I\}\simeq\mathbb{Z}/2\mathbb{Z}$ is the permutation of coordinates $u\leftrightarrow v$. In coordinates $(u,v)=(x,x-y^2)$ (see (11)), the $\mathrm{SL}_2(\mathbb{C})$-action on (S,C) writes

$$\begin{pmatrix} a & b\\ c & d\end{pmatrix}\ :\ (x,y)\mapsto\left(\frac{ax+b}{cx+d},\frac{y}{cx+d}\sqrt{\frac{cx+d}{cx+d-y^2}}\right)$$

where $ad-bc=1$ and the square-root choosen so as $\sqrt{1}=1$ (note that its argument is 1 along $y=0$). The involution writes

$$\begin{pmatrix} \sqrt{-1} & 0\\ 0 & \sqrt{-1}\end{pmatrix}\ :\ (x,y)\mapsto(x-y^2,\sqrt{-1}y).$$

Going to the dual picture, we get a projective structure Π on $(\mathbb{C}^2,0)$ invariant by an action of the same group Γ, and fixing the origin 0. By Bochner Linearization Theorem, the action of the maximal compact subgroup $\Gamma_0\subset\Gamma$ is holomorphically linearizable, and since Γ is just the complexification of Γ_0 (and therefore Zariski dense in Γ) the action of Γ itself is linearizable.

Proposition 5.15. *The unique projective structure* Π *on* $(\mathbb{C}^2,0)$ *which is invariant by the linear action of* $\mathrm{SL}_2(\mathbb{C})$ *is given (up to homothety) by*

$$y''=(xy'-y)^3. \tag{13}$$

Proof. Consider the differential equation defining the projective structure

$$\Pi\ :\ y''=f(x,y,y')=A(x,y)+B(x,y)(y')+C(x,y)(y')^2+D(x,y)(y')^3.$$

The action of a linear map $\varphi(x,y)=(ax+by,cx+dy)$, with $ad-bc=1$, induces a biholomorphism on the contact variety $\mathbb{P}(T\mathbb{C}^2)$ (near the fiber $x=y=0$) given by

$$\tilde\varphi(x,y,z)=\left(ax+by,cx+dy,\frac{c+dz}{a+bz}\right).$$

The geodesic foliation $\mathcal{G}$ defined by the vector field $v=\partial_x+z\partial_y+f(x,y,z)\partial_z$ must be preserved by $\tilde\varphi$ which means that the following two vector fields must be proportional:

$$D\tilde\varphi(x,y,z)\cdot v(x,y,z)=\begin{pmatrix} a+bz\\ c+dz\\ \frac{f(x,y,z)}{(a+bz)^2}\end{pmatrix},\quad X(\tilde\varphi(x,y,z))=\begin{pmatrix} 1\\ \frac{c+dz}{a+bz}\\ f(\tilde\varphi(x,y,z))\end{pmatrix}$$

We thus get

$$f(x,y,z)=(a+bz)^3f(\tilde\varphi(x,y,z)). \tag{14}$$

In fact, this equation is equivalent to $\varphi^*(\omega)=\omega$ where $\omega=A(dx)^3+B(dx)^2(dy)+C(dx)(dy)^2+D(dy)^3$ defines the "inflection" 3-web $\mathcal{W}$, whose directions correspond to inflection points of geodesics. Since the linear action preserves lines, it must indeed preserve the inflection web of the structure Π.

The isotropy group of a point $(x_0,y_0)\neq(0,0)$ under the $\mathrm{SL}_2(\mathbb{C})$-action is a parabolic subgroup fixing all those points along the line $x_0y=y_0x$ through this point and the origin; moreover, the only direction fixed by this subgroup at (x_0,y_0) is along the same line. Therefore, the inflection web $\mathcal{W}$ must be radial: $\omega=\lambda(x,y)(xdy-ydx)^3$. One easily check that $\omega_0=(xdy-ydx)^3$ is invariant by $\mathrm{SL}_2(\mathbb{C})$ and $\lambda(x,y)$ must be invariant, therefore constant. We deduce that

$$\Pi\ :\ y''=\lambda(xy'-y)^3,\quad \lambda\in\mathbb{C}.$$

Since $\Pi \neq \Pi_0$, we have $\lambda \neq 0$ and we can normalize $\lambda = 1$ by homothety. □

Theorem 5.16. *The two pencils of geodesic foliations for the* $\mathrm{SL}_2(\mathbb{C})$*-invariant projective structure* $y'' = (xy' - y)^3$ *are given by*

$$\omega_t^{\pm} = \big(y^2 dx - (xy + \sqrt{-1})dy\big) + t\,\big((xy + \sqrt{-1})dx - x^2 dy\big) = 0, \quad t \in \mathbb{P}^1$$

where $\pm$ *stand for the two determinations of* $\sqrt{-1}$*. For each* t*, we see that the line* $y = tx$ *is a common leaf for the two foliations* $\mathcal{F}_{\omega_t^+}$ *and* $\mathcal{F}_{\omega_t^-}$*.*

Proof. Recall (see Proposition 5.12) that there are exactly 2 pencils of geodesic foliations for this special projective structure. We can just verify by computation that the pencils in the statement are geodesic, however we think that it might be interesting to explain how we found them.

In order to find the elements of the pencils, we will use again the $\mathrm{SL}_2(\mathbb{C})$-invariance. We first construct the two foliations (i.e. for the two pencils) having the geodesic $y = 0$ as a special leaf, i.e. $\omega_0^{\pm}$; then it will be easy to deduce the full pencil by making $\mathrm{SL}_2(\mathbb{C})$ acting on $\omega_0^{\pm}$. In order to characterize $\omega_0^{\pm}$, let us go back to the dual picture (S, C). The two foliations we are looking for come from the two fibers passing through $p_0 = \{x_0 = y_0 = \infty\}$ in C, or equivalently the two fibers $u_0 = \infty$ and $v_0 = \infty$ on $\mathbb{P}^1 \times \mathbb{P}^1$ (here we use coordinates given by formula (11)). These curves are invariant by the action of the Borel subgroup

$$\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \ : \ \left\{ \begin{array}{l} (u_0, v_0) \mapsto \big(a^2 u_0 + ab, a^2 v_0 + ab\big) \\ (x_0, y_0) \mapsto \left(a^2 x_0 + ab, \frac{a y_0}{\sqrt{1 - a y_0^2}}\right) \end{array} \right\}$$

and they are the only one except the diagonal Δ. Therefore, the two foliations we are looking for are the only one that are invariant under the action of the Borel subgroup above, and distinct to the radial foliation. Let $\omega = dy - f(x, y)dx$ be one of these foliations. Setting $(a, b) = (1, t)$, we get $\varphi^t(x, y) = (x + ty, y)$ and

$$(\varphi^t)^*\omega \wedge \omega = 0, \ \forall t \quad \Leftrightarrow \quad y f_x + f^2 = 0 \quad (\text{where } f_x := \frac{df}{dx}).$$

Now, setting $(a, b) = (e^t, 0)$, we get $\varphi^t(x, y) = (e^t x, e^{-t} y)$ and

$$(\varphi^t)^*\omega \wedge \omega = 0, \ \forall t \quad \Leftrightarrow \quad 2f + x f_x - y f_y = 0 \quad (\text{and } f_y := \frac{df}{dy}).$$

Finally, the foliation $\mathcal{F}_\omega$ is geodesic if, and only if, the corresponding surface $\{z = f\}$ in $\mathbb{P}(T\mathbb{C}^2)$ is invariant by the geodesic foliation defined by $v = \partial_x + z\partial_y + (xz - y)^3 \partial_z$. In other words

$$i_v d(z - f)|_{z=f} = 0 \quad \Leftrightarrow \quad (xf - y)^3 - f_x - f f_y = 0.$$

