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MULTIFRACTAL ANALYSIS OF THE BIRKHOFF SUMS OF SAINT-PETERSBURG POTENTIAL

DONG HAN KIM, LINGMIN LIAO, MICHAL RAMS, AND BAO-WEI WANG

Abstract. Let \((0, 1], T\) be the doubling map in the unit interval and \(\varphi\) be the Saint-Petersburg potential, defined by \(\varphi(x) = 2^n\) if \(x \in \left(2^{-n-1}, 2^{-n}\right]\) for all \(n \geq 0\). We consider asymptotic properties of the Birkhoff sum \(S_n(x) = \varphi(x) + \cdots + \varphi(T^{n-1}(x))\). With respect to the Lebesgue measure, the Saint-Petersburg potential is not integrable and it is known that \(\frac{1}{n} \log n S_n(x)\) converges to \(\frac{1}{\log 2}\) in probability. We determine the Hausdorff dimension of the level set \(\{x : \lim_{n \to \infty} S_n(x)/n = \alpha\}\) for \(\alpha > 0\), as well as that of the set \(\{x : \lim_{n \to \infty} S_n(x)/\Psi(n) = \alpha\}\) for \(a > 1, \gamma > 0\). The fast increasing Birkhoff sum of the potential function \(x \mapsto \frac{1}{x}\) is also studied.

1. Introduction

Let \(T\) be the doubling map on the unit interval \((0, 1]\) defined by
\[
Tx = 2x - \left\lfloor 2x \right\rfloor + 1,
\]
where \(\left\lfloor x \right\rfloor\) is the smallest integer larger than or equal to \(x\). Let \(\epsilon_1\) be the function defined by \(\epsilon_1(x) = \left\lfloor 2x \right\rfloor - 1\) and \(\epsilon_n(x) := \epsilon_1(T^{n-1}x)\) for \(n \geq 2\). Then each real number \(x \in (0, 1]\) can be expanded into an infinite series as
\[
x = \frac{\epsilon_1(x)}{2} + \cdots + \frac{\epsilon_n(x)}{2^n} + \cdots.
\]
We call (1.1) the binary expansion of \(x\) and also write it as
\[
x = [\epsilon_1(x)\epsilon_2(x)\ldots].
\]
The Saint-Petersburg potential is a function \(\varphi : (0, 1] \to \mathbb{R}\) defined as
\[
\varphi(x) = 2^n\quad \text{if } x \in (2^{-n-1}, 2^{-n}], \quad \forall n \geq 0.
\]
We remark that the definition of \(\varphi\) is equivalent to
\[
\varphi(x) = 2^n\quad \text{where } n \geq 0 \text{ is the smallest integer such that } \epsilon_{n+1}(x) = 1.
\]
and is also equivalent to
\[
\varphi(x) = 2^n\quad \text{if the binary expansion of } x \text{ begins with } 0^n1,
\]
where \(0^n(0 \geq 0)\) means a block with \(n\) consecutive zeros.

The name of Saint-Petersburg potential is motivated by the famous Saint-Petersburg game in probability theory. The Saint-Petersburg potential is of infinite expectation with respect to the Lebesgue measure. Furthermore, it increases exponentially fast near to the point 0.

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In this paper, we are concerned with the following Birkhoff sums of the Saint-Petersburg potential:

\[ \forall n \geq 1, \quad S_n(x) := \varphi(x) + \varphi(T(x)) + \cdots + \varphi(T^{n-1}(x)), \quad x \in (0, 1). \]

Let

\[ I = \{ x \in (0, 1] : \epsilon_1(x) = 1 \}. \]

Define the hitting time of \( x \in (0, 1] \) to \( I \) as

\[ n(x) := \inf \{ n \geq 0 : T^n x \in I \}. \]

Then

\[ n(x) = n \quad \text{if} \quad x \in \left( \frac{1}{2^n+1}, \frac{1}{2^n} \right], \quad \text{for all} \quad n \geq 0. \]

Using \( n(x) \), we define a new dynamical system \( \hat{T} : (0, 1] \to (0, 1] \) by

\[ \hat{T}(x) = T^{n(x)+1}(x) = 2^{n+1} \left( x - \frac{1}{2^{n+1}} \right) \quad \text{if} \quad x \in \left( \frac{1}{2^{n+1}+1}, \frac{1}{2^n} \right], \quad \text{for all} \quad n \geq 0, \]

called the acceleration of \( T \), in order that \( \varphi \) and \( \varphi \circ \hat{T} \) are independent. Let

\[ \hat{S}_n(x) := \varphi(x) + \varphi(\hat{T}(x)) + \cdots + \varphi(\hat{T}^{n-1}(x)), \quad x \in (0, 1]. \]

The convergence in probability of \( \hat{S}_n(x) \) is well known (e.g., [6, p.253]) which states that for any \( \epsilon > 0 \), the Lebesgue measure \( \lambda \) of

\[ \left\{ x \in (0, 1] : \frac{\hat{S}_n(x)}{n \log n} - \frac{1}{\log 2} \geq \epsilon \right\} \]

tends to 0 as \( n \to \infty \).

Let \( \{ \Psi_n \}_{n \geq 1} \) be an increasing sequence such that \( \Psi_n \to \infty \) as \( n \to \infty \). Then it was shown in [5] that Lebesgue almost surely either

\[ \lim_{n \to \infty} \frac{\hat{S}_n(x)}{\Psi_n} = 0 \quad \text{or} \quad \limsup_{n \to \infty} \frac{\hat{S}_n(x)}{\Psi_n} = \infty, \]

according as

\[ \sum_{n \geq 1} \lambda(\{ x \in (0, 1] : \varphi(x) \geq \Psi_n \}) < \infty \quad \text{or} \quad = \infty. \]

Let \( n_1 = n_1(x) = n(x) + 1 \) and \( n_k = n_k(x) = n_1(\hat{T}^{k-1}x) = n(\hat{T}^{k-1}x) \) for \( k \geq 2 \).

It is direct to see that

\[ \forall \ell \geq 1, \quad S_{n_1 + \cdots + n_k}(x) = 2 \hat{S}_\ell(x) - \ell. \]

Moreover, the ergodicity of \( T \) (of \( \hat{T} \)) implies Lebesgue almost surely

\[ \lim_{\ell \to \infty} \frac{n_1(x) + \cdots + n_k(x)}{\ell} = \int_0^1 (n(x) + 1) d\lambda(x) = 2. \]

Combining these two facts together, we obtain the same convergence results as above if we replace \( \hat{S}_n \) by \( S_n \). In particular, the average \( S_n(x)/(n \log n) \) converges to \( 1/\log 2 \) in probability, and almost surely (with respect to the Lebesgue measure) either

\[ \lim_{n \to \infty} \frac{S_n(x)}{\Psi_n} = 0 \quad \text{or} \quad \limsup_{n \to \infty} \frac{S_n(x)}{\Psi_n} = \infty, \]

according as

\[ \sum_{n \geq 1} \lambda(\{ x \in (0, 1] : \varphi(x) \geq \Psi_n \}) < \infty \quad \text{or} \quad = \infty. \]
where \( \{\Psi_n\}_{n \geq 1} \) is an increasing sequence such that \( \Psi_n \to \infty \) as \( n \to \infty \). Recall that \( \varphi \) has infinite expectation with respect to the Lebesgue measure. We thus have \( S_n(x)/n \) converges to infinity for Lebesgue almost all points.

In this article, we want to further study the asymptotic behavior of the Birkhoff sum \( S_n(x) \) of the Saint-Petersburg potential. We give a complete multifractal analysis of \( S_n(x) \).

