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Stability analysis of a system coupled to a heat equation.

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Abstract: As a first approach to the study of systems coupling finite and infinite dimensional natures, this article addresses the stability of a system of ordinary differential equations coupled with a classic heat equation using a Lyapunov functional technique. Inspired from recent developments in the area of time delay systems, a new methodology to study the stability of such a class of distributed parameter systems is presented here. The idea is to use a polynomial approximation of the infinite dimensional state of the heat equation in order to build an enriched energy functional. A well known efficient integral inequality (Bessel inequality) will allow to obtain stability conditions expressed in terms of linear matrix inequalities. We will eventually test our approach on academic examples in order to illustrate the efficiency of our theoretical results.

Keywords: heat equation, Lyapunov functional, Bessel inequality, polynomial approximation.

1. INTRODUCTION

Coupling a classical finite dimensional system to a partial differential equation (PDE) presents not only interesting theoretical challenges but can also formalize various applicative situations. Effectively, as the solution of the PDE is a state belonging to an infinite dimensional functional space, its coupling with a finite dimensional system brings naturally new difficulties in stability study and/or control of the coupled system. We can also list several specific situations worth being modeled by this king of heterogeneous coupled system. For example, the finite dimensional systems could represent a dynamic controller for a system modeled by a PDE (see Krstic (2009) and references therein). Instead, a system of ordinary differential equations (ODEs) can model a component coupled to a phenomenon described by PDEs as in Daafouz et al. (2014). Conversely, the PDE can model an actuator or sensor’s behavior and the goal could be to study the stabilization of a connected finite dimensional system in spite of it (as e.g. in time delay systems). Actually, the last decade has seen the emergence of number of papers concerning the stability or control of such a coupled system relies on discretization techniques leading to some finite dimensional systems to be studied. The question of convergence of the results is then quite natural and may be complicated to deal with (see Morris (1994)). That’s the reason why several researches have turned to direct approaches: the objective is to determine a Lyapunov functional for the overall system directly, without going through a discretization scheme. This has given rise to many interesting methodologies. Hence, a first one relies on the semi-group theory to model the overall system and it may lead, as in Fridman (2014), to some Linear Operator Inequalities to be solved numerically. Unfortunately, this approach remains quite limited (see Fridman and Orlov (2009)) and works finally only for small dimensional ODE systems since no numerical tools are available to solve these Linear Operator Inequalities. Furthermore, the generic semi-group approach generally fails to develop a constructive approach for the design of Lyapunov functionals.

Another approach considers the design of a Lyapunov functional as usually based on the sum of a classical Lyapunov functional identified for each part of the system under consideration. When dealing with the PDE of a coupled system, its Lyapunov functional is actually the “energy” of the PDE (see Prieur and Mazenc (2012), Papachristodoulou and Peet (2006)). Another example is given is Krstic (2009), where the control of a finite dimensional system connected to an actuator/sensor modeled by a heat equation with Neumann and Dirichlet boundary conditions is considered. The author adopts the backstepping method employed originally in the case of the trans-

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Consider the coupling of a finite dimensional system in the variable \( X \in \mathbb{R}^n \) with a heat partial differential equation in the scalar variable \( u \), in the following way:

\[
\begin{aligned}
\dot{X}(t) &= AX(t) + Bu(1,t) & t &> 0, \\
\partial_t u(x,t) &= \gamma \partial_x x u(x,t), & x &\in (0,1), t > 0, \\
u(0,t) &= CX(t), & t &> 0 \\
\partial_x u(1,t) &= 0, & t &> 0.
\end{aligned}
\]

The state vector of the system is the pair \( (X(t), u(x,t)) \in \mathbb{R}^n \times \mathbb{R} \) and it satisfies the compatible initial datum \( (X(0), u(x,0)) = (X^0, u^0(x)) \) for \( x \in (0,1) \). The thermal diffusivity is denoted \( \gamma \in \mathbb{R}_+ \) and the matrix \( A \in \mathbb{R}^{n \times n} \), the vectors \( B \in \mathbb{R}^{n \times 1} \) and \( C \in \mathbb{R}^{1 \times n} \) are constant.

**Remark 1.** One can imagine different situations that can be translated into the coupled system (1). As a toy problem of more complicated situations, the system we study already allows to face several difficulties inherent to a situation mixing finite and infinite dimensional states. Nevertheless, we can describe two more physical situations that could be simplified as our toy problem: either a finite dimensional system confronted with a thermocouple sensor, or a heat device connected to a finite dimension dynamic controller. Anyway, these are only mere ideas that could link ODEs with a heat PDE and we remain here at a simplified but still challenging level.

**2.2 Existence and regularity of the solutions**

Before anything else, one should know that the partial differential equation \( \partial_t u - \gamma \partial_x x u = 0 \) in (1) of unknown \( u = u(x,t) \) is a classic heat PDE and if the initial datum \( u(\cdot,0) = u^0 \) belongs to \( H^1(0,1) \) and the boundary data are of Dirichlet homogeneous type (i.e. \( u(0,t) = u(1,t) = 0 \)), it has a unique solution \( u \) satisfying

\[
u \in C([0, +\infty); H^1(0,1) \cap L^2(0, +\infty; H^2(0,1))
\]

\[\partial_t u \in L^2(0, +\infty; L^2(0,1)),\]

see e.g. Brezis (1983).

