Multifractal analysis of the Birkhoff sums of Saint-Petersburg potential
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Abstract. Let \((0, 1], T\) be the doubling map in the unit interval and \(\varphi\) be the Saint-Petersburg potential, defined by 
\[ \varphi(x) = 2^n \text{ if } x \in (2^{-n-1}, 2^{-n}] \text{ for all } n \geq 0. \]
We consider asymptotic properties of the Birkhoff sum \(S_n(x) = \varphi(x) + \cdots + \varphi(T^{n-1}(x))\). With respect to the Lebesgue measure, the Saint-Petersburg potential is not integrable and it is known that 
\[ \frac{1}{n} \log n \frac{S_n(x)}{\Psi(n)} \rightarrow 1 \log 2 \text{ in probability}, \]
where \(\Psi(n) = n \log n, n^a, \text{ or } 2^n\) for \(a > 1, \gamma > 0\). We determine the Hausdorff dimension of the level set \(\{x : \lim_{n \to \infty} \frac{S_n(x)}{n} = \alpha\}\) (\(\alpha > 0\)) and the set \(\{x : \lim_{n \to \infty} \frac{S_n(x)}{\Psi(n)} = \alpha\}\) (\(\alpha > 0\)) when \(\Psi(n) = n \log n, n^a, 2^n\) for \(a > 1, \gamma > 0\). The fast increasing Birkhoff sum of the potential function \(x \mapsto \frac{1}{x}\) is also studied.

1. Introduction

Let \(T\) be the doubling map on the unit interval \((0, 1]\) defined by 
\[ Tx = 2x - \lfloor 2x \rfloor + 1, \]
where \(\lfloor x \rfloor\) is the smallest integer larger than or equal to \(x\). Let \(\epsilon_1\) be the function defined by 
\[ \epsilon_1(x) = \lfloor 2x \rfloor - 1 \text{ and } \epsilon_n(x) := \epsilon_1(T^{n-1}x) \text{ for } n \geq 2. \]
Then each real number \(x \in (0, 1]\) can be expanded into an infinite series as 
\[ x = \frac{\epsilon_1(x)}{2} + \cdots + \frac{\epsilon_n(x)}{2^n} + \cdots. \tag{1.1} \]
We call (1.1) the binary expansion of \(x\) and also write it as 
\[ x = [\epsilon_1(x)\epsilon_2(x)\ldots]. \]
The Saint-Petersburg potential is a function \(\varphi : (0, 1] \to \mathbb{R}\) defined as 
\[ \varphi(x) = 2^n \text{ if } x \in (2^{-n-1}, 2^{-n}], \forall n \geq 0. \]
We remark that the definition of \(\varphi\) is equivalent to 
\[ \varphi(x) = 2^n \text{ where } n \geq 0 \text{ is the smallest integer such that } \epsilon_{n+1}(x) = 1. \]
and is also equivalent to 
\[ \varphi(x) = 2^n \text{ if the binary expansion of } x \text{ begins with } 0^n1, \]
where \(0^n(n \geq 0)\) means a block with \(n\) consecutive zeros.

The name of Saint-Petersburg potential is motivated by the famous Saint-Petersburg game in probability theory. The Saint-Petersburg potential is of infinite expectation with respect to the Lebesgue measure. Furthermore, it increases exponentially fast near to the point 0.
In this paper, we are concerned with the following Birkhoff sums of the Saint-Petersburg potential:

$$\forall n \geq 1, \quad S_n(x) := \varphi(x) + \varphi(T(x)) + \cdots + \varphi(T^{n-1}(x)), \quad x \in (0, 1].$$

Let

$$I = \{ x \in (0, 1] : \epsilon_1(x) = 1 \}.$$ 

Define the hitting time of $x \in (0, 1]$ to $I$ as

$$n(x) := \inf \{ n \geq 0 : T^n x \in I \}.$$ 

Then

$$n(x) = n \quad \text{if} \quad x \in \left( \frac{1}{2^{n+1}}, \frac{1}{2^n} \right), \quad \text{for all} \quad n \geq 0.$$ 

Using $n(x)$, we define a new dynamical system $\hat{T} : (0, 1] \to (0, 1]$ by

$$\hat{T}(x) = T^{n(x)+1}(x) = 2^{n(x)+1} \left( x - \frac{1}{2^{n+1}} \right) \quad \text{if} \quad x \in \left( \frac{1}{2^{n+1}}, \frac{1}{2^n} \right), \quad \text{for all} \quad n \geq 0,$$

called the acceleration of $T$, in order that $\varphi$ and $\varphi \circ \hat{T}$ are independent. Let

$$\hat{S}_n(x) := \varphi(x) + \varphi(\hat{T}(x)) + \cdots + \varphi(\hat{T}^{n-1}(x)), \quad x \in (0, 1].$$

The convergence in probability of $\hat{S}_n(x)$ is well known (e.g. [6, p.253]) which states that for any $\epsilon > 0$, the Lebesgue measure $\lambda$ of

$$\left\{ x \in (0, 1] : \frac{\hat{S}_n(x)}{n \log n} - \frac{1}{\log 2} \geq \epsilon \right\}$$

tends to 0 as $n \to \infty$.

Let $\{ \Psi_n \}_{n \geq 1}$ be an increasing sequence such that $\Psi_n \to \infty$ as $n \to \infty$. Then it was shown in [5] that Lebesgue almost surely either

$$\lim_{n \to \infty} \frac{\hat{S}_n(x)}{\Psi_n} = 0 \quad \text{or} \quad \limsup_{n \to \infty} \frac{\hat{S}_n(x)}{\Psi_n} = \infty,$$

according as

$$\sum_{n \geq 1} \lambda(\{ x \in (0, 1] : \varphi(x) \geq \Psi_n \}) < \infty \quad \text{or} \quad = \infty.$$

Let $n_1 = n_1(x) = n(x) + 1$ and $n_k = n_k(x) = n_1(\hat{T}^{k-1}x) = n(\hat{T}^{k-1}x)$ for $k \geq 2$.

It is direct to see that

$$\forall \ell \geq 1, \quad S_{n_1+\cdots+n_\ell}(x) = 2\hat{S}_\ell(x) - \ell.$$ 

Moreover, the ergodicity of $T$ (of $\hat{T}$) implies Lebesgue almost surely

$$\lim_{\ell \to \infty} \frac{n_1(x) + \cdots + n_\ell(x)}{\ell} = \int_0^1 (n(x) + 1)d\lambda(x) = 2.$$ 

Combining these two facts together, we obtain the same convergence results as above if we replace $\hat{S}_n$ by $S_n$. In particular, the average $S_n(x)/(n \log n)$ converges to $1/\log 2$ in probability, and almost surely (with respect to the Lebesgue measure) either

$$\lim_{n \to \infty} \frac{S_n(x)}{\Psi_n} = 0 \quad \text{or} \quad \limsup_{n \to \infty} \frac{S_n(x)}{\Psi_n} = \infty,$$

according as

$$\sum_{n \geq 1} \lambda(\{ x \in (0, 1] : \varphi(x) \geq \Psi_n \}) < \infty \quad \text{or} \quad = \infty,$$
where \( \{\Psi_n\}_{n \geq 1} \) is an increasing sequence such that \( \Psi_n \to \infty \) as \( n \to \infty \). Recall that \( \varphi \) has infinite expectation with respect to the Lebesgue measure. We thus have \( S_n(x)/n \) converges to infinity for Lebesgue almost all points.

In this article, we want to further study the asymptotic behavior of of the Birkhoff sum \( S_n(x) \) of the Saint-Petersburg potential. We give a complete multifractal analysis of \( S_n(x) \).

