Eigenvalue bounds of the Robin Laplacian with magnetic field

Georges Habib, Ayman Kachmar

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Abstract

On a compact Riemannian manifold M with boundary, we give an estimate for the eigenvalues $(\lambda_k(\tau,\alpha))_k$ of the magnetic Laplacian with the Robin boundary conditions. Here, τ is a positive number that defines the Robin condition and α is a real differential 1-form on M that represents the magnetic field. We express those estimates in terms of the mean curvature of the boundary, the parameter τ and a lower bound of the Ricci curvature of M (see Theorems 1.3 and 1.5). The main technique is to use the Bochner formula established in [3] for the magnetic Laplacian and to integrate it over M (see Theorem 1.2). As a direct application, we find the standard Lichnerowicz estimates for both the Neumann and Dirichlet Laplacian, when the parameter τ tends to 0 or to ∞. In the last part, we compare the eigenvalues $\lambda_k(\tau,\alpha)$ with the first eigenvalue $\lambda_1(\tau,0)$ (i.e. without magnetic field) and the Neumann eigenvalues $\lambda_k(0,\alpha)$ (see Theorem 1.7) using the min-max principle.

1 Introduction and Results

Let (M,g) be a Riemannian manifold of dimension n and let α be a smooth real differential 1-form on M. Given two vector fields X,Y in the complexified tangent bundle $TM\otimes\mathbb{C}$, the *magnetic covariant derivative* is defined as $\nabla^\alpha_Y X=\nabla^M_Y X+i\alpha(Y)X$, where ∇^M denotes the Levi-Civita connection on M. It is shown in [3, Lemma 3.2] that ∇^α satisfies the Leibniz rule and the compatibility property with the Riemannian metric g, and is also used to define the *magnetic Hessian* by $\mathrm{Hess}^\alpha f(X,Y)=\langle\nabla^\alpha_X d^\alpha f,Y\rangle$. Here and in all the paper, the product $\langle\cdot,\cdot\rangle$ will denote the Hermitian inner product extended from the metric g to the tangent bundle $TM\otimes\mathbb{C}$ or to the cotangent bundle $T^*M\otimes\mathbb{C}$. We will also use the natural one-to-one isomorphism between $T^*M\otimes\mathbb{C}$ and $TM\otimes\mathbb{C}$ by $w(X)=\langle X,\overline{w^\#}\rangle$ for any $X\in TM\otimes\mathbb{C}$ and $w\in T^*M\otimes\mathbb{C}$.
Given any complex-valued function f on M, the *magnetic Laplacian* is defined as being the trace of the magnetic Hessian

$$\Delta^\alpha f:=-\mathrm{trace}(\mathrm{Hess}^\alpha f)=-\mathrm{div}^\alpha(d^\alpha f)^\#,$$

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where $d^\alpha f := d^M f + if\alpha$ and div^α is the magnetic divergence given for any vector field $X \in TM \otimes \mathbb{C}$ by $\mathrm{div}^\alpha X := \mathrm{div}^M X + i\langle X, \alpha^\# \rangle$.
The study of the spectrum of the magnetic Laplacian has interested many researchers [2, 4, 6, 8, 9, 10] during the last years. For example, the authors in [3] gave an estimate *à la Lichnerowicz* for the first eigenvalue in terms of a lower bound of the Ricci curvature (assumed to be positive) and the infinity norm of the magnetic field $d^M\alpha$. In particular, they deduce a spectral gap between the first eigenvalue (which is not necessarily zero) and the second one. The main technique used in the paper is to establish a Bochner type formula for the magnetic Laplacian Δ^α, to integrate it over the manifold M and to control all the integral terms involving $d^M\alpha$ by its supremum norm. Indeed, they prove

Theorem 1.1. *[3, Thm. 4.1] Let (M, g) be a complete Riemannian manifold of dimension n. Then for all $f \in C^\infty(M, \mathbb{C})$, we have*

$$\begin{aligned} -\frac{1}{2}\Delta^M(|d^\alpha f|^2) &= |\mathrm{Hess}^\alpha f|^2 - \Re\langle d^\alpha f, d^\alpha(\Delta^\alpha f)\rangle + \mathrm{Ric}^M(d^\alpha f, d^\alpha f) \\ &+ i(d^M\alpha(d^\alpha f, \overline{d^\alpha f}) - d^M\alpha(\overline{d^\alpha f}, d^\alpha f)) \\ &+ \frac{i}{2}(\langle \bar{f} d^\alpha f, \delta^M d^M\alpha\rangle - \langle f\overline{d^\alpha f}, \delta^M d^M \alpha\rangle), \end{aligned} \tag{1.1}$$

where δ^M denotes the formal adjoint of d^M on (M, g).