In this article, we are dealing with a coupled system for which we should start with the existence and regularity of the solution \( (X,u) \). A Galerkin method is the key of the proof of such a result (see e.g. Evans (2010)). Actually, in the sake of consistency with the Lyapunov approach we will use in the proof of stability of our coupled system, we give here only the formal idea of this Galerkin energy based method. Hence, let us define the total energy of System (1) by:

\[
E(X(t), u(t)) = |X(t)|_A^2 + \| u(t) \|_{H^1(0,1)}^2.
\]

In the sequel, we will write \( \dot{E}(t) = E(X(t), u(t)) \) in order to simplify the notations.

Easy calculations based on the use of the equations in System (1) and integrations by parts give:

\[
\begin{aligned}
\dot{E}(t) &= X(t)^\top (A^\top + A) X(t) \\
&\quad + u(1,t) (B^\top X(t) + X(t)^\top B) \\
&\quad - 2\gamma \| \partial_x x u(t) \|^2 - 2\gamma \| \partial_x x u(0,t) \| (C + CA) X(t) \\
&\quad - 2\gamma \| \partial_x x u(0,t) \| C B u(1,t) - 2\gamma \| \partial_x x u(t) \|^2.
\end{aligned}
\]
Since we have the Sobolev embeddings $H^1(0, 1) \subset C([0, 1])$ and $H^2(0, 1) \subset C^1([0, 1])$, one can write, omitting the time variable $t$, that $$|u(1)|^2 \leq \|\partial_x u\|^2$$ and $$|\partial_x u(0)|^2 \leq \|\partial_x x u\|^2.$$ Therefore, using Young's inequality $(ab \leq \frac{a^2}{2} + \frac{b^2}{2})$, it's easy to obtain that there exists a constant $K = (A, B, C, \varepsilon) > 0$ such that
\[
\dot{E}(t) \leq K\|X(t)\|^2 + K\|u(1,t)\|^2 - 2\gamma\|\partial_x u(t)\|^2 \\
+ 2\varepsilon\|\partial_x u(t)\|^2 - 2\|\partial_x x u(t)\|^2
\]
\[
\leq K\|X(t)\|^2 + K\|\partial_x u(t)\|^2 + 2(\varepsilon - \gamma)\|\partial_x x u(t)\|^2.
\]
Hence, choosing $0 < \varepsilon < \gamma$, we can either eliminate the last term so that we get $\dot{E}(t) \leq KE(t)$ ensuring the existence of a unique solution $(X, u)$ in the space $\mathbb{R}^n \times C([0, +\infty]; H^1(0, 1))$ or we can move this term to the left hand side of the estimate and get that $u \in L^2(0, +\infty; H^2(0, 1))$ so that we also have, using the heat equation in (1),
\[
\partial_t u \in L^2(0, +\infty; L^2(0, 1)).
\]
These terse explanations allows us to manipulate the solution $(X, u)$ in the appropriate space along this article.

### 2.3 Equilibrium and stability

As proved in the preliminary study Baudouin et al. (2017), if the matrix $A + BC$ is non singular, then system (1) has a unique equilibrium $(X_0 = 0, u_0 \equiv 0) \in \mathbb{R}^n \times H^1(0, 1, \mathbb{R})$. The main result of this article is a possibility, through the verification of tractable conditions, to obtain the following exponential stability around the steady state $(0, 0)$:

**Definition 1.** System (1) is said to be exponentially stable if for all initial conditions $(X^0, u^0) \in \mathbb{R}^n \times H^1(0, 1, \mathbb{R})$, there exist $K > 0$ and $\delta > 0$ such that for all $t > 0$,
\[
E(X(t), u(t)) \leq Ke^{-\delta t} \left(\|X(t)\|^2 + \|u(t)\|^2\right)_{H^1(0, 1)}.
\]

More precisely, our goal is to construct a Lyapunov functional in order to narrow the proof of the stability of the complete infinite dimensional system (1) to the resolution of linear matrix inequalities (LMI).

### 3. MAIN TOOLS

Before stating our main result in the next section, we need to give precise informations about the technical tools we will use in the proof: a Lyapunov functional, some Legendre polynomials and the Bessel inequality.

#### 3.1 Lyapunov functional

Inspired by the complete Lyapunov-Krasovskii functional, which is a necessary and sufficient conditions for stability for delay systems Gu et al. (2003), we consider a Lyapunov functional candidate for system (1) of the form:
\[
V(X(t), u(t)) = X^T(t)PX(t) + 2X^T(t)\int_0^1 Q(x)u(x, t)dx \\
+ \int_0^1 \int_0^1 u^T(x, t)T(x_1, x_2)u(x_2, t)dx_1dx_2 \\
+ \alpha \int_0^1 \|u(x, t)\|^2 dx + \beta \int_0^1 \|u(x, t)\|^2 dx,
\]
where the matrix $P \in \mathcal{S}^n$ and the functions $Q \in C(L^2(0, 1; \mathbb{R}^{n \times m}))$ and $T \in C(L^2(0, 1; \mathbb{S}^m))$ have to be determined.

