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ABSTRACT. Let $((0,1],T)$ be the doubling map in the unit interval and φ be the Saint-Petersburg potential, defined by $\varphi(x) = 2^n$ if $x \in (2^{-n-1}, 2^{-n}]$ for all $n \geq 0$. We consider asymptotic properties of the Birkhoff sum $S_n(x) = \varphi(x) + \cdots + \varphi(T^{n-1}(x))$. With respect to the Lebesgue measure, the Saint-Petersburg potential is not integrable and it is known that $\frac{1}{n \log n} S_n(x)$ converges to $\frac{1}{\log 2}$ in probability. We determine the Hausdorff dimension of the level set $\{x : \lim_{n\to\infty} S_n(x)/n = \alpha\}$ $(\alpha > 0)$, as well as that of the set $\{x : \lim_{n\to\infty} S_n(x)/\Psi(n) = \alpha\}$ $(\alpha > 0)$, when $\Psi(n) = n \log n$, n^a or 2^{n^γ} for $a > 1$, $\gamma > 0$. The fast increasing Birkhoff sum of the potential function $x \mapsto 1/x$ is also studied.

1. Introduction

Let T be the doubling map on the unit interval $(0, 1]$ defined by

$$Tx = 2x - \lceil 2x \rceil + 1,$$

where $\lceil x \rceil$ is the smallest integer larger than or equal to x. Let ϵ_1 be the function defined by $\epsilon_1(x) = \lceil 2x \rceil - 1$ and $\epsilon_n(x) := \epsilon_1(T^{n-1}x)$ for $n \geq 2$. Then each real number $x \in (0, 1]$ can be expanded into *an infinite series* as

$$x = \frac{\epsilon_1(x)}{2} + \cdots + \frac{\epsilon_n(x)}{2^n} + \cdots . \tag{1.1}$$

We call (1.1) the binary expansion of x and also write it as

$$x = [\epsilon_1(x)\epsilon_2(x)\ldots].$$

The Saint-Petersburg potential is a function $\varphi : (0, 1] \to \mathbb{R}$ defined as

$$\varphi(x) = 2^n \text{ if } x \in (2^{-n-1}, 2^{-n}],\ \forall n \geq 0.$$

We remark that the definition of φ is equivalent to

$$\varphi(x) = 2^n \text{ where } n \geq 0 \text{ is the smallest integer such that } \epsilon_{n+1}(x) = 1.$$

and is also equivalent to

$$\varphi(x) = 2^n \text{ if the binary expansion of } x \text{ begins with } 0^n 1,$$

where $0^n (n \geq 0)$ means a block with n consecutive zeros.

The name of Saint-Petersburg potential is motivated by the famous Saint-Petersburg game in probability theory. The Saint-Petersburg potential is of infinite expectation with respect to the Lebesgue measure. Furthermore, it increases exponentially fast near to the point 0.

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In this paper, we are concerned with the following Birkhoff sums of the Saint-Petersburg potential:

$$\forall n \geq 1, \quad S_n(x) := \varphi(x) + \varphi(T(x)) + \cdots + \varphi(T^{n-1}(x)), \qquad x \in (0,1].$$

Let

$$I = \{x \in (0,1] : \epsilon_1(x) = 1\}.$$

Define the hitting time of $x \in (0,1]$ to I as

$$n(x) := \inf\{n \geq 0 : T^n x \in I\}.$$

Then

$$n(x) = n \quad \text{if } x \in \left(\frac{1}{2^{n+1}}, \frac{1}{2^n}\right], \text{ for all } n \geq 0.$$

Using $n(x)$, we define a new dynamical system $\widehat{T} : (0,1] \to (0,1]$ by

$$\widehat{T}(x) = T^{n(x)+1}(x) = 2^{n+1}\Big(x - \frac{1}{2^{n+1}}\Big) \text{ if } x \in \left(\frac{1}{2^{n+1}}, \frac{1}{2^n}\right], \text{ for all } n \geq 0,$$

called the acceleration of T, in order that φ and $\varphi \circ \widehat{T}$ are independent. Let

$$\widehat{S}_n(x) := \varphi(x) + \varphi(\widehat{T}(x)) + \cdots + \varphi(\widehat{T}^{n-1}(x)), \qquad x \in (0,1].$$

The convergence in probability of $\widehat{S}_n(x)$ is well known (e.g. [6, p.253]) which states that for any $\epsilon > 0$, the Lebesgue measure λ of

$$\left\{x \in (0,1] \ : \ \Big|\frac{\widehat{S}_n(x)}{n \log n} - \frac{1}{\log 2}\Big| \geq \epsilon\right\}$$

tends to 0 as $n \to \infty$.

Let $\{\Psi_n\}_{n\geq 1}$ be an increasing sequence such that $\Psi_n \to \infty$ as $n \to \infty$. Then it was shown in [5] that Lebesgue almost surely either

$$\lim_{n\to\infty} \frac{\widehat{S}_n(x)}{\Psi_n} = 0 \quad \text{or} \quad \limsup_{n\to\infty} \frac{\widehat{S}_n(x)}{\Psi_n} = \infty,$$

according as

$$\sum_{n\geq 1} \lambda(\{x \in (0,1] : \varphi(x) \geq \Psi_n\}) < \infty \quad \text{or} \quad = \infty.$$

Let $n_1 = n_1(x) = n(x) + 1$ and $n_k = n_k(x) = n_1(\widehat{T}^{k-1}x) = n(\widehat{T}^{k-1}x)$ for $k \geq 2$. It is direct to see that

$$\forall \ell \geq 1, \quad S_{n_1+\cdots+n_\ell}(x) = 2\widehat{S}_\ell(x) - \ell.$$

Moreover, the ergodicity of T (of $\widehat{T}$) implies Lebesgue almost surely

$$\lim_{\ell\to\infty} \frac{n_1(x) + \cdots + n_\ell(x)}{\ell} = \int_0^1 (n(x)+1)\, d\lambda(x) = 2.$$

Combining these two facts together, we obtain the same convergence results as above if we replace $\widehat{S}_n$ by S_n. In particular, the average $S_n(x)/(n \log n)$ converges to $1/\log 2$ in probability, and almost surely (with respect to the Lebesgue measure) either

$$\lim_{n\to\infty} \frac{S_n(x)}{\Psi_n} = 0 \quad \text{or} \quad \limsup_{n\to\infty} \frac{S_n(x)}{\Psi_n} = \infty,$$

according as

$$\sum_{n\geq 1} \lambda(\{x \in (0,1] : \varphi(x) \geq \Psi_n\}) < \infty \quad \text{or} \quad = \infty,$$

where $\{\Psi_n\}_{n\geq 1}$ is an increasing sequence such that $\Psi_n \to \infty$ as $n \to \infty$. Recall that φ has infinite expectation with respect to the Lebesgue measure. We thus have $S_n(x)/n$ converges to infinity for Lebesgue almost all points.

In this article, we want to further study the asymptotic behavior of of the Birkhoff sum $S_n(x)$ of the Saint-Petersburg potential. We give a complete multifractal analysis of $S_n(x)$.

First, for any $\alpha \geq 1$, we consider the level set

$$E(\alpha) = \left\{ x \in (0,1] : \lim_{n\to\infty} \frac{1}{n} S_n(x) = \alpha \right\}.$$

For $t \in \mathbb{R}$ and $q > 0$, define

$$P(t,q) := \log \sum_{j=1}^{\infty} 2^{-tj-q(2^j-1)}.$$

Then P is a real-analytic function. Furthermore, for each $q > 0$, there is a unique $t(q) > 0$ such that $P(t(q), q) = 0$. This function $q \mapsto t(q)$ is real-analytic, strictly decreasing and convex.

Denote by $\dim_H$ the Hausdorff dimension. The function $\alpha \mapsto \dim_H E(\alpha)$, called the Birkhoff spectrum of the Saint-Petersburg potential φ, is proved to be the Legendre transformation of the function $q \mapsto t(q)$.