In this paper, we are interested in estimating the eigenvalues of the magnetic Laplacian with the Robin boundary condition. That means, we assume on a given compact manifold M with boundary N there exists a complex-valued function f on M satisfying the equation $\Delta^\alpha f = \lambda f$ on M and the boundary condition $(d^\alpha f)(\nu) = \tau f$ for some positive real number τ. Here ν denotes the inward unit normal vector field of N. It a standard fact that the spectrum of such boundary problem is purely discrete and consists of a sequence of eigenvalues $(\lambda_k(\tau, \alpha))_k$ arranged in increasing order counting multiplicities. In order to get the spectral gap between the first two eigenvalues, we shall first integrate the Bochner formula in Theorem 1.1 as in [3] by taking into account the boundary terms. First, we get

Theorem 1.2. *Let (M^n, g) be a compact Riemannian manifold with boundary N and let α be a differential real 1-form on M. Then, we have*

$$\begin{aligned} \int_M |\mathrm{Hess}^\alpha f + \frac{1}{n}(\Delta^\alpha f) g|^2 dv_g =& \frac{n-1}{n}\int_M |\Delta^\alpha f|^2 dv_g - \int_M \mathrm{Ric}^M(d^\alpha f, d^\alpha f) dv_g \\ &+ \int_M \Im m\left((d^M\alpha)(d^\alpha f, \overline{d^\alpha f})\right) dv_g + \int_M |f|^2 |d^M\alpha|^2 dv_g \\ &- (n-1)\int_N H|\langle d^\alpha f, \nu\rangle|^2 dv_g - 2\int_N \Re(\langle \nu, d^\alpha f\rangle \Delta_N^\alpha f) dv_g \\ &- \int_N \langle II(d_N^\alpha f), d_N^\alpha f\rangle dv_g. \end{aligned} \tag{1.2}$$

for all complex valued function $f \in C^\infty(M, \mathbb{C})$.

Here II denotes the second fundamental form of the boundary and H is the mean curvature. Also Δ_N^α is a Laplacian defined on functions on N which is associated to some exterior derivative d_N^α (see Section 2 for the definition).
We apply Theorem 1.2 to a particular solution of the magnetic Robin boundary problem. Under some assumptions on τ, the magnetic field $d^M\alpha$, the Ricci curvature Ric^M, the second fundamental form II and the minimal mean curvature $H_{\min} = \min_M H$, we get two universal bounds on the first two eigenvalues of the magnetic Robin Laplacian. Indeed

Theorem 1.3. *Let (M^n, g) be a compact Riemannian manifold with boundary $\partial M = N$ and let α be a differential 1-form on M and $\tau > 0$. Assume that $\mathrm{Ric}^M \geq k\,(k > 0)$ and that $II + \tau \geq 0$. If α satisfies*

$$k - (n-1)\tau H_{\min} \leq ||d^M\alpha||_\infty \leq \left(1 + 2\sqrt{\frac{n-1}{n}}\right)^{-1} k, \tag{1.3}$$

then we have

$$\lambda_1(\tau,\alpha) \leq a_-(k, ||d^M\alpha||_\infty, n) \quad \text{and} \quad \lambda_2(\tau,\alpha) \geq a_+(k, ||d^M\alpha||_\infty, n),$$

where

$$a_\pm(k, ||d^M\alpha||_\infty, n) = n\frac{(k - ||d^M\alpha||_\infty) \pm \sqrt{(k - ||d^M\alpha||_\infty)^2 - 4(\frac{n-1}{n})||d^M\alpha||_\infty^2}}{2(n-1)},$$

and $H_{\min} = \min_M H$.

Remark 1.4.

- The assumption in (1.3) on the mean curvature is valid when $H_{\min} > 0$, since $\left(1 + 2\sqrt{\frac{n-1}{n}}\right)^{-1} k < k$. Also, when τ is very large, (1.3) becomes an upper bound on $||d^M\alpha||_\infty$, which is a growth condition on the magnetic field with respect to the Ricci curvature.
- It follows from Inequality (1.3) that $(k - ||d^M\alpha||_\infty)^2 - 4(\frac{n-1}{n})||d^M\alpha||_\infty^2 > 0$ and $a_-(k, ||d^M\alpha||_\infty, n) > 0$. This is more transparent in the proof of Theorem 1.3.
- As long as τ is chosen bigger than $\frac{k}{(n-1)H_{\min}}$, the first inequality in (1.3) is clearly satisfied.

In Theorem 1.3, we get the same estimates as in [3, Thm. 1.1] since we shall see that the eigenvalues satisfy the same inequality in [3]. As an application of this theorem, we find the lower bound for the eigenvalues of the Dirichlet Laplacian proved by Reilly in [7]. Indeed, assume that the mean curvature H is nonnegative and consider any closed 1-form α on M. If we let τ tending to $+\infty$ in Theorem 1.3, the inequality (1.3) is clearly satisfied and therefore, one deduces that $\lim_{\tau\to\infty}\lambda_2(\tau,\alpha) \geq \frac{n}{n-1}k$. As the spectrum of the Robin Laplacian tends to the Dirichlet one when $\tau \to \infty$, the result then follows.
Also, the authors in [1, p.14] gave an estimate for the first eigenvalue of the Robin Laplacian in terms of the infimum value of the mean curvature. In fact, they proved that under

the conditions $\mathrm{Ric}^M \geq 0$ and $II+\tau \geq 0$, we have the estimate $\lambda_1(\tau) \geq \tau(n-1)\min_M H - \tau^2$. This estimate is valid when the condition $H_{min} := \min_M H \geq \frac{\tau}{n-1}$ holds. In the presence of a magnetic field the same bound continues to hold for the first eigenvalue, since $\lambda_1(\tau,\alpha) \geq \lambda_1(\tau) := \lambda(\tau, 0)$ for any differential 1-form α, by the celebrated diamagnetic inequality.
Our next result is an estimate of the eigenvalues of the magnetic Robin Laplacian in terms of $H_{\min}$ and a lower bound of the Ricci curvature.

