Can we understand the concepts of the General Relativity and of the Quantum Mechanics based on the principles of the strength of materials? Reflections and proposals

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Reflections and proposals

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Summary

The quantum mechanics, in the infinitesimal world, is the reign of the quantification of energies, and of the probability of presence of particles with the famous wave function ψ associated at the Schrodinger equation. The General relativity, is for it, reserved at the infinitely large world where the mater, the energy density $T_{\mu\nu}$influence the curvature $G_{\mu\nu}$ of the space time at 4 dimensions and reciprocally. Finally, the strength of materials is the mechanic of objects at the human scale (beams, columns, plates, shells) used for the design of structures, without common measures with this two pillars of the of the physics cited earlier. We are going to show in first part of this paper that the results of the quantum mechanics (quantification of energy, shape of the wave function) are similar at the eigen frequencies and eigen modes of a beam. In second part, we will show on simples cases, that the concepts of curvature linked with energy density present in general relativity are equally at the bases of the fundamental equations of the elasticity and of the strength of material. We will demonstrate finally that the stress energy tensor written for small speeds non relativistic is an extension in 4 dimensions of stress tensor of the elasticity theory.

1st part) – Quantum mechanics and beam vibration

1.1) Free vibration of a beam on two supports – natural frequencies and mode of vibration

We consider a beam on two supports, with a span L (m), carried out with a material of elastic modulus E (MPa), that have a section with an inertia I (m^4) and an area S (m^2) and that have a mass by unit of length ($m=\rho S$) in kg/m (cf. figure 1). [4]

The beam is in free vibration and as a deflection along the time $y_{(x,t)}$:

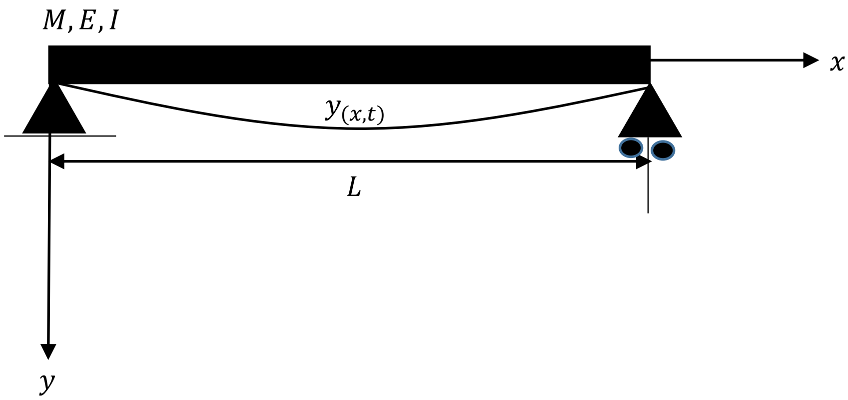


Figure 1 – Beam on 2 supports under its self-weight –

We can write:

$$F = ma = m\frac{\partial^2 y_{(x,t)}}{\partial t^2} = m\ddot{y}_{(x,t)} \quad (1)$$

In addition, we have by connecting the radius of curvature R (m) at the bending moment $M_{(x,t)}$:

$$\frac{1}{R} = \frac{\partial^2 y_{(x,t)}}{\partial x^2} = \frac{M_{(x,t)}}{EI} \quad (2)$$

Knowing that:

$$\frac{\partial M_{(x,t)}}{\partial x} = V_{(x,t)} \quad (3)$$

$$\frac{\partial V_{(x,t)}}{\partial x} = -q_{(x,t)} \quad (4)$$

We obtain by deriving twice respect to x the equation (2):

$$\frac{\partial^2}{\partial x^2}\left(\frac{1}{R}\right) = \frac{\partial^4 y_{(x,t)}}{\partial x^4} = \frac{\partial^2}{\partial x^2}\left(\frac{M_{(x,t)}}{EI}\right) = \frac{-q_{(x,t)}}{EI} \quad (5)$$

So:

$$EI\,\frac{\partial^4 y_{(x,t)}}{\partial x^4} = -q_{(x,t)} \quad (6)$$

By equating equations (1) and (6) in the case of auto-vibration, we obtain the well-known differential equation which controls the self-vibration of a beam on 2 supports:

$$m\frac{\partial^2 y_{(x,t)}}{\partial t^2} = -EI\,\frac{\partial^4 y_{(x,t)}}{\partial x^4} \quad (7)$$

So :

$$\frac{\partial^4 y_{(x,t)}}{\partial x^4} + \left(\frac{m}{EI}\right)\frac{\partial^2 y_{(x,t)}}{\partial t^2} = 0 \quad (8)$$

If we put $y_{(x,t)} = q_{(t)}\phi_{(x)}$ (9)

By replacing in equation (8) above, we obtains:

$$EIq_{(t)}\phi_{(x)}^{iV} + \rho S\ddot{q}_{(t)}\phi_{(x)} = 0 \quad (10)$$

We can therefore separate the variables:

$$\frac{\phi^{IV}}{\phi} = -\frac{\rho S}{EI} \times \frac{\ddot{q}}{q} \quad (11)$$

As this equation has to be verified for all x or t, the above ration have to be constant.

We therefore define:

$$\frac{\phi^{IV}}{\phi} = -\frac{\rho S}{EI} \times \frac{\ddot{q}}{q} = \frac{\alpha^4}{L^4} \quad (12)$$

And we define the pulsation (circular natural frequency) ω thus:

$$\boxed{\omega^2 = \frac{\alpha^4}{L^4} \times \frac{EI}{\rho S} \quad (13)}$$

We can cut in two parts the differential equation (12) in two equations (separation of variables):

$$\begin{cases} \phi_{(x)}^{IV} - \dfrac{\alpha^4}{L^4}\phi_{(x)} = 0 & (14) \\ \ddot{q}_{(t)} + \omega^2 q_{(t)} = 0 & (15) \end{cases}$$

The study of the first equation gives the following general equation:

$$\phi_{(x)} = Asin\left(\alpha\frac{x}{L}\right) + Bcos\left(\alpha\frac{x}{L}\right) + Csinh\left(\alpha\frac{x}{L}\right) + Dcosh\left(\alpha\frac{x}{L}\right) \quad (16)$$

And the successive derivative gives:

$$\phi'_{(x)} = A\frac{\alpha}{L}cos\left(\alpha\frac{x}{L}\right) - B\frac{\alpha}{L}sin\left(\alpha\frac{x}{L}\right) + C\frac{\alpha}{L}cosh\left(\alpha\frac{x}{L}\right) + D\frac{\alpha}{L}sinh\left(\alpha\frac{x}{L}\right) \quad (17)$$

$$\phi''_{(x)} = -A\frac{\alpha^2}{L^2}sin\left(\alpha\frac{x}{L}\right) - B\frac{\alpha^2}{L^2}cos\left(\alpha\frac{x}{L}\right) + C\frac{\alpha^2}{L^2}sinh\left(\alpha\frac{x}{L}\right) + D\frac{\alpha^2}{L^2}cosh\left(\alpha\frac{x}{L}\right) \quad (18)$$

$$\phi'''_{(x)} = -A\frac{\alpha^3}{L^3}cos\left(\alpha\frac{x}{L}\right) + B\frac{\alpha^3}{L^3}sin\left(\alpha\frac{x}{L}\right) + C\frac{\alpha^3}{L^3}cosh\left(\alpha\frac{x}{L}\right) + D\frac{\alpha^3}{L^3}sinh\left(\alpha\frac{x}{L}\right) \quad (19)$$

$$\phi^{IV}{}_{(x)} = A\frac{\alpha^4}{L^4}sin\left(\alpha\frac{x}{L}\right) + B\frac{\alpha^4}{L^4}cos\left(\alpha\frac{x}{L}\right) + C\frac{\alpha^4}{L^4}sinh\left(\alpha\frac{x}{L}\right) + D\frac{\alpha^4}{L^4}cosh\left(\alpha\frac{x}{L}\right) \quad (20)$$