The first term and the two last terms of $V$ are a weighted version of the classical energy $E(t)$ of the system. The term depending on the function $T$ has been recently considered in the literature in Gahlawat and Peet (2017); Ahmadi et al. (2016). The term depending on $Q$ is introduced in order to represent the coupling between the system of ODEs and the heat equation.

Our objective is to specify this Lyapunov functional in order to reduce the proof of the stability of the complete infinite dimensional system (1) to the resolution of LMI. Since a part of the state $(X, u)$ of the system is distributed $(u$ being the solution of a heat equation and depending on a space variable $x$ in addition to the time $t$), it is proposed to impose a special structure for the functions $Q$ and $T$ in order to obtain numerically tractable stability conditions. The two functions will actually be build as projection operators over a finite dimensional orthogonal family: the $N + 1$ first shifted Legendre polynomials.

#### 3.2 Properties of Legendre Polynomials

Let us define here the shifted Legendre polynomials considered over the interval $[0, 1]$ and denoted $\{L_k\}_{k \in \mathbb{N}}$. Instead of giving the explicit formula of these polynomials, we detail here their principal properties. One can find details and proofs in the book by Courant and Hilbert (1989).

to begin with, the family $\{L_k\}_{k \in \mathbb{N}}$ is known to form an orthogonal basis of $L^2(0, 1; \mathbb{R})$ since
\[
\langle L_j, L_k \rangle = \int_0^1 L_j(x) L_k(x) dx = \frac{1}{2k + 1} \delta_{jk},
\]
where $\delta_{jk}$ denotes the Kronecker delta, equal to 1 if $j = k$ and 0 to otherwise. We denote the corresponding norm of this inner scalar product $\|L_k\| = \sqrt{\langle L_k, L_k \rangle} = 1/\sqrt{2k + 1}$. The boundary values are given by:
\[
L_0(0) = (-1)^k, \quad L_k(1) = 1.
\]

The first few shifted Legendre polynomials are: $L_0(x) = 1$, $L_1(x) = 2x - 1$, $L_2(x) = 6x^2 - 6x + 1$.

**Remark 2.** For the record, the classical Legendre polynomials are defined on $[-1, 1]$ as the orthonormalization of the family $\{1, x, x^2, x^3, \ldots\}$ but we need here to work on the interval $[0, 1]$.

Furthermore, the following derivation formula holds:
\[
L'_k(x) = \begin{cases} 0, & k = 0, \\
\sum_{j=0}^{k-1} (2j + 1)(1 - (-1)^{k+j})L_j(x), & k \geq 1,
\end{cases}
\]
from which, denoting
\[
l_{kj} = \begin{cases} (2j + 1)(1 - (-1)^{k+j}) & \text{if } j \leq k - 1, \\
0 & \text{if } j \geq k,
\end{cases}
\]
we deduce that
\[ L_k''(x) = \begin{cases} 0 & \text{if } k = 0 \text{ or } 1, \\ \sum_{j=1}^{k-1} -\ell_k \ell_j L_j(x) & \forall k \geq 2. \end{cases} \] (6)

It is now important to notice that any element \( y \in L^2(0,1) \) can be written
\[ y = \sum_{k \geq 0} \left( y, \frac{L_k}{\|L_k\|} \right) \frac{L_k}{\|L_k\|}. \]

Let us set here
\[ U_N(t) = \text{Vect}_{k=0..N} \left\langle u(t), L_k \right\rangle \text{ in } \mathbb{R}^{N+1}, \]
\[ \mathbf{I}_N = [1 \ldots 1]^\top \text{ in } \mathbb{R}^{N+1}, \]
\[ \mathbf{I}_N^* = [-1 \ldots -1]^\top \text{ in } \mathbb{R}^{N+1}, \]
\[ L_N = (\ell_{ij})_{i,j=0..N} \text{ in } \mathbb{R}^{N+1,N+1}, \]
\[ \mathcal{I}_N = \text{diag}(1,3,\ldots,2N+1) \text{ in } \mathbb{R}^{N+1,N+1}. \] (7)

One should notice that for all \( N \in \mathbb{N}^* \), the \( L_N \) matrices are strictly lower triangular thanks to the definition (5). The following notations, that we will use below, stem from this:
\[
L_N = [L_{1,N} \ 0_{N+1,1}] \text{ with } L_{1,N} \text{ in } \mathbb{R}^{N+1,N}, \\
L_N^2 = [L_{2,N} \ 0_{N+1,2}] \text{ with } L_{2,N} \text{ in } \mathbb{R}^{N+1,N+1}. \] (8)

The following properties will be useful for the stability analysis hereafter.