The combination of the three constraints gives (after eliminating f_x and f_y)

$$\left(f - \frac{y}{x}\right)\left(f - \frac{y^2}{xy + \sqrt{-1}}\right)\left(f - \frac{y^2}{xy - \sqrt{-1}}\right) = 0.$$

The first factor gives the radial foliation and the two other ones yield $\omega_0^{\pm}$. We conclude by applying the one-parameter subgroup $\varphi^t(x, y) = (x, tx + y)$ to each of these foliations and get the two pencils. □

Remark 5.17. The $(+1)$-neighborhood (S, C) considered in this section cannot be embedded in a projective manifold. Indeed, otherwise it would be linearizable by Proposition 2.12. However, the field of meromorphic functions on (S, C) identifies with the field of meromorphic (rational) functions on $\mathbb{P}^1 \times \mathbb{P}^1$, and has therefore transcendance dimension 2 over the field $\mathbb{C}$ of complex numbers. In fact, we cannot globalize the 2-fold cover $(S, C) \to (\mathbb{P}^1 \times \mathbb{P}^1, \Delta)$ since the complement $\mathbb{P}^1 \times \mathbb{P}^1 \setminus \Delta$ is simply connected; morever, it does not contain complete curves so we cannot even extend the cover over a Zariski open set.

Remark 5.18. The $(+4)$-neighborhood of a conic $C \in \mathbb{P}^2$ is not linearizable, otherwise its 2-fold cover ramifying over C would be linearizable as well; but this cover is just $(\mathbb{P}^1 \times \mathbb{P}^1, \Delta)$ divided by $(u, v) \mapsto (v, u)$, which is not linearizable.

5.5. **Two fibrations in general position.** The goal of this section is to show that there are many $(+1)$-neighborhoods with exactly two fibrations; surprisingly, they are easy to classify.

Suppose that the germ of $(+1)$-neighborhood (S, C_0) admits two transverse fibrations $\mathcal{H}_1$ and $\mathcal{H}_2$, such that their tangent locus $T = \mathrm{Tang}(\mathcal{H}_1, \mathcal{H}_2)$ is neither a leaf, nor C_0. Remember (see Proposition 5.2) that T is smooth and intersects C_0 transversely in one point, say $x = 0$. We say that $\mathcal{H}_1$ and $\mathcal{H}_2$ are in *general position* near C_0 if, moreover, the curve T is transversal to $\mathcal{H}_1$ (and therefore $\mathcal{H}_2$). To state our result, denote

$$\mathrm{Diff}^{\geq k}(\mathbb{C}, 0) := \{\varphi(z) \in \mathbb{C}\{z\} \ ; \ \varphi(z) = z + o(z^k)\}$$

the group of germs of diffeomorphisms tangent to the identity at the order $\geq k$ and denote $\mathrm{Diff}^k(\mathbb{C}, 0) := \mathrm{Diff}^{\geq k}(\mathbb{C}, 0) \setminus \mathrm{Diff}^{\geq k+1}(\mathbb{C}, 0)$ the set of those tangent precisely at order k. The group

$$A := \{\varphi(z) = az/(1 + bz) \ : \ a \in \mathbb{C}^*, \ b \in \mathbb{C}\} = \mathrm{PGL}_2(\mathbb{C}) \cup \mathrm{Diff}(\mathbb{C}, 0)$$

acts by conjugacy on each $\mathrm{Diff}^{\geq k}(\mathbb{C}, 0)$ and therefore on $\mathrm{Diff}^k(\mathbb{C}, 0)$.

Theorem 5.19. *Germs of $(+1)$-neighborhoods (S, C_0) that admit two transversal fibrations $\mathcal{H}_1$ and $\mathcal{H}_2$ in general position are in one to one correspondance with the quotient set*

$$\mathrm{Diff}^1(\mathbb{C}, 0)/A.$$

Proof. Like in the proof of Theorem 5.14, consider $h_1, h_2 : S \to \mathbb{P}^1$ the first integrals of $\mathcal{H}_1$ and $\mathcal{H}_2$ whose restrictions on C_0 coincide with the global parametrization $x : C_0 \xrightarrow{\sim} \mathbb{P}^1$. Now, consider the map

$$\Phi : S \to \mathbb{P}^1 \times \mathbb{P}^1 \ ; \ \Phi(p) = (h_1(p), h_2(p))$$

and denote by $\Sigma := \Phi(T)$ the critical locus. In local coordinates (x, y) at p_0 adapted to the first fibration, we have $h_1(x, y) = x$ and we can assume $C_0 : \{y = 0\}$ and $T : \{x = y^k\}$, $k \in \mathbb{Z}_{>0}$. Obviously $k = 1$ if $\mathcal{H}_1$ and $\mathcal{H}_2$ are in general position, but the general case can be treated in the same way. Since $dh_1 \wedge dh_2$ vanish precisely on T, we get

$$\frac{\partial h_2}{\partial y} = (x - y^k)\phi(x, y) \quad \text{with} \quad \phi(0, 0) \neq 0.$$

After integrating, we get

$$h_2(x, y) = x + x\phi_1(x, y) - y^k \phi_2(x, y)$$

where x stands for the integration constant and ϕ_1, ϕ_2 are defined by

$$\frac{\partial \phi_1}{\partial y} = \frac{\partial \phi_2}{\partial y} + \frac{k}{y}\phi_2 = \phi, \quad \text{and} \quad \phi_1(x,0), \phi_2(x,0) \equiv 0. \tag{15}$$

Therefore, $\Sigma = \Phi(T)$ is parametrized by $y \mapsto (y^k, y^k(1+\varphi(y)))$ where

$$\varphi(y) = \phi_1(y^k, y) - \phi_2(y^k, y) = O(y).$$

Clearly, the curve $\Sigma \subset (\mathbb{P}^1 \times \mathbb{P}^1, \Delta)$ only depends on the choice of the global parametrization $x : C_0 \xrightarrow{\sim} \mathbb{P}^1$, not on local coordinates (x, y) choosen near $p_0 \in (S, C_0)$. In coordinates $(U, V) = (u, v - u)$, the curve Σ admits an equation of the form

$$\Sigma \ : \ V^k + \psi_1(U)V^{k-1} + \psi_2(U)V^{k-2} + \cdots + \psi_{k-1}(U)V + \psi_k(U) = 0, \tag{16}$$
$$\text{with} \quad \psi_i = o(U^i), \ \text{for } i = 1, \ldots, k-1 \text{ and } \psi_k = O(U^k).$$

Conversely, let us show that any such curve Σ arise from a $(+1)$-neighborhood (S, C_0) with a pair of fibrations $\mathcal{H}_1$,$\mathcal{H}_2$. Indeed, equation (16) always admits a parametrization $(U, V) = (y^k, y^k\varphi(y))$ with $\varphi(y) = O(y)$, and starting with $\phi(x, y) = \phi(y)$ depending only on y, we get $\varphi = \phi_1 - \phi_2$ which, combined with (15), gives

$$h_1(x, y) = x \quad \text{and} \quad h_2(x, y) = x + x\varphi(y) + \frac{(x - y^k)y}{k}\varphi'(y).$$

We can thus realize the curve Σ as critical set of a local map

$$\Phi = (h_1, h_2) : (S, p_0) \to (\mathbb{P}^1 \times \mathbb{P}^1, q_0).$$

It is then easy to complete this picture in order to form a $(+1)$-neighborhood by setting $(x_0, y_0) = (x, y)$ and

$$(x_\infty, y_\infty) := \left(\frac{1}{h_1}, \frac{1}{h_2} - \frac{1}{h_1}\right) = \left(\frac{1}{x_0}, c\frac{y_0}{x_0} + o(y_0)\right)$$

so that the two fibrations $\mathcal{H}_i : \{dh_i = 0\}$ extend on the whole of the neighborhood, transversal with C_0, and transversal to each other outside p_0. By construction, Σ is the invariant of the bifoliated neighborhood.