Theorem 1.1. *For any $\alpha \geq 1$ we have*

$$\dim_H E(\alpha) = \inf_{q>0} \{t(q) + q\alpha\}.$$

Consequently, $\dim_H E(1) = 0$ and the function $\alpha \mapsto \dim_H E(\alpha)$ is real-analytic, strictly increasing, concave, and has limit 1 as $\alpha \to \infty$.

The Birkhoff spectrum of a continuous potential was obtained for full shifts [14], for topologically mixing subshifts of finite type [4], and for repellers of a topologically mixing $C^{1+\epsilon}$ expanding map [2]. A continuous potential in a compact space is bounded, hence these classical results are all for bounded potentials. Our Theorem 1.1 gives a Birkhoff spectrum for an unbounded function with a singular point. To prove Theorem 1.1, we will transfer our question to a Birkhoff spectrum problem of an interval map with infinitely many branches and we will apply the techniques developed in [9] for continued fraction dynamical system and in [8] for general expanding interval maps with infinitely many branches.

We also study the Birkhoff sums $S_n(x)$ of fast increasing rates. Let $\Psi : \mathbb{N} \to \mathbb{N}$ be an increasing function. For $\beta \in [0, \infty]$, consider the level set

$$E_\Psi(\beta) := \left\{ x \in (0,1] : \lim_{n\to\infty} \frac{1}{\Psi(n)} S_n(x) = \beta \right\}.$$

Theorem 1.2. *If $\Psi(n)$ is one of the following*

$$\Psi(n) = n \log n, \ \Psi(n) = n^a \ (a > 1), \ \Psi(n) = 2^{n^\gamma} \ (0 < \gamma < 1/2),$$

then for any $\beta \in [0, \infty]$, $\dim_H E_\Psi(\beta) = 1$.

If $\Psi(n) = 2^{n^\gamma}$ with $1/2 \leq \gamma < 1$, then for any $\beta \in (0, \infty)$, the set $E_\Psi(\beta)$ is empty, and $\dim_H E_\Psi(\beta) = 1$ for $\beta = 0, +\infty$.

If $\Psi(n) = 2^{n^\gamma}$ with $\gamma \geq 1$, then for any $\beta \in (0, \infty]$, the set $E_\Psi(\beta)$ is empty, and $\dim_H E_\Psi(\beta) = 1$ for $\beta = 0$.

We remark that by the above discussion on the convergence of $S_n(x)$, for all cases in Theorem 1.2, the sets $E_\Psi(0)$ has full measure, and thus obviously has full Hausdorff dimension.

From the definition of $S_n(x)$, we see that for the integer n such that $\epsilon_n(x) = 1$, one has $S_n(x) = S_{n-1}(x) + 1$, which implies for all x, $\liminf_{n\to\infty} \frac{S_n(x)}{S_{n-1}(x)} = 1$. Thus if $\liminf_{n\to\infty} \frac{\Psi(n)}{\Psi(n-1)} > 1$, then for any $\beta \in (0,\infty)$, the set $E_\Psi(\beta)$ is empty. By the definition of $S_n(x)$, we can also check that for all x, for the integer n such that $\epsilon_n(x) = 1$, we have $S_n(x) \le 2^n - 1$. This implies $\liminf_{n\to\infty} S_n(x)/2^n \le 1$ (See also the formula (3.15) in Section 3). Hence, for 'regular' growth functions Ψ we only need to consider exponential and subexponential growth rates.

However, if we pick a point x with dyadic expansion consisting mostly of 0's, with infinitely many 1's but in large distances from each other, then the Birkhoff sum $S_{n_i}(x)$ may grow arbitrarily fast on some subsequence n_i. Thus for any increasing $\Psi(n)$ there exists a point x such that $\limsup_{n\to\infty} S_n(x)/\Psi(n) = \infty$.

Our study on the Saint-Petersburg potential is an attempt of multifractal analysis of unbounded potential functions on the doubling map dynamical system. However, the Saint-Petersburg potential is locally constant and not continuous. One might think of an another unbounded potential function $g : x \mapsto 1/x$ which is close to the Saint-Petersburg potential but is continuous. In fact, our method for studying the fast increasing Birkhoff sum of Saint-Petersburg potential also works for the fast increasing Birkhoff sum of the potential $g : x \mapsto 1/x$.

Denote by $S_n g(x)$ the Birkhoff sum

$$S_n g(x) := g(x) + g(T(x)) + \cdots + g(T^{n-1}(x)), \qquad x \in (0,1].$$

For $\beta \in [0,\infty]$, let

$$F_\Psi(\beta) := \left\{ x \in (0,1] : \lim_{n\to\infty} \frac{1}{\Psi(n)} S_n g(x) = \beta \right\}.$$

Theorem 1.3. *If $\Psi(n)$ is one of the following*

$$\Psi(n) = n \log n,\ \ \Psi(n) = n^a\ \ (a > 1),\ \ \Psi(n) = 2^{n^\gamma}\ \ (0 < \gamma < 1/2),$$

then for any $\beta \in [0,\infty]$, $\dim_H F_\Psi(\beta) = 1$.

If $\Psi(n) = 2^{n^\gamma}$ with $1/2 \le \gamma < 1$, then for any $\beta \in (0,\infty)$, the set $F_\Psi(\beta)$ is empty, and $\dim_H F_\Psi(\beta) = 1$ for $\beta = 0, +\infty$.

If $\Psi(n) = 2^{n^\gamma}$ with $\gamma \ge 1$, then for any $\beta \in (0,\infty]$, the set $F_\Psi(\beta)$ is empty, and $\dim_H F_\Psi(\beta) = 1$ for $\beta = 0$.

We remark that these multifractal analysis on the Birkhoff sums of fast increasing rates have been done for some special potentials in continued fraction dynamical system ([9, 11, 12]).

2. Birkhoff spectrum of the Saint-Petersburg potential

In this section, we will obtain the Birkhoff spectrum of the Saint-Petersburg potential, i.e. the Hausdorff dimension of the following level set:

$$E(\alpha) := \left\{ x \in (0,1] : \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi(T^j x) = \alpha \right\} \quad (\alpha \ge 1).$$

We will transfer our question to a Birkhoff spectrum problem for an interval map with infinitely many branches.

2.1. **Transference lemma.** Recall that the Saint-Petersburg potential φ is given by

$$\varphi(x) = 2^n, \text{ if } \ x = [0^n 1, \cdots]$$

where $x = [\epsilon_1 \epsilon_2, \cdots]$ denotes the digit sequence in the binary expansion of x. Recall also the definition of hitting time $n(x)$ and the acceleration $\widehat{T}$ of the doubling map T in Section 1. Define a new potential function

$$\phi(x) := 2^{n(x)+1} - 1, \ x \in (0, 1].$$

In fact, ϕ is nothing but the function satisfying

$$\phi(x) = \sum_{j=0}^{n(x)} \varphi(T^j x).$$

With the notation $n_1 = n(x) + 1 \geq 1$, and $n_k = n(\widehat{T}^{k-1}x) + 1$ for $k \geq 2$ given in Section 1, we have

$$\phi(\widehat{T}x) = \sum_{j=0}^{n(\widehat{T}x)} \varphi(T^j(\widehat{T}x)) = \sum_{j=n_1}^{n_2-1} \varphi(T^j x).$$

Hence,

$$\sum_{j=0}^{n_1+\cdots+n_\ell-1} \varphi(T^j x) = \sum_{k=0}^{\ell-1} \phi(\widehat{T}^k x) = 2^{n_1} + \cdots + 2^{n_\ell} - \ell. \tag{2.1}$$

Note that the derivative of $\widehat{T}$ satisfies

$$|\widehat{T}'|(x) = 2^{n(x)+1} = 2^{n_1} = \phi(x) + 1. \tag{2.2}$$

We have

$$n_1 + \cdots + n_\ell = \sum_{k=0}^{\ell-1} \log_2 |\widehat{T}'|(\widehat{T}^k x).$$

Recall the set in question:

$$E(\alpha) = \left\{ x \in (0, 1] : \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi(T^j x) = \alpha \right\} \qquad (\alpha \geq 1).$$

Define

$$\widetilde{E}(\alpha) := \left\{ x \in (0, 1] : \lim_{\ell\to\infty} \frac{\sum_{k=0}^{\ell-1} \phi(\widehat{T}^k x)}{\sum_{k=0}^{\ell-1} \log_2 |\widehat{T}'|(\widehat{T}^k x)} = \alpha \right\} \qquad (\alpha \geq 1).$$

The following lemma shows the two level sets are the same.