First, for any \( \alpha \geq 1 \), we consider the level set

\[
E(\alpha) = \left\{ x \in (0, 1] : \lim_{n \to \infty} \frac{1}{n} S_n(x) = \alpha \right\}.
\]

For \( t \in \mathbb{R} \) and \( q > 0 \), define

\[
P(t, q) := \log \sum_{j=1}^{\infty} 2^{-tj-q(2^j-1)}.
\]

Then \( P \) is a real-analytic function. Furthermore, for each \( q > 0 \), there is a unique \( t(q) > 0 \) such that \( P(t(q), q) = 0 \). This function \( q \mapsto t(q) \) is real-analytic, strictly decreasing and convex.

Denote by \( \dim_H \) the Hausdorff dimension. The function \( \alpha \mapsto \dim_H E(\alpha) \), called the Birkhoff spectrum of the Saint-Petersburg potential \( \varphi \), is proved to be the Legendre transformation of the function \( q \mapsto t(q) \).

**Theorem 1.1.** For any \( \alpha \geq 1 \) we have

\[
\dim_H E(\alpha) = \inf_{q > 0} \{ t(q) + q\alpha \}.
\]

Consequently, \( \dim_H E(1) = 0 \) and the function \( \alpha \mapsto \dim_H E(\alpha) \) is real-analytic, strictly increasing, concave, and has limit 1 as \( \alpha \to \infty \).

The Birkhoff spectrum of a continuous potential was obtained for full shifts [14], for topologically mixing subshifts of finite type [4], and for repellers of a topologically mixing C\(^{1+\epsilon} \) expanding map [2]. A continuous potential in a compact space is bounded, hence these classical results are all for bounded potentials. Our Theorem 1.1 gives a Birkhoff spectrum for an unbounded function with a singular point. To prove Theorem 1.1, we will transfer our question to a Birkhoff spectrum problem of an interval map with infinitely many branches and we will apply the techniques developed in [9] for continued fraction dynamical system and in [8] for general expanding interval maps with infinitely many branches.

We also study the Birkhoff sums \( S_n(x) \) of fast increasing rates. Let \( \Psi : \mathbb{N} \to \mathbb{N} \) be an increasing function. For \( \beta \in [0, \infty] \), consider the level set

\[
E_\Psi(\beta) := \left\{ x \in (0, 1] : \lim_{n \to \infty} \frac{1}{\Psi(n)} S_n(x) = \beta \right\}.
\]

**Theorem 1.2.** If \( \Psi(n) \) is one of the following

\[
\Psi(n) = \varphi \log n, \quad \Psi(n) = \varphi^n \quad (\alpha > 1), \quad \Psi(n) = 2^{\alpha^n} \quad (0 < \gamma < 1/2),
\]

then for any \( \beta \in [0, \infty] \), \( \dim_H E_\Psi(\beta) = 1 \).

If \( \Psi(n) = 2^{\alpha^n} \) with \( 1/2 \leq \gamma < 1 \), then for any \( \beta \in (0, \infty) \), the set \( E_\Psi(\beta) \) is empty, and \( \dim_H E_\Psi(\beta) = 1 \) for \( \beta = 0, +\infty \).

If \( \Psi(n) = 2^{\alpha^n} \) with \( \gamma \geq 1 \), then for any \( \beta \in (0, \infty) \), the set \( E_\Psi(\beta) \) is empty, and \( \dim_H E_\Psi(\beta) = 1 \) for \( \beta = 0 \).

We remark that by the above discussion on the convergence of \( S_n(x) \), for all cases in Theorem 1.2, the sets \( E_\Psi(0) \) has full measure, and thus obviously has full Hausdorff dimension.
From the definition of $S_n(x)$, we see that for the integer $n$ such that $\epsilon_n(x) = 1$, one has $S_n(x) = S_{n-1}(x) + 1$, which implies for all $x$, $\liminf_{n\to\infty} \frac{S_n(x)}{S_{n-1}(x)} = 1$. Thus if $\liminf_{n\to\infty} \frac{\Psi(n)}{\Psi(n-1)} > 1$, then for any $\beta \in (0, \infty)$, the set $E_{\Psi}(\beta)$ is empty. By the definition of $S_n(x)$, we can also check that for all $x$, for the integer $n$ such that $\epsilon_n(x) = 1$, we have $S_n(x) \leq 2^n - 1$. This implies $\liminf_{n\to\infty} S_n(x)/2^n \leq 1$ (See also the formula (3.15) in Section 3). Hence, for ‘regular’ growth functions $\Psi$ we only need to consider exponential and subexponential growth rates.

However, if we pick a point $x$ with dyadic expansion consisting mostly of 0’s, with infinitely many 1’s but in large distances from each other, then the Birkhoff sum $S_n(x)$ may grow arbitrarily fast on some subsequence $n_i$. Thus for any increasing $\Psi(n)$ there exists a point $x$ such that $\limsup_{n\to\infty} S_n(x)/\Psi(n) = \infty$.

Our study on the Saint-Petersburg potential is an attempt of multifractal analysis of unbounded potential functions on the doubling map dynamical system. However, the Saint-Petersburg potential is locally constant and not continuous. One might think of another unbounded potential function $g : x \mapsto 1/x$ which is close to the Saint-Petersburg potential but is continuous. In fact, our method for studying the fast increasing Birkhoff sum of Saint-Petersburg potential also works for the fast increasing Birkhoff sum of the potential $g : x \mapsto 1/x$.

Denote by $S_ng(x)$ the Birkhoff sum
$$S_ng(x) := g(x) + g(T(x)) + \cdots + g(T^{n-1}(x)), \quad x \in (0, 1].$$
For $\beta \in [0, \infty]$, let
$$F_{\Psi}(\beta) := \left\{ x \in (0, 1] : \lim_{n\to\infty} \frac{1}{\Psi(n)} S_n g(x) = \beta \right\}.$$

**Theorem 1.3.** If $\Psi(n)$ is one of the following
$$\Psi(n) = n \log n, \quad \Psi(n) = n^\alpha (\alpha > 1), \quad \Psi(n) = 2^{\gamma n} (0 < \gamma < 1/2),$$
then for any $\beta \in [0, \infty]$, $\dim_H F_{\Psi}(\beta) = 1$.
If $\Psi(n) = 2^{\gamma n}$ with $1/2 \leq \gamma < 1$, then for any $\beta \in (0, \infty)$, the set $F_{\Psi}(\beta)$ is empty, and $\dim_H F_{\Psi}(\beta) = 1$ for $\beta = 0, +\infty$.
If $\Psi(n) = 2^{\gamma n}$ with $\gamma \geq 1$, then for any $\beta \in (0, \infty]$, the set $F_{\Psi}(\beta)$ is empty, and $\dim_H F_{\Psi}(\beta) = 1$ for $\beta = 0$.

We remark that these multifractal analysis on the Birkhoff sums of fast increasing rates have been done for some special potentials in continued fraction dynamical system ([9, 11, 12]).

## 2. Birkhoff spectrum of the Saint-Petersburg potential

In this section, we will obtain the Birkhoff spectrum of the Saint-Petersburg potential, i.e. the Hausdorff dimension of the following level set:

$$E(\alpha) := \left\{ x \in (0, 1] : \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi(T^j x) = \alpha \right\}, \quad (\alpha \geq 1).$$