We can now describe the same construction in a more geometrical way. First observe that, in the neighborhood of p_0, Φ is a $(k+1)$-fold cover ramifying over Σ; indeed, the two fibers $\{h_1 = 0\}$ and $\{h_2 = 0\}$ have a contact of order $k+1$ at p_0. In fact, we can extend this ramified cover over the neighborhood of $\Delta \subset \mathbb{P}^1 \times \mathbb{P}^1$ and get

$$\Phi : (S, C_0 \cup C_1 \cup \cdots \cup C_k) \overset{(k+1):1}{\longrightarrow} (\mathbb{P}^1 \times \mathbb{P}^1, \Delta)$$

where $C_0, \ldots, C_k$ are the preimages of Δ, that intersect transversely at p_0. After selecting one of them, say C_0, we get our initial map $\Phi : (S, C_0) \to (\mathbb{P}^1 \times \mathbb{P}^1, \Delta)$. From this point of view, it becomes clear that the lack of unicity comes from the choice C_0 among $C_0, \ldots, C_k$, and the corresponding neighborhoods might be not isomorphic: $(S, C_i) \not\simeq (S, C_0)$.

However, in the case $\mathcal{H}_1$ and $\mathcal{H}_2$ are in general position, then $k = 1$ and the covering, being of degree 2, is automatically galoisian: $(S, C_0) \simeq (S, C_1)$. Then we have a one-to-one correspondance between smooth curves Σ at q_0 having a simple tangency with Δ and $(+1)$-neighborhoods (S, C_0) with $\mathcal{H}_1$, $\mathcal{H}_2$ having a simple tangency at $p_0 = \{x = 0\}$, up to isomorphism preserving the fixed parametrization $x : C_0 \to \mathbb{P}^1$.

Finally, the change of parametrization induces a diagonal action of the group A on $\mathbb{P}^1 \times \mathbb{P}^1$. Viewing Σ as the graph of a germ of diffeomorphism $v = \varphi(u)$, we see that $\varphi(u) = u + cu^2 + \cdots$, $c \neq 0$, and the diagonal action induced by A is an action by conjugacy on φ, whence the result. $\square$

Remark 5.20. If we consider the very special $(+1)$-neighborhood (S, C_0) studied along section 5.4, after specializing to the neighborhood of any deformation $C_\varepsilon \neq C_0$, we get a new $(+1)$-neighborhood (S, C_ε) with two fibrations in general position. This neighborhood does not depend on the choice of ε since $\mathrm{SL}_2(\mathbb{C})$ acts transitively on those rational curves; it is easy to check that it corresponds to the class of the diffeomorphism $\varphi(u) = u/(1-u)$, i.e. to a curve Σ given by a bidegree $(2,2)$ curve (a deformation of Δ).

5.6. **Proof of Theorem 5.1.** Before proving the theorem we make some considerations. First of all, recall (see Proposition 4.8) that there are normal forms

$$\Phi = \left(\frac{1}{x}, \frac{y}{x} + \sum_{V(-2,3)} b_{m,n} x^m y^n\right), \Phi_i = \left(\frac{1}{x}, \frac{y}{x} + \sum_{V(-2,3)} b^i_{m,n} x^m y^n\right), \quad i = 1, 2,$$

compatible with $\mathcal{H}$, $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively, and denote by $\vartheta_i = (\alpha_i, \beta_i, \gamma_i, 1)$ the parameter corresponding to the change of cocycle from Φ to Φ^i, $i = 1, 2$.

$$\begin{tikzcd} & \Phi \arrow[dl, "(\alpha_1,\beta_1,\gamma_1,1)"'] \arrow[dr, "(\alpha_2,\beta_2,\gamma_2,1)"] & \\ \Phi_1 & & \Phi_2 \end{tikzcd}$$

All along the section, we assume moreover $(\alpha_1, \beta_1, \gamma_1) \neq (\alpha_2, \beta_2, \gamma_2)$ and are both $\neq (0,0,0)$ (otherwise two of the three firbations coincide).

Lemma 5.21. *If $b_{-2,3} = 0$, $\vartheta_1 = (\alpha, 0, 0)$ and $\vartheta_2 = (0, \beta, 0)$, $\alpha\beta \neq 0$, then $b_{m,n} = 0$ for every (m,n) and the neighborhood is linearizable.*

Proof. From formula (10), we already get $0 = a^1_{-3,4} = \alpha b_{-3,4}$ and $0 = a^2_{-3,4} = \beta b_{-2,4}$. Assume by induction that $b_{m,k} = 0$ for every $k < n$. Then

$$\Phi = \left(\frac{1}{x}, \frac{y}{x} + (\frac{b_{-2,n}}{x^2} + \ldots + \frac{b_{1-n,n}}{x^{n-1}})y^n + \ldots\right).$$

The change of coordinates sending Φ to Φ_1 takes the form (see Proposition 4.5)

$$\Psi^0_1 = \left(\frac{x}{1+\alpha y} - \alpha b_{-2,n} y^n + o(y^n), \frac{y}{1+\alpha y}\right), \quad \Psi^\infty_1 = (x + \alpha y, y),$$

and we can check by direct computation that

$$\Phi_1 = \left(\frac{1}{x} + \alpha(\frac{b_{-3,n}}{x^2} + \ldots + \frac{b_{1-n,n}}{x^{n-1}})y^n + o(y^n), \frac{y}{x} + o(y)\right)$$

so that we deduce $b_{m,n} = 0$ for $m \neq -2$. On the other hand

$$\Psi^0_2 = (x + \beta y, y), \quad \Psi^\infty_2 = \left(\frac{x}{1+\beta y} + \beta b_{1-n,n} y^n + o(y^n), \frac{y}{1+\beta y}\right),$$

we see in a similar way that $a^2_{-3,n} = \beta b_{-2,n} = 0$. We conclude by induction. $\square$

Lemma 5.22. *If $b_{-2,3} = 0$ and $\vartheta_i = (\alpha_i, 0, 0)$ with $\alpha_1 \neq \alpha_2$ both non zero, then $b_{m,n} = 0$ for every (m,n) and the neighborhood is linearizable.*

Proof. We prove by induction on n, in a very similar way, that $b_{m,n} = 0$ for $m \neq -2$, until $n = 7$: for instance, at each step, we get $a^1_{m,n} = \alpha_1 b_{m,n}$ (we use only two fibrations so far). Then, for $n = 8$ we find that $a^i_{-3,8} = \alpha_i(b_{-3,8} - \alpha_i b_{-2,4}) = 0$. Of course, since $\alpha_1 \neq \alpha_2$ we get both $b_{-3,8} = b_{-2,4} = 0$. For $n > 8$, the induction shows that $b_{-2,n-4} = b_{-3,n} = \cdots = b_{1-n,n} = 0$ and we get the result. □

Proof of Theorem 5.1. We consider the following cases.

Case 1: $\mathrm{Tang}(\mathcal{H}, \mathcal{H}_1) \cap \mathrm{Tang}(\mathcal{H}, \mathcal{H}_2) = \emptyset$**.** We change parametrization $x : C_0 \to \mathbb{P}^1$ so that $\mathrm{Tang}(\mathcal{H}, \mathcal{H}_1) \cap C_0 = \{x = 0\}$ and $\mathrm{Tang}(\mathcal{H}, \mathcal{H}_2) \cap C_0 = \{x = \infty\}$, which implies $\beta_1 = \alpha_2 = 0$. Note that $\alpha_1\beta_2 \neq 0$ because we are not in the case of tangency along C_0.