Lemma 2.1. *For all $\alpha \geq 1$, we have $E(\alpha) = \widetilde{E}(\alpha)$.*

Proof. It is evident that $E(\alpha) \subset \widetilde{E}(\alpha)$, because, as discussed above,

$$\frac{\sum_{k=0}^{\ell-1} \phi(\widehat{T}^k x)}{\sum_{k=0}^{\ell-1} \log_2 |\widehat{T}'|(\widehat{T}^k x)} = \frac{1}{n_1 + \cdots + n_\ell} \sum_{j=0}^{n_1+\cdots+n_\ell-1} \varphi(T^j x). \tag{2.3}$$

Now, we show the other direction. Take an $x \in \widetilde{E}(\alpha)$, express x in its binary expansion

$$x = [0^{n_1-1} 1 0^{n_2-1} 1 \cdots 0^{n_\ell-1} 1 \cdots].$$

In fact, $n_\ell - 1$ is the recurrence time for $n(\widehat{T}^{\ell-1}x)$, for each $\ell \geq 1$.

By (2.1), we have, at present,

$$\lim_{\ell\to\infty}\frac{1}{n_1+\cdots+n_\ell}\sum_{j=0}^{n_1+\cdots+n_\ell-1}\varphi(T^jx)=\alpha.$$

So, we are required to check it holds for all n.

For any $\epsilon>0$, there exists $\ell_0\in\mathbb{N}$ such that, for any $\ell\geq\ell_0$,

$$\alpha-\epsilon\leq\frac{2^{n_1}+\cdots+2^{n_\ell}-\ell}{n_1+\cdots+n_\ell}\leq\alpha+\epsilon. \tag{2.4}$$

For any $n_1+\cdots+n_\ell<n<n_1+\cdots+n_\ell+n_{\ell+1}$ with $\ell\geq\ell_0$, it is trivial that

$$\frac{2^{n_1}+\cdots+2^{n_\ell}-\ell}{n_1+\cdots+n_\ell+n_{\ell+1}}\leq\frac{1}{n}\sum_{j=0}^{n-1}\varphi(T^jx)\leq\frac{2^{n_1}+\cdots+2^{n_\ell}+2^{n_{\ell+1}}-\ell-1}{n_1+\cdots+n_\ell}.$$

Thus, it suffices to show that

$$2^{n_{\ell+1}}=o(n_1+\cdots+n_\ell), \tag{2.5}$$

which also implies

$$n_{\ell+1}=o(n_1+\cdots+n_\ell).$$

Let M_0 be a large integer such that, for all $M\geq M_0$, $2^M\geq 4\alpha M$. So, when $n_{\ell+1}\leq M_0$, there is nothing to prove. So, we always assume $2^{n_{\ell+1}}\geq 4\alpha n_{\ell+1}$.

By (2.4), we have

$$2^{n_1}+\cdots+2^{n_\ell}-\ell\geq(\alpha-\epsilon)(n_1+\cdots+n_\ell),$$
$$2^{n_1}+\cdots+2^{n_\ell}+2^{n_{\ell+1}}-\ell-1\leq(\alpha+\epsilon)(n_1+\cdots+n_\ell+n_{\ell+1}).$$

These give

$$2^{n_{\ell+1}}\leq 2\epsilon(n_1+\cdots+n_\ell)+(\alpha+\epsilon)n_{\ell+1}+1.$$

So, we have

$$2^{n_{\ell+1}}\leq 4\epsilon(n_1+\cdots+n_\ell).$$ □

2.2. **Dimension of $\widetilde{E}(\alpha)$.** Now we calculate the Hausdorff dimension of the set $\widetilde{E}(\alpha)$. At first, we give a notation.

- For each finite word $w\in\bigcup_{n\geq1}\{0,1\}^n$ of length n, a T-*dyadic* cylinder of order n is defined as
$$I_n(w)=\{x\in(0,1]:(\epsilon_1(x),\cdots,\epsilon_n(x))=w\}.$$
- For $(n_1,\cdots,n_\ell)\in(\mathbb{N}\setminus\{0\})^\ell$, a $\widehat{T}$-*dyadic* cylinder of order ℓ is defined as
$$D_\ell(n_1,\cdots,n_\ell)=\{x\in(0,1]:n_k(x)=n_k,\ 1\leq k\leq\ell\}.$$

Proof of Theorem 1.1. To calculate the Hausdorff dimension of $\widetilde{E}(\alpha)$, we construct a suitable measure supported on $\widetilde{E}(\alpha)$. The Gibbs measures derived from the Ruelle-Perron-Frobenius transfer operator are good candidates for such a measure.

In fact, by considering the inverse branches $U_i:x\mapsto\frac{x+1}{2^{i+1}}$ $(i\geq0)$ of $\widehat{T}$, we can code the dynamical system $([0,1],\widehat{T})$ by the conformal infinite iterated function system $(U_i)_{i\geq0}$ which satisfies the open set condition ([10, Section 1]).

Consider the potential function with two parameters

$$\psi_{t,q}:=-t\log|\widehat{T}'|-(\log2)\cdot q\phi\quad(t\in\mathbb{R},\ q>0).$$

Then $(\psi_{t,q}\circ U_i)_{i\geq0}$ is a family of strong Hölder family ([10, Page 30]). Hence, we can define a Ruelle operator

$$\mathcal{L}_{t,q}f(x):=\sum_{y\in\widehat{T}^{-1}x}e^{\psi_{t,q}(y)}f(y),$$

on the Banach space of continuous functions on the corresponding infinite symbolic space ([10, Page 31]).

By the Ruelle-Perron-Frobenius transfer operator theory [10, Theorems 2.9 and 2.10], for any $q > 0$ (to satisfy the condition 2.2 of [10]), we can find an eigenvalue $\lambda_{t,q}$ and an eigenfunction $h_{t,q}$ for $\mathcal{L}_{t,q}$ and an eigenfunction $\nu_{t,q}$ for the conjugate operator $\mathcal{L}^*_{t,q}$. Then the pressure function $P(t,q) = \log \lambda_{t,q}$ and the ergodic Gibbs measure $\mu_{t,q}$ is given by $h_{t,q} \cdot \nu_{t,q}$.

The pressure function can be computed by (see [10, Pages 31 and 48])

$$P(t,q) = \lim_{\ell\to\infty} \frac{1}{\ell} \log \left(\sum_{(n_1,\cdots,n_\ell)\in(\mathbb{N}\setminus\{0\})^\ell} \exp \sup_{x\in D_\ell(n_1,\cdots,n_\ell)} (S_\ell \psi_{t,q}(x)) \right).$$

Note that for all $n \geq 1$,

$$|\widehat{T}'|(x) = 2^n, \text{ and } \phi(x) = 2^n - 1 \text{ if } x \in \left(\frac{1}{2^n}, \frac{1}{2^{n-1}}\right] = D_1(n).$$

Then for $x \in D_\ell(n_1,\cdots,n_\ell)$,

$$S_\ell \psi_{t,q}(x) = -(\log 2)\cdot t \cdot \sum_{j=1}^{\ell} n_j + (\log 2)\cdot q \cdot \sum_{j=1}^{\ell} (2^{n_j} - 1).$$

Thus

$$\begin{aligned} &\sum_{(n_1,\cdots,n_\ell)\in(\mathbb{N}\setminus\{0\})^\ell} \exp \sup_{x\in D_\ell(n_1,\cdots,n_\ell)} (S_\ell \psi_{t,q}(x)) \\ = \;& \sum_{n_1=1}^{\infty}\cdots\sum_{n_\ell=1}^{\infty} \left(\prod_{j=1}^{\ell} 2^{-tn_j} \cdot \prod_{j=1}^{\ell} 2^{q(2^{n_j}-1)} \right) = \left(\sum_{j=1}^{\infty} 2^{-tj-q(2^j-1)} \right)^{\ell}. \end{aligned}$$