**Property 1.** Let \( u \in C(\mathbb{R}_+; L^2(0,1)) \) satisfy the heat equation and its boundary conditions in (1). The following formula holds:
\[
\text{Vect}_{k=0..N} \left\langle \partial_x u(t), L_k \right\rangle = \underbrace{L_N U_N(t) + \mathbf{I}_N u(1,t) - \mathbf{I}_N^* C X(t)}_{\text{9}} - \underbrace{L_{1,N} U_{N-1}(t) + L_N u(1,t) - \mathbf{I}_N^* C X(t)}_{\text{10}}
\]

\[
= \left[ \begin{array}{c} X(t) \\ u(1,t) \\ L_{1,N} \end{array} \right]^\top \left[ \begin{array}{c} 0 \\ \ell_{1} \\ \mathbf{I}_N \end{array} \right] U_{N-1}(t).
\]

**Proof:** We obtain easily, using an integration by parts, and the first derivation formula (4) of the Legendre polynomials, that
\[
\left\langle \partial_x u(t), L_0 \right\rangle = u(1,t) - u(0,t), \quad \forall k \geq 1 \\
\left\langle \partial_x u(t), L_k \right\rangle = \sum_{j=1}^{k-1} \ell_{kj} \left\langle u(t), L_j \right\rangle + u(1,t) - u(0,t)(-1)^k.
\]

Using the notations introduced in (7) we obtain equation (9) and one can deduce (10) from (8). \( \square \)

**Property 2.** Let \( u \in C(\mathbb{R}_+; L^2(0,1)) \) satisfy the heat equation and its boundary conditions in (1). The following time derivative formula holds if \( \partial_t u \in C(\mathbb{R}_+; L^2(0,1))):\]
\[
\frac{1}{\gamma} \frac{d}{dt} U_N(t) = \frac{1}{\gamma} \text{Vect}_{k=0..N} \left\langle \partial_t u(t), L_k \right\rangle
\]
\[
= L_N^2 U_N(t) + L_N \mathbf{I}_N^* C X(t) - L_N \mathbf{I}_N u(1,t) - \mathbf{I}_N^* u_x(0,t)(11)
\]
\[
= L_{2,N} U_{N-2}(t) + L_N \mathbf{I}_N^* C X(t) - L_N \mathbf{I}_N u(1,t) - \mathbf{I}_N^* u_x(0,t)
\]
\[
= \left[ \begin{array}{c} L_N \mathbf{I}_N^* C \\ -L_N \mathbf{I}_N \\ L_{2,N} \end{array} \right] \left[ \begin{array}{c} X(t) \\ u(1,t) \\ u_x(0,t) \\ \mathbf{I}_N^* u_x(0,t) \end{array} \right]^\top.
\] (12)

**Proof:** We obtain easily, using the heat equation and integrations by parts, along with formulas (3) and (6) of the Legendre polynomials, that
\[
\frac{d}{dt} \left\langle u(t), L_0 \right\rangle = -\gamma u_x(0,t),
\]
\[
\frac{d}{dt} \left\langle u(t), L_1 \right\rangle = 2\gamma u(0,t) - 2\gamma u(1,t) + \gamma u_x(0,t),
\]
\[
\frac{d}{dt} \left\langle u(t), L_k \right\rangle = \gamma \sum_{j=1}^{k-1} \ell_{jk} \left\langle u(t), L_j \right\rangle + \gamma u(0,t) \sum_{j=0}^{k-1} \ell_{jk} (-1)^j - \gamma u(1,t) \sum_{j=0}^{k-1} \ell_{jk} (-1)^j
\]
\[
-\gamma u_x(0,t)(-1)^k, \quad \forall k \geq 2.
\]

The notations introduced in (7) allow to conclude to equation (11). It is then easy to deduce (12) from (8). \( \square \)

**Remark 3.** It is important to notice here that the main reason for the choice of a base of polynomials to truncate the infinite dimensional state \( u \) is the fact that the derivation matrices \( L_N \) and \( L_N^2 \) are strictly lower triangular. It has interesting consequences on the stability study of the whole system (1) and is the cornerstone to obtain a hierarchy of tractable LMIs, in the same vein as in Seuret and Gouaisbaut (2015).

### 3.3 Bessel-Legendre Inequality

When constructing the projection of the infinite dimensional state \( u \) on the \( N+1 \) first Legendre polynomials to build the approximate vector state \( U_N \) of dimension \( N+1 \), capturing an estimate of the approximation error is crucial. The following lemma provides a useful information.

**Lemma 1.** Let \( u \in C(\mathbb{R}_+; L^2(0,1)) \). The following integral inequality holds for all \( N \in \mathbb{N} \):
\[
||u(t)||^2 \geq U_N(t)^\top \mathcal{I}_N U_N(t). \] (13)

**Proof:** Estimate (13) can be called the Bessel-Legendre Inequality. Since \( u(t) = \sum_{k=0}^{N} \left\langle u(t), L_k \right\rangle \|L_k\|^2 \), using the orthogonality of the Legendre polynomials and the fact that \( \|L_k\|^2 = \langle L_k, L_k \rangle = 1/(2k+1) \), we easily get
\[
\int_0^1 |u(x,t)|^2 dx = \sum_{k=0}^{N} \|\langle u(t), L_k \rangle\|^2 \|L_k\|^2 \\
\geq \sum_{k=0}^{N} (2k+1) \|\langle u(t), L_k \rangle\|^2.
\]