If $b_{-2,3} \neq 0$, we can assume after change of coordinate $\vartheta = (0, 0, 0, \theta)$ that $b_{-2,3} = 1$. We consider the equations $a^1_{m,n} = 0$ and $a^2_{3,n} = 0$ for $n \leq 7$, we can solve and express all $b_{m,n}$, $n \leq 7$, in terms of α_1, γ_1, β_2 and γ_2. We replace them in the remaining coefficients $a^2_{m,n}$ for $n \leq 7$, $m \neq 3$, and use Groebner basis in order to rewrite the ideal

$$\langle a^2_{-4,5}, a^2_{-4,6}, a^2_{-5,6}, a^2_{-4,7}, a^2_{-5,7}, a^2_{-6,7} \rangle = \langle \gamma_2, \alpha_1\beta_2 \rangle$$

but this implies that $\alpha_1\beta_2 = 0$, which is not possible.

If $b_{-2,3} = 0$ with a similar argument we arrive in $\gamma_1 = \gamma_2 = 0$ and thus we conclude by lemma 5.21.

Case 2: $\mathrm{Tang}(\mathcal{H}, \mathcal{H}_1) \cap \mathrm{Tang}(\mathcal{H}, \mathcal{H}_2) \cap C_0 = \{x = 0\}$**.** In this case we can assume $\vartheta_i = (\alpha_i, 0, \gamma_i, 1)$ with $\alpha_1\alpha_2 \neq 0$. Suppose first (α_1, γ_1) not parallel to (α_2, γ_2).

If $b_{-2,3} = 1$, we use equations $a^1_{m,n} = 0$, $3 \leq m < n \leq 7$, and $a^2_{-3,5} = a^2_{-3,6} = a^2_{-3,7} = 0$ in order to find all $b_{m,n}$ with $n \leq 7$ except $b_{-2,7}$ in terms of α_1, γ_1, α_2 and γ_2. Replacing them in $a^2_{-6,7}$ we obtain $\gamma_1 = 0$ or $\gamma_1 = 2$. The former case implies, by using $a^2_{-3,4}$, that $\gamma_2 = 2$ and this give us the equation $a^2_{-4,6} = 6\alpha_1 = 0$, impossible. On the other hand, the last case gives us $a^2_{-5,7} = 12(\alpha_1\gamma_2 - \alpha_2\gamma_1) = 0$, and this also contradicts our hypothesis. We conclude that we are never in this case.

If $b_{-2,3} = 0$, we use equations $a^1_{m,n} = 0$, $3 \leq i < j \leq 7$, and $a^2_{-3,5} = a^2_{-3,6} = a^2_{-3,7} = 0$ in order to find all $b_{m,n}$ with $n \leq 7$ except $b_{-2,7}$ in terms of α_1, γ_1, α_2 and γ_2. Replacing them in $a^2_{-3,4}$, $a^2_{-4,5}$ and $a^2_{-4,6}$, we arrive in $\gamma_1 = \gamma_2 = 0$. We are in the hypothesis of lemma 5.22 and therefore we conclude the theorem.

Finally we consider the case $(\alpha_2, \gamma_2) = \lambda(\alpha_1, \gamma_1)$, for $\lambda \neq 0, 1$.

If $b_{-2,3} = 0$, from $a^1_{-3,4} = a^2_{-3,4} = 0$ we obtain $\gamma_1 = \gamma_2 = 0$, and then we are able to apply lemma 5.22.

If $b_{-2,3} = 1$, again from $a^1_{-3,4} = a^2_{-3,4} = 0$, we obtain $\gamma_1 = \gamma_2 = 0$ and $a^1_{-3,4} = \alpha_1 b_{-3,4} = 0$. Now, from $a^1_{-3,5} = a^1_{-4,5} = 0$ we also get $b_{-3,5} = b_{-4,5} = 0$. Therefore $a^1_{-3,6} = \alpha_1(b_{-3,6} - \alpha_1) = 0$ and $a^2_{-3,6} = \alpha_2(b_{-3,6} - \alpha_2) = 0$, which is a contradiction since $\alpha_1 \neq \alpha_2$. □

6. Automorphism group

Let Π be a projective structure defined on some neighborhood U of $0 \in \mathbb{C}^2$ by

$$y'' = f(x, y, y') \tag{17}$$

$$\text{with} \quad f(x, y, z) = A(x, y) + B(x, y)z + C(x, y)z^2 + D(x, y)z^3$$

A local diffeomorphism Ψ in U (fixing 0 or not) is an automorphism of the projective structure Π, if it sends geodesics to geodesics of (17). As a consequence, Ψ acts also on the dual neighborhood (U^*, C_0), inducing a diffeomorphism $\check{\Psi}$ from the neighborhood of C_0 onto the neighborhood of (itself or another) $(+1)$-rational curve C inside U^*. Conversely, such a diffeomorphism $\check{\Psi}$ in U^*, between neighborhoods of $(+1)$-rational curves, induces an automorphism Ψ of the projective structure Π as above. Lie showed that the pseudo-group of automorphisms of the projective structure forms a Lie pseudo-group denoted by $\mathrm{Aut}(\Pi)$. Vector fields whose local flow belong to this pseudo-group are called infinitesimal symmetries and form a Lie algebra denoted by $\mathfrak{aut}(\Pi)$. Elements of $\mathfrak{aut}(\Pi)$ obviously correspond to germs of holomorphic vector fields on the dual (germ of) neighborhood (U^*, C_0), and we denote by $\mathfrak{aut}(U^*, C_0)$ the corresponding Lie algebra. Clearly, we have

$$\mathfrak{aut}(\Pi) \simeq \mathfrak{aut}(U^*, C_0).$$

In [22], Lie gives a classification of the possible infinitesimal symmetry algebras for projective structures, showing that they must be isomorphic to one of the following algebras

$$\{0\}, \quad \mathbb{C}, \quad \mathrm{aff}(\mathbb{C}), \quad \mathrm{sl}_2(\mathbb{C}) \quad \text{or} \quad \mathrm{sl}_3(\mathbb{C}) \tag{18}$$

where $\mathrm{aff}(\mathbb{C})$ is the non commutative 2-dimensional Lie algebra corresponding to the affine group acting on the line: it is spanned by X and Y satisfying $[X, Y] = X$.

In their paper [6], R. Bryant, G. Manno and V. Matveev classified two-dimensional local metrics (U, g) whose underlying projective structure (U, Π_g) is such that $\dim \mathfrak{aut}(\Pi_g) = 2$. This problem was settled by Lie himself. As a biproduct, they provide in [6, section 2.3] a list of almost unique normal forms for generic local projective structures (U, Π) with $\dim \mathfrak{aut}(\Pi) = 2$. In section 6.2, we give a precise statement of this, completed with other possible dimensions $\dim \mathfrak{aut}(\Pi) = 1, 2, 3, 8$.

6.1. Preliminaries.

Let us start with some considerations in the case the projective structure (U, Π) is invariant by one (regular) vector field, say ∂_y.

Lemma 6.1. *Let $X = \partial_y$ be a non trivial symmetry of a projective structure (U, Π) and let $\check{X}$ be the dual vector field on (U^*, C_0). Then the differential equation for the projective structure takes the form*

$$y'' = A(x) + B(x)(y') + C(x)(y')^2 + D(x)(y')^3 \tag{19}$$

and we have the following possibilities:

- *$D(0) \neq 0$ and $\check{X}$ is regular on (U^*, C_0), with exactly one tangency with C_0;*
- *$D(0) = 0$ but $D \not\equiv 0$ and $\check{X}$ has an isolated singular point on (U^*, C_0);*
- *$D \equiv 0$ and $\check{X}$ has a curve Γ of singular points on (U^*, C_0); moreover, Γ is transversal to C_0 and the saturated foliation $\mathcal{F}_{\check{X}}$ defines a fibration transversal to C_0.*

Proof. The normal form (19) follows from a straithforward computation. Clearly, $\check{X}$ has a singular point at $p \in U^*$ if, and only if, the corresponding geodesic in U is a trajectory of X (i.e. is X-invariant). Therefore, up to shrinking the neighborhoods U and U^*, we have 3 possibilities:

- $D \neq 0$ on U and $\check{X}$ is regular on (U^*, C_0);
- D vanishes exactly along $x = 0$ which is therefore geodesic, and $\check{X}$ has an isolated singular point at the corresponding point on (U^*, C_0);
- $D \equiv 0$, the foliation $dx = 0$ defined by X is geodesic, and $\check{X}$ has a curve Γ of singular points on (U^*, C_0).