We will transfer our question to a Birkhoff spectrum problem for an interval map with infinitely many branches.
2.1. Transference lemma. Recall that the Saint-Petersburg potential \( \varphi \) is given by
\[
\varphi(x) = 2^n, \text{ if } x = [0^n, \cdots]
\]
where \( x = [\epsilon_1\epsilon_2, \cdots] \) denotes the digit sequence in the binary expansion of \( x \). Recall also the definition of hitting time \( n(x) \) and the acceleration \( \hat{T} \) of the doubling map \( T \) in Section 1. Define a new potential function
\[
\phi(x) := 2^{n(x) + 1} - 1, \ x \in (0, 1].
\]
In fact, \( \phi \) is nothing but the function satisfying
\[
\phi(x) = \sum_{j=0}^{n(x)} \varphi(T^j x).
\]
With the notation \( n_1 = n(x) + 1 \geq 1 \), and \( n_k = n(\hat{T}^{k-1} x) + 1 \) for \( k \geq 2 \) given in Section 1, we have
\[
\phi(\hat{T} x) = \sum_{j=0}^{n(\hat{T} x)} \varphi(T^j (\hat{T} x)) = \sum_{j=n_1}^{n_2-1} \varphi(T^j x).
\]
Hence,
\[
\sum_{j=0}^{n_1 + \cdots + n_k-1} \varphi(T^j x) = \sum_{k=0}^{\ell-1} \phi(\hat{T}^k x) = 2^{n_1} + \cdots + 2^{n_k} - \ell. \tag{2.1}
\]
Note that the derivative of \( \hat{T} \) satisfies
\[
|\hat{T}'(x)| = 2^{n(x)+1} = 2^{n_1} = \phi(x) + 1. \tag{2.2}
\]
We have
\[
n_1 + \cdots + n_\ell = \sum_{k=0}^{\ell-1} \log_2 |\hat{T}'(\hat{T}^k x)|.
\]
Recall the set in question:
\[
E(\alpha) = \left\{ x \in (0, 1] : \lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi(T^j x) = \alpha \right\} \quad (\alpha \geq 1).
\]
Define
\[
\tilde{E}(\alpha) := \left\{ x \in (0, 1] : \lim_{\ell \to \infty} \frac{\sum_{k=0}^{\ell-1} \phi(\hat{T}^k x)}{\sum_{k=0}^{\ell-1} \log_2 |\hat{T}'(\hat{T}^k x)|} = \alpha \right\} \quad (\alpha \geq 1).
\]
The following lemma shows the two level sets are the same.

**Lemma 2.1.** For all \( \alpha \geq 1 \), we have \( E(\alpha) = \tilde{E}(\alpha) \).

**Proof.** It is evident that \( E(\alpha) \subseteq \tilde{E}(\alpha) \), because, as discussed above,
\[
\frac{\sum_{k=0}^{\ell-1} \phi(\hat{T}^k x)}{\sum_{k=0}^{\ell-1} \log_2 |\hat{T}'(\hat{T}^k x)|} = \frac{1}{n_1 + \cdots + n_\ell} \sum_{j=0}^{n_1 + \cdots + n_k-1} \varphi(T^j x). \tag{2.3}
\]
Now, we show the other direction. Take an \( x \in \tilde{E}(\alpha) \), express \( x \) in its binary expansion
\[
x = [0^{n_1-1}10^{n_2-1}1\cdots0^{n_k-1}1\cdots].
\]
In fact, \( n_\ell - 1 \) is the recurrence time for \( n(\hat{T}^{\ell-1} x) \), for each \( \ell \geq 1 \).
By \eqref{eq:2.1}, we have, at present,

\[
\lim_{\ell \to \infty} \frac{1}{n_1 + \cdots + n_\ell} \sum_{j=0}^{n_1 + \cdots + n_\ell - 1} \varphi(T^j x) = \alpha.
\]

So, we are required to check it holds for all \(n\).

For any \(\epsilon > 0\), there exists \(\ell_0 \in \mathbb{N}\) such that, for any \(\ell \geq \ell_0\),

\[
\alpha - \epsilon \leq \frac{2^{n_1} + \cdots + 2^{n_\ell} - \ell}{n_1 + \cdots + n_\ell} \leq \alpha + \epsilon. \tag{2.4}
\]

For any \(n_1 + \cdots + n_\ell < n < n_1 + \cdots + n_\ell + n_{\ell+1}\) with \(\ell \geq \ell_0\), it is trivial that

\[
\frac{2^{n_1} + \cdots + 2^{n_\ell} - \ell}{n_1 + \cdots + n_\ell + n_{\ell+1}} \leq \frac{1}{n} \sum_{j=0}^{n_1 + \cdots + n_\ell - 1} \varphi(T^j x) \leq \frac{2^{n_1} + \cdots + 2^{n_\ell} + 2^{n_{\ell+1}} - \ell - 1}{n_1 + \cdots + n_\ell}.
\]

Thus, it suffices to show that

\[
2^{n_{\ell+1}} = o(n_1 + \cdots + n_\ell), \tag{2.5}
\]

which also implies

\[
n_{\ell+1} = o(n_1 + \cdots + n_\ell).
\]

Let \(M_0\) be a large integer such that, for all \(M \geq M_0\), \(2^M \geq 4\alpha M\). So, when \(n_{\ell+1} \leq M_0\), there is nothing to prove. So, we always assume \(2^{n_{\ell+1}} \geq 4\alpha n_{\ell+1}\).

By \eqref{eq:2.4}, we have

\[
2^{n_1} + \cdots + 2^{n_\ell} - \ell \geq (\alpha - \epsilon)(n_1 + \cdots + n_\ell),
\]

\[
2^{n_1} + \cdots + 2^{n_\ell} + 2^{n_{\ell+1}} - \ell - 1 \leq (\alpha + \epsilon)(n_1 + \cdots + n_\ell + n_{\ell+1}).
\]

These give

\[
2^{n_{\ell+1}} \leq 2\epsilon(n_1 + \cdots + n_\ell) + (\alpha + \epsilon)n_{\ell+1} + 1.
\]

So, we have

\[
2^{n_{\ell+1}} \leq 4\epsilon(n_1 + \cdots + n_\ell).
\]

\section{2. Dimension of \(\widehat{E}(\alpha)\).}

Now we calculate the Hausdorff dimension of the set \(\widehat{E}(\alpha)\). At first, we give a notation.

- For each finite word \(w \in \bigcup_{n \geq 1} \{0,1\}^n\) of length \(n\), a \(T\)-dyadic cylinder of order \(n\) is defined as
  \[
  I_n(w) = \{x \in \{0,1\} : (\epsilon_1(x), \ldots, \epsilon_n(x)) = w\}.
  \]

- For \((n_1, \ldots, n_\ell) \in (\mathbb{N} \setminus \{0\})^\ell\), a \(\widehat{T}\)-dyadic cylinder of order \(\ell\) is defined as
  \[
  D_{\ell}(n_1, \ldots, n_\ell) = \{x \in \{0,1\} : n_k(x) = n_k, 1 \leq k \leq \ell\}.
  \]

\begin{proof}[Proof of Theorem \ref{thm:1.1}]

To calculate the Hausdorff dimension of \(\widehat{E}(\alpha)\), we construct a suitable measure supported on \(\widehat{E}(\alpha)\). The Gibbs measures derived from the Ruelle-Perron-Frobenius transfer operator are good candidates for such a measure.

In fact, by considering the inverse branches \(U_i : x \mapsto \frac{x_i}{2^i} (i \geq 0)\) of \(\widehat{T}\), we can code the dynamical system \((\{0,1\}, \widehat{T})\) by the conformal infinite iterated function system \((U_i)_{i \geq 0}\) which satisfies the open set condition \([10, \text{Section 1}]\).

Consider the potential function with two parameters

\[
\psi_{t,q} := -t \log |\widehat{T}'| - (\log 2) \cdot q \phi \quad (t \in \mathbb{R}, q > 0).
\]

Then \((\psi_{t,q} \circ U_i)_{i \geq 0}\) is a family of strong Hölder family \([10, \text{Page 30}]\). Hence, we can define a Ruelle operator

\[
\mathcal{L}_{t,q}f(x) := \sum_{y \in \widehat{T}^{-1}x} e^{\psi_{t,q}(y)} f(y),
\]
on the Banach space of continuous functions on the corresponding infinite symbolic space ([10, Page 31]).

By the Ruelle-Perron-Frobenius transfer operator theory [10, Theorems 2.9 and 2.10], for any \( q > 0 \) (to satisfy the condition 2.2 of [10]), we can find an eigenvalue \( \lambda_{t,q} \) and an eigenfunction \( h_{t,q} \) for \( \mathcal{L}_{t,q} \) and an eigenfunction \( u_{t,q} \) for the conjugate operator \( \mathcal{L}_{t,q}^* \). Then the pressure function \( P(t,q) = \log \lambda_{t,q} \) and the ergodic Gibbs measure \( \mu_{t,q} \) is given by \( h_{t,q} \cdot u_{t,q} \).