First, for any \( \alpha \geq 1 \), we consider the level set

\[
E(\alpha) = \left\{ x \in (0,1] : \lim_{n \to \infty} \frac{1}{n} S_n(x) = \alpha \right\}.
\]

For \( t \in \mathbb{R} \) and \( q > 0 \), define

\[
P(t,q) := \log \sum_{j=1}^{\infty} 2^{-tj-q(2^j-1)}.
\]

Then \( P \) is a real-analytic function. Furthermore, for each \( q > 0 \), there is a unique \( t(q) > 0 \) such that \( P(t(q),q) = 0 \). This function \( q \mapsto t(q) \) is real-analytic, strictly decreasing and convex.

Denote by \( \dim_H \) the Hausdorff dimension. The function \( \alpha \mapsto \dim_H E(\alpha) \), called the Birkhoff spectrum of the Saint-Petersburg potential \( \varphi \), is proved to be the Legendre transformation of the function \( q \mapsto t(q) \).

**Theorem 1.1.** For any \( \alpha \geq 1 \) we have

\[
\dim_H E(\alpha) = \inf_{q>0} \left\{ t(q) + q\alpha \right\}.
\]

Consequently, \( \dim_H E(1) = 0 \) and the function \( \alpha \mapsto \dim_H E(\alpha) \) is real-analytic, strictly increasing, concave, and has limit 1 as \( \alpha \to \infty \).

The Birkhoff spectrum of a continuous potential was obtained for full shifts [14], for topologically mixing subshifts of finite type [4], and for repellers of a topologically mixing \( C^{1+\epsilon} \) expanding map [2]. A continuous potential in a compact space is bounded, hence these classical results are all for bounded potentials. Our Theorem 1.1 gives a Birkhoff spectrum for an unbounded function with a singular point. To prove Theorem 1.1, we will transfer our question to a Birkhoff spectrum problem of an interval map with infinitely many branches and we will apply the techniques developed in [9] for continued fraction dynamical systems and in [8] for general expanding interval maps with infinitely many branches.

We also study the Birkhoff sums \( S_n(x) \) of fast increasing rates. Let \( \Psi : \mathbb{N} \to \mathbb{N} \) be an increasing function. For \( \beta \in [0,\infty] \), consider the level set

\[
E_{\Psi}(\beta) := \left\{ x \in (0,1] : \lim_{n \to \infty} \frac{1}{\Psi(n)} S_n(x) = \beta \right\}.
\]

**Theorem 1.2.** If \( \Psi(n) \) is one of the following

\[
\Psi(n) = n \log n, \quad \Psi(n) = n^\alpha (\alpha > 1), \quad \Psi(n) = 2^{n^\gamma} (0 < \gamma < 1/2),
\]

then for any \( \beta \in [0,\infty] \), \( \dim_H E_{\Psi}(\beta) = 1 \).

If \( \Psi(n) = 2^{n^\gamma} \) with \( 1/2 \leq \gamma < 1 \), then for any \( \beta \in (0,\infty) \), the set \( E_{\Psi}(\beta) \) is empty, and \( \dim_H E_{\Psi}(\beta) = 1 \) for \( \beta = 0, +\infty \).

If \( \Psi(n) = 2^{n^\gamma} \) with \( \gamma \geq 1 \), then for any \( \beta \in (0,\infty] \), the set \( E_{\Psi}(\beta) \) is empty, and \( \dim_H E_{\Psi}(\beta) = 1 \) for \( \beta = 0 \).

We remark that by the above discussion on the convergence of \( S_n(x) \), for all cases in Theorem 1.2, the sets \( E_{\Psi}(0) \) has full measure, and thus obviously has full Hausdorff dimension.
From the definition of $S_n(x)$, we see that for the integer $n$ such that $\epsilon_n(x) = 1$, one has $S_n(x) = S_{n-1}(x) + 1$, which implies for all $x$, $\liminf_{n \to \infty} \frac{S_n(x)}{S_{n-1}(x)} = 1$. Thus if $\liminf_{n \to \infty} \frac{\Psi(n)}{\Psi(n-1)} > 1$, then for any $\beta \in (0, \infty)$, the set $E_\Psi(\beta)$ is empty. By the definition of $S_n(x)$, we can also check that for all $x$, for the integer $n$ such that $\epsilon_n(x) = 1$, we have $S_n(x) \leq 2^n - 1$. This implies $\liminf_{n \to \infty} S_n(x)/2^n \leq 1$ (See also the formula (3.15) in Section 3). Hence, for ‘regular’ growth functions $\Psi$ we only need to consider exponential and subexponential growth rates.

However, if we pick a point $x$ with dyadic expansion consisting mostly of 0’s, with infinitely many 1’s but in large distances from each other, then the Birkhoff sum $S_n(x)$ may grow arbitrarily fast on some subsequence $n_i$. Thus for any increasing $\Psi(n)$ there exists a point $x$ such that $\limsup_{n \to \infty} S_n(x)/\Psi(n) = \infty$.

Our study on the Saint-Petersburg potential is an attempt of multifractal analysis of unbounded potential functions on the doubling map dynamical system. However, the Saint-Petersburg potential is locally constant and not continuous. One might think of an another unbounded potential function $g : x \mapsto 1/x$ which is close to the Saint-Petersburg potential but is continuous. In fact, our method for studying the fast increasing Birkhoff sum of Saint-Petersburg potential also works for the fast increasing Birkhoff sum of the potential $g : x \mapsto 1/x$.

Denote by $S_n g(x)$ the Birkhoff sum

$$S_n g(x) := g(x) + g(T(x)) + \cdots + g(T^{n-1}(x)), \quad x \in (0, 1].$$

For $\beta \in [0, \infty)$, let

$$F_\Psi(\beta) := \left\{ x \in (0, 1] : \lim_{n \to \infty} \frac{1}{\Psi(n)} S_n g(x) = \beta \right\}.$$

**Theorem 1.3.** If $\Psi(n)$ is one of the following

$$\Psi(n) = n \log n, \quad \Psi(n) = n^\alpha (\alpha > 1), \quad \Psi(n) = 2^n \gamma (0 < \gamma < 1/2),$$

then for any $\beta \in [0, \infty]$, $\dim_H F_\Psi(\beta) = 1$.

If $\Psi(n) = 2^n \gamma$ with $1/2 \leq \gamma < 1$, then for any $\beta \in (0, \infty)$, the set $F_\Psi(\beta)$ is empty, and $\dim_H F_\Psi(\beta) = 1$ for $\beta = 0, +\infty$.

If $\Psi(n) = 2^n \gamma$ with $\gamma \geq 1$, then for any $\beta \in (0, \infty]$, the set $F_\Psi(\beta)$ is empty, and $\dim_H F_\Psi(\beta) = 1$ for $\beta = 0$.

We remark that these multifractal analysis on the Birkhoff sums of fast increasing rates have been done for some special potentials in continued fraction dynamical system ([9, 11, 12]).