The 4 boundary conditions allows to write:

$\phi_{(x=0)} = 0$

$\phi_{(x=L)} = 0$

$\overset{II}{\phi}_{(x=0)} = 0$

$\overset{II}{\phi}_{(x=L)} = 0$

$$\phi_{(x=0)} = Asin\left(\alpha\frac{0}{L}\right) + Bcos\left(\alpha\frac{0}{L}\right) + Csinh\left(\alpha\frac{0}{L}\right) + Dcosh\left(\alpha\frac{0}{L}\right) \quad (21)$$

$\phi_{(x=0)} = 0A + B + 0C + D = 0$ (22)

$$\phi_{(x=L)} = Asin\left(\alpha\frac{L}{L}\right) + Bcos\left(\alpha\frac{L}{L}\right) + Csinh\left(\alpha\frac{L}{L}\right) + Dcosh\left(\alpha\frac{L}{L}\right) \quad (23)$$

$\phi_{(x=L)} = Asin(\alpha) + Bcos(\alpha) + Csinh(\alpha) + Dcosh(\alpha) = 0$ (24)

$$\phi''_{(x=0)} = -A\frac{\alpha^2}{L^2}sin\left(\alpha\frac{0}{L}\right) - B\frac{\alpha^2}{L^2}cos\left(\alpha\frac{0}{L}\right) + C\frac{\alpha^2}{L^2}sinh\left(\alpha\frac{0}{L}\right) + D\frac{\alpha^2}{L^2}cosh\left(\alpha\frac{0}{L}\right) \quad (25)$$

$\phi''_{(x=0)} = 0A - B\frac{\alpha^2}{L^2} + 0C + D\frac{\alpha^2}{L^2} = 0$ (26)

$$\phi''_{(x=L)} = -A\frac{\alpha^2}{L^2}sin\left(\alpha\frac{L}{L}\right) - B\frac{\alpha^2}{L^2}cos\left(\alpha\frac{L}{L}\right) + C\frac{\alpha^2}{L^2}sinh\left(\alpha\frac{L}{L}\right) + D\frac{\alpha^2}{L^2}cosh\left(\alpha\frac{L}{L}\right) \quad (27)$$

$\phi''_{(x=L)} = -A\frac{\alpha^2}{L^2}sin(\alpha) - B\frac{\alpha^2}{L^2}cos(\alpha) + C\frac{\alpha^2}{L^2}sinh(\alpha) + D\frac{\alpha^2}{L^2}cosh(\alpha) = 0$ (28)

We can put these equations on the form of a matrix:

$$\begin{bmatrix} 0 & B & 0 & D \\ Asin\alpha & Bcos\alpha & Csinh\alpha & Dcosh\alpha \\ 0 & -B\frac{\alpha^2}{L^2} & 0 & D\frac{\alpha^2}{L^2} \\ -A\frac{\alpha^2 sin\alpha}{L^2} & -B\frac{\alpha^2 cos\alpha}{L^2} & C\frac{\alpha^2 sinh\alpha}{L^2} & D\frac{\alpha^2 cosh\alpha}{L^2} \end{bmatrix} \quad (29)$$

The determinant of this matrix have to be null.

We notice that from the equations (1) and (3) that B = D = 0

$$\phi_{(x)} = Asin\left(\alpha\frac{x}{L}\right) + Bcos\left(\alpha\frac{x}{L}\right) + Csinh\left(\alpha\frac{x}{L}\right) + Dcosh\left(\alpha\frac{x}{L}\right) \quad (30)$$

$$\phi_{(x)} = Asin\left(\alpha\frac{x}{L}\right) + Csinh\left(\alpha\frac{x}{L}\right) \quad (31)$$

And we have: $-\frac{\rho S}{EI} \times \frac{\ddot{q}}{q} = \frac{\alpha^4}{L^4}$ (32)

The two equations (2) and (4) gives:

$$\begin{cases} \phi_{(x=L)} = Asin(\alpha) + Csinh(\alpha) = 0 \quad (33) \\ \phi''_{(x=L)} = -A\frac{\alpha^2}{L^2}sin(\alpha) + C\frac{\alpha^2}{L^2}sinh(\alpha) = 0 \quad (34) \end{cases}$$

So:

$$\begin{cases} \phi_{(x=L)} = Asin(\alpha) + Csinh(\alpha) = 0 \quad (35) \\ \phi''_{(x=L)} = -Asin(\alpha) + Csinh(\alpha) = 0 \quad (36) \end{cases}$$

The sum of these two equations gives:

$$2Csinh(\alpha) = 0 => \ \alpha = 0 \ ou \ C = 0 \ (37)$$

The subtraction of these two equations gives:

$2Asin(\alpha) = 0 => \ \alpha = n\pi$ And so C = 0 (38)

Is the Eigen value of the system.

We obtain the following equation of the Eigen mode:

$$\phi_{(x)} = Asin\left(\frac{n\pi x}{L}\right) \quad (39)$$

That we write in function of n (quantification):

The Eigen mode is so written:

$$\phi_{n(x)} = A_n sin\left(\frac{n\pi x}{L}\right) \quad (40)$$

The norm of the Eigen mode is so written:

$$\int_0^L \phi_n(x)\,\phi_n(x) = \int_0^L A_n^2 sin^{2\left(\frac{n\pi x}{L}\right)} dx \ \ (41)$$

We replace du sin²(ax) by $\frac{1}{2}(1 - \cos(2ax))$ (42)

$$\int_0^L \phi_n(x)\,\phi_n(x) = A_n^2 \int_0^L \frac{1}{2}\left(1 - cos\left(\frac{2n\pi x}{L}\right)\right) dx \ \ (43)$$

We define $u = \frac{2n\pi x}{L}$

$$du = \frac{2n\pi}{L} dx$$

$$dx = \frac{L}{2n\pi} du$$

$$\int_0^L \phi_n(x)\,\phi_n(x) = A_n^2 \int_0^L \frac{1}{2} dx + A_n^2 \frac{1}{2} \int_0^L \frac{L}{2n\pi} cosu \, du \quad (44)$$

The antiderivative of cosu is sinu :

$$\int_0^L \phi_n(x)\,\phi_n(x) = A_n^2 \left[\left(\frac{x}{2} - \frac{L}{4n\pi} sin\left(\frac{2n\pi x}{L}\right)\right)\right]_0^L = A_n^2 \left(\frac{L}{2} - 0 - (0 - 0)\right) = \frac{A_n^2 L}{2} \quad (45)$$

We note that if we chose$A_n = \sqrt{\frac{2}{L}}$ (46)) the modes $\phi_n(x)$ would form an orthonormal basis for the canonical scalar product.
In the case of normed Eigen vector we have so:

$$\boxed{\phi_{n(x)} = \sqrt{\frac{2}{L}} sin\left(\frac{n\pi x}{L}\right) \quad (47)}$$

We have to study the second equation now :

$$\ddot{q}_{(t)} + \omega^2 q_{(t)} = 0$$

We fix $q_{(t)}$as following:

$$q_{(t)} = acos\omega t + bsin\omega t$$

$$\dot{q}_{(t)} = -a\omega sin\omega t + b\omega cos\omega t$$

$$\ddot{q}_{(t)} = -a\omega^2 cos\omega t - b\omega^2 sin\omega t$$

At the time t = 0, the system is in static, and so the speed is null:

$$\dot{q}_{(t=0)} = -a\omega sin0 + b\omega cos0$$

That imply b = 0:

And so the final deflection of the beam is:

$$y_{(x,t)} = q_{(t)}\phi_{(x)} \quad (9)$$

$$\boxed{y_{n(x,t)} = a\sqrt{\frac{2}{L}} sin\left(\frac{n\pi x}{L}\right)\cos(\omega t) \quad (9bis)}$$

Note

We find this result by writing that the determinant of the matrix of the components must be zero.