Hence

$$P(t,q) = \log \sum_{j=1}^{\infty} 2^{-tj-q(2^j-1)}.$$

Now we calculate the local dimension of the Gibbs measure $\mu_{t,q}$. Let $D_\ell(x)$ be the $\widehat{T}$-dyadic cylinder containing x of order ℓ. By the Gibbs property of $\mu_{t,q}$,

$$\begin{aligned} \frac{\log \mu_{t,q}(D_\ell(x))}{\log |D_\ell(x)|} &= \frac{S_\ell \psi_{t,q}(x) - \ell P(t,q)}{-S_\ell \log |\widehat{T}'|(x)} \\ &= \frac{-tS_\ell \log |\widehat{T}'|(x) - (\log 2)\cdot q S_\ell \phi(x) - \ell P(t,q)}{-S_\ell \log |\widehat{T}'|(x)} \\ &= t + q\frac{S_\ell \phi(x)}{S_\ell \log_2 |\widehat{T}'|(x)} + \frac{\ell P(t,q)}{S_\ell \log |\widehat{T}'|(x)}. \end{aligned} \tag{2.6}$$

• Upper bound. For each $q > 0$, let $t(q)$ be the the number such that $P(t(q),q) = 0$. (The existence of $t(q)$ comes from the facts that $P(t,q)$ is real-analytic and that for fixed $q > 0$, $P(t,q) > 0$ when $t \to -\infty$ and $P(t,q) < 0$ when $t \to +\infty$.) Then for all $x \in \widetilde{E}(\alpha)$, we have

$$\begin{aligned} \liminf_{r\to 0} \frac{\log \mu_{t,q}(B(x,r))}{\log r} &\leq \liminf_{\ell\to\infty} \frac{\log \mu_{t,q}(B(x,|D_\ell(x)|))}{\log |D_\ell(x)|} \\ &\leq \liminf_{\ell\to\infty} \frac{\log \mu_{t,q}(D_\ell(x))}{\log |D_\ell(x)|} = t(q) + q\alpha, \end{aligned}$$

where for the second inequality the trivial inclusion $D_\ell(x) \subset B(x, |D_\ell(x)|)$ is used. By Billingsley Lemma (see e.g. [3, Proposition 4.9.]), this gives an upper bound of the Hausdorff dimension of $\widetilde{E}(\alpha)$. Thus we have

$$\dim_H \widetilde{E}(\alpha) \le \inf_{q>0}\{t(q) + q\alpha\}.$$

• Lower bound. By the real-analyticity of $P(t,q)$ and the implicit function theorem, the function $q \mapsto t(q)$ is also real-analytic. Thus there exists q_0 such that the following infimum is attained

$$\inf_{q>0}\{t(q) + q\alpha\}.$$

Then we have

$$t'(q_0) + \alpha = 0. \tag{2.7}$$

To prove the lower bound, we first show two claims.

Claim (A): The measure $\mu_{t(q_0),q_0}$ is supported on E_α.

On the one hand, since $P(t(q), q) = 0$,

$$\frac{\partial P}{\partial t} t'(q) + \frac{\partial P}{\partial q} = 0. \tag{2.8}$$

On the other hand, by the ergodicity of the measure $\mu_{t,q}$, we have for $\mu_{t,q}$ almost all x,

$$\lim_{\ell\to\infty} \frac{S_\ell \phi(x)}{S_\ell \log_2 |\widehat{T}'|(x)} = \frac{\int \phi d\mu_{t,q}}{\int \log|\widehat{T}'| d\mu_{t,q}} \cdot \log 2.$$

By Ruelle-Perron-Frobenius transfer operator theory ([10], Proposition 6.5),

$$\int (\log 2)\cdot \phi d\mu_{t,q} = -\frac{\partial P}{\partial q} \quad \text{and} \quad \int \log|\widehat{T}'| d\mu_{t,q} = -\frac{\partial P}{\partial t}.$$

Thus by (2.8) and then (2.7), for $\mu_{t(q_0),q_0}$ almost all x,

$$\lim_{\ell\to\infty} \frac{S_\ell \phi(x)}{S_\ell \log_2 |\widehat{T}'|(x)} = \frac{\frac{\partial P}{\partial q}}{\frac{\partial P}{\partial t}} = -t'(q_0) = \alpha.$$

This shows Claim (A).

Claim (B): For $\mu_{t(q_0),q_0}$ almost all x,

$$\lim_{n\to\infty} \frac{\log \mu_{t(q_0),q_0}(I_n(x))}{\log 2^{-n}} = t(q_0) + q_0\alpha,$$

where $I_n(x)$ is the T-dyadic cylinder of order n containing x.

On the one hand, by (2.6) and then by (2.8) and (2.7), one has for $\mu_{t(q_0),q_0}$ almost all x

$$\lim_{\ell\to\infty} \frac{\log \mu_{t(q_0),q_0}(D_\ell(x))}{\log |D_\ell(x)|} = t(q_0) + q_0 \frac{\frac{\partial P}{\partial q}}{\frac{\partial P}{\partial t}} = t(q_0) + q_0\alpha. \tag{2.9}$$

On the other hand, note that for any $x \in E(\alpha)$, if the binary expansion of x is $x = [0^{n_1-1}10^{n_2-1}1\ldots]$, then for any $\delta > 0$, for ℓ large enough,

$$(\alpha - \delta)\ell \le 2^{n_1} + \cdots + 2^{n_\ell} - \ell = S_\ell \phi(x) \le (\alpha+\delta)\ell.$$

Hence

$$n_\ell = O(\log \ell),$$

which implies

$$\lim_{\ell\to\infty} \frac{\log|D_\ell(x)|}{\log|D_{\ell+1}(x)|} = \lim_{\ell\to\infty} \frac{n_1 + \cdots + n_\ell}{n_1 + \cdots + n_\ell + n_{\ell+1}} = 1. \tag{2.10}$$

Thus

$$\lim_{n\to\infty} \frac{\log \mu_{t(q_0),q_0}(I_n(x))}{\log 2^{-n}} = \lim_{\ell\to\infty} \frac{\log \mu_{t(q_0),q_0}(D_\ell(x))}{\log |D_\ell(x)|}.$$

This shows Claim (B).

To conclude the desired lower bound, we apply the classical mass distribution principle (see [3, Proposition 4.2]). Since the Hausdorff dimension will not be changed if we replace the δ-coverings by T-dyadic cylinder coverings (see [3, Section 2.4]), the lower bound of the Hausdorff dimension can be given by the mass transference principle on T-dyadic cylinders. By the above two claims and Egorov's theorem, for any $\eta > 0$, there exists an integer N_0 such that the set

$$\Big\{x \in E_\alpha : \mu(I_n(x)) \le |I_n(x)|^{t(q_0)+q_0\alpha-\eta},\ n \ge N\Big\}$$

is of $\mu_{t(q_0),q_0}$ positive measure. So, it implies that

$$\dim_H E_\alpha \ge t(q_0) + q_0\alpha - \eta.$$

Note that ([10, Lemma 7.5]) the function $q \mapsto t(q)$ is a decreasing convex function such that

$$t(0) = 1, \quad \lim_{q\to\infty} (t(q) + q) = 0,$$

and

$$\lim_{q\to 0^+} t'(q) = -\infty, \quad \lim_{q\to+\infty} t'(q) = -1.$$

Therefore, we have proved for any $\alpha \in (1, +\infty)$

$$\dim_H(\widetilde{E}(\alpha)) = \inf_{q>0}\{t(q) + q\alpha\},$$

which is Legendre transformation. All the properties stated in Theorem 1.1 are satisfied by the function $\alpha \mapsto \dim_H(\widetilde{E}(\alpha))$ which is the same function as $\alpha \mapsto \dim_H(E(\alpha))$ by Lemma 2.1.