2. **Birkhoff spectrum of the Saint-Petersburg potential**

In this section, we will obtain the Birkhoff spectrum of the Saint-Petersburg potential, i.e. the Hausdorff dimension of the following level set:

$$E(\alpha) := \left\{ x \in (0, 1] : \lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi(T^j x) = \alpha \right\} \quad (\alpha \geq 1).$$

We will transfer our question to a Birkhoff spectrum problem for an interval map with infinitely many branches.
2.1. Transference lemma. Recall that the Saint-Petersburg potential $\varphi$ is given by

$$\varphi(x) = 2^n, \text{ if } x = [0^n1, \cdots]$$

where $x = [\epsilon_1\epsilon_2, \cdots]$ denotes the digit sequence in the binary expansion of $x$. Recall also the definition of hitting time $n(x)$ and the acceleration $\hat{T}$ of the doubling map $T$ in Section 1. Define a new potential function

$$\phi(x) := 2^{n(x)+1} - 1, \ x \in (0, 1].$$

In fact, $\phi$ is nothing but the function satisfying

$$\phi(x) = \sum_{j=0}^{n(x)} \varphi(T^j x).$$

With the notation $n_1 = n(x) + 1 \geq 1$, and $n_k = n(\hat{T}^{k-1}x) + 1$ for $k \geq 2$ given in Section 1, we have

$$\phi(\hat{T}x) = \sum_{j=0}^{n(\hat{T}x)} \varphi(T^j(\hat{T}x)) = \sum_{j=n_1}^{n_2-1} \varphi(T^j x).$$

Hence,

$$\sum_{j=0}^{n_1 + \cdots + n_{\ell} - 1} \varphi(T^j x) = \sum_{k=0}^{\ell - 1} \phi(\hat{T}^k x) = 2^{n_1} + \cdots + 2^{n_{\ell}} - \ell. \quad (2.1)$$

Note that the derivative of $\hat{T}$ satisfies

$$|\hat{T}'(x)| = 2^{n(x)+1} = 2^{n_1} = \phi(x) + 1. \quad (2.2)$$

We have

$$n_1 + \cdots + n_{\ell} = \sum_{k=0}^{\ell - 1} \log_2 |\hat{T}'(\hat{T}^k x)|.$$

Recall the set in question:

$$E(\alpha) = \left\{ x \in (0, 1] : \lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi(T^j x) = \alpha \right\} \quad (\alpha \geq 1).$$

Define

$$\tilde{E}(\alpha) := \left\{ x \in (0, 1] : \lim_{\ell \to \infty} \frac{\sum_{k=0}^{\ell - 1} \phi(\hat{T}^k x)}{\sum_{k=0}^{\ell - 1} \log_2 |\hat{T}'(\hat{T}^k x)|} = \alpha \right\} \quad (\alpha \geq 1).$$

The following lemma shows the two level sets are the same.

**Lemma 2.1.** For all $\alpha \geq 1$, we have $E(\alpha) = \tilde{E}(\alpha)$.

**Proof.** It is evident that $E(\alpha) \subset \tilde{E}(\alpha)$, because, as discussed above,

$$\frac{\sum_{k=0}^{\ell - 1} \phi(\hat{T}^k x)}{\sum_{k=0}^{\ell - 1} \log_2 |\hat{T}'(\hat{T}^k x)|} = \frac{1}{n_1 + \cdots + n_{\ell}} \sum_{j=0}^{n_1 + \cdots + n_{\ell} - 1} \varphi(T^j x). \quad (2.3)$$

Now, we show the other direction. Take an $x \in \tilde{E}(\alpha)$, express $x$ in its binary expansion

$$x = [0^{n_1-1}10^{n_2-1}1\cdots0^{n_{\ell}-1}1\cdots].$$

In fact, $n_{\ell} - 1$ is the recurrence time for $n(\hat{T}^{\ell-1}x)$, for each $\ell \geq 1$. 

By (2.1), we have, at present,

$$\lim_{\ell \to \infty} \frac{1}{n_1 + \cdots + n_\ell} \sum_{j=0}^{n_1 + \cdots + n_\ell - 1} \varphi(T^j x) = \alpha.$$ 

So, we are required to check it holds for all $n$.

For any $\epsilon > 0$, there exists $\ell_0 \in \mathbb{N}$ such that, for any $\ell \geq \ell_0$,

$$\alpha - \epsilon \leq \frac{2^{n_1} + \cdots + 2^{n_\ell} - \ell}{n_1 + \cdots + n_\ell} \leq \alpha + \epsilon.$$ 

(2.4)

For any $n_1 + \cdots + n_\ell < n < n_1 + \cdots + n_\ell + n_{\ell+1}$ with $\ell \geq \ell_0$, it is trivial that

$$\frac{2^{n_1} + \cdots + 2^{n_\ell} - \ell}{n_1 + \cdots + n_\ell + n_{\ell+1}} \leq 1 - \frac{1}{n} \sum_{j=0}^{n-1} \varphi(T^j x) \leq \frac{2^{n_1} + \cdots + 2^{n_\ell} + 2^{n_{\ell+1}} - \ell - 1}{n_1 + \cdots + n_\ell}.$$ 

Thus, it suffices to show that

$$2^{n_{\ell+1}} = o(n_1 + \cdots + n_\ell),$$ 

(2.5)

which also implies

$$n_{\ell+1} = o(n_1 + \cdots + n_\ell).$$

Let $M_0$ be a large integer such that, for all $M \geq M_0$, $2^M \geq 4\alpha M$. So, when $n_{\ell+1} \leq M_0$, there is nothing to prove. So, we always assume $2^{n_{\ell+1}} \geq 4\alpha n_{\ell+1}$.

By (2.4), we have

$$2^{n_1} + \cdots + 2^{n_\ell} - \ell \geq (\alpha - \epsilon)(n_1 + \cdots + n_\ell),$$

$$2^{n_1} + \cdots + 2^{n_\ell} + 2^{n_{\ell+1}} - \ell - 1 \leq (\alpha + \epsilon)(n_1 + \cdots + n_\ell + n_{\ell+1}).$$

These give

$$2^{n_{\ell+1}} \leq 2\epsilon(n_1 + \cdots + n_\ell) + (\alpha + \epsilon)n_{\ell+1} + 1.$$ 

So, we have

$$2^{n_{\ell+1}} \leq 4\epsilon(n_1 + \cdots + n_\ell).$$

\[ \square \]

2.2. Dimension of $\hat{E}(\alpha)$. Now we calculate the Hausdorff dimension of the set $\hat{E}(\alpha)$. At first, we give a notation.

- For each finite word $w \in \bigcup_{n \geq 1}\{0,1\}^n$ of length $n$, a $T$-dyadic cylinder of order $n$ is defined as
  
  $$I_n(w) = \{ x \in (0,1) : (\epsilon_1(x), \cdots, \epsilon_n(x)) = w \}.$$ 

- For $(n_1, \cdots, n_\ell) \in (\mathbb{N} \setminus \{0\})^\ell$, a $T$-dyadic cylinder of order $\ell$ is defined as
  
  $$D_{\ell}(n_1, \cdots, n_\ell) = \{ x \in (0,1) : n_k(x) = n_k, \ 1 \leq k \leq \ell \}.$$ 

Proof of Theorem 1.1. To calculate the Hausdorff dimension of $\hat{E}(\alpha)$, we construct a suitable measure supported on $\hat{E}(\alpha)$. The Gibbs measures derived from the Ruelle-Perron-Frobenius transfer operator are good candidates for such a measure.

In fact, by considering the inverse branches $U_i : x \mapsto \frac{x - i}{2^n}$ $(i \geq 0)$ of $\hat{T}$, we can code the dynamical system $([0,1], \hat{T})$ by the conformal infinite iterated function system $(U_i)_{i \geq 0}$ which satisfies the open set condition ([10, Section 1]).

Consider the potential function with two parameters

$$\psi_{t,q} := -t \log |\hat{T}'| - (\log 2) \cdot q \phi \quad (t \in \mathbb{R}, \ q > 0).$$

Then $\psi_{t,q} \circ U_i$ is a family of strong Hölder family ([10, Page 30]). Hence, we can define a Ruelle operator

$$\mathcal{L}_{t,q} f(x) := \sum_{y \in \hat{T}^{-1} x} e^{\psi_{t,q}(y)} f(y),$$
on the Banach space of continuous functions on the corresponding infinite symbolic space ([10, Page 31]).

By the Ruelle-Perron-Frobenius transfer operator theory [10, Theorems 2.9 and 2.10], for any $q > 0$ (to satisfy the condition 2.2 of [10]), we can find an eigenvalue $\lambda_{t,q}$ and an eigenfunction $h_{t,q}$ for $\mathcal{L}_{t,q}$ and an eigenfunction $\nu_{t,q}$ for the conjugate operator $\mathcal{L}_{t,q}^*$. Then the pressure function $P(t,q) = \log \lambda_{t,q}$ and the ergodic Gibbs measure $\mu_{t,q}$ is given by $h_{t,q} \cdot \nu_{t,q}$.