Determination of circular natural frequency:

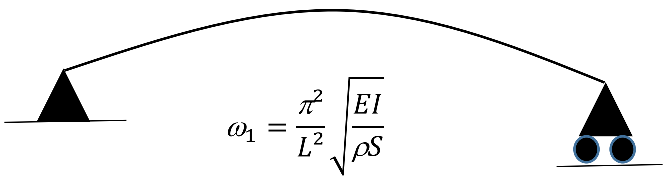
$$\omega^2 = \frac{\alpha^4}{L^4} \times \frac{EI}{\rho S} \quad (48)$$

With α = nπ   (49)

$$\omega^2 = \frac{n^4\pi^4}{L^4} \times \frac{EI}{\rho S} \quad (50) \qquad => \omega = \frac{n^2\pi^2}{L^2}\sqrt{\frac{EI}{\rho S}} \quad (51)$$

With these equations, we can define the different Eigen mode of vibration of the beam (cf. figure 2).

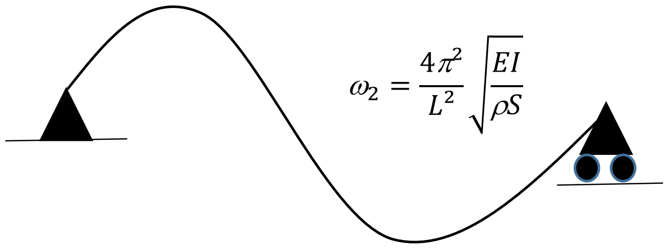
1st mode :



$$\omega_1 = \frac{\pi^2}{L^2}\sqrt{\frac{EI}{\rho S}}$$

$$\phi_{1(x)} = \sqrt{\frac{2}{L}} sin\left(\frac{\pi x}{L}\right)$$

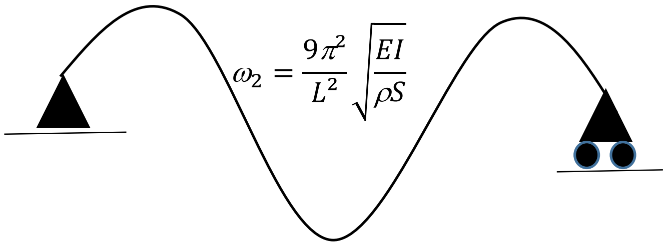
2nd mode :



$$\omega_2 = \frac{4\pi^2}{L^2}\sqrt{\frac{EI}{\rho S}}$$

$$\phi_{2(x)} = \sqrt{\frac{2}{L}}\,sin\left(\frac{2\pi x}{L}\right)$$

3rd mode :



$$\omega_2 = \frac{9\pi^2}{L^2}\sqrt{\frac{EI}{\rho S}}$$

$$\phi_{3(x)} = \sqrt{\frac{2}{L}}\,sin\left(\frac{3\pi x}{L}\right)$$

Figure 2 – Eigen mode and circular natural frequencies of a beam on two supports –

1.2) Particle in a potential well with an infinite dimension

We consider a particle in a potential well as defined below (cf. figure 3).

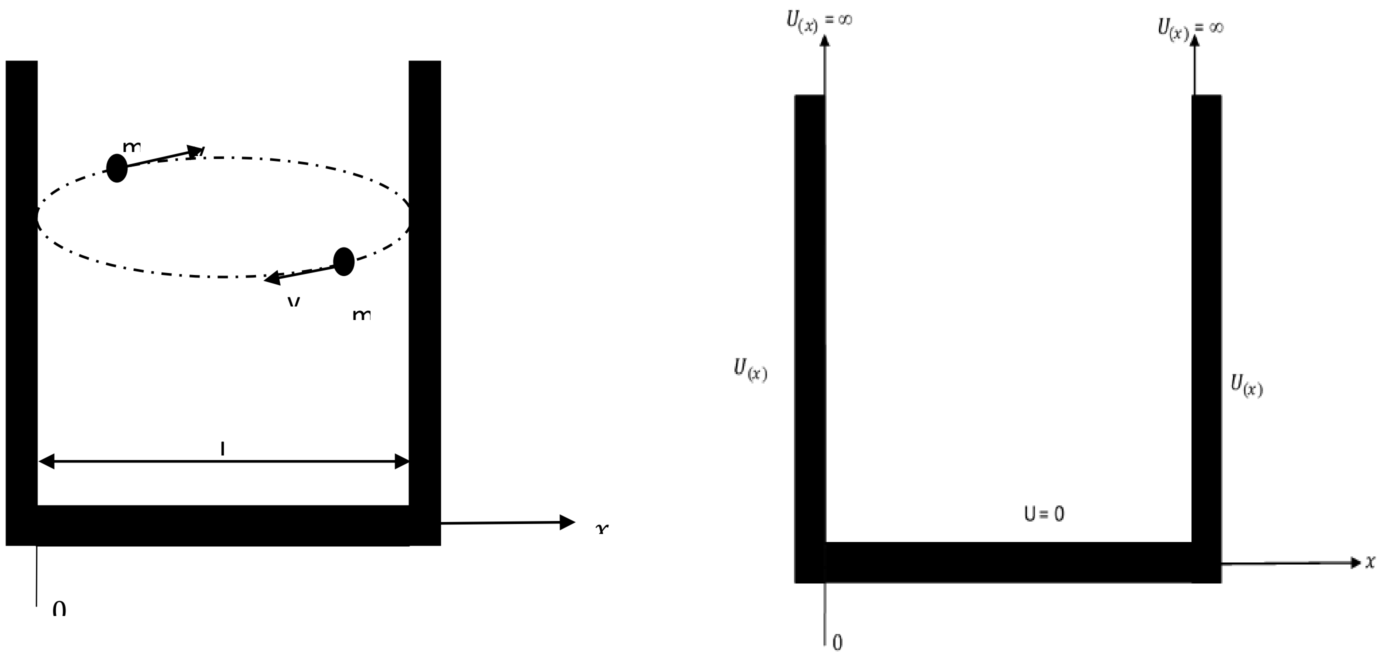


Figure 3 – Particle in movement in a potential well –

The Schrodinger equation independent of the time can be written as following:

$$\left(\frac{\hbar^2}{2m}\right)\frac{\partial^2 \psi_{(x)}}{\partial x^2} + E_m\,\psi_{(x)} = 0 \qquad (52)$$

This equation can be also written on the form curvature = k energy

$$\frac{d^2\Psi_{(x)}}{dx^2} = -\left(\frac{2m}{\hbar^2}\Psi_{(x)}\right)E_m \quad (52\ bis)$$