Theorem 1.5. *Let (M^n, g) be a compact Riemannian manifold with boundary $\partial M = N$ and let α be a differential 1-form on M and $\tau > 0$. Assume that $\mathrm{Ric}^M \geq k$ $(k > 0)$, $II+\tau \geq 0$ and $H_{\min} \geq \frac{1}{n}$. If $k \geq (n-1)\tau H_{\min}$ and α satisfies*

$$||d^M\alpha||_\infty \leq \inf\left(\frac{\sqrt{\Delta_2} - (k - 2(\frac{n-1}{n})\tau)}{4\frac{n-1}{n} - 1}, k - (n-1)\tau H_{\min}\right),$$

where

$$\Delta_2 = 4\frac{n-1}{n}\left(k^2 - 4\frac{n-1}{n}k\tau + \frac{n-1}{n}\tau^2 - (n-1)\tau^2 H_{\min}(1 - 4\frac{n-1}{n})\right),$$

then we have

$$\lambda_1(\tau,\alpha) \leq b_-(k, ||d^M\alpha||_\infty, n) \quad \text{and} \quad \lambda_2(\tau,\alpha) \geq b_+(k, ||d^M\alpha||_\infty, n),$$

where $b_\pm(k, ||d^M\alpha||_\infty, n)$ are given by

$$n\frac{(k - ||d^M\alpha||_\infty) \pm \sqrt{\Delta}}{2(n-1)},$$

with

$$\Delta = \left(1 - 4\frac{n-1}{n}\right)||d^M\alpha||_\infty^2 - 2\left(k - 2\left(\frac{n-1}{n}\right)\tau\right)||d^M\alpha||_\infty + k^2 - 4\frac{n-1}{n}k\tau + 4\frac{(n-1)^2}{n}\tau^2 H_{\min}.$$

Remark 1.6.

- Note that the assumption on $d^M\alpha$ in Theorem 1.5 guarantees that the quantities Δ, $b_\pm(k, ||d^M\alpha||_\infty, n)$ and Δ_2 are non-negative. This is more apparent in the proof of Theorem 1.5.
- In the case where $H_{\min} \leq \frac{1}{n}$, we can get the same results as in Theorem 1.5 but choosing $k \geq 2(n-1)\tau\left(\frac{1}{n} + \sqrt{\frac{1}{n}(\frac{1}{n} - H_{\min})}\right)$. If we let τ tending to zero, we find the same bound for the eigenvalues of the Neumann Laplacian proved in [5].
- When we are working with a Riemannian manifold with minimal boundary (that is, $H = 0$), the estimate in [1] cited previously does not give any information on the spectrum of the Robin Laplacian. However, the estimate in Theorem 1.5 does.

In the last part of this paper, we present two-sided estimates of all the eigenvalues $\lambda_k(\tau,\alpha)$ in terms of $\lambda_1(\tau) := \lambda_1(\tau,0)$ and the Neumann eigenvalues $\lambda_k^N(\alpha) := \lambda_k(0,\alpha)$, using a variational argument (see Theorem 1.7 below). These estimates yield a quantitative measurement of the diamagnetism (i.e. the quantity $\lambda(\tau,\alpha)-\lambda_1(\alpha)$).
To state this theorem, we need the quantity $C(\tau)$ that we introduce below. Let $f_\tau : M \to \mathbb{R}$ be the first normalized eigenfunction of the Robin Laplacian (without magnetic field) and let us define the following constant

$$C(\tau) = \frac{\min\limits_{x\in M} f_\tau^2(x)}{\max\limits_{x\in M} f_\tau^2(x)} > 0\,. \tag{1.4}$$

Note that $C(0)=1$ and $\lim\limits_{\tau\to+\infty} C(\tau)=0$ and the function f_τ can be selected in a unique manner so that $f_\tau>0$. We have

Theorem 1.7. *For all $\tau>0$ and $k\geq 1$,*

$$\lambda_1(\tau)+C(\tau)\lambda_k^N(\alpha) \leq \lambda_k(\tau,\alpha) \leq \lambda_1(\tau)+\frac{1}{C(\tau)}\lambda_k^N(\alpha)\,.$$

Remark 1.8.

1. Using the existing estimates on the Neumann eigenvalues $\lambda_k^N(\alpha)$ (see e.g. [2]), we deduce immediately estimates on the Robin eigenvalues $\lambda_k(\tau,\alpha)$.
2. **(Zero magnetic field)** Assume that α is closed and not exact. Combining the result in [8] and the estimates in Theorem 1.7, we deduce that $\lambda_1(\tau,\alpha)=\lambda_1(\tau)$ if and only if the flux of α satsifies

   $$\Phi_c^\alpha := \oint_c \alpha \in \mathbb{Z}$$

   for every closed curve $c\subset M$.

The rest of the paper is organized as follows. Section 2 is devoted to the lengthy proof of Theorem 1.2. In Section 3, we prove Theorems 1.3 and 1.5. Finally, we present the proof of Theorem 1.7 in Section 4.