In the first case, the restriction $\check{X}|_{C_0}$ cannot be identically tangent to $TC_0 \simeq \mathcal{O}_{\mathbb{P}^1}(2)$, otherwise it would have a singular point; it thus defines a non trivial section of $NC_0 \simeq \mathcal{O}_{\mathbb{P}^1}(1)$ which must have a single zero, meaning that $\check{X}$ has a single tangency with C_0.

The second case we are not interested, since moving to a nearby point of U, we can assume that we are in the first case.

In the third case, each trajectory of X is geodesic, so is the induced foliation $dx = 0$. By duality, this foliation defines a cross section to C_0 consisting of singular points of $\check{X}$. The saturated foliation $\mathcal{F}_{\check{X}}$ is locally defined by $\check{X}$ oustide Γ, and by the vector field $\frac{1}{f}\check{X}$ near Γ, where f is a reduced equation of Γ. But $\frac{1}{f}\check{X}$ induces a non zero section of NC_0 (near Γ) since it must be less vanishing; $\frac{1}{f}\check{X}$ is therefore transversal to C_0, and so is $\mathcal{F}_{\check{X}}$. □

Lemma 6.2 (Cartan [8, p.78-83])**.** *Let $\mathcal{W} = \mathcal{F}_0 \boxtimes \mathcal{F}_1 \boxtimes \mathcal{F}_\infty$ be a regular 3-web on $(\mathbb{C}, 0)$, and let $\mathfrak{aut}(\mathcal{W})$ be the Lie algebra of vector fields whose flow preserve $\mathcal{W}$. If $\mathfrak{aut}(\mathcal{W}) \neq 0$ then we are in one of the two cases, up to change of coordinates:*

- *$\mathfrak{aut}(\mathcal{W}) = \mathbb{C}\partial_y$ and $\mathcal{W} = dy \boxtimes (dy - dx) \boxtimes (dy + f(x)(dy - dx))$, with f analytic, not of the form $f(x) = ae^{bx}$, $a, b \in \mathbb{C}$;*
- *$\mathfrak{aut}(\mathcal{W}) = \mathbb{C}\langle \partial_x, \partial_y, x\partial_x + y\partial_y \rangle$ and $\mathcal{W} = dy \boxtimes (dy - dx) \boxtimes dx$.*

Lemma 6.3. *Under assumptions and notations of Lemma 6.1, assume that we are in the last case $D \equiv 0$. If the singular set Γ of $\check{X}$ is a fiber of the fibration defined by $\mathcal{F}_{\check{X}}$ on (U^*, C_0), then (U, Π) is linearizable.*

Proof. Take 3 different fibers of $\mathcal{F}_{\check{X}}$ different from Γ, they define a 3-web $\mathcal{W} = \mathcal{F}_0 \boxtimes \mathcal{F}_1 \boxtimes \mathcal{F}_\infty$ which is invariant by X. By Cartan's Lemma 6.2, we can assume $X = \partial_y$ and $\mathcal{W} = dy \boxtimes (dy - dx) \boxtimes (dy + f(x)(dy - dx))$ so that the flat structure of Π is defined by the pencil $dy + zf(x)(dy - dx) = 0$. But the foliation $dx = 0$ defined by X is, by assumption, belonging to the pencil, which means that Π is also defined by the hexagonal 3-web $dy \boxtimes (dy - dx) \boxtimes dx$, thus linearizable. □

6.2. **Classification of projective structures with Lie symmetries.** The problem of this section is to classify those local projective structures (U, Π) having non trivial Lie symmetry, i.e. such that $\dim \mathfrak{aut}(\Pi) > 0$, up to change of coordinates. However, in this full generality, the problem is out of reach; indeed, it includes for instance the problem of classification of germs of holomorphic vector fields (with arbitrary complicated singular points), which is still challenging. Instead of this, and in the spirit of Lie's work, we produce a list of possible normal forms up to change of coordinates for such a (U, Π) at a generic point $p \in U$, i.e. outside a closed analytic subset consisting of singular features. For instance, a non trivial vector field is regular at a generic point and can be rectified to ∂_y; we only consider

this constant vector field in the case $\dim \mathfrak{aut}(\Pi) = 1$. The following resumes some results of [6, section 2.3].

Theorem 6.4. *Let (U, Π) be a projective structure with $\mathfrak{aut}(\Pi) \neq \{0\}$. Then, at the neighborhood of a generic point $p \in U$, the pair $(\Pi, \mathfrak{aut}(\Pi))$ can be reduced by local change of coordinate to one of the following normal forms:*

(i) $\mathfrak{aut}(\Pi) = \mathbb{C} \cdot \partial_y$ *and* $(A, B, C, D) =$
  (i.a) $(A(x), B(x), 0, 1)$ *with* $A, B \in \mathbb{C}\{x\}$*;*
  (i.b) $(A(x), 0, e^x, 0)$ *with* $A \in \mathbb{C}\{x\}$*;*
(ii) $\mathfrak{aut}(\Pi) = \mathbb{C}\langle \partial_y, \partial_x + y\partial_y \rangle$ *and* $(A, B, C, D) =$
  (ii.a) $(\alpha e^x, \beta, 0, e^{-2x})$ *with* $\alpha, \beta \in \mathbb{C}$, $(\alpha, \beta) \neq (0, 2), (0, \frac{1}{2})$*;*
  (ii.b) $(\alpha e^x, 0, e^{-x}, 0)$ *with* $\alpha \in \mathbb{C}$*;*
(iii) $\mathfrak{aut}(\Pi) = \mathbb{C}\langle \partial_y, \partial_x + y\partial_y, y\partial_x + \frac{y^2}{2}\partial_y \rangle$ *and* $(A, B, C, D) = (0, \frac{1}{2}, 0, e^{-2x})$*;*
(iv) $\mathfrak{aut}(\Pi) = \mathrm{sl}_3(\mathbb{C})$ *and* $(A, B, C, D) = (0, 0, 0, 0)$.

These normal forms are unique, except for the case (i.a)*, which is unique up to the $\mathbb{C}^*$-action:*

$$(A(x), B(x), 0, 1) \xrightarrow{\lambda \in \mathbb{C}^*} (\lambda^3 A(\lambda^2 x), \lambda^2 B(\lambda^2 x), 0, 1).$$

Remark 6.5. The normal forms for $\mathfrak{aut}(\Pi)$ in the statement correspond to the list of transitive local actions of Lie algebras listed in (18), except that $\mathrm{sl}_2(\mathbb{C})$ has also exotic representations generated by

$$X = \partial_y, \quad Y = \partial_x + y\partial_y \quad \text{and} \quad Z = (y + c_1 e^x)\partial_x + (\frac{y^2}{2} + c_2 e^{2x})\partial_y, \quad c_1, c_2 \in \mathbb{C}.$$

Only the standart one occurs as symmetry algebra of a projective structure.

Remark 6.6. Case (iii) corresponds to the special structure studied in section 5.4

$$\Pi_0 \ : \ y'' = (xy' - y)^3$$

at the neighborhood of any point $p \neq (0, 0)$ and is invariant under the standart action of $\mathrm{sl}_2(\mathbb{C})$

$$\mathfrak{aut}(\Pi_0) = \mathbb{C}\langle x\partial_y, \frac{1}{2}(-x\partial_x + y\partial_y), -\frac{1}{2}y\partial_x \rangle$$

However, at $p = (0, 0)$, the Lie algebra is singular, which is excluded from the list of Theorem 6.4. Case (iv) corresponds to the linearizable case $y'' = 0$.