The pressure function can be computed by (see [10, Pages 31 and 48])

$$P(t,q) = \lim_{\ell \to \infty} \frac{1}{\ell} \log \left( \sum_{(n_1, \ldots, n_\ell) \in (\mathbb{N}\setminus\{0\})^\ell} \exp \sup_{x \in D_\ell(n_1, \ldots, n_\ell)} (S_\ell \psi_{t,q}(x)) \right).$$

Note that for all $n \geq 1$,

$$|\tilde{T}|(x) = 2^n,$$

and $\phi(x) = 2^n - 1$ if $x \in \left(\frac{1}{2^n}, \frac{1}{2^n-1}\right) = D_1(n)$.

Then for $x \in D_\ell(n_1, \ldots, n_\ell)$,

$$S_\ell \psi_{t,q}(x) = -(\log 2) \cdot \ell \cdot \sum_{j=1}^{\ell} n_j + (\log 2) \cdot q \cdot \sum_{j=1}^{\ell} (2^{n_j} - 1).$$

Thus

$$\sum_{(n_1, \ldots, n_\ell) \in (\mathbb{N}\setminus\{0\})^\ell} \exp \sup_{x \in D_\ell(n_1, \ldots, n_\ell)} (S_\ell \psi_{t,q}(x))$$

$$= \prod_{n_1=1}^{\infty} \cdots \prod_{n_\ell=1}^{\infty} \left( \prod_{j=1}^{\ell} 2^{-t n_j} \cdot \prod_{j=1}^{\ell} 2^{q(2^{n_j} - 1)} \right) = \left( \sum_{j=1}^{\infty} 2^{-t_j - q(2^j - 1)} \right)^\ell.$$

Hence

$$P(t,q) = \log \sum_{j=1}^{\infty} 2^{-t_j - q(2^j - 1)}.$$

Now we calculate the local dimension of the Gibbs measure $\mu_{t,q}$. Let $D_\ell(x)$ be the $\tilde{T}$-dyadic cylinder containing $x$ of order $\ell$. By the Gibbs property of $\mu_{t,q},$

$$\frac{\log \mu_{t,q}(D_\ell(x))}{\log |D_\ell(x)|} = \frac{S_\ell \psi_{t,q}(x) - \ell P(t,q)}{-S_\ell \log |\tilde{T}|(x)}$$

$$= \frac{-\ell S_\ell \log |\tilde{T}|(x) - (\log 2) \cdot q S_\ell \phi(x) - \ell P(t,q)}{-S_\ell \log |\tilde{T}|(x)}$$

$$= t + q \frac{S_\ell \phi(x)}{S_\ell \log |\tilde{T}|(x)} + \frac{\ell P(t,q)}{S_\ell \log |\tilde{T}|(x)}.$$

**Upper bound.** For each $q > 0$, let $t(q)$ be the the number such that $P(t(q),q) = 0$. (The existence of $t(q)$ comes from the facts that $P(t,q)$ is real-analytic and that for fixed $q > 0$, $P(t,q) > 0$ when $t \to -\infty$ and $P(t,q) < 0$ when $t \to +\infty$.) Then for all $x \in \tilde{E}(\alpha)$, we have

$$\liminf_{r \to 0} \frac{\log \mu_{t,q}(B(x,r))}{\log r} \leq \liminf_{\ell \to \infty} \frac{\log \mu_{t,q}(B(x,|D_\ell(x)|))}{\log |D_\ell(x)|}$$

$$\leq \liminf_{\ell \to \infty} \frac{\log \mu_{t,q}(D_\ell(x))}{\log |D_\ell(x)|} = t(q) + qa,$$
where for the second inequality the trivial inclusion $D_\ell(x) \subset B(x, |D_\ell(x)|)$ is used. By Billingsley Lemma (see e.g. [3, Proposition 4.9.]), this gives an upper bound of the Hausdorff dimension of $\tilde{E}(\alpha)$. Thus we have

$$\dim_H \tilde{E}(\alpha) \leq \inf_{q > 0} \{t(q) + q\alpha\}.$$

- **LOWER BOUND.** By the real-analyticity of $P(t, q)$ and the implicit function theorem, the function $q \mapsto t(q)$ is also real-analytic. Thus there exists $q_0$ such that the following infimum is attained

$$\inf_{q > 0} \{t(q) + q\alpha\}.$$

Then we have

$$t'(q_0) + \alpha = 0. \quad (2.7)$$

To prove the lower bound, we first show two claims.

Claim (A): The measure $\mu_{t(q_0), q_0}$ is supported on $E_\alpha$.

On the one hand, since $P(t(q), q) = 0$,

$$\frac{\partial P}{\partial t} t'(q) + \frac{\partial P}{\partial q} = 0. \quad (2.8)$$

On the other hand, by the ergodicity of the measure $\mu_{t, q}$, we have for $\mu_{t, q}$ almost all $x$,

$$\lim_{\ell \to \infty} \frac{S_\ell \phi(x)}{S_\ell \log_2 |T'|(x)} = \frac{\int \phi d\mu_{t, q}}{\int \log |T'| d\mu_{t, q}} \cdot \log 2.$$

By Ruelle-Perron-Frobenius transfer operator theory ([10], Proposition 6.5),

$$\int (\log 2) \cdot \phi d\mu_{t, q} = -\frac{\partial P}{\partial q} \quad \text{and} \quad \int \log |T'| d\mu_{t, q} = -\frac{\partial P}{\partial t}.$$

Thus by (2.8) and then (2.7), for $\mu_{t(q_0), q_0}$ almost all $x$,

$$\lim_{\ell \to \infty} \frac{S_\ell \phi(x)}{S_\ell \log_2 |T'|(x)} = -t'(q_0) = \alpha.$$

This shows Claim (A).

Claim (B): For $\mu_{t(q_0), q_0}$ almost all $x$,

$$\lim_{n \to \infty} \frac{\log \mu_{t(q_0), q_0}(I_n(x))}{\log 2^{-n}} = t(q_0) + q_0\alpha,$$

where $I_n(x)$ is the $T$-dyadic cylinder of order $n$ containing $x$.

On the one hand, by (2.6) and then by (2.8) and (2.7), one has for $\mu_{t(q_0), q_0}$ almost all $x$

$$\lim_{\ell \to \infty} \frac{\log \mu_{t(q_0), q_0}(D_\ell(x))}{\log |D_\ell(x)|} = t(q_0) + q_0 \frac{\partial P}{\partial t} = t(q_0) + q_0\alpha. \quad (2.9)$$

On the other hand, note that for any $x \in E(\alpha)$, if the binary expansion of $x$ is $x = [0^{n_1-1}10^{n_2-1}1\ldots]$, then for any $\delta > 0$, for $\ell$ large enough,

$$(\alpha - \delta)\ell \leq 2^{n_1} + \cdots + 2^{n_\ell} - \ell = S_\ell \phi(x) \leq (\alpha + \delta)\ell.$$

Hence

$$n_\ell = O(\log \ell),$$

which implies

$$\lim_{\ell \to \infty} \frac{\log |D_\ell(x)|}{\log |D_{\ell+1}(x)|} = \lim_{\ell \to \infty} \frac{n_1 + \cdots + n_\ell}{n_1 + \cdots + n_\ell + n_{\ell+1}} = 1. \quad (2.10)$$
Thus

\[
\lim_{n \to \infty} \frac{\log \mu_{t(q_0),q_0}(I_n(x))}{\log 2^{-n}} = \lim_{t \to \infty} \frac{\log \mu_{t(q_0),q_0}(D_t(x))}{\log |D_t(x)|}.
\]

This shows Claim (B).