2 Proof of Theorem 1.2

In this section, we will prove Theorem 1.2 by integrating all the terms in the Bochner formula. First, with the help of the Stokes formula the integral of the l.h.s. of Equation (1.1) is equal to

$$-\frac{1}{2}\int_M \Delta^M(|d^\alpha f|^2)dv_g = -\frac{1}{2}\int_N g(d^M(|d^\alpha f|^2),\nu))dv_g = -\int_N \Re\langle\nabla_\nu^M d^\alpha f, d^\alpha f\rangle dv_g.$$

Now, we will compute the term $\Re\langle\nabla_\nu^M d^\alpha f, d^\alpha f\rangle$ pointwise by decomposing the vectors over a local orthonormal frame $\{e_i\}_{i=1,\cdots,n-1}$ of T_xN at some point $x\in N$. Indeed, using the definition of the operator d^α, we write

$$\begin{aligned}
\langle \nabla^M_\nu d^\alpha f, d^\alpha f\rangle &= \sum_{i=1}^{n-1}(\nabla^M_\nu d^\alpha f)(e_i)\langle e_i, d^\alpha f\rangle + (\nabla^M_\nu d^\alpha f)(\nu)\langle \nu, d^\alpha f\rangle \\
&= \sum_{i=1}^{n-1}(\nabla^M_\nu d^M f)(e_i)\langle e_i, d^\alpha f\rangle + i\nu(f)\sum_{i=1}^{n-1}\alpha(e_i)\langle e_i, d^\alpha f\rangle \\
&\quad + if\sum_{i=1}^{n-1}(\nabla^M_\nu \alpha)(e_i)\langle e_i, d^\alpha f\rangle + (\nabla^M_\nu d^\alpha f)(\nu)\langle \nu, d^\alpha f\rangle \\
&= \sum_{i=1}^{n-1}(\nabla^M_{e_i} d^M f)(\nu)\langle e_i, d^\alpha f\rangle + i\nu(f)\sum_{i=1}^{n-1}\alpha(e_i)\langle e_i, d^\alpha f\rangle \\
&\quad + if\sum_{i=1}^{n-1}(d^M \alpha)(\nu, e_i)\langle e_i, d^\alpha f\rangle + if\sum_{i=1}^{n-1}(\nabla^M_{e_i}\alpha)(\nu)\langle e_i, d^\alpha f\rangle \\
&\quad + (\nabla^M_\nu d^\alpha f)(\nu)\langle \nu, d^\alpha f\rangle.
\end{aligned}$$

In the last equality, we just use the fact that the hessian of the function f is a symmetric 2-tensor. We then proceed

$$\begin{aligned}
\langle \nabla^M_\nu d^\alpha f, d^\alpha f\rangle &= \sum_{i=1}^{n-1}e_i(\nu(f))\langle e_i, d^\alpha f\rangle - \sum_{i=1}^{n-1}(d^M f)(\nabla^M_{e_i}\nu)\langle e_i, d^\alpha f\rangle + i\nu(f)\sum_{i=1}^{n-1}\alpha(e_i)\langle e_i, d^\alpha f\rangle \\
&\quad + if\sum_{i=1}^{n-1}(d^M\alpha)(\nu, e_i)\langle e_i, d^\alpha f\rangle + if\sum_{i=1}^{n-1}e_i(\alpha(\nu))\langle e_i, d^\alpha f\rangle \\
&\quad - if\sum_{i=1}^{n-1}\alpha(\nabla^M_{e_i}\nu)\langle e_i, d^\alpha f\rangle + (\nabla^M_\nu d^\alpha f)(\nu)\langle \nu, d^\alpha f\rangle \\
&= \langle d^N(\nu(f)), d^\alpha f\rangle + \sum_{i=1}^{n-1}(d^M f)(II(e_i))\langle e_i, d^\alpha f\rangle + i\nu(f)\sum_{i=1}^{n-1}\alpha(e_i)\langle e_i, d^\alpha f\rangle \\
&\quad + if\sum_{i=1}^{n-1}(d^M\alpha)(\nu, e_i)\langle e_i, d^\alpha f\rangle + if\langle d^N(\alpha(\nu)), d^\alpha f\rangle \\
&\quad + if\sum_{i=1}^{n-1}\alpha(II(e_i))\langle e_i, d^\alpha f\rangle + (\nabla^M_\nu d^\alpha f)(\nu)\langle \nu, d^\alpha f\rangle.
\end{aligned}$$

As α is a 1-form on M, we can write it at any point of the boundary by $\alpha = \alpha^T + \alpha(\nu)\nu$. We then define the operator d_N^α by $d_N^\alpha h := d^N h + ih\alpha^T$ for any complex-valued function $h \in C^\infty(N, \mathbb{C})$. Hence, the above equality becomes

$$\begin{aligned}
\langle \nabla^M_\nu d^\alpha f, d^\alpha f\rangle =& \langle d_N^\alpha(\nu(f)), d^\alpha f\rangle + \langle II(d_N^\alpha f), d^\alpha f\rangle + if\langle \nu \lrcorner d^M\alpha, d^\alpha f\rangle \\
&+ if\langle d^N(\alpha(\nu)), d^\alpha f\rangle + (\nabla^M_\nu d^\alpha f)(\nu)\langle \nu, d^\alpha f\rangle.
\end{aligned}$$