The formulation of Lemma 1 stems from the last notation in (7). \( \square \)
4. STABILITY ANALYSIS

4.1 Exponential stability result

Following the previous developments, \( N \) being a prescribed positive integer, we introduce an approximate state of size \( n + N + 1 \), composed by the state of the ODE system \( X \) and the projection of the infinite dimensional state \( u \) over the set of the Legendre polynomial of degree less than \( N \). In other words, the approximate finite dimensional state vector is given by

\[
\begin{bmatrix}
X(t) \\
U_N(t)
\end{bmatrix} = \begin{bmatrix}
\text{Vec}_{k=0..N}[u(t), L_k]
\end{bmatrix}.
\]

The main objective of this article is to provide the following stability result for the coupled system (1), which is based on an approximate Lyapunov functional and the use of Property 2 and Lemma 1.

**Theorem 1.** Consider system (1) with a given thermal diffusivity \( \gamma > 0 \). If there exist an integer \( N \geq 0 \), such that there exists \( \delta > 0, \alpha > 0, P \in S_n, \bar{Q} \in R^{n,(N+1)m} \) and \( T \in S_{(N+1)m} \) satisfying the following LMIs

\[
\Phi_N = \begin{bmatrix}
P & Q \\
Q^T & T
\end{bmatrix} > 0,
\]

\[
\Psi_N(\gamma) = \tilde{\Psi}_N - \alpha \gamma \Psi_{N,2} - 2 \beta \gamma \Psi_{N,3} < 0,
\]

where

\[
\tilde{\Psi}_N = \begin{bmatrix}
\Psi_{11} PB - \gamma QL_N 1_N & \Psi_{13} & \Psi_{14} \\
* & 0 & -\beta B^T C^T \\
* & * & 0 - \gamma T^T T
\end{bmatrix}
\]

\[
\Psi_{N,2} = \begin{bmatrix}
\Psi_{12} & \Psi_{14} \\
0 & \Psi_{22}
\end{bmatrix}
\]

\[
\Psi_{N,3} = \begin{bmatrix}
L_{N+1} & \Psi_{14} \\
0 & \Psi_{22}
\end{bmatrix}
\]

with

\[
\Psi_{11} = \text{He}(PA + \gamma QL_N 1_N C),
\]

\[
\Psi_{13} = -\gamma Q_1 L_N - \alpha \gamma C^T - \beta A^T C^T,
\]

\[
\Psi_{14} = A^T Q + \gamma C^T 1_N^T L_N T + \gamma Q L_N^2,
\]

\[
\Psi_{22} = B^T Q - \gamma T^T T,
\]

\[
\Psi_{44} = \text{He}(\gamma L_N T),
\]

then the coupled system (1) is exponentially stable. Indeed, there exist constants \( K > 0 \) and \( \delta > 0 \) such that:

\[
E(t) \leq Ke^{-\delta t} (\|X(t)\|_n^2 + \|u(t)\|^2), \forall t > 0.
\]

**Remark 4.** One can point out the robustness of the approach with respect to the triplet \( (A, B, \gamma) \), meaning that we could have \( A, B \) and \( \gamma \) uncertain, switched of time-varying...without losing the stability property. It suffices indeed then to test these LMIs at the vertices of a polytope defining the uncertainties of the triplet.

In order to reveal the approximate state \( U_N \) in the candidate Lyapunov functional \( V \) written in section 3.1, we select the functions \( Q \) and \( T \) as follows:

\[
Q(x) = \sum_{k=0}^{N} Q_k L_k(x), \quad T(x_1, x_2) = \sum_{i=0}^{N} \sum_{j=0}^{N} T_{ij} L_i(x_1)L_j(x_2)
\]

where \( \{Q_k\}_{i=0..N} \) belong to \( R^n \) and \( \{T_{ij}\}_{i,j=0..N} \) to \( R \). Therefore we can write

\[
V_N(t) = V(X(t), u(t)) = \left[ X(t) U_N(t) \right]^T \left[ \begin{array}{cc}
P & Q \\
Q^T & T
\end{array} \right] \left[ \begin{array}{c}
X(t) \\
U_N(t)
\end{array} \right] + \alpha \int_0^1 |u(x,t)|^2 dx + \beta \int_0^1 |\partial_x u(x,t)|^2 dx,
\]

where

\[
Q = [Q_0 \ldots Q_N] \in R^{n,N+1},
\]

\[
T = (T_{jk})_{j,k=0..N} \in R^{N+1,N+1}.
\]

In the following subsection, conditions for exponential stability of the origin of system (1) can be obtained using the LMI framework. More particularly, we aim at proving that the functional \( V_N \) is positive definite and satisfies

\[
V_N(t) + 2\lambda V_N(t) \leq 0
\]

for a prescribed \( \delta > 0 \) and under LMIs to be determined.