The dimensional equation is so the following:

$$\frac{1}{m^2} = \frac{kg}{\left(\frac{kgm^2s}{s^2}\right)^2} \times \frac{kgm^2}{s^2}$$

$$\frac{1}{m^2} = \frac{s^2}{kgm^4} \times \frac{kgm^2}{s^2}$$

$$\frac{1}{m^2} = \frac{s^2}{kgm} \times \frac{U}{m^3}$$

The boundary conditions are therefore the following:

The particle is present in the potential well:
$0 \leq x \leq L$

$$\left(\frac{\hbar^2}{2m}\right)\frac{\partial^2\psi_{(x)}}{\partial x^2} + E_m\psi_{(x)} = 0$$

The particle follows the Schrodinger equation

The particle is not present in the potential well:
$x < 0$ ou $x > L$

$$\psi_{(x)} = 0$$

there is no particle

Otherwise, $\psi_{(x)}$ is a continuous function. The consequence is so: $\psi_{(0)} = \psi_{(L)} = 0$

So:

$$\frac{d^2\psi_{(x)}}{dx^2} + k^2\psi_{(x)} = 0 \quad (53)$$

With:

$$k^2 = \frac{2mE_m}{\hbar^2} \quad (54)$$

The solution of the wave function for this differential equation is of the following shape:

$$\psi_{(x)} = Asin(kx) + Bcos(kx) \quad (55)$$

We suppose that the energy E_m is positive.

By use of the boundary condition we can find the constant A and B:

In $x = 0$, $\psi_{(0)} = 0$:

That this imply that $B = 0$

$$\psi_{(x)} = Asin(kx)$$

In $x = L$, $\psi_{(L)} = 0$:

$$\psi_{(L)} = Asin(kL) = 0$$

It is so necessary that with A and k different from 0 that $kL = n\pi$

That imply that:

$$k = \frac{n\pi}{L} \quad (56)$$

With n an integer =1,2,3……

As we have posed:

$$k^2 = \frac{2mE_m}{ħ^2}$$

The result is :

$$k^2 = \frac{2mE_m}{ħ^2} = \frac{n^2\pi^2}{L^2} \quad (57)$$

We finally obtain the quantified values of the energy:

$$\boxed{E_{m,n} = \frac{n^2\pi^2ħ^2}{2mL^2} \quad (58)}$$

With n = 1,2,3,…

So, the Eigen values of the energy are quantified.

We can finally write the expression of the wave function:

$$\psi_{(x)} = Asin\left(\frac{n\pi x}{L}x\right)$$

We are looking for now the value of the constant A:

As the particle has to be somewhere in the quantum well

The function $\left\|\psi_{(x)}\right\|^2 = 1$ to satisfy the density of probability (the particle must be somewhere in the box between 0 and L).

$$\int_0^L \left\|\psi_{(x)}\right\|^2 dx = 1 \quad (59)$$

$$\int_0^L A^2 sin^2\left(\frac{n\pi x}{L}\right)dx \quad = \frac{A^2}{2}\int_0^L \left\{1 - cos\left(\frac{2n\pi x}{L}\right)\right\}dx \quad = \frac{A^2}{2}\left[x - \frac{L}{2n\pi}sin\left(\frac{2n\pi x}{L}\right)\right]_0^L$$

$$= \frac{A^2}{2}\left(L - \frac{L}{2n\pi}sin\left(\frac{2\pi nL}{L}\right)\right) = \frac{A^2L}{2} = 1$$

$$\text{So } A = \sqrt{\frac{2}{L}} \quad (60)$$

And the solution of the problem in the quantum well, the wave function, is so:

$$\boxed{\psi_{(x)} = \sqrt{\frac{2}{L}}sin\left(\frac{n\pi x}{L}x\right) \quad (61)}$$

Wave function probability to be present

$$\psi_{(x)} = \sqrt{\frac{2}{L}}sin\left(\frac{n\pi x}{L}x\right) \qquad \left\|\psi_{(x)}\right\|^2 = \frac{2}{L}sin^2\left(\frac{n\pi x}{L}x\right) \quad (62)$$

Not about the Energy:

By the following classic notation:$E = hv = \frac{h}{T}$

And with $T = \frac{2\pi}{\omega}$

$$E = \frac{\omega h}{2\pi} = ħ\omega$$

$$ħ\omega = \frac{n^2\pi^2}{L^2}\left(\frac{ħ^2}{2m}\right)$$

$$\boxed{\omega = \frac{n^2\pi^2}{L^2}\left(\frac{ħ}{2m}\right) \quad (63)}$$

1.3 Conclusion of this first part

Table 1 below shows the perfect analogy between the vibrations of a beam on two supports (resulting from the strength of the materials) and the quantification of the energy of a particle in a quantum well (derived from quantum mechanics).

<table>
<tr><td rowspan="2">Case studied</td><td colspan="2">Eigen value</td><td colspan="2">Eigen Modes</td></tr>
<tr><td>Natural circular Frequencies</td><td>Quantified energy</td><td>Eigen mode of oscillation of a beam</td><td>Expression of the Wave function</td></tr>
<tr><td>Beam on two supports</td><td>$\omega = \frac{n^2\pi^2}{L^2}\left(\sqrt{\frac{EI}{\rho S}}\right)$
$\frac{\phi^{IV}}{\phi} = -\frac{\rho S}{EI} \times \frac{\ddot{q}}{q} = \frac{\alpha^4}{L^4}$
$\phi^{IV}_{(x)} - \frac{\alpha^4}{L^4}\phi_{(x)} = 0$
$y_{(x,t)} = q_{(t)}\phi_{(x)}$</td><td>Without interests</td><td>$\phi_{n(x)} = \sqrt{\frac{2}{L}}sin\left(\frac{n\pi x}{L}\right)$
$\frac{\phi^{IV}}{\phi} = -\frac{\rho S}{EI} \times \frac{\ddot{q}}{q} = \frac{\alpha^4}{L^4}$
$\frac{d^2q_{(t)}}{dt^2} + \left(\frac{\alpha^4 EI}{L^4\rho S}\right)q_{(t)} = 0$
$y_{(x,t)} = q_{(t)}\phi_{(x)}$</td><td>Without interests</td></tr>
<tr><td>Particle in the potential well</td><td>With the notation :$E = h\nu = \frac{h}{T}$
$\omega = \frac{n^2\pi^2}{L^2}\left(\frac{\hbar}{2m}\right)$
$\frac{d^2\Psi_{(x)}}{dx^2} + \left(\frac{2mE_m}{\hbar^2}\right)\Psi_{(x)} = 0$</td><td>$E = \frac{n^2\pi^2}{L^2}\left(\frac{\hbar^2}{2m}\right)$</td><td>Without interests</td><td>$\Psi_{n(x)} = \sqrt{\frac{2}{L}}sin\left(\frac{n\pi x}{L}\right)$
$\frac{d^2\Psi_{(x)}}{dx^2} + \left(\frac{2mE_m}{\hbar^2}\right)\Psi_{(x)} = 0$</td></tr>
</table>

Tableau 1 – Analogy between the natural circular frequencies of vibration of a beam on 2 supports and the quantified energy of a particle in a quantum well

The circular natural frequencies ω of a beam in strength of materials have an analogy with the jump of quantified energy E_m of a particle in a quantum well. $\left(\sqrt{\frac{EI}{\rho S}}\right)_{RDM} \Rightarrow \left(\frac{\hbar}{2m}\right)_{quantique}$

The Eigen modes s ϕ of vibration of a beam in strength of materials have an analogy with the Eigen mode of the wave function in the quantum well.