Therefore after integrating, we deduce that

$$\begin{aligned}
-\frac{1}{2}\int_M \Delta^M(|d^\alpha f|^2)dv_g = -\int_N \Re(&\langle d_N^\alpha(\nu(f)), d^\alpha f\rangle + \langle II(d_N^\alpha f), d^\alpha f\rangle + if\langle \nu \lrcorner d^M\alpha, d^\alpha f\rangle \\
&+ if\langle d^N(\alpha(\nu)), d^\alpha f\rangle + (\nabla^M_\nu d^\alpha f)(\nu)\langle \nu, d^\alpha f\rangle)dv_g. \qquad (2.1)
\end{aligned}$$

In the second step, we want to integrate the term $\Re\langle d^\alpha f, d^\alpha(\Delta^\alpha f)\rangle$ in the r.h.s. of Theorem 1.1. First, recall the Stokes formula on complex functions: For all $h \in C^\infty(M, \mathbb{C})$ and smooth complex valued 1-form β, one has

$$\int_M \langle d^M h, \beta\rangle dv_g = \int_M h\overline{\delta^M \beta} dv_g - \int_N h\langle \nu, \beta\rangle dv_g.$$

Therefore according to this formula, one can easily get that

$$\int_M \langle d^\alpha h, \beta\rangle dv_g = \int_M h\overline{\delta^\alpha \beta} dv_g - \int_N h\langle \nu, \beta\rangle dv_g,$$

where the adjoint δ^α of d^α is given by $\delta^\alpha = \delta^M - i\langle\cdot, \alpha\rangle$ [3, Def. 2.1]. Here we mention that $\delta^\alpha X = -\mathrm{trace}(\nabla^\alpha X)$, where ∇^α is the magnetic covariant derivative defined previously. Hence, by taking $h = \Delta^\alpha f$ and $\beta = d^\alpha f$, we deduce

$$\int_M \langle d^\alpha(\Delta^\alpha f), d^\alpha f\rangle dv_g = \int_M |\Delta^\alpha f|^2 dv_g - \int_N (\Delta^\alpha f)\langle \nu, d^\alpha f\rangle dv_g. \tag{2.2}$$

Now we want to evaluate the term $\Delta^\alpha f$ in the second integral of the r.h.s. of the equality above. Using the compatibility equations in [3, Lem. 3.2] and taking an orthonormal frame $\{e_i\}_{i=1,\cdots,n-1}$ of TN with $\nabla^N_{e_i} e_i = 0$ at some point, we compute

$$\begin{aligned}
\Delta^\alpha f &= -\sum_{i=1}^{n-1}\langle \nabla^\alpha_{e_i}(d^\alpha f), e_i\rangle - \langle \nabla^\alpha_\nu(d^\alpha f), \nu\rangle \\
&= -\sum_{i=1}^{n-1} e_i(\langle d^\alpha f, e_i\rangle) + \sum_{i=1}^{n-1}\langle d^\alpha f, \nabla^\alpha_{e_i} e_i\rangle - \langle \nabla^\alpha_\nu(d^\alpha f), \nu\rangle \\
&= -\sum_{i=1}^{n-1} e_i(\langle d^\alpha f, e_i\rangle) + \sum_{i=1}^{n-1}\langle d^\alpha f, \nabla^M_{e_i} e_i + i\alpha(e_i)e_i\rangle - \langle \nabla^\alpha_\nu(d^\alpha f), \nu\rangle \\
&= -\sum_{i=1}^{n-1} e_i(\langle d^\alpha f, e_i\rangle) + \sum_{i=1}^{n-1}\langle d^\alpha f, II(e_i, e_i)\nu + i\alpha(e_i)e_i\rangle - \langle \nabla^\alpha_\nu(d^\alpha f), \nu\rangle \\
&= -\sum_{i=1}^{n-1} e_i(\langle d^\alpha_N f, e_i\rangle) + (n-1)H\langle d^\alpha f, \nu\rangle + \sum_{i=1}^{n-1}\langle d^\alpha_N f, i\alpha(e_i)e_i\rangle - \langle \nabla^\alpha_\nu(d^\alpha f), \nu\rangle \\
&= \Delta^\alpha_N f + (n-1)H\langle d^\alpha f, \nu\rangle - \langle \nabla^\alpha_\nu(d^\alpha f), \nu\rangle,
\end{aligned}$$

where $\Delta^\alpha_N := \delta^\alpha_N d^\alpha_N$, with $\delta^\alpha_N = \delta^N - i(\cdot, \alpha^T)$. We notice that δ^α_N is the L^2-adjoint of d^α_N on N. Plugging the expression of $\Delta^\alpha f$ above into Equation (2.2), we find

$$\begin{aligned}
\int_M \langle d^\alpha(\Delta^\alpha f), d^\alpha f\rangle dv_g &= \int_M |\Delta^\alpha f|^2 dv_g - \int_N (\Delta^\alpha_N f)\langle \nu, d^\alpha f\rangle dv_g - (n-1)\int_N H|\langle d^\alpha f, \nu\rangle|^2 dv_g \\
&\quad + \int_N \langle \nabla^\alpha_\nu(d^\alpha f), \nu\rangle\langle \nu, d^\alpha f\rangle dv_g. \\
&= \int_M |\Delta^\alpha f|^2 dv_g - \int_N (\Delta^\alpha_N f)\langle \nu, d^\alpha f\rangle dv_g - (n-1)\int_N H|\langle d^\alpha f, \nu\rangle|^2 dv_g \\
&\quad + \int_N \langle \nabla^M_\nu(d^\alpha f), \nu\rangle\langle \nu, d^\alpha f\rangle dv_g + \int_N i\alpha(\nu)|\langle \nu, d^\alpha f\rangle|^2 dv_g.
\end{aligned} \tag{2.3}$$