To conclude the desired lower bound, we apply the classical mass distribution principle (see [3, Proposition 4.2]). Since the Hausdorff dimension will not be changed if we replace the \( \delta \)-coverings by \( T \)-dyadic cylinder coverings (see [3, Section 2.4]), the lower bound of the Hausdorff dimension can be given by the mass transference principle on \( T \)-dyadic cylinders. By the above two claims and Egorov’s theorem, for any \( \eta > 0 \), there exists an integer \( N_0 \) such that the set

\[
\left\{ x \in E_\alpha : \mu(I_n(x)) \leq |I_n(x)|^{t(q_0)+q_0\alpha-\eta}, \ n \geq N \right\}
\]

is of \( \mu_{t(q_0),q_0} \) positive measure. So, it implies that

\[
\dim_H E_\alpha \geq t(q_0) + q_0\alpha - \eta.
\]

Note that ([10, Lemma 7.5]) the function \( q \mapsto t(q) \) is a decreasing convex function such that

\[
t(0) = 1, \quad \lim_{q \to \infty} (t(q) + q) = 0,
\]

and

\[
\lim_{q \to 0^+} t'(q) = -\infty, \quad \lim_{q \to +\infty} t'(q) = -1.
\]

Therefore, we have proved for any \( \alpha \in (1, +\infty) \)

\[
\dim_H(\bar{E}(\alpha)) = \inf_{q>0} \{t(q) + q\alpha\},
\]

which is Legendre transformation. All the properties stated in Theorem 1.1 are satisfied by the function \( \alpha \mapsto \dim_H(\bar{E}(\alpha)) \) which is the same function as \( \alpha \mapsto \dim_H(E(\alpha)) \) by Lemma 2.1.

For the end point \( \alpha = 1 \), it suffices to note that the level set \( E(1) \) is nothing but the set of numbers with frequency of the digit 1 in its binary expansion being 1. Thus the Hausdorff dimension of \( E(1) \) is 0. Hence, the Legendre transformation formula for the Hausdorff dimension of \( E(\alpha) \) \((\alpha > 1)\) also holds for \( \alpha = 1 \). \( \square \)

3. Fast increasing Birkhoff sum

At first, we give two simple observations.

**Lemma 3.1.** Let \( W \) be an integer such that \( 2^t \leq W < 2^{t+1} \) for some positive integer \( t \). For any \( 0 \leq n \leq t \), among the integers between \( W \) and \( W(1+2^{-n}) \), there is one \( V = V(W,n) \) whose binary expansion of \( V \) has at most \( n + 2 \) digits 1 and ends with at least \( t - n \) zeros.

**Proof.** By the assumption, we have \( 2^{-n}W \geq 2^{t-n} \). Thus among the \( 2^{-n}W \) consecutive integers from \( W \) to \( W(1+2^{-n}) \) there is at least one integer which is divisible by \( 2^{t-n} \) which means there is an integer \( \ell \geq 1 \) such that

\[
W \leq \ell 2^{t-n} \leq W(1 + 2^{-n}).
\]

Let \( V = \ell 2^{t-n} \) and note that \( V \) is an integer whose binary expansion ends with at least \( t - n \) zeros. Since \( \ell 2^{t-n} \leq W(1 + 2^{-n}) \leq 2^{t+1} \), we conclude that \( \ell 2^{t-n} \) has at most \( (t+2) - (t - n) = n + 2 \) digits 1 in its binary expansion. \( \square \)

In the follows, the base of the logarithm is taken to be 2.
Lemma 3.2. For each integer $W$, and any integer $n \leq \log W$, we can find a word $w$ with length $|w| \leq (n + 2)(2 + \log W)$ and for any $x \in I_{|w|}(w)$

$$W \leq \sum_{j=0}^{\lfloor \log W \rfloor - 1} \varphi(T^j x) \leq W(1 + 2^{-n}).$$

Proof. Let $V$ be an integer given in Lemma 3.1. Then $W \leq V \leq W(1 + 2^{-n})$.

Moreover, if we write this number $V$ in binary expansion:

$$V = 2^{t_1} + \cdots + 2^{t_p},$$

one has that $\lfloor \log W \rfloor + 1 \geq t_1 > \cdots > t_p \geq \lfloor \log W \rfloor - n$ and $p \leq n + 2$. Consider the word $w = (10^{t_1-1}, 10^{t_2-1}, \cdots, 10^{t_p-1})$ here the word $10^{t_p-1}$ is 1 when $t_p = 0$. Then we can check that the length of $w$ satisfies

$$|w| = (t_1 + 1) + \cdots + (t_p + 1) \leq p(t_1 + 1) \leq (n + 2)(2 + \log W),$$

and for any $x \in I_{|w|}(w)$,

$$\sum_{j=0}^{\lfloor \log W \rfloor - 1} \varphi(T^j x) = V.$$

Hence, the proof is completed. □

We also need the following lemmas whose proofs are left for the reader.

Lemma 3.3. For any $m \geq 1$, define

$$F_m = \{x \in (0, 1] : \epsilon_{km}(x) = 1, \text{ for all } k \geq 1\}.$$

Then $\dim_H F_m = \frac{m-1}{m}$.

Lemma 3.4. [13, Lemma 4] Given a subset $J$ of positive integers and an infinite sequence $\{a_k\}_{k=1}^{\infty}$ of 0’s and 1’s, let

$$E(J, \{a_k\}_{k=1}^{\infty}) = \{x \in (0, 1] : \epsilon_k(x) = a_k, \text{ for all } k \in J\}.$$

If the density of $J$ is zero, that is,

$$\lim_{n \to \infty} \frac{1}{n} \text{Card}\{k \leq n : k \in J\} = 0$$

then $\dim_H E(J, \{a_k\}_{k=1}^{\infty}) = 1$.

Before the proof Theorem 1.2, we show the following lemma.

Lemma 3.5. Let $\Psi : \mathbb{N} \to \mathbb{N}$ be an increasing function such that $\Psi(n)/n \to \infty$ as $n \to \infty$. Assume that there exists a subsequence $N_k$ satisfying the following conditions

$$N_k - N_{k-1} \to \infty, \quad \Psi(N_k) - \Psi(N_{k-1}) \to \infty, \quad \text{(3.1)}$$

and

$$\frac{\Psi(N_{k-1})}{\Psi(N_k)} \to 1, \quad \frac{\log (\Psi(N_k) - \Psi(N_{k-1}))}{N_k - N_{k-1}} \to 0, \quad \text{(3.2)}$$

as $k \to \infty$. Then the set

$$E_{\Psi}(1) = \{x \in (0, 1] : \lim_{n \to \infty} \frac{1}{\Psi(n)} S_n(x) = 1\}$$

has Hausdorff dimension 1.
Proof. Fix a large integer \( m \) and write
\[
U = \left\{ u = (\epsilon_1, \ldots, \epsilon_m) : \epsilon_m = 1, \epsilon_i \in \{0, 1\}, i \neq m \right\}.
\]

To avoid the abuse of notation, by the first assumption of (3.1), we assume \( N_k - N_{k-1} > m \) for all \( k \geq 1 \) by setting \( N_0 = 0 \) and \( \Psi(N_0) = 0 \).

For each \( k \geq 1 \), we write
\[
W_k := \Psi(N_k) - \Psi(N_{k-1})
\]
and let \( \{n_k\} \) be a sequence of integers tending to \( \infty \) such that
\[
n_k \leq \log W_k, \quad n_k \cdot \frac{\log (\Psi(N_k) - \Psi(N_{k-1}))}{N_k - N_{k-1}} \to 0.
\]

By the second assumptions of (3.1) and (3.2), this sequence of \( n_k \geq 0 \) do exist.