The pressure function can be computed by (see [10, Pages 31 and 48])

\[
P(t,q) = \lim_{\ell \to \infty} \frac{1}{\ell} \log \left( \sum_{(n_1, \ldots, n_{\ell}) \in (\mathbb{N} \setminus \{0\})^\ell} \exp \sup_{x \in D_\ell(n_1, \ldots, n_{\ell})} (S_\ell \psi_{t,q}(x)) \right).
\]

Note that for all \( n \geq 1 \),
\[
|\hat{T}^\ell|(x) = 2^n, \quad \text{and} \quad \phi(x) = 2^n - 1 \quad \text{if} \quad x \in \left( \frac{1}{2^n}, \frac{1}{2^{n-1}} \right] = D_1(n).
\]

Then for \( x \in D_\ell(n_1, \ldots, n_{\ell}) \),
\[
S_\ell \psi_{t,q}(x) = -\left( \log 2 \right) \cdot \ell \cdot \sum_{j=1}^\ell n_j + \left( \log 2 \right) \cdot q \cdot \sum_{j=1}^\ell 2^{n_j} - 1.
\]

Thus
\[
\sum_{(n_1, \ldots, n_{\ell}) \in (\mathbb{N} \setminus \{0\})^\ell} \exp \sup_{x \in D_\ell(n_1, \ldots, n_{\ell})} (S_\ell \psi_{t,q}(x))
= \sum_{n_1=1}^\infty \cdots \sum_{n_{\ell}=1}^\infty \left( 2^{-1} \prod_{j=1}^\ell 2^{n_j} \right) \cdot \left( 2^{n_{\ell}} - 1 \right).
\]

Hence
\[
P(t,q) = \log \sum_{j=1}^\infty 2^{-t_j - q(2^j - 1)}.
\]

Now we calculate the local dimension of the Gibbs measure \( \mu_{t,q} \). Let \( D_\ell(x) \) be the \( \hat{T}^\ell \)-dyadic cylinder containing \( x \) of order \( \ell \). By the Gibbs property of \( \mu_{t,q} \),
\[
\frac{\log \mu_{t,q}(D_\ell(x))}{\log |D_\ell(x)|} = \frac{S_\ell \psi_{t,q}(x) - \ell P(t,q)}{-S_\ell \log |\hat{T}^\ell|(x)}
= \frac{-\ell S_\ell \log |\hat{T}^\ell|(x) - \left( \log 2 \right) \cdot q S_\ell \phi(x) - \ell P(t,q)}{-S_\ell \log |\hat{T}^\ell|(x)}
= t + q \frac{S_\ell \phi(x)}{S_\ell \log |\hat{T}^\ell|(x)} + \frac{\ell P(t,q)}{S_\ell \log |\hat{T}^\ell|(x)}.
\]

**Upper bound.** For each \( q > 0 \), let \( t(q) \) be the the number such that \( P(t(q), q) = 0 \). (The existence of \( t(q) \) comes from the facts that \( P(t,q) \) is real-analytic and that for fixed \( q > 0 \), \( P(t,q) > 0 \) when \( t \to -\infty \) and \( P(t,q) < 0 \) when \( t \to +\infty \).) Then for all \( x \in \hat{E}(\alpha) \), we have
\[
\liminf_{\ell \to \infty} \frac{\log \mu_{t,q}(B(x,r))}{\log r} \leq \liminf_{\ell \to \infty} \frac{\log \mu_{t,q}(B(x,|D_\ell(x)|))}{\log |D_\ell(x)|}
\leq \liminf_{\ell \to \infty} \frac{\log \mu_{t,q}(D_\ell(x))}{\log |D_\ell(x)|} = t(q) + qa,
\]
where for the second inequality the trivial inclusion \( D_\ell(x) \subset B(x, |D_\ell(x)|) \) is used. By Billingsley Lemma (see e.g. [3, Proposition 4.9.]), this gives an upper bound of the Hausdorff dimension of \( \tilde{E}(\alpha) \). Thus we have

\[
\dim_H \tilde{E}(\alpha) \leq \inf_{q > 0} \{ t(q) + qa \}.
\]

- **Lower bound.** By the real-analyticity of \( P(t, q) \) and the implicit function theorem, the function \( q \mapsto t(q) \) is also real-analytic. Thus there exists \( q_0 \) such that the following infimum is attained

\[
\inf_{q > 0} \{ t(q) + qa \}.
\]

Then we have

\[
t'(q_0) + \alpha = 0. \tag{2.7}
\]

To prove the lower bound, we first show two claims.

Claim (A): The measure \( \mu_{t(q_0), q_0} \) is supported on \( E_\alpha \).

On the one hand, since

\[
P(t(q), q) = 0,
\]

\[
\frac{\partial P}{\partial t} \frac{t'(q)}{t(q)} + \frac{\partial P}{\partial q} = 0. \tag{2.8}
\]

On the other hand, by the ergodicity of the measure \( \mu_{t,q} \), we have for \( \mu_{t,q} \) almost all \( x \),

\[
\lim_{\ell \to \infty} \frac{S_\ell \phi(x)}{S_\ell \log_2 |T'|(x)} = \frac{\int \phi \mathrm{d} \mu_{t,q}}{\int \log |T'| \mathrm{d} \mu_{t,q}} \cdot \log 2.
\]

By Ruelle-Perron-Frobenius transfer operator theory ([10], Proposition 6.5),

\[
\int (\log 2) \cdot \phi \mathrm{d} \mu_{t,q} = -\frac{\partial P}{\partial q} \text{ and } \int \log |T'| \mathrm{d} \mu_{t,q} = -\frac{\partial P}{\partial t}.
\]

Thus by (2.8) and then (2.7), for \( \mu_{t(q_0), q_0} \) almost all \( x \),

\[
\lim_{\ell \to \infty} \frac{S_\ell \phi(x)}{S_\ell \log_2 |T'|(x)} = \frac{\partial P}{\partial q} = -t'(q_0) = \alpha.
\]

This shows Claim (A).

Claim (B): For \( \mu_{t(q_0), q_0} \) almost all \( x \),

\[
\lim_{n \to \infty} \frac{\log \mu_{t(q_0), q_0}(I_n(x))}{\log 2^{-n}} = t(q_0) + q_0 \alpha,
\]

where \( I_n(x) \) is the \( T \)-dyadic cylinder of order \( n \) containing \( x \).

On the one hand, by (2.6) and then by (2.8) and (2.7), one has for \( \mu_{t(q_0), q_0} \) almost all \( x \)

\[
\lim_{\ell \to \infty} \frac{\log \mu_{t(q_0), q_0}(D_\ell(x))}{\log |D_\ell(x)|} = t(q_0) + q_0 \frac{\partial P}{\partial t} = t(q_0) + q_0 \alpha. \tag{2.9}
\]

On the other hand, note that for any \( x \in E(\alpha) \), if the binary expansion of \( x \) is \( x = [0^{n_1-1}10^{n_2-1}1 \ldots] \), then for any \( \delta > 0 \), for \( \ell \) large enough,

\[
(\alpha - \delta)\ell \leq 2^n_1 + \cdots + 2^n_{\ell} - \ell = S_\ell \phi(x) \leq (\alpha + \delta)\ell.
\]

Hence

\[
n_\ell = O(\log \ell),
\]

which implies

\[
\lim_{\ell \to \infty} \frac{\log |D_\ell(x)|}{\log |D_{\ell+1}(x)|} = \lim_{\ell \to \infty} \frac{n_1 + \cdots + n_\ell}{n_1 + \cdots + n_\ell + n_{\ell+1}} = 1. \tag{2.10}
\]
Thus
\[
\lim_{n \to \infty} \frac{\log \mu_{t(q_0), q_0}(I_n(x))}{\log 2^{-n}} = \lim_{t \to \infty} \frac{\log \mu_{t(q_0), q_0}(D_t(x))}{\log |D_t(x)|}.
\]

This shows Claim (B).