The last step is to compute the term $\frac{i}{2}\int_M \langle \bar f d^\alpha f, \delta^M d^M\alpha\rangle dv_g$ and its conjugate in Theorem 1.1. For this, we proceed as in [3, p.17] to get

$$\begin{aligned}
\frac{i}{2}\int_M \langle \bar f d^\alpha f, \delta^M d^M\alpha\rangle dv_g &= \frac{i}{2}\int_M \langle d^M(\bar f d^\alpha f), d^M\alpha\rangle dv_g + \frac{i}{2}\int_N \langle \bar f d^\alpha f, \nu\lrcorner d^M\alpha\rangle dv_g \\
&= \frac{i}{2}\int_M (d^M\alpha)(\overline{d^\alpha f}, d^\alpha f) dv_g - \frac{1}{2}\int_M |f|^2|d^M\alpha|^2 dv_g \\
&\quad +\frac{i}{2}\int_N \langle \bar f d^\alpha f, \nu\lrcorner d^M\alpha\rangle dv_g.
\end{aligned} \tag{2.4}$$

Now, we have all the ingredients to integrate Equation (1.1) over M. In fact, using Equations (2.1), (2.3) and (2.4), we find that

$$\begin{aligned}
&-\int_N \Re(\langle d^\alpha_N(\nu(f)), d^\alpha f\rangle + \langle II(d^\alpha_N f), d^\alpha f\rangle + if\langle \nu\lrcorner d^M\alpha, d^\alpha f\rangle + if\langle d^N(\alpha(\nu)), d^\alpha f\rangle \\
&+(\nabla^M_\nu d^\alpha f)(\nu)\langle \nu, d^\alpha f\rangle) dv_g = \int_M |\mathrm{Hess}^\alpha f|^2 dv_g - \int_M |\Delta^\alpha f|^2 dv_g + \int_N \Re((\Delta^\alpha_N f)\langle \nu, d^\alpha f\rangle) dv_g \\
&+(n-1)\int_N H|\langle d^\alpha f, \nu\rangle|^2 dv_g - \int_N \Re(\langle \nabla^M_\nu(d^\alpha f), \nu\rangle\langle \nu, d^\alpha f\rangle) dv_g + \int_M \mathrm{Ric}^M(d^\alpha f, d^\alpha f) dv_g \\
&+\frac{i}{2}\int_M \Bigg(\underbrace{(d^M\alpha)(d^\alpha f, \overline{d^\alpha f}) - (d^M\alpha)(\overline{d^\alpha f}, d^\alpha f)}_{2i\Im m((d^M\alpha)(d^\alpha f, \overline{d^\alpha f}))}\Bigg) dv_g - \int_M |f|^2|d^M\alpha|^2 dv_g \\
&+\frac{i}{2}\int_N \Bigg(\underbrace{\langle \bar f d^\alpha f, \nu\lrcorner d^M\alpha\rangle - \langle f\overline{d^\alpha f}, \nu\lrcorner d^M\alpha\rangle}_{=-2i\Im m f\langle \nu\lrcorner d^M\alpha, d^\alpha f\rangle}\Bigg) dv_g.
\end{aligned}$$

By writing $d^\alpha f = d^\alpha_N f + (\nu(f) + if\alpha(\nu))\nu$ at any point of the boundary, the first integral in the l.h.s. reduces to

$$\begin{aligned}
\int_N \Re\langle d^\alpha_N(\nu(f)), d^\alpha f\rangle dv_g &= \int_N \Re\langle d^\alpha_N(\nu(f)), d^\alpha_N f\rangle dv_g \\
&= \int_N \Re(\nu(f)\overline{\delta^\alpha_N d^\alpha_N f}) dv_g = \int_N \Re(\nu(f)\overline{\Delta^\alpha_N f}) dv_g \\
&= \int_N \Re(\langle d^\alpha f - i\alpha f, \nu\rangle \overline{\Delta^\alpha_N f}) dv_g \\
&= \int_N \Re(\langle \nu, d^\alpha f\rangle \Delta^\alpha_N f) dv_g - \int_N \Re(i\alpha(\nu) f\, \overline{\Delta^\alpha_N f}) dv_g.
\end{aligned}$$