Now for \( W_k \) and \( n_k \), let \( w_k \) be the word given in Lemma 3.2. Then the length \( a_k \) of \( w_k \) satisfies
\[
a_k \leq (n_k + 2)(2 + \log W_k)
= (n_k + 2)(2 + \log (\Psi(N_k) - \Psi(N_{k-1}))) = o(N_k - N_{k-1})
\]
and for any \( x \in I_{n_k}(w_k) \),
\[
W_k \leq \sum_{j=0}^{a_k-1} \varphi(T^j x) \leq W_k(1 + 2^{-n_k}).
\]

Define \( t_k, \ell_k \) to be the integers satisfying
\[
N_k - N_{k-1} - a_k = t_k m + \ell_k,
\]
for some \( 0 \leq \ell_k < m \).

Let \( w_k (k \geq 1) \) be given as the above. We define a Cantor subset of \( E_k(1) \) as follows.

**Level 1 of the Cantor subset.** Define
\[
E_1 = \left\{ I_{N_k}(u_1, \ldots, u_{t_1}, 1^{\ell_1}, w_1) : u_i \in U, 1 \leq i \leq t_1 \right\}.
\]

For simplicity, we use \( I_{N_k}(U_1) \) to denote a general cylinder in \( E_1 \).

**Level 2 of the Cantor subset.** This level is composed by sublevels for each cylinder \( I_{N_k}(U_1) \in E_1 \). Fix an element \( I_{N_k} = I_{N_k}(U_1) \in E_1 \). Define
\[
E_2(I_{N_k}(U_1)) = \left\{ I_{N_k}(U_1, u_1, \ldots, u_{t_2}, 1^{\ell_2}, w_2) : u_i \in U, 1 \leq i \leq t_2 \right\}.
\]

Then
\[
E_2 = \bigcup_{I_{N_k} \in E_1} E_2(I_{N_k}).
\]

For simplicity, we use \( I_{N_2}(U_2) \) to denote a general cylinder in \( E_2 \).

**From Level \( k \) to \( k+1 \).** Fix \( I_{N_k}(U_k) \in E_k \). Define
\[
E_{k+1}(I_{N_k}(U_k)) = \left\{ I_{N_k+1}(U_k, u_1, \ldots, u_{t_{k+1}}, 1^{\ell_{k+1}}, w_{k+1}) : u_i \in U, 1 \leq i \leq t_{k+1} \right\}.
\]

Then
\[
E_{k+1} = \bigcup_{I_{N_k} \in E_k} E_{k+1}(I_{N_k}).
\]

Up to now we have constructed a sequence of nested sets \( \{E_k\}_{k \geq 1} \). Set
\[
F = \bigcap_{k \geq 1} E_k.
\]
We claim that 
\[ F \subset E(\Psi). \]
In fact, for all \( x \in F \), by construction, for each \( k \geq 1 \),
\[
\sum_{n=N_k}^{N_{k+1}-1} \varphi(T^n x) = \sum_{n=N_k}^{N_{k+1}+t_k} \varphi(T^n x) + \sum_{n=N_k+t_k}^{N_{k+1}} \varphi(T^n x)
\]
\[ = t_k O(2^m) + W_k (1 + O(2^{-n_k})) = O \left( \frac{(N_k - N_{k-1})2^m}{m} \right) + (\Psi(N_k) - \Psi(N_{k-1}))(1 + O(2^{-n_k})). \]

Since \( n_k \to \infty \) which implies \( 2^{-n_k} \to 0 \) as \( k \to \infty \), we have
\[
\sum_{n=0}^{N_k-1} \varphi(T^n x) = \Psi(N_k)(1 + o(1)) + O \left( \frac{N_k 2^m}{m} \right).
\]

By the assumption \( \Psi(n)/n \to \infty \) as \( n \to \infty \), we then deduce
\[
\sum_{n=0}^{N_k-1} \varphi(T^n x) = \Psi(N_k) + o(\Psi(N_k)),
\]

Thus
\[
\lim_{k \to \infty} \frac{\sum_{n=0}^{N_k-1} \varphi(T^n x)}{\Psi(N_k)} = 1. \quad (3.5)
\]

While, for each \( N_k < n \leq N_k \)
\[
\sum_{n=0}^{N_k-1} \frac{\varphi(T^n x)}{\Psi(N_k)} \leq \frac{\sum_{n=0}^{N_k-1} \varphi(T^n x)}{\Psi(N_k)} \leq \frac{\sum_{n=0}^{N_k-1} \varphi(T^n x)}{\Psi(N_{k-1})}.
\]

So by the first assumption of (3.2), we deduce from (3.5) that
\[
\lim_{n \to \infty} \frac{1}{\Psi(n)} S_n(x) = 1.
\]

This proves \( x \in E_{\Psi}(1) \) and hence \( F \subset E_{\Psi}(1) \).

In the following, we will construct a Hölder continuous function from \( F \) to \( F_m \).

Recall that
\[
F_m = \{ x \in [0,1] : \epsilon_{km}(x) = 1, \text{ for all } k \geq 1 \}.
\]

Define
\[
f : F \to F_m \\
x \mapsto y
\]
where \( y \) is obtained by eliminating the digits \( \{ \epsilon_{N_k-t_k-a_k,1}, \ldots, \epsilon_{N_k} \} \) in the binary expansion of \( x \).

Now we calculate the Hölder exponent of \( f \).

Take two points \( x_1, x_2 \in F \) closed enough. Let \( n \) be the smallest integer such that \( \epsilon_n(x_1) \neq \epsilon_n(x_2) \) and \( k \) be the integer such that \( N_k < n \leq N_{k+1} \). Note that by the construction of \( F \), the digits sequence
\[
\{ \epsilon_{N_k-t_k-a_k,1}, \ldots, \epsilon_{N_k} \} \] and \( \{ \epsilon_{N_k+tm} \}_{1 \leq t \leq t_k+1} \]
are the same for all \( x \in F \). So we must have
\[
N_k < n < N_{k+1} - \ell_{k+1} - a_{k+1}. \quad (3.6)
\]
Suppose now $\Psi(n)$ in (3.2), by (3.7)

Thus, it follows and $y_k \ell_m$.

Recall that we can choose $N$ large enough such that

$$n \leq m.$$ 

Then we have

$$|x_1 - x_2| \geq \frac{1}{2n+m}.$$ 

Again by the construction and the definition of the map $f$, we have $y_1 = f(x_1)$ and $y_2 = f(x_2)$ have common digits up to the position $n-1-(\ell_1+a_1)\cdots (\ell_k+a_k)$.

Thus, it follows

$$|f(x_1) - f(x_2)| \leq \frac{1}{2n-1-(\ell_1+a_1)\cdots (\ell_k+a_k)}.$$ 

Recall that $\ell_k < m$ and $a_1 + \cdots + a_k = o(N_k)$ (see (3.3)) and also that $N_k/k \to \infty$ as $k \to \infty$ (by (3.1)). We have

$$1 \geq \frac{n-1-(\ell_1+a_1)\cdots (\ell_k+a_k)}{n+m} \geq \frac{n-1-km-o(N_k)}{n+m} = 1 + o(1),$$

which implies that $f$ is $(1-\eta)$-Hölder for any $\eta > 0$. Thus

$$\dim_H F \geq (1-\eta)\dim_H F_m.$$ 

By Lemma 3.3, we then have

$$\dim_H F \geq (1-\eta)\frac{m-1}{m}.$$ 

By the arbitrariness of $\eta > 0$ and letting $m \to \infty$, we conclude that $\dim_H E(\Psi) = 1$. This finishes the proof. 

\textit{Proof of Theorem 1.2.} In all the three parts of Theorem 1.2, the case of $\beta = 0$ is a direct consequence of Theorem 1.1.

(i). Assume that $\Psi$ is one of the functions $\Psi(n) = n \log n$, $\Psi(n) = n^\alpha$ ($a > 1$), $\Psi(n) = 2^n$ with $0 < \alpha < 1/2$.