Proof of Theorem 6.4. We essentially follow [6, section 2.3]. Let us start with the case $\mathfrak{aut}(\Pi) = \mathbb{C}\langle X \rangle$. At a generic point $p \in U$, the vector field X is regular and we can choose local coordinates such that $X = \partial_y$. One easily deduce that the equation (17) for the projective structure, being X-invariant, takes the form $y'' = f(x, y, y')$ with

$$f(x, y, z) = A(x) + B(x)z + C(x)z^2 + D(x)z^3. \tag{20}$$

The normalizing coordinates for X are unique up to a change of the form

$$\Phi : (x, y) \mapsto (\psi(x), y + \phi(x)), \quad \psi(0) = \phi(0) = 0, \ \psi'(0) \neq 0.$$

The projective structure $\Phi^*\Pi$ is defined by

$$f(x, y, z) = \widehat{A}(x) + \widehat{B}(x)z + \widehat{C}(x)(z)^2 + \widehat{D}(x)(z)^3$$

where (we decompose for simplicity)

$$\text{when } \Phi = (\psi(x), y), \text{ then } \begin{cases} \widehat{A} = A \circ \psi \cdot (\psi')^2 \\ \widehat{B} = B \circ \psi \cdot \psi' + \frac{\psi''}{\psi'} \\ \widehat{C} = C \circ \psi \\ \widehat{D} = \frac{D \circ \psi}{\psi'} \end{cases} \tag{21}$$

$$\text{when } \Phi = (x, y + \phi(x)), \text{ then } \begin{cases} \widehat{A} = A + B\phi' + C(\phi')^2 + D(\phi')^3 - \phi'' \\ \widehat{B} = B + 2C\phi' + 3D(\phi')^2 \\ \widehat{C} = C + 3D\phi' \\ \widehat{D} = D \end{cases} \tag{22}$$

If $D \not\equiv 0$, then we can assume at a generic point that $D \neq 0$. We can normalize $\widehat{D} = 1$ by setting $\psi^{-1}(x) := \int_0^x \frac{d\zeta}{D(\zeta)}$ in the first change, and then normalize $\widehat{C} = 0$ by setting $\phi'(x) = -C/3$ (which does not change $D = 1$): $(A(x), B(x), 0, 1)$.

Since we are interested in the Lie algebra, more than a given vector field, then we can also change $\Phi(x, y) = (x, ay)$ with $a \neq 0$ and get the form

$$\text{when } \Phi = (x, ay), \text{ then } (\widehat{A}, \widehat{B}, \widehat{C}, \widehat{D}) = (a^{-1}A, \ \ B, \ \ aC, \ \ a^2 D). \tag{23}$$

Finally, $\Phi = (a^2 x, ay)$, a combination of (21) and (23), yields the new normal form

$$(a^3 A(a^2 x), a^2 B(a^2 x), 0, 1)$$

whence the $\mathbb{C}^*$-action of the statement.

Suppose now $D \equiv 0$. If C would be constant then, by (2), $L_1 = L_2 = 0$ and Π is linearizable. Passing to a generic point, we can assume $C'(0) \neq 0$ and use changes (21) and (23) to normalize $C = e^x$. Finally by using (22), we arrive in the unique desired normal form $(A(x), 0, e^x, 0)$. In this case the equation is never linearizable, since by (2) we get $(L_1, L_2) = (0, 2e^{2x})$.

Now we study the case $\mathfrak{aut}(\Pi) = \mathbb{C}\langle X, Y\rangle$, with $[X, Y] = X$. By [6, Lemma 1], we know that, at a generic point, we can find coordinates where $X = \partial_y$, $Y = \partial_x + y\partial_y$. The invariance of the projective structure by both the flows of X and Y yields

$$(A, B, C, D) = (\alpha e^x, \beta, \gamma e^{-x}, \delta e^{-2x}),$$

were α, β, γ and δ are constants. The normalizing coordinates for the Lie algebra are unique up to a change

$$\Phi = (x, ay + be^x)$$

with $a, b \in \mathbb{C}$, $a \neq 0$. If $\delta \neq 0$, we obtain a unique normal form $(\alpha e^x, \beta, 0, e^{-2x})$. By [6, Lemma 4], the cases $(\alpha, \beta) = (0, 2)$ and $(0, \frac{1}{2})$ have more symmetries: they respectively correspond to the $\mathrm{sl}_3(\mathbb{C})$ and $\mathrm{sl}_2(\mathbb{C})$ cases.

When $\delta = 0$, we shall have $\gamma \neq 0$ (otherwise Π would be linearizable), and we can normalize $(\widehat{A}, \widehat{B}, \widehat{C}, \widehat{D}) = (\alpha e^x, 0, e^{-x}, 0)$, with $\alpha \in \mathbb{C}$; this normal form is unique.

The case $\mathfrak{aut}(\Pi) = \mathrm{sl}_2(\mathbb{C})$ follows directly from [6, Lemma 4]. □

In Theorem 6.4, normal forms (i) contain some models with larger symmetry Lie algebra, and we end the section by determining them. First of all, the projective structure is linearizable when Liouville invariants L_1 and L_2 given by (2) are identically zero, and we get:

$$(L_1, L_2) = (-3A'(x), -3B'(x)) \text{ for model (i.a), and}$$
$$(L_1, L_2) = (-e^{-x}, -2e^{-2x}) \text{ for model (i.b).}$$

Linearizability only occur in model (i.a) when A and B are simultaneously constant.

In the case $\mathfrak{aut}(\Pi) = \mathrm{aff}(\mathbb{C})$, there must exists a vector field $v \in \mathfrak{aut}(\Pi)$ such that

$$[\partial_y, v] = \partial_y \quad \text{or} \quad c \cdot v, \quad c \in \mathbb{C}.$$

This implies that v takes the respective form

$$v = \alpha(x)\partial_x + (y + \beta(x))\partial_y \quad \text{or} \quad e^{cy}(\alpha(x)\partial_x + \beta(x)\partial_y).$$

We can furthermore assume $\alpha(0) \neq 0$ so that the local action is transitive together with ∂_y; moreover, $c \neq 0$, otherwise the two vector fields commute, which is excluded in the non linearizable case. Let us firstly discuss the case of normal form (i.a). In the case where ∂_y is stabilized by $\mathfrak{aut}(\Pi)$, by using [6, formula (3)] for $v = \alpha\partial_x + (y + \beta)\partial_y$, one easily deduce that

$$v = 2(x + \alpha_0)\partial_x + (y + \beta_0)\partial_y \quad \text{and} \quad (A, B, C, D) = \left(\frac{\gamma_0}{(x + \alpha_0)^{3/2}}, \frac{\delta_0}{(x + \alpha_0)}, 0, 1\right),$$

with $\alpha_0, \beta_0, \gamma_0, \delta_0 \in \mathbb{C}$, $\alpha_0 \neq 0$. The second case $v = e^{cy}(\alpha(x)\partial_x + \beta(x)\partial_y)$ is less explicit. The invariance of the projective structure in normal form (i.a) allows us to express everything in terms of $\alpha(x)$ and its derivatives:

$$\beta = \frac{\alpha' - c^2\alpha}{2c} \quad \text{and} \quad (A, B, C, D) = \left(\frac{\alpha''' - c^4\alpha'}{4c^3\alpha}, -\frac{3\alpha'' + c^4\alpha}{4c^2\alpha}, 0, 1\right),$$

and finally yields the following differential equation for α

$$c^4(\alpha\alpha'' - (\alpha')^2) - 3c^2(\alpha\alpha''' - \alpha'\alpha'') + 2\alpha\alpha'''' + \alpha'\alpha''' - 3(\alpha'')^2.$$

Once we know the 3-jet of α, then we can deduce all the coefficients by mean of this differential equation. Mind that we can set $\alpha(0) = 1$ so that we get a 4-parameter family of projective structures, taking into account the constant c, that can further be normalized to $c = 1$ by using the $\mathbb{C}^*$-action. Equivalently, the family of projective structures is given by the solutions of the system of differential equations

$$A' = \frac{27cA^2 + 9AB' - 3c(B + c^2)B' + c(4B + c^2)(B + c^2)^2}{6(B + c^2)}$$

$$B'' = -\frac{27cA(cA - B') - 12(B')^2 - 9c^2(B + c^2)B' + c^2(4B + c^2)(B + c^2)^2}{6(B + c^2)}$$

and we can recover α and β by:

$$\frac{\alpha'}{\alpha} = -\frac{3cA + B'}{B + c^2} \quad \text{and} \quad \beta = \frac{\alpha' - c^2\alpha}{2c}.$$

For normal forms (i.b), the discussion is similar, easier though, and one find $v = -\partial_x + (y + c)\partial_y$, $c \in \mathbb{C}$, with projective structure $(\alpha_0 e^{-x}, 0, e^x, 0)$.