Using the fact that δ^α_N is the L^2-adjoint of d^α_N and that $d^\alpha_N(f_1 f_2) = f_2 d^N f_1 + f_1 d^\alpha_N f_2$ for any complex valued functions f_1 and f_2 on N, the above equality becomes

$$\begin{aligned}\int_N \Re\langle d^\alpha_N(\nu(f)), d^\alpha f\rangle dv_g &= \int_N \Re(\langle \nu, d^\alpha f\rangle \Delta^\alpha_N f) dv_g - \int_N \Re\langle d^\alpha_N f, d^\alpha_N\, (i\alpha(\nu)f)\rangle dv_g \\ &= \int_N \Re(\langle \nu, d^\alpha f\rangle \Delta^\alpha_N f) dv_g + \int_N \Re(i\langle d^\alpha_N f, f d^N(\alpha(\nu)) + \alpha(\nu) d^\alpha_N f\rangle) dv_g \\ &= \int_N \Re(\langle \nu, d^\alpha f\rangle \Delta^\alpha_N f) dv_g + \int_N \Re(i\bar{f}\langle d^\alpha_N f, d^N(\alpha(\nu))\rangle) dv_g \\ &\quad + \int_N \alpha(\nu) \underbrace{\Re(i\langle d^\alpha_N f, d^\alpha_N f\rangle)}_{=0} dv_g \\ &= \int_N \Re(\langle \nu, d^\alpha f\rangle \Delta^\alpha_N f) dv_g - \int_N \Re(if\langle d^N(\alpha(\nu)), d^\alpha f\rangle) dv_g.\end{aligned}$$

Therefore, we deduce

$$\begin{aligned}-2\int_N \Re(\langle \nu, d^\alpha f\rangle \Delta^\alpha_N f) dv_g - \int_N \langle II(d^\alpha_N f), d^\alpha_N f\rangle dv_g = \\ \int_M |\mathrm{Hess}^\alpha f|^2 dv_g - \int_M |\Delta^\alpha f|^2 dv_g + (n-1)\int_N H|\langle d^\alpha f, \nu\rangle|^2 dv_g + \int_M \mathrm{Ric}^M(d^\alpha f, d^\alpha f) dv_g \\ - \int_M \Im m\left((d^M\alpha)(d^\alpha f, \overline{d^\alpha f})\right) dv_g - \int_M |f|^2 |d^M\alpha|^2 dv_g.\end{aligned}$$

The proof of the proposition then follows. □

3 Proof of Theorems 1.3 and 1.5

In the following, we will give a proof of both Theorems 1.3 and 1.5. For this, we consider an eigenfunction f of the Robin Laplacian associated to the eigenvalue $\lambda(\tau, \alpha)$, that is $\Delta^\alpha f = \lambda(\tau,\alpha) f$ with $\nu(f) + if\alpha(\nu) = \tau f$ for some positive τ. We then apply Equality (1.2) to the eigenfunction f. First, we have

$$\int_N \Re(\langle \nu, d^\alpha f\rangle \Delta^\alpha_N f) dv_g = \tau \int_N \Re(\bar{f}\Delta^\alpha_N f) dv_g = \tau\int_N \Re(f\overline{\Delta^\alpha_N f}) dv_g = \tau \int_N |d^\alpha_N f|^2 dv_g.$$

Also, the following inequality

$$\int_M \Im m\left((d^M\alpha)(d^\alpha f, \overline{d^\alpha f})\right) dv_g \leq ||d^M\alpha||_\infty \int_M |d^\alpha f|^2 dv_g,$$

holds. Therefore, as the r.h.s. of Equality (1.2) is nonnegative, we get after using the conditions $\mathrm{Ric}^M \geq k$ and $II + \tau \geq 0$ that

$$\begin{aligned} 0 \;\leq\; & \frac{n-1}{n}\lambda(\tau,\alpha)^2 \int_M |f|^2 dv_g - (k - ||d^M\alpha||_\infty)\int_M |d^\alpha f|^2 dv_g + ||d^M\alpha||^2_\infty \int_M |f|^2 dv_g \\ & -(n-1)\tau^2 \int_N H|f|^2 dv_g - \tau\int_N |d^\alpha_N f|^2 dv_g.\end{aligned}$$

Since f is an eigenfunction of the Laplacian, one has

$$\int_M |d^\alpha f|^2 dv_g = \lambda(\tau,\alpha)\int_M |f|^2 dv_g - \tau\int_N |f|^2 dv_g.$$

Hence, the above inequality reduces to

$$\begin{aligned} 0 \ \leq\ & \frac{n-1}{n}\lambda(\tau,\alpha)^2\int_M |f|^2 dv_g - (k-||d^M\alpha||_\infty)\lambda(\tau,\alpha)\int_M |f|^2 dv_g + (k-||d^M\alpha||_\infty)\tau\int_N |f|^2 dv_g \\ & +||d^M\alpha||^2_\infty\int_M |f|^2 dv_g - (n-1)\tau^2 H_{\min}\int_N |f|^2 dv_g - \tau\int_N |d^\alpha_N f|^2 dv_g. \end{aligned}$$

By grouping the terms and using the fact that the last term is nonpositive, we find at the end

$$\begin{aligned} 0 \ \leq\ & \left(\frac{n-1}{n}\lambda(\tau,\alpha)^2 - (k-||d^M\alpha||_\infty)\lambda(\tau,\alpha) + ||d^M\alpha||^2_\infty\right)\int_M |f|^2 dv_g \\ & +\tau\left(k-||d^M\alpha||_\infty - (n-1)\tau H_{\min}\right)\int_N |f|^2 dv_g. \end{aligned}$$

Now two cases occur depending on the sign of the term $(k-||d^M\alpha||_\infty) - (n-1)\tau H_{\min}$. If this last term is nonpositive, we deduce as in [3, Eq. 62] the inequality

$$0 \leq \frac{n-1}{n}\lambda(\tau,\alpha)^2 - (k-||d^M\alpha||_\infty)\lambda(\tau,\alpha) + ||d^M\alpha||^2_\infty.$$