2nd part) Relation between curvature and energy in 1 and 2 dimensions (beam and slab in strength of materials) and in 4 dimensions (General relativity)

2.1) Case of the beam on two simple supports

Considering the same beam as that defined in figure 1.

The fundamental relation connecting the curvature (1/R) at the bending moment $M_{(x)}$and at the second derivative of the deflection $y_{(x)}$ writes:

$$\frac{d^2y}{dx^2} = -\frac{M_{(x)}}{EI} = \frac{1}{R} \qquad (64)$$

In this expression, $y_{(x)}$ is the deflection of the beam (m), $M_{(x)}$ the bending moment (N.m), E the young modulus of the material that constitute the beam (N/m²), I the inertia in(m⁴) and R the curvature radius in (m).

The exact expression of the curvature is given in the expression (65). The part in cube root is negligible.

$$\frac{1}{R} = \frac{\frac{d^2y}{dx^2}}{\left(\sqrt{1+\left(\frac{dy}{dx}\right)^2}\right)^3} \quad (65)$$

In addition the elastic bending energy of a beam can be written as:

$$U = \frac{1}{2}\int_0^L \frac{M_{(x)}^2}{EI}dx \quad (66)$$

Considering to simplify a beam under a constant bending moment M at each extremity (cf. figure 4).

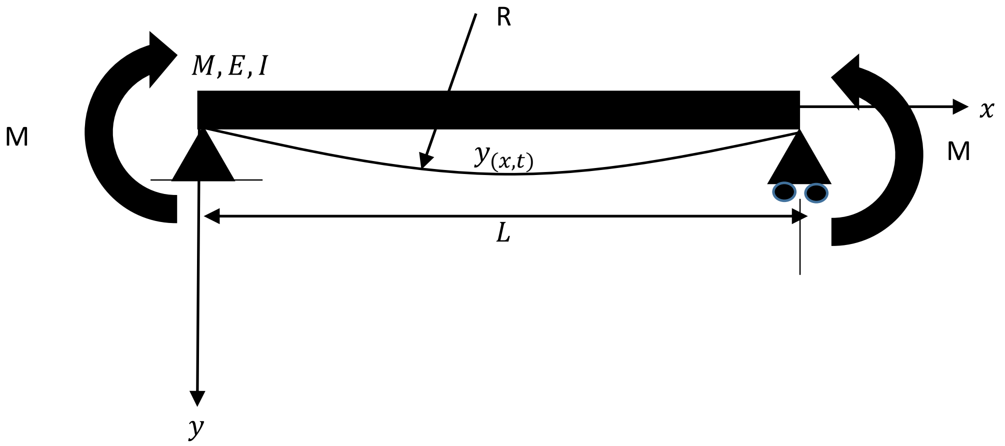


Figure 4 – Beam on two supports under a bending moment M at each extremity –

The bending moment equation (pure bending) is the following:

$$M_{(x)} = M \quad (67)$$

From the equation (64) w obtain:

$$\frac{d^2y}{dx^2} = -\frac{M}{EI} = \frac{1}{R} \quad (68)$$

$$M_{(x)} = M = -\frac{EI}{R} \quad (69)$$

By introducing the expression (69) in the expression of bending energy (66):

$$U = \frac{1}{2}\int_0^L \frac{(EI)^2}{R^2 EI}dx = \frac{1}{2}\int_0^L \frac{EI}{R^2}dx$$

With the curvature that is constant, we obtain:

$$U = \frac{1}{2}\frac{EIL}{R^2}$$

So in pure bending:

$$\boxed{\frac{1}{R^2} = \frac{2}{EI}\left(\frac{U}{L}\right) \quad (70)}$$

We obtain thus a relation between the curvature and the strain energy of a beam

Note:
By integral two times from x the expression (68) and considering that the deflection have to be null on each support, we obtain the expression of the deflection.

$$y_{(x)} = -\frac{M}{2EI}x^2 + \frac{ML}{2EI}x \quad (71)$$

And we find the well-known result of the strength of materials:

$$y_{(L/2)} = -\frac{ML^2}{8EI} + \frac{ML^2}{4EI} = \frac{ML^2}{8EI} \quad (72)$$

$$\frac{d^2y_{(x)}}{dx^2} - \frac{M}{EI} = \frac{1}{R}$$

2.2) Case of a thin plate of thickness h

Considering a thin plane h, of sides Δx et Δy sunder a bending moments M_x (cf. figure 5) :

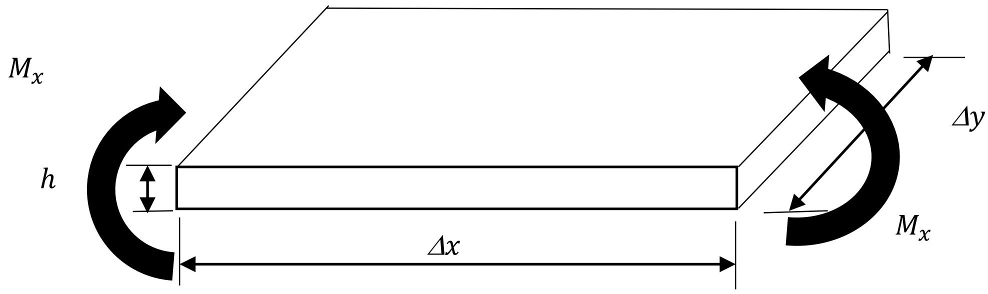


Figure 5 – Plate under a bending moment on its sides –

M_x Represent a bending moment by unit of length:

Under the bending moment, the plate is in bending and takes a curvature of radius R as indicated at the figure 6 below:

:

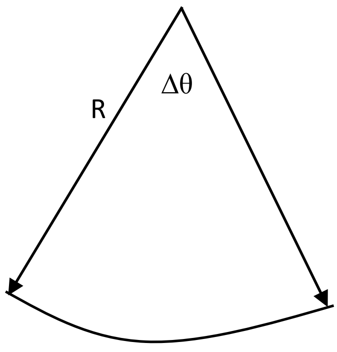


Figure 6 – Relation between the radius of curvature R of the plate and each mean fiber –

We have following the figure 6: $R_x \Delta\theta = ds \approx \Delta x \quad (73)$

And we have a relation connecting the curvature 1/R and the defection w of the plate:

$$\frac{1}{R_x} = \frac{\partial^2 w_{(x,y)}}{\partial x^2} \quad (74)$$

By introducing the expression (74) in the expression (73) we obtain:

$$\Delta\theta = \frac{\Delta x}{R_x} = -\frac{\partial^2 w_{(x,y)}}{\partial x^2}\Delta x \quad (75)$$

The elastic bending energy of the plate can be written so:

$$\Delta U = \frac{1}{2}M_x \Delta y \Delta\theta \quad (76)$$

$$\Delta U_x = -\frac{1}{2}M_x \frac{\partial^2 w_{(x,y)}}{\partial x^2}\Delta x\Delta y \quad (77)$$

The contribution of energy following y can be written following the same approach if a bending M_y is now applied on the side Δx :

$$\Delta U_y = -\frac{1}{2}M_y \frac{\partial^2 w_{(x,y)}}{\partial y^2}\Delta x\Delta y \quad (78)$$