Finally, for the case $\mathfrak{aut}(\Pi) = \mathrm{sl}_2(\mathbb{C})$, we just note that ∂_y must be contained in a 2-dimensional affine Lie subalgebra, and we are in a particular case of the previous one.

6.3. **Symmetries of flat projective structures.** Here, we classify those projective structures having simultaneously a flat structure (see section 3) and Lie symmetries. In other words, we describe which models in the list of Theorem 6.4 have a flat structure, and how many. Recal (see section 5) that a given projective structure, if not linearizable, has at most 2 flat structures.

Theorem 6.7. *Let (U,Π) be a flat projective structure with Lie symmetries: $\mathfrak{aut}(\Pi)\neq\{0\}$. Then, at the neighborhood of a generic point $p\in U$, the pair $(\Pi,\mathfrak{aut}(\Pi))$ and pencil of geodesic foliations $\mathcal{F}_z:\omega_0+z\omega_\infty$ can be reduced by local change of coordinate to one of the following normal forms:*

(i) $\mathfrak{aut}(\Pi)=\mathbb{C}\cdot\partial_y$, $(A,B,C,D)=$

(i.a.1) $(0,0,1+g',g)$ *and* $\mathcal{F}_z:\underbrace{e^y(dx+g(x)dy)}_{\omega_0}+z\underbrace{dy}_{\omega_\infty}$;

(i.a.2) $(0,0,g',1)$ *and* $\mathcal{F}_z:\underbrace{-(dx+(g(x)+y)dy)}_{\omega_0}+z\underbrace{dy}_{\omega_\infty}$;

(i.b) $(0,0,g',0)$ *and* $\mathcal{F}_z:\underbrace{dx+g(x)dy}_{\omega_0}+z\underbrace{dy}_{\omega_\infty}$;

(ii) $\mathfrak{aut}(\Pi)=\mathbb{C}\langle\partial_y,\partial_x+y\partial_y\rangle$ *and* $(A,B,C,D)=$

(ii.a) $(\alpha e^x,\beta,0,e^{-2x})$ *with* $\alpha,\beta\in\mathbb{C}$ *belonging to the cubic nodal curve*

$$\begin{cases}\mathbb{C} & \to \quad \Gamma=\{27\alpha^2+4\beta^3-12\beta^2+9\beta-2=0\}\subset\mathbb{C}^2\\ \gamma & \mapsto \quad (\gamma(2\gamma^2-1),2-3\gamma^2)\end{cases}\tag{24}$$

and the corresponding flat structure is given by

$$\mathcal{F}_z:\underbrace{e^x(\gamma y+(2\gamma^2-1)e^x)dx-(y+2\gamma e^x)dy}_{\omega_0}+z\underbrace{(dy-\gamma e^x dx)}_{\omega_\infty}.$$

Here, we exclude the cases $\alpha=0$ corresponding to (iii) and (iv) below.

(ii.b.1) $\left(\frac{1-\lambda^2}{4}e^x,0,e^{-x},0\right)$ *with* $\lambda\in\mathbb{C}^*$, *and*

$$\mathcal{F}_z:\underbrace{e^{\lambda x}\left[dy-\left(\frac{1-\lambda}{2}\right)e^x dx\right]}_{\omega_0}+z\underbrace{\left[dy-\left(\frac{1+\lambda}{2}\right)e^x dx\right]}_{\omega_\infty};$$

(ii.b.2) $(\frac{e^x}{4},0,e^{-x},0)$ *and* $\mathcal{F}_z:\underbrace{(1-\frac{x}{2})e^x dx+xdy}_{\omega_0}+z\underbrace{(dy-\frac{1}{2}e^x dx)}_{\omega_\infty}$;

(iii) $\mathfrak{aut}(\Pi)=\mathbb{C}\langle\partial_y,\partial_x+y\partial_y,y\partial_x+\frac{y^2}{2}\partial_y\rangle$ *and* $(A,B,C,D)=(0,\frac{1}{2},0,e^{-2x})$;

(iv) $\mathfrak{aut}(\Pi)=\mathrm{sl}_3(\mathbb{C})$ *and* $(A,B,C,D)=(0,0,0,0)$.

Case (iii) corresponds to the case (ii.a) with $\gamma=\pm\frac{1}{\sqrt{2}}$; the two values of γ provide the two flat structures for Π in this case. Case (iv) corresponds to the case (ii.a) with $\gamma=0$; in that case, all flat structures are described in example 3.3.

Lemma 6.8. *Let (U,Π) be a projective structure with Lie symmetry $X=\partial_y$ and flat structure $\mathcal{F}_{\omega_z}$, with $\omega_z=\omega_0+z\omega_\infty$. If (U,Π) is not linearizable, then*

- *the flow ϕ_X^t of X must preserve the flat structure,*
- *the flow ϕ_X^t must preserve at least one foliation of the pencil, say $\mathcal{F}_{\omega_\infty}$,*
- *no element of the pencil $\mathcal{F}_{\omega_z}$ can coincide with the foliation $\mathcal{F}_X:\{dx=0\}$, and after change of coordinate $y\mapsto y+\phi(x)$, we may furthermore assume $\omega_\infty=dy$.*

In particular, at the neighborhood of a generic point $p\in U$, we may furthermore assume $\omega_\infty=dy$ in convenient coordinates.

Proof. The vector field induces an action on geodesics, and therefore on the dual space (U^*,C_0); denote by $\check{X}$ the infinitesimal generator. Let $\mathcal{H}$ be the transverse

fibration corresponding to the falt structure. If $\mathcal{H}$ is not invariant by the flow $\phi^t_{\check{X}}$, then we deduce a 1-parameter family $\mathcal{H}_t = (\phi^t_{\check{X}})_*\mathcal{H}$ and deduce from Theorem 5.1 that Π is linearizable, contradiction. Therefore, $\check{X}$ preserves $\mathcal{H}$ and acts on the space of leaves $\simeq \mathbb{P}^1_z$. In particular, it has a fixed point, corresponding to an X-invariant foliation in the pencil, say $\mathcal{F}_{\omega_\infty}$.