To conclude the desired lower bound, we apply the classical mass distribution principle (see [3, Proposition 4.2]). Since the Hausdorff dimension will not be changed if we replace the \( \delta \)-coverings by \( T \)-dyadic cylinder coverings (see [3, Section 2.4]), the lower bound of the Hausdorff dimension can be given by the mass transference principle on \( T \)-dyadic cylinders. By the above two claims and Egorov’s theorem, for any \( \eta > 0 \), there exists an integer \( N_0 \) such that the set
\[
\{ x \in E_\alpha : \mu(I_n(x)) \leq |I_n(x)|^{t(q_0)+q_0 \alpha-\eta}, \ n \geq N \}
\]
is a of \( \mu_{t(q_0), q_0} \) positive measure. So, it implies that
\[
\dim_H E_\alpha \geq t(q_0) + q_0 \alpha - \eta.
\]

Note that ([10, Lemma 7.5]) the function \( q \mapsto t(q) \) is a decreasing convex function such that
\[
t(0) = 1, \ \lim_{q \to \infty} (t(q) + q) = 0,
\]

and
\[
\lim_{q \to 0^+} t'(q) = -\infty, \ \lim_{q \to +\infty} t'(q) = -1.
\]

Therefore, we have proved for any \( \alpha \in (1, +\infty) \)
\[
\dim_H(\bar{E}(\alpha)) = \inf_{q>0} \{ t(q) + q \alpha \},
\]

which is Legendre transformation. All the properties stated in Theorem 1.1 are satisfied by the function \( \alpha \mapsto \dim_H(\bar{E}(\alpha)) \) which is the same function as \( \alpha \mapsto \dim_H(E(\alpha)) \) by Lemma 2.1.

For the end point \( \alpha = 1 \), it suffices to note that the level set \( E(1) \) is nothing but the set of numbers with frequency of the digit 1 in its binary expansion being 1. Thus the Hausdorff dimension of \( E(1) \) is 0. Hence, the Legendre transformation formula for the Hausdorff dimension of \( E(\alpha) \) \((\alpha > 1)\) also holds for \( \alpha = 1 \).

\[
\square
\]

3. Fast increasing Birkhoff sum

At first, we give two simple observations.

**Lemma 3.1.** Let \( W \) be an integer such that \( 2^t \leq W < 2^{t+1} \) for some positive integer \( t \). For any \( 0 \leq n \leq t \), among the integers between \( W \) and \( W(1+2^{-n}) \), there is one \( V = V(W, n) \) whose binary expansion of \( V \) has at most \( n + 2 \) digits 1 and ends with at least \( t - n \) zeros.

**Proof.** By the assumption, we have \( 2^{-n}W \geq 2^{t-n} \). Thus among the \( 2^{-n}W \) consecutive integers from \( W \) to \( W(1 + 2^{-n}) \) there is at least one integer which is divisible by \( 2^{t-n} \) which means there is an integer \( \ell \geq 1 \) such that
\[
W \leq \ell 2^{k-n} \leq W(1 + 2^{-n}).
\]

Let \( V = \ell 2^{k-n} \) and note that \( V \) is an integer whose binary expansion ends with at least \( t - n \) zeros. Since \( \ell 2^{t-n} \leq W(1 + 2^{-n}) < 2^{t+2} \), we conclude that \( \ell 2^{t-n} \) has at most \( (t + 2) - (t - n) = n + 2 \) digits 1 in its binary expansion.

\[
\square
\]

In the follows, the base of the logarithm is taken to be 2.
Lemma 3.2. For each integer $W$, and any integer $n \leq \log W$, we can find a word $w$ with length
\[ |w| \leq (n + 2)(2 + \log W) \]
and for any $x \in I_{|w|}(w)$
\[ W \leq \sum_{j=0}^{|w|-1} \varphi(T^j x) \leq W(1 + 2^{-n}). \]

Proof. Let $V$ be an integer given in Lemma 3.1. Then $W \leq V \leq W(1 + 2^{-n})$. Moreover if we write this number $V$ in binary expansion:
\[ V = 2^{t_1} + \cdots + 2^{t_p}, \]
one has that
\[ \lfloor \log W \rfloor + 1 \geq t_1 > \cdots > t_p \geq \lfloor \log W \rfloor - n \]
and
\[ p \leq n + 2. \]
Consider the word
\[ w = (10^{t_1-1}, 10^{t_2-1}, \ldots, 10^{t_p-1}) \]
here the word $10^{t_p-1}$ is 1 when $t_p = 0$. Then we can check that the length of $w$ satisfies
\[ |w| = (t_1 + 1) + \cdots + (t_p + 1) \leq p(t_1 + 1) \leq (n + 2)(2 + \log W), \]
and for any $x \in I_{|w|}(w)$,
\[ \sum_{j=0}^{|w|-1} \varphi(T^j x) = V. \]
Hence, the proof is completed. \qed

We also need the following lemmas whose proofs are left for the reader.

Lemma 3.3. For any $m \geq 1$, define
\[ F_m = \{ x \in (0, 1) : \epsilon_{km}(x) = 1, \text{ for all } k \geq 1 \}. \]
Then $\dim_H F_m = \frac{m-1}{m}$.

Lemma 3.4. [13, Lemma 4] Given a subset $J$ of positive integers and an infinite sequence $\{a_k\}_{k=1}^\infty$ of 0’s and 1’s, let
\[ E(J, \{a_k\}_{k=1}^\infty) = \{ x \in (0, 1) : \epsilon_k(x) = a_k, \text{ for all } k \in J \}. \]
If the density of $J$ is zero, that is,
\[ \lim_{n \to \infty} \frac{1}{n} \text{Card}\{ k \leq n : k \in J \} = 0 \]
then $\dim_H E(J, \{a_k\}_{k=1}^\infty) = 1$.

Before the proof Theorem 1.2, we show the following lemma.

Lemma 3.5. Let $\Psi : \mathbb{N} \to \mathbb{N}$ be an increasing function such that $\Psi(n)/n \to \infty$ as $n \to \infty$. Assume that there exists a subsequence $N_k$ satisfying the following conditions
\[ N_k - N_{k-1} \to \infty, \quad \Psi(N_k) - \Psi(N_{k-1}) \to \infty, \quad (3.1) \]
and
\[ \frac{\Psi(N_{k-1})}{\Psi(N_k)} \to 1, \quad \frac{\log (\Psi(N_k) - \Psi(N_{k-1}))}{N_k - N_{k-1}} \to 0, \quad (3.2) \]
as $k \to \infty$. Then the set
\[ E_\Psi(1) = \{ x \in (0, 1) : \lim_{n \to \infty} \frac{1}{\Psi(n)} S_n(x) = 1 \} \]
has Hausdorff dimension 1.
Proof. Fix a large integer \( m \) and write

\[
U = \left\{ u = (\epsilon_1, \cdots, \epsilon_m) : \epsilon_m = 1, \epsilon_i \in \{0, 1\}, i \neq m \right\}.
\]

To avoid the abuse of notation, by the first assumption of (3.1), we assume \( N_k = N_{k-1} \gg m \) for all \( k \geq 1 \) by setting \( N_0 = 0 \) and \( \Psi(N_0) = 0 \).

For each \( k \geq 1 \), we write

\[
W_k := \Psi(N_k) - \Psi(N_{k-1})
\]

and let \( \{n_k\} \) be a sequence of integers tending to \( \infty \) such that

\[
n_k \leq \log W_k, \quad n_k \cdot \frac{\log (\Psi(N_k) - \Psi(N_{k-1}))}{N_k - N_{k-1}} \to 0.
\]

By the second assumptions of (3.1) and (3.2), this sequence of \( n_k \geq 0 \) do exist.