The contribution due at the torsion can be also written:

$$\Delta U_{xy} = \frac{1}{2}2M_{xy} \frac{\partial^2 w_{(x,y)}}{\partial y \partial x}\Delta x\Delta y \quad (79)$$

Finally the strain energy of the plate can be defined:

$$\Delta U = \Delta U_x + \Delta U_y + \Delta U_{xy} \quad (80)$$

$$\Delta U = -\frac{1}{2}M_x \frac{\partial^2 w_{(x,y)}}{\partial x^2}\Delta x\Delta y - \frac{1}{2}M_y \frac{\partial^2 w_{(x,y)}}{\partial y^2}\Delta x\Delta y + \frac{1}{2}2M_{xy} \frac{\partial^2 w_{(x,y)}}{\partial y\partial x}\Delta x\Delta y \quad (81)$$

So :

$$\Delta U = \frac{1}{2}\left[-M_x \frac{\partial^2 w_{(x,y)}}{\partial x^2} + 2M_{xy}\frac{\partial^2 w_{(x,y)}}{\partial y\partial x} - M_y \frac{\partial^2 w_{(x,y)}}{\partial y^2}\right]\Delta x\Delta y \quad (82)$$

By using the well-known relation connecting the bending moment at the second derivative of the deflection:

$$M_x = -D\left(\frac{\partial^2 w_{(x,y)}}{\partial x^2} + v\frac{\partial^2 w_{(x,y)}}{\partial y^2}\right) \quad (83)\ , M_y = -D\left(\frac{\partial^2 w_{(x,y)}}{\partial y^2} + v\frac{\partial^2 w_{(x,y)}}{\partial x^2}\right) \quad (84)$$

$$M_x = D\left(\frac{1}{R_x} + v\frac{1}{R_y}\right) \quad (85), M_y = D\left(\frac{1}{R_y} + v\frac{1}{R_x}\right) \quad (86)$$

$$M_{xy} = D(1-v)\frac{\partial^2 w_{(x,y)}}{\partial x\partial y} \quad (87)$$

And by reporting these expressions into (82) we obtain:

$$\Delta U = \frac{1}{2}\left[D\left(\frac{\partial^2 w_{(x,y)}}{\partial x^2} + v\frac{\partial^2 w_{(x,y)}}{\partial y^2}\right)\frac{\partial^2 w_{(x,y)}}{\partial x^2} + 2D(1-v)\frac{\partial^2 w_{(x,y)}}{\partial x\partial y}\frac{\partial^2 w_{(x,y)}}{\partial y\partial x} + D\left(\frac{\partial^2 w_{(x,y)}}{\partial y^2} + v\frac{\partial^2 w_{(x,y)}}{\partial x^2}\right)\frac{\partial^2 w_{(x,y)}}{\partial y^2}\right]\Delta x\Delta y \quad (88)$$

By developing the expression (88):

$$\Delta U = \frac{1}{2}\left[D\left(\frac{\partial^2 w_{(x,y)}}{\partial x^2}\right)^2 \nu D\frac{\partial^2 w_{(x,y)}}{\partial x^2}\frac{\partial^2 w_{(x,y)}}{\partial y^2} + 2D\frac{\partial^2 w_{(x,y)}}{\partial x\partial y}\frac{\partial^2 w_{(x,y)}}{\partial y\partial x} - 2\nu D\frac{\partial^2 w_{(x,y)}}{\partial x\partial y}\frac{\partial^2 w_{(x,y)}}{\partial y\partial x} + D\left(\frac{\partial^2 w_{(x,y)}}{\partial y^2}\right)^2 + \nu D\frac{\partial^2 w_{(x,y)}}{\partial x^2}\frac{\partial^2 w_{(x,y)}}{\partial y^2}\right]\Delta x\Delta y$$

$$\Delta U = \frac{D}{2}\left[\left(\frac{\partial^2 w_{(x,y)}}{\partial x^2}\right)^2 + 2\nu\frac{\partial^2 w_{(x,y)}}{\partial x^2}\frac{\partial^2 w_{(x,y)}}{\partial y^2} + 2(1-\nu)\left(\frac{\partial^2 w_{(x,y)}}{\partial x\partial y}\right)^2 + \left(\frac{\partial^2 w_{(x,y)}}{\partial y^2}\right)^2\right]\Delta x\Delta y$$

$$\Delta U = \frac{D}{2}\left[\left(\frac{\partial^2 w_{(x,y)}}{\partial x^2}\right)^2 + 2\nu\frac{\partial^2 w_{(x,y)}}{\partial x^2}\frac{\partial^2 w_{(x,y)}}{\partial y^2} + 2(1-\nu)\left(\frac{\partial^2 w_{(x,y)}}{\partial x\partial y}\right)^2 + \left(\frac{\partial^2 w_{(x,y)}}{\partial y^2}\right)^2\right]\Delta x\Delta y \quad (89)$$

We obtain a relation between the curvature and the density of strain energy of the plate:

$$\boxed{\left[\left(\frac{1}{R_x}\right)^2 + \left(\frac{1}{R_y}\right)^2 + 2(1-\nu)\left\{\left(\frac{1}{R_{xy}}\right)^2\right\} + 2\nu\left\{\frac{1}{R_x}\frac{1}{R_y}\right\}\right] = \frac{24(1-\nu^2)}{Eh^3}\times\frac{\Delta U}{\Delta x\Delta y} \quad (90)}$$

Note:

The expression below allow to reformulate and to simplify the expression (89):

$$-2(1-\nu)\left[\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - \left(\frac{\partial^2 w}{\partial x\partial y}\right)^2\right] = -2\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} + 2\left(\frac{\partial^2 w}{\partial x\partial y}\right)^2 + 2\nu\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - 2\nu\left(\frac{\partial^2 w}{\partial x\partial y}\right)^2 \quad (91)$$

$$\left(\frac{\partial^2 w_{(x,y)}}{\partial x^2} + \frac{\partial^2 w_{(x,y)}}{\partial y^2}\right)^2 = \left(\frac{\partial^2 w_{(x,y)}}{\partial x^2}\right)^2 + \left(\frac{\partial^2 w_{(x,y)}}{\partial y^2}\right)^2 + 2\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} \quad (92)$$

The sum of the two expressions (91) et (92) above gives again the expression (89) :

$$\left(\frac{\partial^2 w_{(x,y)}}{\partial x^2}\right)^2 + \left(\frac{\partial^2 w_{(x,y)}}{\partial y^2}\right)^2 + \cancel{2\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}} - \cancel{2\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}} + 2\left(\frac{\partial^2 w}{\partial x\partial y}\right)^2 + 2\nu\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - 2\nu\left(\frac{\partial^2 w}{\partial x\partial y}\right)^2$$

Therefore the expression (89) can be written from the two expression (91) and (92):

$$\Delta U = \frac{D}{2}\left[\left(\frac{\partial^2 w_{(x,y)}}{\partial x^2} + \frac{\partial^2 w_{(x,y)}}{\partial y^2}\right)^2 - 2(1-\nu)\left[\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - \left(\frac{\partial^2 w}{\partial x\partial y}\right)^2\right]\right]\Delta x\Delta y \quad (93)$$

With for the bending rigidity D:

$$D = \frac{Eh^3}{12(1-\nu^2)} \quad (94)$$

$$\Delta U = \frac{D}{2}\left[\left(-\frac{1}{R_x} - \frac{1}{R_y}\right)^2 - 2(1-\nu)\left[\frac{1}{R_x}\frac{1}{R_y} - \left(\frac{1}{R_{xy}}\right)^2\right]\right]\Delta x\Delta y \quad (95)$$