4.2 Proof of the Stability Theorem

The proof consists in showing that, if the LMI conditions (14) and (15) are verified for a given \( N \geq 0 \), then there exist three positive scalars \( \varepsilon_1, \varepsilon_2 \) and \( \varepsilon_3 \) such that for all \( t > 0 \),

\[
\varepsilon_1 E(t) \leq V_N(t) \leq \varepsilon_2 E(t),
\]

\[
V_N(t) \leq -\varepsilon_3 E(t).
\]

Indeed, on the one hand, its suffices to notice that we obtain directly from (22) and (23)

\[
V_N(t) + \frac{\varepsilon_3}{\varepsilon_2} V_N(t) \leq 0
\]

so that

\[
\frac{d}{dt} (V_N(t)e^{-\varepsilon_3 t/\varepsilon_2}) \leq 0
\]

and integrating in time, we get \( V_N(t) \leq V_N(0)e^{-\varepsilon_3 t/\varepsilon_2} \) for all \( t > 0 \). On the other hand, with the help of (22), we can finally write

\[
\varepsilon_1 E(t) \leq V_N(t) \leq V_N(0)e^{-\varepsilon_3 t/\varepsilon_2} \leq \varepsilon_2 E(0)e^{-\varepsilon_3 t/\varepsilon_2},
\]

allowing to conclude (20).

**Existence of \( \varepsilon_1 \):** Since \( \alpha > 0, \beta > 0 \) and \( \Phi_N > 0 \), there exists a sufficiently small \( \varepsilon_1 > 0 \) such that

\[
\varepsilon_1 \leq \alpha, \quad \varepsilon_1 \leq \beta \quad \text{and} \quad \Phi_N = \begin{bmatrix}
P & Q \\
Q^T & T
\end{bmatrix} > \begin{bmatrix}
I_n & 0 \\
0 & 0
\end{bmatrix}.
\]

Since \( \varepsilon_1 \leq \alpha \) and \( \varepsilon_1 < \beta \), we obtain a lower bound of \( V_N \) depending on the energy function \( E(t) \):

\[
V_N(t) \geq \varepsilon_1 (\|X(t)\|_n^2 + \|u(t)\|^2) + \beta |\partial_x u(t)|^2 \geq \varepsilon_1 E(t).
\]

**Existence of \( \varepsilon_2 \):** There exists a sufficiently large scalar \( \lambda > 0 \) such that

\[
\begin{bmatrix}
P & Q \\
Q^T & T
\end{bmatrix} \preceq \lambda \begin{bmatrix}
I_n & 0 \\
0 & I_n
\end{bmatrix},
\]

yielding

\[
V_N(t) \leq \lambda (\|X(t)\|_n^2 + \lambda U_N(t) I_N U_N(t) + \alpha \|u(t)\|^2 + \beta |\partial_x u(t)|^2.
\]
Applying Lemma 1 to the second term of the right-hand side ensures that
\[ V_N(t) \leq \varepsilon_2 E(t) \]
with \( \varepsilon_2 = \max\{\lambda_{\max}\left(\frac{P}{Q} T\right) + \alpha, \beta\} \).

**Existence of \( \varepsilon_2 \):** In order to prove now that (23) relies on the solvability of the LMI (15), we need to define an augmented approximate state vector of size \( n + N + 3 \) given by
\[ \xi_N(t) = \begin{bmatrix} X(t) \\ u(1, t) \\ u_x(0, t) \\ U_N(t) \end{bmatrix}. \]

**Step 1:** Let us split the computation of \( \bar{V}_N \) into three terms, namely \( \bar{V}_{N,1}, \bar{V}_{N,2} \) and \( \bar{V}_{N,3} \) corresponding to each term of \( V_N \) in (21). We omit the variable \( t \) in the sequel.

On the one hand, using the first equation in system (1) and Property 2, we have:
\[
\frac{d}{dt}\begin{bmatrix} X \\ U_N \end{bmatrix} = 
\begin{bmatrix} AX + Bu(1) \\
\gamma L_N^2 U_N + \gamma L_N \Gamma_N CX - \gamma L_N \Gamma_N u(1) - \gamma \Gamma_N u_x(0) \end{bmatrix}
\]
so that we can calculate
\[
\bar{V}_{N,1} = \frac{d}{dt}\begin{bmatrix} X^T \\ P \\ Q^T \\ T \end{bmatrix} = \xi_N^T \Psi_{N,1}(\gamma) \xi_N
\]
with
\[
\Psi_{N,1} = \begin{bmatrix} 
\Psi_{11} PB - \gamma Q L_N \Gamma_N - \gamma Q \Gamma_N \Psi_{14} \\
0 & 0 & 0 & -\gamma \Gamma_N^T T \\
* & * & 0 & 0 \\
* & * & * & \Psi_{44} 
\end{bmatrix}
\]
where \( \Psi_{11}, \Psi_{14}, \Psi_{24} \) and \( \Psi_{44} \) are defined in (17).