For the end point $\alpha = 1$, it suffices to note that the level set $E(1)$ is nothing but the set of numbers with frequency of the digit 1 in its binary expansion being 1. Thus the Hausdorff dimension of $E(1)$ is 0. Hence, the Legendre transformation formula for the Hausdorff dimension of $E(\alpha)$ ($\alpha > 1$) also holds for $\alpha = 1$. □

3. Fast increasing Birkhoff sum

At first, we give two simple observations.

Lemma 3.1. *Let W be an integer such that $2^t \le W < 2^{t+1}$ for some positive integer t. For any $0 \le n \le t$, among the integers between W and $W(1+2^{-n})$, there is one $V = V(W, n)$ whose binary expansion of V has at most $n + 2$ digits 1 and ends with at least $t - n$ zeros.*

Proof. By the assumption, we have $2^{-n}W \ge 2^{t-n}$. Thus among the $2^{-n}W$ consecutive integers from W to $W(1 + 2^{-n})$ there is at least one integer which is divisible by 2^{t-n} which means there is an integer $\ell \ge 1$ such that

$$W \le \ell 2^{k-n} \le W(1 + 2^{-n}).$$

Let $V = \ell 2^{t-n}$ and note that V is an integer whose binary expansion ends with at least $t - n$ zeros. Since $\ell 2^{t-n} \le W(1 + 2^{-n}) < 2^{t+2}$, we conclude that $\ell 2^{t-n}$ has at most $(t + 2) - (t - n) = n + 2$ digits 1 in its binary expansion. □

In the follows, the base of the logarithm is taken to be 2.

Lemma 3.2. *For each integer W, and any integer $n \le \log W$, we can find a word w with length*

$$|w| \le (n+2)(2+\log W)$$

and for any $x \in I_{|w|}(w)$

$$W \le \sum_{j=0}^{|w|-1} \varphi(T^j x) \le W(1+2^{-n}).$$

Proof. Let V be an integer given in Lemma 3.1. Then $W \le V \le W(1+2^{-n})$. Moreover if we write this number V in binary expansion:

$$V = 2^{t_1} + \cdots + 2^{t_p},$$

one has that $\lfloor \log W \rfloor + 1 \ge t_1 > \cdots > t_p \ge \lfloor \log W \rfloor - n$ and $p \le n+2$. Consider the word

$$w = (10^{t_1-1}1, 10^{t_2-1}1, \cdots, 10^{t_p-1}1)$$

here the word $10^{t_p-1}1$ is 1 when $t_p = 0$. Then we can check that the length of w satisfies

$$|w| = (t_1+1) + \cdots + (t_p+1) \le p(t_1+1) \le (n+2)(2+\log W),$$

and for any $x \in I_{|w|}(w)$,

$$\sum_{j=0}^{|w|-1} \varphi(T^j x) = V.$$

Hence, the proof is completed. □

We also need the following lemmas whose proofs are left for the reader.

Lemma 3.3. *For any $m \ge 1$, define*

$$F_m = \{x \in (0,1] : \epsilon_{km}(x) = 1, \text{ for all } k \ge 1\}.$$

Then $\dim_H F_m = \frac{m-1}{m}$.

Lemma 3.4. [13, Lemma 4] *Given a subset $\mathbb{J}$ of positive integers and an infinite sequence $\{a_k\}_{k=1}^{\infty}$ of 0's and 1's, let*

$$E(\mathbb{J}, \{a_k\}_{k=1}^{\infty}) = \Big\{x \in (0,1] : \epsilon_k(x) = a_k, \text{for all } k \in \mathbb{J}\Big\}.$$

If the density of $\mathbb{J}$ is zero, that is,

$$\lim_{n\to\infty} \frac{1}{n} \mathit{Card}\Big\{k \le n : k \in \mathbb{J}\Big\} = 0$$

then $\dim_H E(\mathbb{J}, \{a_k\}_{k=1}^{\infty}) = 1$.

Before the proof Theorem 1.2, we show the following lemma.

Lemma 3.5. *Let $\Psi : \mathbb{N} \to \mathbb{N}$ be an increasing function such that $\Psi(n)/n \to \infty$ as $n \to \infty$. Assume that there exists a subsequence N_k satisfying the following conditions*

$$N_k - N_{k-1} \to \infty, \ \Psi(N_k) - \Psi(N_{k-1}) \to \infty, \tag{3.1}$$

and

$$\frac{\Psi(N_{k-1})}{\Psi(N_k)} \to 1, \quad \frac{\log\big(\Psi(N_k) - \Psi(N_{k-1})\big)}{N_k - N_{k-1}} \to 0, \tag{3.2}$$

as $k \to \infty$. Then the set

$$E_\Psi(1) = \Big\{x \in (0,1] : \lim_{n\to\infty} \frac{1}{\Psi(n)} S_n(x) = 1\Big\}$$

has Hausdorff dimension 1.

Proof. Fix a large integer m and write

$$\mathcal{U} = \Big\{u = (\epsilon_1, \cdots, \epsilon_m) : \epsilon_m = 1, \epsilon_i \in \{0,1\}, i \neq m\Big\}.$$

To avoid the abuse of notation, by the first assumption of (3.1), we assume $N_k - N_{k-1} \gg m$ for all $k \geq 1$ by setting $N_0 = 0$ and $\Psi(N_0) = 0$.

For each $k \geq 1$, we write

$$W_k := \Psi(N_k) - \Psi(N_{k-1})$$

and let $\{n_k\}$ be a sequence of integers tending to ∞ such that

$$n_k \leq \log W_k, \quad n_k \cdot \frac{\log\big(\Psi(N_k) - \Psi(N_{k-1})\big)}{N_k - N_{k-1}} \to 0.$$

By the second assumptions of (3.1) and (3.2), this sequence of $n_k \geq 0$ do exist.

Now for W_k and n_k, let w_k be the word given in Lemma 3.2. Then the length a_k of w_k satisfies

$$\begin{aligned} a_k \leq &(n_k + 2)(2 + \log W_k) \\ = &(n_k + 2)\left(2 + \log(\Psi(N_k) - \Psi(N_{k-1}))\right) = o(N_k - N_{k-1}) \end{aligned} \tag{3.3}$$

and for any $x \in I_{a_k}(w_k)$,

$$W_k \leq \sum_{j=0}^{a_k - 1} \varphi(T^j x) \leq W_k(1 + 2^{-n_k}). \tag{3.4}$$

Define t_k, ℓ_k to be the integers satisfying

$$N_k - N_{k-1} - a_k = t_k m + \ell_k,$$

for some $0 \leq \ell_k < m$.

Let w_k $(k \geq 1)$ be given as the above. We define a Cantor subset of $E_\Psi(1)$ as follows.

Level 1 of the Cantor subset. Define

$$E_1 = \Big\{I_{N_1}(u_1, \cdots, u_{t_1}, 1^{\ell_1}, w_1) : u_i \in \mathcal{U}, 1 \leq i \leq t_1\Big\}.$$

For simplicity, we use $I_{N_1}(U_1)$ to denote a general cylinder in E_1.

Level 2 of the Cantor subset. This level is composed by sublevels for each cylinder $I_{N_1}(U_1) \in E_1$. Fix an element $I_{N_1} = I_{N_1}(U_1) \in E_1$. Define

$$E_2(I_{N_1}(U_1)) = \Big\{I_{N_2}(U_1, u_1, \cdots, u_{t_2}, 1^{\ell_2}, w_2) : u_i \in \mathcal{U}, 1 \leq i \leq t_2\Big\}.$$

Then

$$E_2 = \bigcup_{I_{N_1} \in E_1} E_2(I_{N_1}).$$

For simplicity, we use $I_{N_2}(U_2)$ to denote a general cylinder in E_2.