If we present the expression to put in first plane the curvature in one side and the energy of the other side:

$$\boxed{\left[\left(-\frac{1}{R_x} - \frac{1}{R_y}\right)^2 - 2(1-\nu)\left[\frac{1}{R_x}\frac{1}{R_y} - \left(\frac{1}{R_{xy}}\right)^2\right]\right] = \left(\frac{2}{D}\right)\frac{\Delta U}{\Delta x\Delta y} \quad (96)}$$

We are developing now this expression in the case of the pure flexion:

The radius of curvature are all equal. The radius of curvature due to the torsion is null.

$$\frac{\partial^2 w}{\partial x \partial y} = 0, \frac{1}{R_x} = \frac{1}{R_y} = \frac{1}{R} \quad (97)$$

By introducing these expressions in the equation (96) we obtain:

$$\Delta U = \frac{D}{2}\left[\left(-\frac{2}{R}\right)^2 - 2(1-v)\left[\frac{1}{R^2}\right]\right]\Delta x \Delta y$$

$$\Delta U = \frac{D}{2}\left[\frac{4}{R^2} - \frac{2}{R^2} + 2v\frac{1}{R^2}\right]\Delta x \Delta y$$

$$\Delta U = \frac{D}{2}\left[\frac{2}{R^2} + 2v\frac{1}{R^2}\right]\Delta x \Delta y$$

$$\Delta U = \frac{D}{R^2}[1+v]\Delta x \Delta y$$

So:

$$\boxed{\frac{1}{R^2} = \frac{1}{D(1+v)}\frac{\Delta U}{\Delta x \Delta y} \quad (98)}$$

If we take into account of the bending rigidity (94) and if we define fix $\Delta x . \Delta y = A$

$$\frac{1}{R^2} = \frac{12(1-v^2)}{Eh^3(1+v)}\frac{U}{A}$$

$$\frac{1}{R^2} = \frac{12(1-v^2)}{Eh^2(1+v)} \times \frac{U}{Ah}$$

$$\frac{1}{R^2} = \frac{12(1+v)(1-v)}{Eh^2(1+v)} \times \frac{U}{Ah}$$

We obtain in the case of the pure bending:

Energy density

Curvature

$$\boxed{\frac{1}{R^2} = \frac{12(1-v)}{Eh^2} \times \frac{U}{Ah} \quad (99)}$$

Proportionality factor

And the dimensional equation is:

$$\frac{1}{m^2} = \frac{1}{\frac{kgm}{s^2 m^2} \times m^2} \times \frac{U}{m^3}$$

$$\frac{1}{m^2} = \frac{s^2}{kgm} \times \frac{U}{V}$$

2.3) Case of the General relativity

2.3.1) Expression Einstein Equation

The Einstein equation has to be written as following:

$$G_{\mu v} = \kappa T_{\mu v} \quad (100)$$

In this equation, the curvature and the energy of the space time are linked (cf. figure 7).

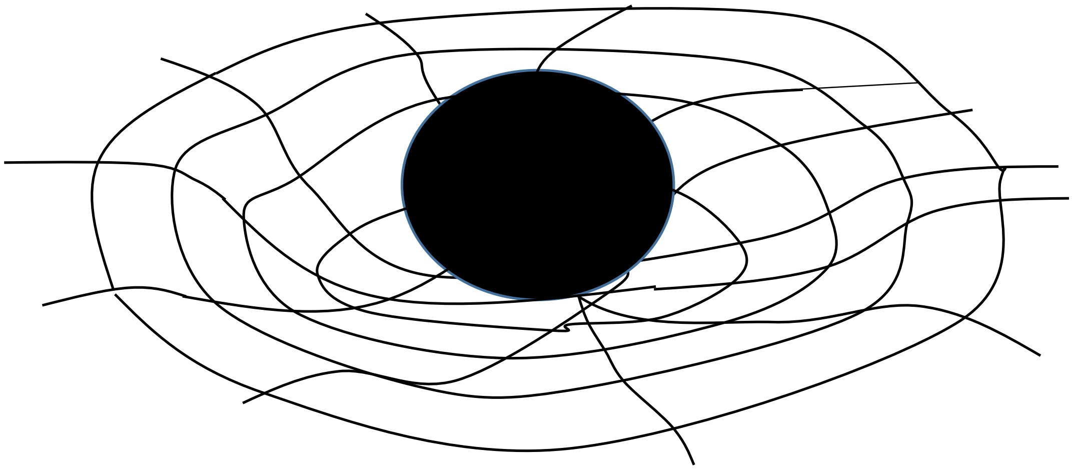


Figure 7 – Symbolic view of the curvature of the space time by projection in a plane space –

With:

$G_{\mu v}$ is the Einstein tensor.

$$\kappa = \frac{8\pi G}{c^4} \quad (101)$$

In addition the dimensional equation of κ is the following:

$$\kappa = \frac{L^3T^4}{MT^2L^4} = \frac{T^2}{ML} = \frac{s^2}{kgm} \quad (102)$$

The developed equation of Einstein becomes:

$$R_{\mu v} - \frac{1}{2} g_{\mu v} R + \Lambda g_{\mu v} = \frac{8\pi G}{c^4} T_{\mu v} \quad (103)$$

$$\frac{1}{m^2} = \frac{s^2}{kgm} \times \frac{U}{V}$$

2.3.2) Details on the curvature tensor

The Ricci tensor is obtained by contraction of the Riemann tensor on the indices λ:

$R_{\mu v} = R^{\lambda}_{\mu v \lambda}$

R is a contraction of the tensor $R_{\mu v}$.

The curvature tensor or Riemann tensor is written as following:

$R^{\lambda}{}_{\mu v \alpha} = \Gamma^{\lambda}_{\mu\alpha,v} - \Gamma^{\lambda}_{\mu v,\alpha} + \Gamma^{\lambda}_{v\eta}\Gamma^{\eta}_{\mu\alpha} - \Gamma^{\lambda}_{\alpha\eta}\Gamma^{\eta}_{\mu v} \quad (104)$

Or in another way:

$$R^{\lambda}{}_{\mu\nu\alpha} = \frac{\partial \Gamma^{\lambda}_{\mu\alpha}}{\partial x^{\nu}} - \frac{\partial \Gamma^{\lambda}_{\mu\nu}}{\partial x^{\alpha}} + \Gamma^{\lambda}_{\nu\eta}\Gamma^{\eta}_{\mu\alpha} - \Gamma^{\lambda}_{\alpha\eta}\Gamma^{\eta}_{\mu\nu} \quad (104\ bis)$$

By definition the definition of the Christoffel symbols is the following:

$$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2} g^{\lambda\rho}\left(g_{\mu\rho,\nu} + g_{\nu\rho,\mu} - g_{\mu\nu,\rho}\right) = \frac{1}{2} g^{\lambda\rho}\left(\frac{\partial g_{\mu\rho}}{\partial x^{\nu}} + \frac{\partial g_{\nu\rho}}{\partial x^{\mu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\rho}}\right) \quad (105)$$