On the other hand, using the heat equation in (1), and an integration by parts, we get both
\[
\bar{V}_{N,2} = \alpha \int_0^1 \partial_t \left( |u(x)|^2 \right) dx = 2\alpha \int_0^1 u(x) \partial_x u(x) dx
\]
\[= 2\alpha \gamma \int_0^1 u(x) \partial_x u(x) dx\]
\[= -2\alpha \gamma \int_0^1 \left| \partial_x u(x) \right|^2 dx + 2\alpha \gamma \left| \partial_x u \right|_0^1 \]
\[= -2\alpha \gamma \left| \partial_x u \right|^2 - 2\alpha \gamma C X u_x(0) \]
and
\[
\bar{V}_{N,3} = \beta \int_0^1 \partial_t \left( \left| \partial_x u(x) \right|^2 \right) dx = 2\beta \int_0^1 \partial_x u(x) \partial_x u(x) dx
\]
\[= -2\beta \int_0^1 \partial_t u(x) \partial_x u(x) dx + 2\beta \left[ \partial_t u \right]_0^1 \]
\[= -2\beta \left| \partial_t u \right|^2 - 2\beta \partial_t u(0) \partial_x u(0) \]
\[= -2\beta \left| \partial_t u \right|^2 - 2\beta \partial_x u(0) (AX + Bu(1)). \]

Merging the expressions of \( \bar{V}_{N,1}, \bar{V}_{N,2} \) and \( \bar{V}_{N,3} \), we can write
\[
\dot{V}_N = \xi_N^T \Psi_N(\gamma) \xi_N - 2\alpha \gamma \left| \partial_x u \right|^2 - 2\beta \left| \partial_t u \right|^2
\]
\[= -2\alpha \gamma C X u_x(0) - 2\beta \partial_t u(0) (AX + Bu(1)) \]
\[= \xi_N^T \Psi_N(\gamma) \xi_N - 2\alpha \gamma \left| \partial_x u \right|^2 - 2\beta \left| \partial_t u \right|^2 \quad (25) \]

where \( \Psi_N(\gamma) \) is defined in (16).

**Step 2:** Let us explain here how we can deal with the terms \( \left| \partial_x u(t) \right|^2 \) and \( \left| \partial_t u(t) \right|^2 \).

Following the proof of Lemma 1, up to the order \( N + 1 \), we can write, using an integration by parts and the derivation formula in Property 1 of the Legendre polynomial
\[
\left| \partial_x u(t) \right|^2 \geq \sum_{k=0}^{N+1} (2k + 1) \left| \partial_x u, L_k \right|^2
\]
so that with \( \Psi_N \) defined in (18),
\[\geq \Psi_{N,2} \xi_N(t). \quad (26) \]

Similarly, using Property 2 and Lemma 1 up to the order \( N + 2 \), we have
\[
\frac{1}{\gamma} \left| \partial_t u(t) \right|^2 \geq \frac{1}{\gamma} \frac{dU_N^T}{dt} L_{N+2} \frac{dU_N}{dt}
\]
\[\geq \xi_{N,3} \xi_N(t). \quad (27) \]

**Step 3:** Since we assume the LMI (15) that writes \( \Psi_N < 0 \), there exists \( \varepsilon > 0 \) such that
\[
\Psi_N < -\varepsilon \begin{bmatrix} I_n & 0 & 0 & 0 \\
0 & 2 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 
\end{bmatrix}.
\]

For instance, \( \varepsilon = \lambda_{\min}(-\Psi_N)/2 \).

Therefore, we can write from (25), (26) and (27), choosing \( \varepsilon_3 = \min\{\frac{\varepsilon_3}{\alpha \gamma}, \lambda_{\min}(-\Psi_N)/2\} \),
\[
\dot{V}_N(t) \leq \xi_N^T(t) \Psi_N(\gamma) \xi_N(t) - \alpha \gamma \left| \partial_x u(t) \right|^2 - 2\beta \left| \partial_t u(t) \right|^2
\]
\[\leq -3\varepsilon_3 \left| \partial_x u(t) \right|^2
\]
\[\leq \xi_N^T(t) \left( \Psi_N - \alpha \gamma \Psi_N - 2\beta \gamma \Psi_N \right) \xi_N(t)
\]
\[\leq -3\varepsilon_3 \left| \partial_x u(t) \right|^2
\]
\[\leq -\xi_N^T(t) \Psi_N(\gamma) \xi_N(t) - 3\varepsilon_3 \left| \partial_x u(t) \right|^2
\]
\[\leq \xi_N^T(t) \left( \Psi_N - \alpha \gamma \Psi_N - 2\beta \gamma \Psi_N \right) \xi_N(t)
\]
\[\leq -3\varepsilon_3 \left| \partial_x u(t) \right|^2
\]

Finally, since one can easily prove that for any \( u \in H^1(0,1) \),
\[\left| u \right|^2 \leq 2u(1)^2 + 2\left| \partial_x u \right|^2,
\]
we obtain
\[ \dot{V}_N(t) \leq -\varepsilon_3 \| X(t) \|_n^2 - \varepsilon_3 \| u(t) \|_n^2 - \varepsilon_3 \| \partial_x u(t) \|_n^2 \] (28)
which is precisely (23).

One can therefore conclude to the exponential stability of system (1).