Therefore, the same result holds as in [3, Thm. 1.1] and thus we finish the proof of Theorem 1.3. To prove Theorem 1.5, we choose α such that $k-(n-1)\tau H_{\min} \geq ||d^M\alpha||_\infty$. Recall that the condition of k allows to consider such an α. Hence, we find after using $\int_N |f|^2 dv_g \leq \int_M |f|^2 dv_g$ that

$$0 \leq \frac{n-1}{n}\lambda(\tau,\alpha)^2 - (k-||d^M\alpha||_\infty)\lambda(\tau,\alpha) + ||d^M\alpha||^2_\infty + (k-||d^M\alpha||_\infty)\tau - (n-1)\tau^2 H_{\min}.$$

The discriminant of this polynomial is equal to

$$\Delta = (1-4\frac{n-1}{n})||d^M\alpha||^2_\infty - 2(k-2\left(\frac{n-1}{n}\right)\tau)||d^M\alpha||_\infty + k^2 - 4\frac{n-1}{n}k\tau + 4\frac{(n-1)^2}{n}\tau^2 H_{\min},$$

which in turn has a discriminant equal to

$$\Delta_2 = 4\frac{n-1}{n}\left(k^2 - 4\frac{n-1}{n}k\tau + \frac{n-1}{n}\tau^2 - (n-1)\tau^2 H_{\min}(1-4\frac{n-1}{n})\right).$$

Since $H_{\min} \geq \frac{1}{n}$, this last discriminant is positive (just compute its discriminant again). Moreover, a straightforward computation shows that $\sqrt{\Delta_2} \geq k - 2(\frac{n-1}{n})\tau$. Therefore, if we take α such that the following inequality

$$||d^M\alpha||_\infty \leq \frac{\sqrt{\Delta_2} - (k-2(\frac{n-1}{n})\tau)}{4\frac{n-1}{n} - 1},$$

holds, we finish the proof of the theorem. □

4 Proof of Theorem 1.7

Let f be the function defined by $f = uf_\tau$, where $u : M \to \mathbb{C}$ is a complex valued function on M and f_τ is a normalized eigenfunction of the Robin Laplacian associated to the first eigenvalue $\lambda_1(\tau)$. Then, we compute

$$\begin{aligned}
\int_M |(d^M + i\alpha)f|^2 dv_g &= \int_\Omega |ud^M f_\tau + f_\tau(d^M u + i\alpha u)|^2 dv_g \\
&= \int_M |u|^2 |d^M f_\tau|^2 dv_g + \int_M f_\tau^2 |(d^M + i\alpha)u|^2 dv_g \\
&\quad +2\int_M f_\tau \Re\langle ud^M f_\tau, d^M u + i\alpha u\rangle dv_g \\
&= \int_M f_\tau \delta^M(|u|^2 d^M f_\tau) dv_g - \tau \int_N |u|^2 f_\tau^2 dv_g + \int_M f_\tau^2 |(d^M + i\alpha)u|^2 dv_g \\
&\quad + \int_M \Re\langle d^M(f_\tau^2), \bar{u} d^M u\rangle dv_g \\
&= \int_M f_\tau |u|^2 \delta^M(d^M f_\tau) dv_g - \int_M f_\tau g(d^M(|u|^2), d^M(f_\tau)) dv_g - \tau \int_N |u|^2 f_\tau^2 dv_g \\
&\quad + \int_M f_\tau^2 |(d^M + i\alpha)u|^2 + \int_M \Re\langle d^M(f_\tau^2), \bar{u} d^M u\rangle dv_g \\
&= \lambda_1(\tau) \int_M f_\tau^2 |u|^2 dv_g - \int_M f_\tau g(d^M(|u|^2), d^M(f_\tau)) dv_g - \tau \int_N |u|^2 f_\tau^2 dv_g \\
&\quad + \int_M f_\tau^2 |(d^M + i\alpha)u|^2 dv_g + \int_M \Re\langle d^M(f_\tau^2), \bar{u} d^M u\rangle dv_g.
\end{aligned}$$

Now, it is easy to see that one has pointwise

$$f_\tau g(d^M(|u|^2), d^M(f_\tau)) = f_\tau \langle \bar{u} d^M u + u d^M \overline{u}, d^M(f_\tau)\rangle = \Re\langle d^M(f_\tau^2), \bar{u} d^M u\rangle.$$

Consequently, we deduce that

$$\frac{\int_M |d^\alpha f|^2 dv_g + \tau \int_N f^2 dv_g}{||f||^2} = \lambda_1(\tau) + \frac{\int_M f_\tau^2 |d^\alpha u|^2 \, dv_g}{\int_M |u|^2 f_\tau^2 \, dv_g}.$$

Now the proof follows from the variational min-max principle. Indeed, the definition of $C(\tau)$ in (1.4) yields

$$C(\tau) \frac{\int_M f_\tau^2 |d^\alpha u|^2 \, dv_g}{\int_M |u|^2 f_\tau^2 \, dv_g} \leq \frac{\int_M f_\tau^2 |d^\alpha u|^2 \, dv_g}{\int_M |u|^2 f_\tau^2 \, dv_g} \leq \frac{1}{C(\tau)} \frac{\int_M |d^\alpha u|^2 \, dv_g}{\int_M |u|^2 \, dv_g},$$

which finishes the proof. □

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