(i$_1$). $0 < \beta < \infty$. It suffices to consider the dimension of $E_\Psi(1)$ i.e. $\beta = 1$, since for other $\beta \in (0, \infty)$, we need only replace $\Psi(n)$ by $\beta \Psi(n)$.

To show $\dim_H E_\Psi(1) = 1$, we can apply Lemma 3.5 directly. If $\Psi(n) = n \log n$, we can choose $N_k = k^2$. For $\Psi(n) = n^\alpha$ ($a > 1$), we can also choose $N_k = k^2$.

Suppose now $\Psi(n) = 2^n$ with $0 < \gamma < 1/2$. Let $\delta > 0$ be small such that

$$\frac{\gamma}{1-\gamma} + \delta \gamma < 1$$

(3.7)

which is possible since $\gamma < 1/2$. Take

$$N_k = \lfloor k^{\frac{1}{1-\gamma}+\delta} \rfloor.$$ 

(3.8)

Then we have

$$N_{k+1} - N_k \approx k^{\frac{1}{1-\gamma}+\delta},$$

and

$$\log(\Psi(N_{k+1}) - \Psi(N_k)) \approx \log(\Psi(N_k)(N_{k+1} - N_k)) \approx N_k^{\gamma} + \log(N_{k+1} - N_k) \approx N_k^{\gamma}.$$ 

Here we write $A \approx B$ when $A/B \to 1$. This shows the validity of (3.1). Moreover,

$$\frac{\log(\Psi(N_{k+1}) - \Psi(N_k))}{N_{k+1} - N_k} \approx \frac{k^{\frac{1}{1-\gamma}+\delta} + \gamma \delta}{k^{\frac{1}{1-\gamma}+\delta}} = k^{-\delta(1-\gamma)} \to 0 \ (k \to \infty).$$ 

Thus the second assumption of (3.2) is satisfied. At last, for the first assumption in (3.2), by (3.7)

$$\frac{\Psi(N_{k-1})}{\Psi(N_k)} \approx 2^{(k-1)\frac{1}{1-\gamma}+\delta - k^{\frac{1}{1-\gamma}+\delta}} \to 1.$$
Hence Lemma 3.5 applies.

(I.II). If \( \beta = \infty \), we may choose \( \tilde{\Psi}(n) = 2^n \) for some \( 0 < \eta < \frac{1}{2} \) such that \( E_{\tilde{\Psi}}(1) \subset E_{\tilde{\Psi}}(\infty) \). Then \( \dim_H E_{\tilde{\Psi}}(\infty) = 1 \) follows from (I.I).

(II). Now suppose that \( \Psi(n) = 2^n \) with \( 1/2 \leq \gamma < 1 \).

Let \( \beta \in (0, \infty) \). We will prove that \( E_{\Psi}(\beta) \) is empty. On the contrary, suppose there is \( x \in E_{\Psi}(\beta) \), which has binary expansion
\[
x = [0^{n_1-1}10^{n_2-1} \ldots 0^{n_k-1} \ldots].
\]

(3.10)

Then, by (2.1) we have
\[
\frac{S_{n_1+n_2+\ldots+n_\ell}(x)}{\Psi(n_1+n_2+\ldots+n_\ell)} = \frac{2^{n_1} + 2^{n_2} + \ldots + 2^{n_\ell} - \ell}{2(n_1+n_2+\ldots+n_\ell)^\gamma} \to \beta,
\]
\[
\frac{S_{n_1+n_2+\ldots+n_\ell+1}(x)}{\Psi(n_1+n_2+\ldots+n_\ell+1)} = \frac{2^{n_1} + 2^{n_2} + \ldots + 2^{n_\ell} - \ell + 2^{n_{\ell+1}-1}}{2(n_1+n_2+\ldots+n_\ell+1)^\gamma} \to \beta.
\]

(3.11)

Since
\[
\frac{2(n_1+n_2+\ldots+n_\ell)^\gamma}{2(n_1+n_2+\ldots+n_\ell+1)^\gamma} \to 1,
\]
by dividing the two limits of (3.11), we deduce that
\[
\frac{2^{n_1} + 2^{n_2} + \ldots + 2^{n_{\ell+1}-1}}{2^{n_1} + 2^{n_2} + \ldots + 2^{n_\ell} - \ell} = 1 + \frac{2^{n_{\ell+1}-1}}{2^{n_1} + 2^{n_2} + \ldots + 2^{n_\ell} - \ell} \to 1,
\]
which implies that
\[
\frac{S_{n_1+n_2+\ldots+n_\ell+1}(x)}{S_{n_1+n_2+\ldots+n_\ell}(x)} = 1 + \frac{2^{n_{\ell+1}-1}}{2^{n_1} + 2^{n_2} + \ldots + 2^{n_\ell} - \ell} \to 1.
\]

Combining with (3.11), we get
\[
1 \leftarrow \frac{\Psi(n_1+\ldots+n_{\ell+1})}{\Psi(n_1+\ldots+n_\ell)} = \frac{2(n_1+n_2+\ldots+n_\ell+n_{\ell+1})^\gamma}{2(n_1+n_2+\ldots+n_\ell)^\gamma}.
\]

Thus
\[
(n_1 + n_2 + \ldots + n_{\ell+1})^\gamma - (n_1 + n_2 + \ldots + n_\ell)^\gamma
\]
\[
= (n_1 + n_2 + \ldots + n_{\ell})^\gamma \left( 1 + \frac{n_{\ell+1}}{n_1 + n_2 + \ldots + n_\ell} \right)^\gamma - 1 \approx \gamma n_{\ell+1} \to 0.
\]

Therefore, for any \( \varepsilon > 0 \), there exists \( k_0 \geq 1 \) such that for all \( j > k_0 \),
\[
n_j < \varepsilon(n_1 + n_2 + \ldots + n_{j-1})^{1-\gamma}.
\]

(3.12)

Then for any \( k_0 < j \leq \ell \)
\[
n_j < \varepsilon(n_1 + n_2 + \ldots + n_{\ell})^{1-\gamma}.
\]

This implies
\[
S_{n_1+n_2+\ldots+n_\ell}(x) = 2^{n_1} + 2^{n_2} + \ldots + 2^{n_\ell} - \ell
\]
\[
\leq M + \ell 2^{(n_1+n_2+\ldots+n_\ell)^{1-\gamma} - \ell},
\]
with \( M := 2^{n_1} + \cdots + 2^{n_{k_0}} \). Thus we have
\[
\frac{S_{n_1+n_2+\ldots+n_\ell}(x)}{\Psi(n_1+n_2+\ldots+n_\ell)} < \frac{M + \ell 2^{(n_1+n_2+\ldots+n_\ell)^{1-\gamma} - \ell}}{2^{(n_1+n_2+\ldots+n_\ell)^\gamma}}.
\]

(3.13)

By observing \( n_j \geq 1 \), we deduce that the upper bound of (3.13) converges to 0 for \( 1/2 \leq \gamma < 1 \), a contradiction to (3.11). Hence \( E_{\Psi}(\beta) \) is an empty set.
(II). \( \beta = \infty \). Fix \( \delta \in (\gamma, 1) \) and take a large integer \( K \) such that \( 2^{K\delta} > 1 \). Consider the set of points such that at every position \( 2^k, k > K \) in their binary expansions, they have a string of zeros of length \( 2^{k\delta} \), i.e.

\[
E := \left\{ x \in (0, 1] : \epsilon_{2^k+1} = \cdots = \epsilon_{2^k+2^{k\delta}} = 0, \text{ for all } k \geq K \right\}.
\]

On the one hand, \( E \subset E_\psi(\infty) \), since for any \( n \in (2^k, 2^{k+1}] \) for some \( k \geq K \),

\[
S_n(x) > 2^{2^{k\delta}} \geq 2^{(n/2)^\delta} \gg 2^n.
\]

On the other hand, the set \( E \) has dimension 1 guaranteed by Lemma 3.4.