From Level k to $k+1$. Fix $I_{N_k}(U_k) \in E_k$. Define

$$E_{k+1}(I_{N_k}(U_k)) = \Big\{I_{N_{k+1}}(U_k, u_1, \cdots, u_{t_{k+1}}, 1^{\ell_{k+1}}, w_{k+1}) : u_i \in \mathcal{U}, 1 \leq i \leq t_{k+1}\Big\}.$$

Then

$$E_{k+1} = \bigcup_{I_{N_k} \in E_k} E_{k+1}(I_{N_k}).$$

Up to now we have constructed a sequence of nested sets $\{E_k\}_{k \geq 1}$. Set

$$F = \bigcap_{k \geq 1} E_k.$$

We claim that
$$F \subset E(\Psi).$$
In fact, for all $x \in F$, by construction, for each $k \geq 1$,
$$\begin{aligned}
&\sum_{n=N_{k-1}}^{N_k-1} \varphi(T^n x) \\
=&\sum_{n=N_{k-1}}^{N_{k-1}+t_k m-1} \varphi(T^n x) + \sum_{n=N_{k-1}+t_k m}^{N_{k-1}+t_k m+\ell_k-1} \varphi(T^n x) + \sum_{n=N_{k-1}+t_k m+\ell_k}^{N_k-1} \varphi(T^n x) \\
=&t_k O(2^m) + \ell_k + W_k(1+O(2^{-n_k})) \\
=&O\left(\frac{(N_k - N_{k-1})2^m}{m}\right) + (\Psi(N_k) - \Psi(N_{k-1}))(1+O(2^{-n_k})).
\end{aligned}$$
Since $n_k \to \infty$ which implies $2^{-n_k} \to 0$ as $k \to \infty$, we have
$$\sum_{n=0}^{N_k-1} \varphi(T^n x) = \Psi(N_k)\big(1+o(1)\big) + O\left(\frac{N_k 2^m}{m}\right).$$
By the assumption $\Psi(n)/n \to \infty$ as $n \to \infty$, we then deduce
$$\sum_{n=0}^{N_k-1} \varphi(T^n x) = \Psi(N_k) + o(\Psi(N_k)),$$
Thus
$$\lim_{k\to\infty} \frac{\sum_{n=0}^{N_k-1} \varphi(T^n x)}{\Psi(N_k)} = 1. \tag{3.5}$$
While, for each $N_{k-1} < N \leq N_k$
$$\frac{\sum_{n=0}^{N_{k-1}-1} \varphi(T^n x)}{\Psi(N_k)} \leq \frac{\sum_{n=0}^{N-1} \varphi(T^n x)}{\Psi(N)} \leq \frac{\sum_{n=0}^{N_k-1} \varphi(T^n x)}{\Psi(N_{k-1})}.$$
So by the first assumption of (3.2), we deduce from (3.5) that
$$\lim_{n\to\infty} \frac{1}{\Psi(n)} S_n(x) = 1.$$
This proves $x \in E_\Psi(1)$ and hence $F \subset E_\Psi(1)$.

In the following, we will construct a Hölder continuous function from F to F_m. Recall that
$$F_m = \big\{x \in (0,1] : \epsilon_{km}(x) = 1, \text{ for all } k \geq 1\big\}.$$
Define
$$\begin{aligned} f : F &\to F_m \\ x &\mapsto y \end{aligned}$$
where y is obtained by eliminating the digits $\{(\epsilon_{N_k-\ell_k-a_k+1}, \cdots, \epsilon_{N_k})\}_{k\geq 1}$ in the binary expansion of x. Now we calculate the Hölder expoent of f.

Take two points $x_1, x_2 \in F$ closed enough. Let n be the smallest integer such that $\epsilon_n(x_1) \neq \epsilon_n(x_2)$ and k be the integer such that $N_k < n \leq N_{k+1}$. Note that by the construction of F, the digits sequence
$$\{(\epsilon_{N_k-\ell_k-a_k+1}, \cdots, \epsilon_{N_k})\}_{k\geq 1} \text{ and } \{\epsilon_{N_k+tm}\}_{1\leq t\leq t_{k+1}}$$
are the same for all $x \in F$. So we must have
$$N_k < n < N_{k+1} - \ell_{k+1} - a_{k+1}. \tag{3.6}$$

Since n is strictly less than $N_{k+1} - \ell_{k+1} - a_{k+1}$ and $\epsilon_{N_k+tm}(x_1) = \epsilon_{N_k+tm}(x_2) = 1$ for all $1 \le t \le t_{k+1}$, thus, at most m steps after the position n, saying n', $\epsilon_{n'}(x_1) = \epsilon_{n'}(x_2) = 1$. So it follows that

$$|x_1 - x_2| \ge \frac{1}{2^{n+m}}.$$

Again by the construction and the definition of the map f, we have $y_1 = f(x_1)$ and $y_2 = f(x_2)$ have common digits up to the position $n-1-(\ell_1+a_1)-\cdots-(\ell_k+a_k)$. Thus, it follows

$$|f(x_1) - f(x_2)| \le \frac{1}{2^{n-1-(\ell_1+a_1)-\cdots-(\ell_k+a_k)}}.$$

Recall that $\ell_k < m$ and $a_1 + \cdots + a_k = o(N_k)$ (see (3.3)) and also that $N_k/k \to \infty$ as $k \to \infty$ (by (3.1)). We have

$$1 \ge \frac{n-1-(\ell_1+a_1)-\cdots-(\ell_k+a_k)}{n+m} \ge \frac{n-1-km-o(N_k)}{n+m} = 1 + o(1),$$

which implies that f is $(1-\eta)$-Hölder for any $\eta > 0$. Thus

$$\dim_H F \ge (1-\eta) \dim_H F_m.$$

By Lemma 3.3, we then have

$$\dim_H F \ge (1-\eta)\frac{m-1}{m}.$$

By the arbitrariness of $\eta > 0$ and letting $m \to \infty$, we conclude that $\dim_H E(\Psi) = 1$. This finishes the proof. □

Proof of Theorem 1.2. In all the three parts of Theorem 1.2, the case of $\beta = 0$ is a direct consequence of Theorem 1.1.

(I). Assume that Ψ is one of the functions $\Psi(n) = n \log n$, $\Psi(n) = n^a$ $(a > 1)$, $\Psi(n) = 2^{n^\gamma}$ with $0 < \gamma < 1/2$.

(I_1). $0 < \beta < \infty$. It suffices to consider the dimension of $E_\Psi(1)$ i.e. $\beta = 1$, since for other $\beta \in (0, \infty)$, we need only replace $\Psi(n)$ by $\beta\Psi(n)$.

To show $\dim_H E_\Psi(1) = 1$, we can apply Lemma 3.5 directly. If $\Psi(n) = n \log n$, we can choose $N_k = k^2$. For $\Psi(n) = n^a$ $(a > 1)$, we can also choose $N_k = k^2$. Suppose now $\Psi(n) = 2^{n^\gamma}$ with $0 < \gamma < 1/2$. Let $\delta > 0$ be small such that

$$\frac{\gamma}{1-\gamma} + \delta\gamma < 1 \tag{3.7}$$

which is possible since $\gamma < 1/2$. Take

$$N_k = \lfloor k^{\frac{1}{1-\gamma}+\delta} \rfloor. \tag{3.8}$$

Then we have

$$N_{k+1} - N_k \approx k^{\frac{\gamma}{1-\gamma}+\delta}, \tag{3.9}$$

and

$$\begin{aligned} \log(\Psi(N_{k+1}) - \Psi(N_k)) &\approx \log(\Psi'(N_k)(N_{k+1} - N_k)) \\ &\approx N_k^\gamma + \log(N_{k+1} - N_k) \approx N_k^\gamma. \end{aligned}$$

Here we write $A \approx B$ when $A/B \to 1$. This shows the validity of (3.1). Moreover,

$$\frac{\log(\Psi(N_{k+1}) - \Psi(N_k))}{N_{k+1} - N_k} \approx \frac{k^{\frac{\gamma}{1-\gamma}+\gamma\delta}}{k^{\frac{\gamma}{1-\gamma}+\delta}} = k^{-\delta(1-\gamma)} \to 0 \ (k \to \infty).$$

Thus the second assumption of (3.2) is satisfied. At last, for the first assumption in (3.2), by (3.7)

$$\frac{\Psi(N_{k-1})}{\Psi(N_k)} = 2^{(k-1)^{\frac{\gamma}{1-\gamma}+\delta\gamma} - k^{\frac{\gamma}{1-\gamma}+\delta\gamma}} \to 1.$$

Hence Lemma 3.5 applies.