Assume for contradiction that the foliation $\mathcal{F}_X$, defined by $dx = 0$, coincides with one of the $\mathcal{F}_{\omega_z}$'s; since it is X-invariant, we can assume $z = \infty$. Therefore, we are in the third case of Lemma 6.1: $\check{X}$ has a curve $\Gamma \subset U^*$ of singular points transversal to C_0. Moreover, Γ is $\mathcal{H}$-invariant and $\check{X}$ defines another transverse fibration $\mathcal{F}_{\check{X}}$. If Γ is invariant by $\mathcal{F}_{\check{X}}$, then Lemma 6.3 implies that (U, Π) is linearizable, contradiction. Therefore, Γ is not invariant by $\mathcal{F}_{\check{X}}$, and in particular, the fibrations $\mathcal{F}_{\check{X}}$ and $\mathcal{H}$ do not coincide. Consider the tangency set $\Sigma := \mathrm{tang}(\mathcal{F}_{\check{X}}, \mathcal{H})$. Since $\mathcal{H}$ is $\check{X}$-invariant, Γ must be $\check{X}$-invariant. Clearly, Σ is not contained in the singular set $\mathrm{sing}(\check{X}) = \Gamma$ and $\mathcal{H}$ is thus $\mathcal{F}_{\check{X}}$-invariant. Again, this means that Γ is a common fiber of $\mathcal{F}_{\check{X}}$ and $\mathcal{H}$, and Lemma 6.3 implies that (U, Π) is linearizable, contradiction. □

Proof of Theorem 6.7. Let us start with the case $\mathfrak{aut}(\Pi) = \mathbb{C} \cdot X$ with $X = \partial_y$, and assume (U, Π) not linearizable. Then, by Lemma 6.8, X preserves the pencil of foliation and acts on the parameter space $z \in \mathbb{P}^1$ fixing $z = \infty$. More precisely, we can assume $\omega_\infty = dy$ and that the action on the pencil is induced by one of the following vector fields

$$z\partial_z, \quad \partial_z \quad \text{or} \quad 0.$$

In the first case, we must have that $(\phi^t_X)_*\omega_z$ is proportional to $\omega_{e^t z}$ for any $t, z \in \mathbb{C}$; since $\omega_z = \omega_0 + z\omega_\infty$ and ω_∞ is ϕ^t_X-invariant, we deduce

$$(\phi^t_X)_*\omega_0 = e^{-t}\omega_0.$$

This implies that $\omega_0 = e^y(f(x)dx + g(x)dy)$ for some functions $f, g \in \mathbb{C}\{x\}$, $f(0) \neq 0$. After taking $\int f(x)dx$ as a new coordinate, we get the normal form

$$\omega_0 = e^y(dx + g(x)dy).$$

We easily derive the projective structure Π by derivating "ω_0/ω_∞":

$$0 = \left(e^y\left(\frac{1}{y'} + g\right)\right)' = e^y y'\left(\frac{1}{y'} + g\right) + e^y\left(-\frac{y''}{(y')^2} + g'\right)$$

$$\Rightarrow y'' = (1 + g')(y')^2 + (g)(y')^3, \quad \text{i.e. } (A, B, C, D) = (0, 0, 1 + g', g).$$

If the action is now induced by ∂_z, then we get

$$(\phi^t_X)_*\omega_0 = \omega_0 + t\omega_\infty$$

which gives $\omega_0 = f(x)dx + (g(x) - y)dy$, where again we can normalize $f \equiv 1$ which gives the projective structure

$$\omega_0 = -(dx + (g(x) + y)dy \quad \text{and} \quad (A, B, C, D) = (0, 0, g', 1).$$

Finally, when the action is trivial on the parameter space z, we get that ω_0 is also invariant, i.e. of the form $f(x)dx + g(x)dy$; we can again normalize $f \equiv 1$ and get

$$\omega_0 = dx + g(x)dy \quad \text{and} \quad (A, B, C, D) = (0, 0, g', 0).$$

Let us now consider the case $\mathfrak{aut}(\Pi) = \mathbb{C}\langle X, Y\rangle$ with $X = \partial_y$ and $Y = \partial_x + y\partial_y$, and still assume (U, Π) not linearizable. Like before, the Lie algebra preserves the pencil $\mathcal{F}_z$ and induces an action on the parameter space of the form

$$(X, Y)|_z = (\partial_z, z\partial_z), \quad (0, \lambda z\partial_z), \quad (0, \partial_z), \quad \text{or} \quad (0, 0),$$

with $\lambda \in \mathbb{C}^*$. We note that we cannot normalize $\lambda = 1$ by homothecy since Y has to satisfy $[X, Y] = X$ in the (x, y)-variables; different values of λ will correspond to different projective structures.

In any case for $(X, Y)|_z$, $\mathcal{F}_\infty$ is fixed, and this means that we can write

$$\omega_\infty = d(y - \gamma e^x)$$

for some $\gamma \in \mathbb{C}$. Indeed, the invariance by X means that the leaves of $\mathcal{F}_\infty$ are ∂_y-translates of the leaf $y = f(x)$ passing through the origin, i.e. we can choose $\omega_\infty = d(y - f(x))$; then, the invariance by Y gives the special form $f(x) = \gamma e^x$. Here, we have used Lemma 6.8 to insure that, maybe passing to a generic point $p \in U$, we can assume that $\mathcal{F}_\infty$ is not vertical at p.

If $(X, Y)|_z = (\partial_z, z\partial_z)$, then we can check that

$$\omega_0 = (\alpha e^{2x} + \gamma e^x y)dx + (\beta e^x - y)dy$$

for some constants $\alpha, \beta \in \mathbb{C}$. This normalization is unique up to a change of coordinate of the form $\Phi = (x, ay + be^x)$ preserving the Lie algebra; this allow to reduce the corresponding projective structure Π into the normal form (ii.a) of Theorem 6.4, yielding after straightforward computation the formulae (ii.a) of Theorem 6.7.

If $(X, Y)|_z = (0, \lambda z\partial_z)$, then we find

$$\omega_0 = e^{\lambda x}\left(\alpha e^x dx + \beta e^x dy\right)$$

which gives after normalization

$$\omega_z = e^{\lambda x}\left[dy - \left(\frac{1-\lambda}{2}\right)e^x dx\right] + z\left[dy - \left(\frac{1+\lambda}{2}\right)e^x dx\right]$$

$$\text{and} \quad (A, B, C, D) = \left(\frac{1-\lambda^2}{4}e^x, 0, e^{-x}, 0\right).$$

If $(X, Y)|_z = (0, \partial_z)$, then we find

$$\omega_0 = (\alpha + \gamma x)e^x dx + (\beta - x)dy$$

which gives after normalization

$$\omega_z = (1 - \frac{x}{2})e^x dx + xdy + z(dy - \frac{1}{2}e^x) \quad \text{and} \quad (A, B, C, D) = \left(\frac{e^x}{4}, 0, e^{-x}, 0\right).$$

Finally, if $(X, Y)|_z = (0, 0)$, then we find

$$\omega_0 = \alpha e^x dx + \beta dy$$

which gives after normalization $(A, B, C, D) = (0, 1, 0, 0)$, which is linearizable. □

Remark 6.9. Projective structures of Theorem 6.7 (i) can be put in normal form as in Theorem 6.4. For instance, in the case (i.a.1), using change(21) , one easily get the following form

$$\omega_z = e^{f(x)y}(dx + dy) + zdy \quad \text{and} \quad (A, B, C, D) = (0, f', 1 + f', 1).$$

The final normalization (22) is not so explicit, but turning the other way round, we can easily check that a normalized projective structure $(A, B, 0, 1)$ comes from such a flat structure iff it satisfies

$$A^2 = -\frac{(4B^2 + 5B - 3B' + 1)^2}{3(4B + 1)}$$

and in this case, $f(x)$ (and the flat structure) is given by

$$f' = \frac{1}{2} + \frac{1}{2}\sqrt{-3(4B + 1)}.$$

In a very similar way, the projective structure $(A, B, 0, 1)$ comes from the case (i.a.2) iff

$$A^2 = -\frac{(4B^2 - 3B')^2}{108B}$$

and in this case, $g(x)$ (and the flat structure) is given by

$$g' = \sqrt{-3B}.$$

Finally, one easily check by similar computations that any normal form (i.b) of Theorem 6.4, i.e. $(A(x), 0, e^x, 0)$, is also flat, i.e. comes from (i.b) of Theorem 6.7.

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