(III). Suppose that \( \Psi(n) = 2^n \gamma \) with \( \gamma \geq 1 \) and let \( \beta \in (0, +\infty) \). Assume that there exists \( x \in E_\psi(\beta) \) for some \( \beta \in (0, +\infty) \). Write the binary expansion of \( x \) as \((3.10)\). Then by \((2.1)\),

\[
\frac{S_{n_1+n_2+\cdots+n_{\ell}}(x)}{\Psi(n_1+n_2+\cdots+n_{\ell})} = \frac{2^{n_1} + 2^{n_2} + \cdots + 2^{n_{\ell}} - \ell}{2^{(n_1+n_2+\cdots+n_{\ell})\gamma}} \to \beta,
\]

\[
\frac{S_{n_1+n_2+\cdots+n_{\ell}-1}(x)}{\Psi(n_1+n_2+\cdots+n_{\ell}-1)} = \frac{2^{n_1} + 2^{n_2} + \cdots + 2^{n_{\ell}} - \ell - 1}{2^{(n_1+n_2+\cdots+n_{\ell}-1)\gamma}} \to \beta.
\]

However,

\[
\frac{2^{n_1} + 2^{n_2} + \cdots + 2^{n_{\ell}} - \ell}{2^{n_1} + 2^{n_2} + \cdots + 2^{n_{\ell}} - \ell - 1} \to 1 \quad \text{but} \quad \frac{2^{(n_1+n_2+\cdots+n_{\ell})\gamma}}{2^{(n_1+n_2+\cdots+n_{\ell}-1)\gamma}} \geq 2,
\]

which is a contradiction. Hence \( E_\psi(\beta) \) is empty when \( \beta \in (0, +\infty) \).

When \( \beta = +\infty \), by \((2.1)\), we have

\[
\liminf_{n \to \infty} \frac{S_n(x)}{2^n} \leq 1.
\]

So,

\[
\liminf_{n \to \infty} \frac{S_n(x)}{\Psi(n)} \leq 1.
\]

This shows that \( E_\psi(\infty) \) is also empty.

\[\square\]

4. The potential \( 1/x \)

In fact, the techniques in Section 3 can be applied to the continuous potential \( g : x \mapsto 1/x \) on \([0, 1]\) which has a singularity at 0. 

**Proof of Theorem 1.3.** We first show that if \( \Psi(n) \) is one of the following

\[
\Psi(n) = n \log n, \quad \Psi(n) = n^a (a > 1), \quad \Psi(n) = 2^n \gamma (0 < \gamma < 1/2),
\]

then for any \( \beta \in [0, \infty] \), \( \dim_F F_\psi(\beta) = 1 \).

We note that if \( x \in (0, 1] \) has binary expansion \( x = [0^n 1^s \ldots] \), then \( \varphi(x) = 2^n \) and

\[
2^n \leq g(x) \leq 2^n + 2^{n-s+1} = 2^n(1 + 2^{-s+1}).
\]

In Lemma 3.2, for an integer \( W \), and for any integer \( n \leq \log W \), we can construct instead of the words \( w = (10^{t_1-1}, 10^{t_2-1}, \ldots, 10^{t_s-1}) \), the following word

\[
w = (10^{t_1-1}1^{s+1}, 10^{t_2-1}1^{s+1}, \ldots, 10^{t_s-1}1^{s+1}),(4.1)
\]

Then the length of the word satisfies

\[
|w| = \sum_{i=1}^p (t_i + s + 1) \leq p(t_1 + s + 1) \leq (n + 2)(\log W + s + 2).
\]
By (4.1), for any \( x \in I_{|w|}(w) \),
\[ W + s(n + 2) \leq \sum_{j=0}^{[w]-1} g(T^j x) \leq W(1 + 2^{-n}) \cdot (1 + 2^{-s}) + 2s(n + 2). \]

For each \( k \geq 1 \), we still write
\[ W_k := \Psi(N_k) - \Psi(N_{k-1}) \]
and let \( n_k, s_k \) be a sequence of integers tending to \( \infty \) such that
\[ \frac{n_k}{N_k - N_{k-1}} \rightarrow 0, \quad (4.3) \]
\[ \frac{n_k \cdot s_k}{N_k - N_{k-1}} \rightarrow 0, \quad (4.4) \]
and
\[ \frac{n_k \cdot s_k}{\Psi(N_k) - \Psi(N_{k-1})} \rightarrow 0. \quad (4.5) \]

By (3.1) and (3.2), these two sequences of \( n_k \geq 0, s_k \geq 0 \) do exist.

Now for \( W_k \) and \( n_k, s_k \), let \( w_k \) be the word given as above. Then by (4.3) and (4.4), the length \( a_k \) of \( w_k \) satisfies
\[ a_k \leq (n_k + 2)(\log W_k + s_k + 2) \]
\[ = (n_k + 2)(\log(\Psi(N_k) - \Psi(N_{k-1})) + s_k + 2) \]
\[ = o(N_k - N_{k-1}) \quad (4.6) \]
and for any \( x \in I_{n_k}(w_k) \),
\[ W_k + s_k(n_k + 2) \leq \sum_{j=0}^{n_k-1} g(T^j x) \leq W_k(1 + 2^{-n_k}) \cdot (1 + 2^{-s_k}) + 2s_k(n_k + 2). \quad (4.7) \]

Hence by (4.5) we still have the same estimation:
\[ \sum_{n=0}^{N_k-1} g(T^n x) = \Psi(N_k) + o(\Psi(N_k)) \]
and the rest of the proof is the same as (I) of the proof of Theorem 1.2.

We can repeat the same arguments in Section 3 and show that for potential \( g \), the set \( F_g(\beta) \) is empty if \( \beta \in (0, \infty), \Psi(n) = 2^n (1/2 \leq \gamma < 1) \) or \( \beta \in (0, \infty], \Psi(n) = 2^n (\gamma \geq 1) \).

In fact, by definition, if there exists \( x \in F_g(\beta) \), with its binary expansion
\[ x = [0^{n_1-1}10^{n_2-1}1 \ldots 0^{n_\ell-1}10^{n_{\ell+1}-1}1 \ldots], \]
then
\[ \frac{S_{n_1+n_2+\ldots+n_\ell}g(x)}{\Psi(n_1 + n_2 + \ldots + n_\ell)} \rightarrow \beta, \quad \frac{S_{n_1+n_2+\ldots+n_\ell+1}g(x)}{\Psi(n_1 + n_2 + \ldots + n_\ell + 1)} \rightarrow \beta. \]

Thus
\[ \frac{S_{n_1+n_2+\ldots+n_\ell}g(x)}{S_{n_1+n_2+\ldots+n_\ell+1}g(x)} \rightarrow 1. \]

Observing \( \varphi \leq g \leq 2\varphi \), we have
\[ \frac{2^n}{S_{n_1+n_2+\ldots+n_\ell}g(x)} \rightarrow 0, \]
which then implies
\[
\frac{S_{n_1+n_2+\cdots+n_\ell} g(x)}{S_{n_1+n_2+\cdots+n_{\ell+1}} g(x)} \to 1.
\]
By the definition of \( x \in F_\Psi(\beta) \), we have
\[
\frac{\Psi(n_1 + n_2 + \cdots + n_{\ell+1})}{\Psi(n_1 + n_2 + \cdots + n_\ell)} \to 1.
\]
This further implies the same inequality with (3.12) and the rest of proof is the
same as (I1) and (III) by noting \( \varphi \leq y \leq 2\varphi \).

For \( \Psi(n) = 2^n (1/2 \leq \gamma < 1) \), we can also prove \( \dim_H F_\Psi(\infty) = 1 \) by the same
proof as (II2). \qed

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