(I_2). If $\beta=\infty$, we may choose $\widetilde{\Psi}(n)=2^{n^\eta}$ for some $0<\eta<\frac{1}{2}$ such that $E_{\widetilde{\Psi}}(1)\subset E_\Psi(\infty)$. Then $\dim_H E_\Psi(\infty)=1$ follows from (I_1).

(II). Now suppose that $\Psi(n)=2^{n^\gamma}$ with $1/2\le\gamma<1$.

(II_1). Let $\beta\in(0,\infty)$. We will prove that $E_\Psi(\beta)$ is empty. On the contrary, suppose there is $x\in E_\Psi(\beta)$, which has binary expansion

$$x=[0^{n_1-1}10^{n_2-1}1\cdots0^{n_\ell-1}1\cdots]. \tag{3.10}$$

Then, by (2.1) we have

$$\begin{aligned}\frac{S_{n_1+n_2+\cdots+n_\ell}(x)}{\Psi(n_1+n_2+\cdots+n_\ell)}&=\frac{2^{n_1}+2^{n_2}+\cdots+2^{n_\ell}-\ell}{2^{(n_1+n_2+\cdots+n_\ell)^\gamma}}\to\beta,\\ \frac{S_{n_1+n_2+\cdots+n_\ell+1}(x)}{\Psi(n_1+n_2+\cdots+n_\ell+1)}&=\frac{2^{n_1}+2^{n_2}+\cdots+2^{n_\ell}-\ell+2^{n_{\ell+1}-1}}{2^{(n_1+n_2+\cdots+n_\ell+1)^\gamma}}\to\beta.\end{aligned} \tag{3.11}$$

Since

$$\frac{2^{(n_1+n_2+\cdots+n_\ell)^\gamma}}{2^{(n_1+n_2+\cdots+n_\ell+1)^\gamma}}\to1,$$

by dividing the two limits of (3.11), we deduce that

$$\frac{2^{n_1}+2^{n_2}+\cdots+2^{n_\ell}-\ell+2^{n_{\ell+1}-1}}{2^{n_1}+2^{n_2}+\cdots+2^{n_\ell}-\ell}=1+\frac{2^{n_{\ell+1}-1}}{2^{n_1}+2^{n_2}+\cdots+2^{n_\ell}-\ell}\to1,$$

which implies that

$$\frac{S_{n_1+n_2+\cdots+n_{\ell+1}}(x)}{S_{n_1+n_2+\cdots+n_\ell}(x)}=1+\frac{2^{n_{\ell+1}}-1}{2^{n_1}+2^{n_2}+\cdots+2^{n_\ell}-\ell}\to1.$$

Combining with (3.11), we get

$$1\leftarrow\frac{\Psi(n_1+\cdots+n_{\ell+1})}{\Psi(n_1+\cdots+n_\ell)}=\frac{2^{(n_1+n_2+\cdots+n_\ell+n_{\ell+1})^\gamma}}{2^{(n_1+n_2+\cdots+n_\ell)^\gamma}}.$$

Thus

$$\begin{aligned}&(n_1+n_2+\cdots+n_\ell+n_{\ell+1})^\gamma-(n_1+n_2+\cdots+n_\ell)^\gamma\\ &=(n_1+n_2+\cdots+n_\ell)^\gamma\left(\left(1+\frac{n_{\ell+1}}{n_1+n_2+\cdots+n_\ell}\right)^\gamma-1\right)\\ &\approx\frac{\gamma n_{\ell+1}}{(n_1+n_2+\cdots+n_\ell)^{1-\gamma}}\to0.\end{aligned}$$

Therefore, for any $\varepsilon>0$, there exists $k_0\ge1$ such that for all $j>k_0$,

$$n_j<\varepsilon(n_1+n_2+\cdots+n_{j-1})^{1-\gamma}. \tag{3.12}$$

Then for any $k_0<j\le\ell$

$$n_j<\varepsilon(n_1+n_2+\cdots+n_\ell)^{1-\gamma}.$$

This implies

$$\begin{aligned}S_{n_1+n_2+\cdots+n_\ell}(x)&=2^{n_1}+2^{n_2}+\cdots+2^{n_\ell}-\ell\\ &\le M+\ell2^{\epsilon(n_1+n_2+\cdots+n_\ell)^{1-\gamma}}-\ell,\end{aligned}$$

with $M:=2^{n_1}+\cdots+2^{n_{k_0}}$. Thus we have

$$\frac{S_{n_1+n_2+\cdots+n_\ell}(x)}{\Psi(n_1+n_2+\cdots+n_\ell)}<\frac{M+\ell2^{\epsilon(n_1+n_2+\cdots+n_\ell)^{1-\gamma}}-\ell}{2^{(n_1+n_2+\cdots+n_\ell)^\gamma}}. \tag{3.13}$$

By observing $n_j\ge1$, we deduce that the upper bound of (3.13) converges to 0 for $1/2\le\gamma<1$, a contradiction to (3.11). Hence $E_\Psi(\beta)$ is an empty set.

(II$_2$). $\beta = \infty$. Fix $\delta \in (\gamma, 1)$ and take a large integer K such that $2^{K\delta} > 1$. Consider the set of points such that at every position $2^k, k > K$ in their binary expansions, they have a string of zeros of length $2^{\delta k}$, i.e.

$$E := \Big\{x \in (0,1] : \epsilon_{2^k+1} = \cdots = \epsilon_{2^k+\lfloor 2^{k\delta}\rfloor} = 0, \text{ for all } k \geq K\Big\}.$$

On the one hand, $E \subset E_\psi(\infty)$, since for any $n \in (2^k, 2^{k+1}]$ for some $k \geq K$,

$$S_n(x) > 2^{2^{k\delta}} \geq 2^{(n/2)^\delta} \gg 2^{n^\gamma}.$$

On the other hand, the set E has dimension 1 guaranteed by Lemma 3.4.

(III). Suppose that $\Psi(n) = 2^{n^\gamma}$ with $\gamma \geq 1$ and let $\beta \in (0, +\infty]$. Assume that there exists $x \in E_\Psi(\beta)$ for some $\beta \in (0, +\infty)$. Write the binary expansion of x as (3.10). Then by (2.1),

$$\begin{aligned} \frac{S_{n_1+n_2+\cdots+n_\ell}(x)}{\Psi(n_1+n_2+\cdots+n_\ell)} &= \frac{2^{n_1}+2^{n_2}+\cdots+2^{n_\ell}-\ell}{2^{(n_1+n_2+\cdots+n_\ell)^\gamma}} \to \beta, \\ \frac{S_{n_1+n_2+\cdots+n_\ell-1}(x)}{\Psi(n_1+n_2+\cdots+n_\ell-1)} &= \frac{2^{n_1}+2^{n_2}+\cdots+2^{n_\ell}-\ell-1}{2^{(n_1+n_2+\cdots+n_\ell-1)^\gamma}} \to \beta. \end{aligned} \tag{3.14}$$

However,

$$\frac{2^{n_1}+2^{n_2}+\cdots+2^{n_\ell}-\ell}{2^{n_1}+2^{n_2}+\cdots+2^{n_\ell}-\ell-1} \to 1 \quad \text{but} \quad \frac{2^{(n_1+n_2+\cdots+n_\ell)^\gamma}}{2^{(n_1+n_2+\cdots+n_\ell-1)^\gamma}} \geq 2,$$

which is a contradiction. Hence $E_\Psi(\beta)$ is empty when $\beta \in (0, +\infty)$.

When $\beta = +\infty$, by (2.1), we have

$$\liminf_{n\to\infty} \frac{S_n(x)}{2^n} \leq 1. \tag{3.15}$$

So,

$$\liminf_{n\to\infty} \frac{S_n(x)}{\Psi(n)} \leq 1.$$

This shows that $E_\Psi(\infty)$ is also empty. □

4. The potential $1/x$

In fact, the techniques in Section 3 can be applied to the continuous potential $g : x \mapsto 1/x$ on $(0,1]$ which has a singularity at 0.