Now for \( W_k \) and \( n_k \), let \( w_k \) be the word given in Lemma 3.2. Then the length \( a_k \) of \( w_k \) satisfies

\[
a_k \leq (n_k + 2)(2 + \log W_k)
\]

\[
= (n_k + 2)(2 + \log N_k - \Psi(N_{k-1})) = o(N_k - N_{k-1})
\]

(3.3)

and for any \( x \in I_{n_k}(w_k) \),

\[
W_k \leq \sum_{j=0}^{a_k-1} \varphi(T^j x) \leq W_k(1 + 2^{-n_k}).
\]

(3.4)

Define \( t_k, \ell_k \) to be the integers satisfying

\[
N_k - N_{k-1} - a_k = t_k m + \ell_k,
\]

for some \( 0 \leq \ell_k < m \).

Let \( w_k \) \( (k \geq 1) \) be given as the above. We define a Cantor subset of \( E_1 \) as follows.

**Level 1 of the Cantor subset.** Define

\[
E_1 = \left\{ I_{N_1}(u_1, \cdots, u_{t_1}, 1^{t_1}, w_1) : u_i \in U, 1 \leq i \leq t_1 \right\}.
\]

For simplicity, we use \( I_{N_1}(U_1) \) to denote a general cylinder in \( E_1 \).

**Level 2 of the Cantor subset.** This level is composed by sublevels for each cylinder \( I_{N_1}(U_1) \in E_1 \). Fix an element \( I_{N_2} = I_{N_1}(U_1) \in E_1 \). Define

\[
E_2(I_{N_1}(U_1)) = \left\{ I_{N_2}(U_1, u_1, \cdots, u_{t_2}, 1^{t_2}, w_2) : u_i \in U, 1 \leq i \leq t_2 \right\}.
\]

Then

\[
E_2 = \bigcup_{I_{N_1} \in E_1} E_2(I_{N_1}).
\]

For simplicity, we use \( I_{N_2}(U_2) \) to denote a general cylinder in \( E_2 \).

**From Level \( k \) to \( k + 1 \).** Fix \( I_{N_k}(U_k) \in E_k \). Define

\[
E_{k+1}(I_{N_k}(U_k)) = \left\{ I_{N_{k+1}}(U_k, u_1, \cdots, u_{t_{k+1}}, 1^{t_{k+1}}, w_{k+1}) : u_i \in U, 1 \leq i \leq t_{k+1} \right\}.
\]

Then

\[
E_{k+1} = \bigcup_{I_{N_k} \in E_k} E_{k+1}(I_{N_k}).
\]

Up to now we have constructed a sequence of nested sets \( \{E_k\}_{k \geq 1} \). Set

\[
F = \bigcap_{k \geq 1} E_k.
\]
We claim that
\[ F \subset E(\Psi). \]
In fact, for all \( x \in F \), by construction, for each \( k \geq 1 \),
\[
\sum_{n=N_k-1}^{N_k+1} \varphi(T^n x) = \sum_{n=N_k-1}^{N_k+t_k m + 1} \varphi(T^n x) + \sum_{n=N_k-1}^{N_k+t_k m} \varphi(T^n x) + \sum_{n=N_k-1}^{N_k-1} \varphi(T^n x) = t_k O(2^{m(1+\varepsilon_k)}) + W_k(1 + O(2^{-n_k})) = O(\left(\frac{(N_k - N_{k-1}) 2^{m}}{m}\right) + (\Psi(N_k) - \Psi(N_{k-1}))(1 + O(2^{-n_k})).
\]
Since \( n_k \to \infty \) which implies \( 2^{-n_k} \to 0 \) as \( k \to \infty \), we have
\[
\sum_{n=0}^{N_k-1} \varphi(T^n x) = \Psi(N_k) + o(1) + O\left(\frac{N_k 2^m}{m}\right).
\]
By the assumption \( \Psi(n)/n \to \infty \) as \( n \to \infty \), we then deduce
\[
\sum_{n=0}^{N_k-1} \varphi(T^n x) = \Psi(N_k) + o(\Psi(N_k)).
\]
Thus
\[
\lim_{k \to \infty} \frac{\sum_{n=0}^{N_k-1} \varphi(T^n x)}{\Psi(N_k)} = 1. \tag{3.5}
\]
While, for each \( N_{k-1} < N \leq N_k \)
\[
\frac{\sum_{n=0}^{N_k-1} \varphi(T^n x)}{\Psi(N_k)} \leq \frac{\sum_{n=0}^{N_k-1} \varphi(T^n x)}{\Psi(N_k-1)} \leq \frac{\sum_{n=0}^{N_k-1} \varphi(T^n x)}{\Psi(N_k-1)}.
\]
So by the first assumption of (3.2), we deduce from (3.5) that
\[
\lim_{n \to \infty} \frac{1}{\Psi(n)} S_n(x) = 1.
\]
This proves \( x \in E_\Psi(1) \) and hence \( F \subset E_\Psi(1) \).

In the following, we will construct a Hölder continuous function from \( F \) to \( F_m \).
Recall that
\[ F_m = \{ x \in (0,1] : \epsilon_k m(x) = 1, \text{ for all } k \geq 1 \}. \]
Define
\[ f : F \to F_m \]
\[ x \mapsto y \]
where \( y \) is obtained by eliminating the digits \( \{\epsilon_{N_k-a_{k-1}} \cdots \epsilon_{N_k}\} \) in the binary expansion of \( x \).
Now we calculate the Hölder exponent of \( f \).
Take two points \( x_1, x_2 \in F \) closed enough. Let \( n \) be the smallest integer such that \( \epsilon_n(x_1) \neq \epsilon_n(x_2) \) and \( k \) be the integer such that \( N_k < n \leq N_{k+1} \). Note that by the construction of \( F \), the digits sequence
\[ \{\epsilon_{N_k-a_{k-1}} \cdots \epsilon_{N_k}\} \] and \( \{\epsilon_{N_k+tm}\} \)
are the same for all \( x \in F \). So we must have
\[ N_k < n < N_{k+1} - \ell_{k+1} - a_{k+1}. \tag{3.6} \]
Since $n$ is strictly less than $N_{k+1} - \ell_k - a_{k+1}$ and $e_{N_k + tm}(x_1) = e_{N_k + tm}(x_2) = 1$ for all $1 \leq t \leq k+1$, thus, at most $m$ steps after the position $n$, saying $n'$, $e_n(x_1) = e_{n'}(x_2) = 1$. So it follows that
\[
|x_1 - x_2| \geq \frac{1}{2^{n+m}}.
\]
Again by the construction and the definition of the map $f$, we have $y_1 = f(x_1)$ and $y_2 = f(x_2)$ have common digits up to the position $n-1-(\ell_1 + a_1) - \cdots - (\ell_k + a_k)$. Thus, it follows
\[
|y_1 - y_2| \geq \frac{1}{2^{n-1-(\ell_1 + a_1) - \cdots - (\ell_k + a_k)}}.
\]
Recall that $\ell_k < m$ and $a_1 + \cdots + a_k = o(N_k)$ (see (3.3)) and also that $N_k/k \to \infty$ as $k \to \infty$ (by (3.1)). We have
\[
1 \geq \frac{n - 1 - (\ell_1 + a_1) - \cdots - (\ell_k + a_k)}{n + m} \geq \frac{n - 1 - km - o(N_k)}{n + m} = 1 + o(1),
\]
which implies that $f$ is $(1-\eta)$-Hölder for any $\eta > 0$. Thus
\[
\dim_H F \geq (1-\eta)\dim_H F_m.
\]
By Lemma 3.3, we then have
\[
\dim_H F \geq (1-\eta) \frac{m-1}{m}.
\]
By the arbitrariness of $\eta > 0$ and letting $m \to \infty$, we conclude that $\dim_H E(\Psi) = 1$. This finishes the proof. \hfill \Box

**Proof of Theorem 1.2.** In all the three parts of Theorem 1.2, the case of $\beta = 0$ is a direct consequence of Theorem 1.1.