5. NUMERICAL EXAMPLE

Consider system (1) with the following value
\[
A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -10 & K & 0 & 0 \\ 5 & -15 & 0 & -0.25 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} K \\ 0 \\ 0 \\ 0 \end{bmatrix}^T.
\]

This data triplet \((A, B, C)\) has already been considered in the context of time delay systems where the delayed matrix is \(A_d = BC\). The main motivation for studying this TDS arises from the fact the stability region has a very complicated shape, that is hard to detect using a Lyapunov-Krasovskii functional approach. We will see that the stability region is difficult to detect for our system (1) as well.

In order to illustrate the potential of Theorem 1, we have generated Figure 1, depicting in the plan \((K, \gamma)\) and in logarithmic scales, for which values of \(N\) solutions to the LMI problem (14-15) have been found. The white area corresponds to values of \((K, \gamma)\) for which no solutions have been obtained for \(N < 13\). The darkest area corresponds to the stability region obtained with \(N = 0\) in Theorem 1. The general tendency presented in Figure 1 is that for large values of \(\gamma\), stability is guaranteed. However, for small values of \(\gamma\), peculiar stability regions are detected. One can see that increasing \(N\) in Theorem 1 allows to enlarge the stability regions as illustrated in the hierarchical structure of LMIs (14-15). Interestingly, Figure 1 also detects two instability zones, where (14) or (15) are not solvable, even for larger values of \(N\).

Remark 5. Figure 1 has also the interest of illustrating the hierarchy that our approach suggests. One sees clearly the progression of the guaranteed domain of stability with the increase of \(N\).

In order to illustrate the stability regions depicted in Figure 1, several temporal simulations of the coupled-system have been provided in Figure 2. They correspond to system (1) with the same numerical values \((A, B, C)\) and the particular choice of \(K = 100\). This selection of \(K\) is relevant since there is an interval of values of \(\gamma\) included in \([0.1, 0.2]\) such that the LMIs conditions of Theorem 1 are not verified even for large values of \(N\). Under the initial conditions
\[
u^0(x) = CX^0 - 20 x (x - 2) + 10 (1 - \cos(8 \pi x)),
\]
\[X^0 = [0 \ 1 \ -1 \ 0]
\]
and noting that this is compatible with the requirements \(u^0(0) = CX^0\) and \(\partial_x u^0(1) = 0\), three simulations are provided with

Fig. 1. Stability region in the plan \((K, \gamma)\), obtained using Theorem 1 for \(N = 0, \ldots, 12\).

Fig. 2. Figure representing the evolution of the state \((X, u)\) with respect to time with \(K = 100\) and for several values of \(\gamma\).
are very slowly converging to the origin, with however a
presented in Figure 2(c), shows simulations results which
N
for any
no solutions to the conditions of Theorem 1 can be found
of this system diverges. This is consistent with the fact that
Figure 2(b) is consistent with Figure 1, since the solution
to the equilibrium. The illustration of the second case
by Theorem 1 with
N
approximation.

Simulations of the coupled ODE - Heat PDE have been
performed using classical tools available in the literature.
The ODE has been discretized using a Runge-Kutta algo-
rithm of order 4 with a principal step \( \delta \). The PDE have
been simulated by performing a backward in time central
order difference in space with a step \( \delta \), and
\( \delta = 1/20 \) to ensure the numerical stability of the
approximation.

Figure 2(a) obviously shows the stable behaviors detected
by Theorem 1 with \( N = 0 \), with a quite fast convergence
to the equilibrium. The illustration of the second case
Figure 2(b) is consistent with Figure 1, since the solution
of this system diverges. This is consistent with the fact that
no solutions to the conditions of Theorem 1 can be found
for any \( N \leq 12 \). More interestingly, the last situation,
presented in Figure 2(c), shows simulations results which
are very slowly converging to the origin, with however a
lightly damped oscillatory behavior of the state of the ODE
and of the PDE close to the boundary \( x = 0 \). On the other side, the state function \( u(x,t) \) for sufficiently
large values of \( x \) is clearly smooth and converges slowly to
the origin. This slow convergent behavior of the solution
for case (c) can illustrate the fact that the order \( N \)
for which the conditions of Theorem 1 are verified is quite
large: \( N \geq 5 \).

6. DISCUSSION AND CONCLUSION

This article has provided a new and fruitful approach
to numerically check the exponential stability of coupled
ODE - Heat PDE systems. Our approach relies on the
efficient construction of specific Lyapunov functionals
allowing to derive diffusion parameter-dependent stability
conditions. These tractable conditions of stability are ex-
pressed in terms of LMIs and obtained using the Bessel
inequality. This work is a first contribution in the study of
coupled ODE-Heat PDE systems using this framework and
has the ambition to provide a method that could prove to
be robust and useful in more intricate situations, such as
other parabolic PDEs (e.g. reaction-diffusion, Kuramoto-
Sivashinski...), or vectorial infinite dimensional state
u to handle MIMO systems. A very interesting but challenging
question is also the study of the convergence of our result
when the order \( N \) of truncation grows. We would like to
prove that if the stability of the coupled system holds, then
there exists an order \( N \) for which our LMIs are verified.
Future research will also include the study of the
robustness of our technique with respect to the whole data
quadruplet \((A, B, C, \gamma)\).

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