Proof of Theorem 1.3. We first show that if $\Psi(n)$ is one of the following

$$\Psi(n) = n\log n, \ \Psi(n) = n^a \ (a > 1), \ \Psi(n) = 2^{n^\gamma} \ (0 < \gamma < 1/2),$$

then for any $\beta \in [0, \infty]$, $\dim_H F_\Psi(\beta) = 1$.

We note that if $x \in (0,1]$ has binary expansion $x = [0^n 1^s \ldots]$, then $\varphi(x) = 2^n$ and

$$2^n \leq g(x) \leq 2^n + 2^{n-s+1} = 2^n(1 + 2^{-s+1}). \tag{4.1}$$

In Lemma 3.2, for an integer W, and for any integer $n \leq \log W$, we can construct instead of the words $w = (10^{t_1-1}1, 10^{t_2-1}1, \cdots, 10^{t_p-1}1)$, the following word

$$w = (10^{t_1-1}1^{s+1}, 10^{t_2-1}1^{s+1}, \cdots, 10^{t_p-1}1^{s+1}).$$

Then the length of the word satisfies

$$|w| = \sum_{i=1}^{p}(t_i + s + 1) \leq p(t_1 + s + 1) \leq (n+2)(\log W + s + 2). \tag{4.2}$$

By (4.1), for any $x \in I_{|w|}(w)$,

$$W + s(n+2) \leq \sum_{j=0}^{|w|-1} g(T^j x) \leq W(1+2^{-n}) \cdot (1+2^{-s}) + 2s(n+2).$$

For each $k \geq 1$, we still write

$$W_k := \Psi(N_k) - \Psi(N_{k-1})$$

and let n_k, s_k be a sequence of integers tending to ∞ such that

$$n_k \cdot \frac{\log\big(\Psi(N_k) - \Psi(N_{k-1})\big)}{N_k - N_{k-1}} \to 0, \tag{4.3}$$

$$\frac{n_k \cdot s_k}{N_k - N_{k-1}} \to 0, \tag{4.4}$$

and

$$\frac{n_k \cdot s_k}{\Psi(N_k) - \Psi(N_{k-1})} \to 0. \tag{4.5}$$

By (3.1) and (3.2), these two sequences of $n_k \geq 0$, $s_k \geq 0$ do exist.

Now for W_k and n_k, s_k, let w_k be the word given as above. Then by (4.3) and (4.4), the length a_k of w_k satisfies

$$\begin{aligned} a_k \leq &(n_k+2)(\log W_k + s_k + 2) \\ = &(n_k+2)\left(\log(\Psi(N_k) - \Psi(N_{k-1})) + s_k + 2\right) \\ = &o(N_k - N_{k-1}) \end{aligned} \tag{4.6}$$

and for any $x \in I_{a_k}(w_k)$,

$$\begin{aligned} W_k + s_k(n_k+2) &\leq \sum_{j=0}^{a_k-1} g(T^j x) \\ &\leq W_k(1+2^{-n_k}) \cdot (1+2^{-s_k}) + 2s_k(n_k+2). \end{aligned} \tag{4.7}$$

Hence by (4.5) we still have the same estimation:

$$\sum_{n=0}^{N_k-1} g(T^n x) = \Psi(N_k) + o(\Psi(N_k))$$

and the rest of the proof is the same as (I) of the proof of Theorem 1.2.

We can repeat the same arguments in Section 3 and show that for potential g, the set $F_\Psi(\beta)$ is empty if $\beta \in (0,\infty), \Psi(n) = 2^{n^\gamma} (1/2 \leq \gamma < 1)$ or $\beta \in (0,\infty], \Psi(n) = 2^{n^\gamma} (\gamma \geq 1)$.

In fact, by definition, if there exists $x \in F_\Psi(\beta)$, with its binary expansion

$$x = [0^{n_1-1}10^{n_2-1}1 \cdots 0^{n_\ell-1}10^{n_{\ell+1}-1}1 \cdots],$$

then

$$\frac{S_{n_1+n_2+\cdots+n_\ell} g(x)}{\Psi(n_1+n_2+\cdots+n_\ell)} \to \beta, \quad \frac{S_{n_1+n_2+\cdots+n_\ell+1} g(x)}{\Psi(n_1+n_2+\cdots+n_\ell+1)} \to \beta.$$

Thus

$$\frac{S_{n_1+n_2+\cdots+n_\ell} g(x)}{S_{n_1+n_2+\cdots+n_\ell+1} g(x)} \to 1.$$

Observing $\varphi \leq g \leq 2\varphi$, we have

$$\frac{2^{n_{\ell+1}}}{S_{n_1+n_2+\cdots+n_\ell} g(x)} \to 0,$$

which then implies
$$\frac{S_{n_1+n_2+\cdots+n_\ell} g(x)}{S_{n_1+n_2+\cdots+n_{\ell+1}} g(x)} \to 1.$$
By the definition of $x \in F_\Psi(\beta)$, we have
$$\frac{\Psi(n_1+n_2+\cdots+n_{\ell+1})}{\Psi(n_1+n_2+\cdots+n_\ell)} \to 1.$$
This further implies the same inequality with (3.12) and the rest of proof is the same as (II_1) and (III) by noting $\varphi \le g \le 2\varphi$.

For $\Psi(n) = 2^{n^\gamma}$ $(1/2 \le \gamma < 1)$, we can also prove $\dim_H F_\Psi(\infty) = 1$ by the same proof as (II_2). □

References

[1] J. Aaronson, An Introduction to Infinite Ergodic Theory, Mathematical Surveys and Monographs, 50. American Mathematical Society, Providence, RI, 1997. xii+284 pp.
[2] L. Barreira, B. Saussol, *Variational principles and mixed multifractal spectra*, Trans. Amer. Math. Soc. 353 (2001), no. 10, 3919–3944.
[3] K. J. Falconer, Fractal Geometry, Mathematical Foundations and Application, Wiley, 1990.
[4] A.-H. Fan, D.-J. Feng, and J. Wu, *Recurrence, dimension and entropy*, J. London Math. Soc. (2) 64 (2001), no. 1, 229–244,
[5] W. Feller, *A Limit Theoerm for Random Variables with Infinite Moments*, Amer. J. Math., 68 (1946), 257–262.,
[6] W. Feller, An Introduction to Probability Theory and its Applications, Vol. 1, 3rd Edition, John Wiley, 1968.
[7] A.-H. Fan, L. Liao, B.-W. Wang, J. Wu, *On Khintchine exponents and Lyapunov exponents of continued fractions*, Ergodic Theory Dynam. Systems 29 (2009), no. 1, 73–109.
[8] A.-H. Fan, T. Jordan, L. Liao, M. Rams, *Multifractal analysis for expanding interval maps with infinitely many branches*, Trans. Amer. Math. Soc. 367 (2015), 1847–1870.
[9] A.-H. Fan, L. Liao, B.-W. Wang, J. Wu, *On the fast Khintchine spectrum in continued fractions*, Monats. Math., 171 (2013), 329–340.
[10] P. Hanus, R. D. Mauldin and M. Urbański, *Thermodynamic formalism and multifractal analysis of conformal infinite iterated function systems*, Acta. Math. Hungar. 96 (2002), no. 1–2, 27–98.
[11] L. Liao, M. Rams, *Subexponentially increasing sum of partial quotients in continued fraction expansions*, Math. Proc. Camb. Phil. Soc., 160 (2016), 401–412.
[12] L. Liao, M. Rams, *Upper and lower fast Khintchine spectra in continued fractions*, Monats. Math., 180 (2016), 65–81.
[13] J. Ma, S. Wen, Z. Wen, *Egoroff's theorem and maximal run length,* Monatsh. Math. 151 (2007), no. 4, 287–292.
[14] E. Olivier, *Analyse multifractale de fonctions continues*, C. R. Acad. Sci. Paris Sér. I Math. 326 (1998), no. 10, 1171–1174.
[15] W. Philipp, *Limit theorems for sums of partial quotients of continued fractions*, Monats. Math., 105 (1988), 195–206.
[16] V. Sprindžuk, Metric Theory of Diophantine Approximations, V. H. Winston & Sons, Washington, D.C., 1979.

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