(1). Assume that $\Psi$ is one of the functions $\Psi(n) = n \log n$, $\Psi(n) = n^\alpha$ ($\alpha > 1$), $\Psi(n) = 2^{n^\gamma}$ with $0 < \gamma < 1/2$.

(1). $0 < \beta < \infty$. It suffices to consider the dimension of $E_\Psi(1)$ i.e. $\beta = 1$, since for other $\beta \in (0,\infty)$, we need only replace $\Psi(n)$ by $\beta \Psi(n)$.

To show $\dim_H E_\Psi(1) = 1$, we can apply Lemma 3.5 directly. If $\Psi(n) = n \log n$, we can choose $N_k = k^2$. For $\Psi(n) = n^\alpha$ ($\alpha > 1$), we can also choose $N_k = k^2$. Suppose now $\Psi(n) = 2^{n^\gamma}$ with $0 < \gamma < 1/2$. Let $\delta > 0$ be small such that
\[
\frac{\gamma}{1-\gamma} + \delta \gamma < 1
\]
which is possible since $\gamma < 1/2$. Take
\[
N_k = \lfloor k^{\frac{1}{1-\gamma} + \delta} \rfloor.
\]
Then we have
\[
N_{k+1} - N_k \approx k^{\frac{1}{1-\gamma} + \delta},
\]
and
\[
\log(\Psi(N_{k+1}) - \Psi(N_k)) \approx \log(\Psi(N_k)(N_{k+1} - N_k)) \approx N_k^\gamma + \log(N_{k+1} - N_k) \approx N_k^\gamma.
\]
Here we write $A \approx B$ when $A/B \to 1$. This shows the validity of (3.1). Moreover,
\[
\frac{\log(\Psi(N_{k+1}) - \Psi(N_k))}{N_{k+1} - N_k} \approx \frac{k^{\frac{1}{1-\gamma} + \gamma \delta} \approx k^{-\delta(1-\gamma)} \to 0 (k \to \infty).}
\]
Thus the second assumption of (3.2) is satisfied. At last, for the first assumption in (3.2), by (3.7)
\[
\frac{\Psi(N_{k+1})}{\Psi(N_k)} = 2(k-1)^{\frac{1}{1-\gamma} + \delta} \to 1.
\]
Hence Lemma 3.5 applies.

(I). If $\beta = \infty$, we may choose $\tilde{\Psi}(n) = 2^{n^\gamma}$ for some $0 < \eta < \frac{1}{2}$ such that $E_{\tilde{\Psi}}(1) \subset E_{\Psi}(\infty)$. Then $\dim_H E_{\Psi}(\infty) = 1$ follows from (I).

(II). Now suppose that $\Psi(n) = 2^n$ with $1/2 \leq \gamma < 1$.

(II). Let $\beta \in (0, \infty)$. We will prove that $E_{\Psi}(\beta)$ is empty. On the contrary, suppose there is $x \in E_{\Psi}(\beta)$, which has binary expansion

$$x = [0^{n_1 - 1}10^{n_2 - 1} \ldots 0^{n_k - 1} \ldots].$$

(3.10)

Then, by (2.1) we have

$$\frac{S_{n_1 + n_2 + \ldots + n_\ell + 1}}{\Psi(n_1 + n_2 + \ldots + n_\ell)} = \frac{2^{n_1} + 2^{n_2} + \ldots + 2^{n_\ell} - \ell}{2^{n_1 + n_2 + \ldots + n_\ell + 1}} \to \beta,$$

$$\frac{S_{n_1 + n_2 + \ldots + n_\ell + 1}}{\Psi(n_1 + n_2 + \ldots + n_\ell + 1)} = \frac{2^{n_1} + 2^{n_2} + \ldots + 2^{n_\ell} - \ell + 2^n}{2^{n_1 + n_2 + \ldots + n_\ell + 1}} \to \beta.$$

(3.11)

Since

$$\frac{2^{n_1 + n_2 + \ldots + n_\ell + 1}}{2^{n_1 + n_2 + \ldots + n_\ell}} \to 1,$$

by dividing the two limits of (3.11), we deduce that

$$\frac{2^{n_1} + 2^{n_2} + \ldots + 2^{n_\ell} - \ell + 2^n}{2^{n_1 + n_2 + \ldots + n_\ell} + 2^{n_1 + n_2 + \ldots + n_\ell + 1}} = 1 + \frac{2^n}{2^{n_1} + 2^{n_2} + \ldots + 2^{n_\ell} - \ell} \to 1,$$

which implies that

$$\frac{S_{n_1 + n_2 + \ldots + n_\ell + 1}}{S_{n_1 + n_2 + \ldots + n_\ell}} = 1 + \frac{2^{n_\ell + 1} - 1}{2^{n_1 + n_2 + \ldots + n_\ell} - \ell} \to 1.$$}

Combining with (3.11), we get

$$1 \leftarrow \frac{\Psi(n_1 + \ldots + n_{\ell + 1})}{\Psi(n_1 + \ldots + n_{\ell})} = \frac{2^{n_1 + n_2 + \ldots + n_{\ell + 1} + 1}}{2^{n_1 + n_2 + \ldots + n_{\ell}} + 1}.$$

Thus

$$(n_1 + n_2 + \ldots + n_{\ell + 1})^\gamma - (n_1 + n_2 + \ldots + n_{\ell})^\gamma$$

$$= (n_1 + n_2 + \ldots + n_{\ell})^\gamma \left( \frac{n_{\ell + 1}}{n_1 + n_2 + \ldots + n_{\ell}} \right)^\gamma - 1$$

$$\approx \frac{\gamma n_{\ell + 1}}{(n_1 + n_2 + \ldots + n_{\ell})^{1-\gamma}} \to 0.$$}

Therefore, for any $\varepsilon > 0$, there exists $k_0 \geq 1$ such that for all $j > k_0$,

$$n_j < \varepsilon (n_1 + n_2 + \ldots + n_{j-1})^{1-\gamma}.$$}

(3.12)

Then for any $k_0 < j \leq \ell$

$$n_j < \varepsilon (n_1 + n_2 + \ldots + n_{\ell})^{1-\gamma}.$$}

This implies

$$S_{n_1 + n_2 + \ldots + n_\ell}(x) = 2^{n_1} + 2^{n_2} + \ldots + 2^{n_\ell} - \ell \leq M + \ell 2^{(n_1 + n_2 + \ldots + n_\ell)^{1-\gamma} - \ell},$$

with $M := 2^{n_1} + \ldots + 2^{n_{k_0}}$. Thus we have

$$\frac{S_{n_1 + n_2 + \ldots + n_\ell}}{\Psi(n_1 + n_2 + \ldots + n_\ell)} < \frac{M + \ell 2^{(n_1 + n_2 + \ldots + n_\ell)^{1-\gamma} - \ell}}{2^{(n_1 + n_2 + \ldots + n_\ell)^{1-\gamma}}}.$$}

(3.13)

By observing $n_j \geq 1$, we deduce that the upper bound of (3.13) converges to 0 for $1/2 \leq \gamma < 1$, a contradiction to (3.11). Hence $E_{\Psi}(\beta)$ is an empty set.
(II). \( \beta = \infty \). Fix \( \delta \in (\gamma, 1) \) and take a large integer \( K \) such that \( 2K^\delta > 1 \). Consider the set of points such that at every position \( 2^k, k > K \) in their binary expansions, they have a string of zeros of length \( 2^k \), i.e.

\[
E := \left\{ x \in (0, 1] : \epsilon_{2^k+1} = \cdots = \epsilon_{2^{k+|2^k\delta|}} = 0, \text{ for all } k \geq K \right\}.
\]

On the one hand, \( E \subset E_\Phi(\infty) \), since for any \( n \in (2^k, 2^{k+1}] \) for some \( k \geq K \),

\[
S_n(x) > 2^{2k^\delta} \geq 2^{(n/2)^\delta} \gg 2^n.
\]

On the other hand, the set \( E \) has dimension 1 guaranteed by Lemma 3.4.

(III). Suppose that \( \Psi(n) = 2^n\gamma \) with \( \gamma \geq 1 \) and let \( \beta \in (0, +\infty] \). Assume that there exists \( x \in E_\Phi(\beta) \) for some \( \beta \in (0, +\infty) \). Write the binary expansion of \( x \) as (3.10). Then by (2.1),

\[
\frac{S_{n_1+n_2+\cdots+n_\ell}(x)}{\Psi(n_1+n_2+\cdots+n_\ell)} = \frac{2^{n_1} + 2^{n_2} + \cdots + 2^{n_\ell} - \ell}{2^{(n_1+n_2+\cdots+n_\ell)\gamma}} \to \beta, \quad \frac{S_{n_1+n_2+\cdots+n_\ell-1}(x)}{\Psi(n_1+n_2+\cdots+n_\ell-1)} = \frac{2^{n_1} + 2^{n_2} + \cdots + 2^{n_\ell} - 1}{2^{(n_1+n_2+\cdots+n_\ell-1)\gamma}} \to \beta.
\]

However,

\[
\frac{2^{n_1} + 2^{n_2} + \cdots + 2^{n_\ell} - \ell}{2^{n_1} + 2^{n_2} + \cdots + 2^{n_\ell} - 1} \to 1 \quad \text{but} \quad \frac{2^{(n_1+n_2+\cdots+n_\ell)\gamma}}{2^{(n_1+n_2+\cdots+n_\ell-1)\gamma}} \geq 2,
\]

which is a contradiction. Hence \( E_\Phi(\beta) \) is empty when \( \beta \in (0, +\infty) \).

When \( \beta = +\infty \), by (2.1), we have

\[
\liminf_{n \to \infty} \frac{S_n(x)}{2^n} \leq 1.
\]

So,

\[
\liminf_{n \to \infty} \frac{S_n(x)}{\Psi(n)} \leq 1.
\]

This shows that \( E_\Phi(\infty) \) is also empty. \( \square \)

4. The potential \( 1/x \)

In fact, the techniques in Section 3 can be applied to the continuous potential \( g : x \mapsto 1/x \) on \( [0, 1] \) which has a singularity at 0.

**Proof of Theorem 1.3.** We first show that if \( \Psi(n) \) is one of the following

\[
\Psi(n) = n \log n, \quad \Psi(n) = n^\alpha (\alpha > 1), \quad \Psi(n) = 2^n \gamma (0 < \gamma < 1/2),
\]

then for any \( \beta \in [0, \infty], \dim_H F_\Psi(\beta) = 1 \).

We note that if \( x \in (0, 1] \) has binary expansion \( x = [0^n1^s \ldots] \), then \( \varphi(x) = 2^n \) and

\[
2^n \leq g(x) \leq 2^n + 2^{n-s+1} = 2^n(1 + 2^{-s+1}). \quad (4.1)
\]

In Lemma 3.2, for an integer \( W \), and for any integer \( n \leq \log W \), we can construct instead of the words \( w = (10^t1^{-1}, 10^t5^{-1}, \ldots, 10^t9^{-1}) \), the following word

\[
w = (10^t_11^{s+1}, 10^t_25^{s+1}, \ldots, 10^t_{s-1}1^{s+1}).
\]

Then the length of the word satisfies

\[
|w| = \sum_{i=1}^{p} (t_i + s + 1) \leq p(t_1 + s + 1) \leq (n + 2)(\log W + s + 2). \quad (4.2)
\]
By (4.1), for any \( x \in I_{|w|}(w) \),
\[
W + s(n + 2) \leq \sum_{j=0}^{[w]-1} g(T^j x) \leq W(1 + 2^{-n}) \cdot (1 + 2^{-s}) + 2s(n + 2).
\]

For each \( k \geq 1 \), we still write
\[
W_k := \Psi(N_k) - \Psi(N_{k-1})
\]
and let \( n_k, s_k \) be a sequence of integers tending to \( \infty \) such that
\[
n_k \cdot \log \left( \frac{\Psi(N_k) - \Psi(N_{k-1})}{N_k - N_{k-1}} \right) \to 0, \tag{4.3}
\]
\[
n_k \cdot s_k \to 0, \tag{4.4}
\]
and
\[
\frac{n_k \cdot s_k}{\Psi(N_k) - \Psi(N_{k-1})} \to 0. \tag{4.5}
\]

By (3.1) and (3.2), these two sequences of \( n_k \geq 0, s_k \geq 0 \) do exist.

Now for \( W_k \) and \( n_k, s_k \), let \( w_k \) be the word given as above. Then by (4.3) and (4.4), the length \( a_k \) of \( w_k \) satisfies
\[
a_k \leq (n_k + 2)(\log W_k + s_k + 2)
= (n_k + 2)(\log(\Psi(N_k) - \Psi(N_{k-1})) + s_k + 2) \tag{4.6}
= o(N_k - N_{k-1})
\]
and for any \( x \in I_{a_k}(w_k) \),
\[
W_k + s_k(n_k + 2) \leq \sum_{j=0}^{a_k-1} g(T^j x)
\leq W_k(1 + 2^{-n_k}) \cdot (1 + 2^{-s_k}) + 2s_k(n_k + 2). \tag{4.7}
\]

Hence by (4.5) we still have the same estimation:
\[
\sum_{n=0}^{N_k-1} g(T^n x) = \Psi(N_k) + o(\Psi(N_k))
\]
and the rest of the proof is the same as (I) of the proof of Theorem 1.2.

We can repeat the same arguments in Section 3 and show that for potential \( g \), the set \( F_g(\beta) \) is empty if \( \beta \in (0, \infty), \Psi(n) = 2^n \gamma (1/2 \leq \gamma < 1) \) or \( \beta \in (0, \infty], \Psi(n) = 2^n \gamma (\gamma \geq 1) \).

In fact, by definition, if there exists \( x \in F_g(\beta) \), with its binary expansion
\[
x = [0^{a_1-1}0^{a_2-1}1 \cdots 0^{a_{\ell}-1}10^{a_{\ell+1}-1}1 \cdots],
\]
then
\[
\frac{S_{n_1+n_2+\cdots+n_\ell} g(x)}{\Psi(n_1 + n_2 + \cdots + n_\ell)} \to \beta, \quad \frac{S_{n_1+n_2+\cdots+n_\ell+1} g(x)}{\Psi(n_1 + n_2 + \cdots + n_\ell + 1)} \to \beta.
\]
Thus
\[
\frac{S_{n_1+n_2+\cdots+n_\ell} g(x)}{S_{n_1+n_2+\cdots+n_\ell+1} g(x)} \to 1.
\]

Observing \( \varphi \leq g \leq 2\varphi \), we have
\[
\frac{2^{n_{\ell+1}}}{S_{n_1+n_2+\cdots+n_\ell} g(x)} \to 0,
\]
which then implies
\[ \frac{S_{n_1+n_2+\cdots+n_\ell}g(x)}{S_{n_1+n_2+\cdots+n_{\ell+1}}g(x)} \to 1. \]
By the definition of \( x \in F_\Psi(\beta) \), we have
\[ \frac{\Psi(n_1+n_2+\cdots+n_{\ell+1})}{\Psi(n_1+n_2+\cdots+n_{\ell})} \to 1. \]
This further implies the same inequality with (3.12) and the rest of proof is the same as (I1) and (II) by noting \( \varphi \leq g \leq 2\varphi \).

For \( \Psi(n) = 2^n (1/2 \leq \gamma < 1) \), we can also prove \( \dim_H F_\Psi(\infty) = 1 \) by the same proof as (II2).

\[ \square \]

References