$g_{\mu\nu}$ is the metric,

The coefficient of the metric are issued of the differential distance (special relativity):

$$ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2 \quad (106)$$

So, considering the Einstein convention of summation:

$$ds^2 = g_{\mu\nu}dx^{\mu}dx^{\nu} \quad (107)$$

$$g_{\mu\nu} = \eta_{\mu\nu}\frac{\partial \xi^{\alpha}}{\partial x^{\mu}}\frac{\partial \xi^{\beta}}{\partial x^{\nu}} \quad (108)$$

$$\xi^0 = ct; \xi^1 = x; \xi^2 = y; \xi^3 = z \quad (109)$$

$$\eta_{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \quad (110)$$

Λ is the cosmologic constant (possible candidate to explain the dark energy and dark matter)

The $\Gamma^{\lambda}_{\mu\nu}$ are the first derivative of the metric $g_{\mu\nu}$:

For example in coordinates (t,r,θ,φ) :

$$\Gamma^{\lambda}_{r\lambda,r} = \Gamma^{t}_{rt,r} + \Gamma^{r}_{rr,r} + \Gamma^{\theta}_{r\theta,r} + \Gamma^{\varphi}_{r\varphi,r} = \frac{\partial \Gamma^{t}_{rt}}{\partial r} + \frac{\partial \Gamma^{r}_{rr}}{\partial r} + \frac{\partial \Gamma^{\theta}_{r\theta}}{\partial r} + \frac{\partial \Gamma^{\varphi}_{r\varphi}}{\partial r} \quad (111)$$

So the $\Gamma^{\lambda}_{r\lambda,r}$ are the second derivatives of the metrics $g_{\mu\nu}$ that we can by analogy compare in one and two dimensions with the expressions of curvatures given in the expression (70) , (90), (96) and (99).

So $G_{\mu\nu}$ has the dimensional value of a curvature in 1/m²

For example the first terms $R^{\lambda}{}_{\mu\nu\alpha}$ (see 104 bis)

$$\frac{\partial \Gamma^{\lambda}_{\mu\alpha}}{\partial x^{\nu}} = \frac{\partial\left\{\frac{1}{2} g^{\lambda\rho}\left(\frac{\partial g_{\mu\rho}}{\partial x^{\nu}} + \frac{\partial g_{\nu\rho}}{\partial x^{\mu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\rho}}\right)\right\}}{\partial x^{\nu}}$$

2.3.3) Detail on the stress energy tensor

We have showing below that the component of the stress energy tensor $T_{\mu v}$ have the dimension of an energy density

G = 6.6726x10^{-11} m^{3}kg^{-1}s^{-2} 1J = 1W.s = 1 N.m = 1kg.m^2.s^{-2} kg

$$Curvature = \frac{8\pi G}{c^4} \times \frac{Energy}{Volume} = \frac{8\pi G}{c^2}\frac{Mass}{Volume} \quad (112)$$

1/L^2 (L/T)^4 (L^3)

The stress energy tensor is a matrix where the component are given below (113):

$$T_{\mu v} = \begin{bmatrix} \frac{m\gamma^2c^2}{V} & \rho\gamma^2cv_x & \rho\gamma^2cv_y & \rho\gamma^2cv_z \\ \rho\gamma^2cv_x & \rho\gamma^2v_xv_x & \rho\gamma^2v_xv_y & \rho\gamma^2v_xv_z \\ \rho\gamma^2cv_y & \rho\gamma^2v_yv_x & \rho\gamma^2v_yv_y & \rho\gamma^2v_yv_z \\ \rho\gamma^2cv_z & \rho\gamma^2v_zv_x & \rho\gamma^2v_zv_y & \rho\gamma^2v_zv_z \end{bmatrix} \quad (113)$$

Where γ is le factor of Lorentz (114) (cf. special relativity), the energy density $\rho = \frac{m}{V}$ with m the mass and V unitary volume, c the speed of light, v_i a speed in the direction i.

The factor of Lorentz becomes from the Lorentz transformation that imply that the speed of light stay constant in all the referential.

$$\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{1}{\sqrt{1-\beta^2}} \quad (114)$$

We show that at low velocity (in a non-relativistic situation) and in 3 dimensions, this tensor with 16 components in four-dimensional space-time actually includes the stress tensor (9 components) of the continuum mechanics from which derive the strength of materials [11].

The stress tensor in two dimensions can be written as following:

$$\sigma_{ij} = \begin{bmatrix} \sigma_{xx} & \tau_{xy} \\ \tau_{xy} & \sigma_{yy} \end{bmatrix} \quad (115)$$

And in 3 dimensions:

$$\sigma_{ij} = \begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix} \quad (116)$$

In theory of elasticity, that come from the continuum mechanics, the relation between the stress tensor σ_{ij} and the applied force Q_i on a surface of normal n_j can be written as following:

$$Q_i = \sigma_{ij}n_j \quad (117)$$

In a field of a variational approach, the stress tensor can be written as follow:

$$\sigma_{ij} = \frac{\Delta Q_i}{\Delta n_j}\ with\ \Delta_{nj} \to 0 \qquad (118)$$

So, with m the mass, ρ the density of mass energy, V the volume and a_i the acceleration, we have:

$$\sigma_{ij} = \frac{\Delta Q_i}{\Delta n_j} = \frac{\Delta(m \times a_i)}{\Delta n_j} = \frac{\Delta(\rho.V.a_i)}{\Delta n_j} \qquad (119)$$

If we make the hypothesis that the variation of the force is due only to the variation of the Volume V in function of the time t we obtain with:

$$a_i = \frac{v_i}{\Delta t} \quad (120)$$

$$\sigma_{ij} = \frac{\Delta(\rho.V.a_i)}{\Delta n_j} = \rho\frac{1}{\Delta n_j}\left(\frac{\Delta V}{\Delta t}\right)v_i \quad (121)$$

We can of course define the volume V:

$$V = \Delta x_i.\Delta x_j.\Delta x_k \quad (122)$$

Thus we obtain:

$$\sigma_{ij} = \rho\frac{1}{\Delta n_j}\left(\frac{\Delta x_i.\Delta x_j.\Delta x_k}{\Delta t}\right)v_i \quad (123)$$

We can replace the n_j by its value:

$$n_j = \Delta x_i.\Delta x_k \quad (124)$$

$$\sigma_{ij} = \rho\frac{v_i}{\Delta t}\left(\frac{\Delta x_i.\Delta x_j.\Delta x_k}{\Delta x_i.\Delta x_k}\right) \quad (125)$$

After simplification we obtain:

$$\sigma_{ij} = \rho v_i\left(\frac{\Delta x_j}{\Delta t}\right) \quad (126)$$

By definition of a speed, we have:

$$v_j = \left(\frac{\Delta x_j}{\Delta t}\right) \quad (127)$$

We obtain finally the expression of the stress tensor at low speed in function of the energy density ρ and based on the multiplication of the velocity v_i and v_j:

$$\sigma_{ij} = \rho v_i v_j \ (128)$$

The stress energy tensor becomes from the product of the density of energy and the multiplication of the four velocity (4 dimension of the space time) issued from the general relativity.

$$T_{\mu v} = \rho u_\mu u_v \ (129)$$

With for the four velocity:

$$u_\mu = \begin{vmatrix} \gamma c \\ \gamma v_x \\ \gamma v_y \\ \gamma v_z \end{vmatrix} \quad (130)$$

Under low speed γ=1 the stress energy tensor becomes:

$$T_{\mu v} = \begin{bmatrix} \frac{mc^2}{V} & \rho c v_x & \rho c v_y & \rho c v_z \\ \rho c v_x & \rho v_x v_x & \rho v_x v_y & \rho v_x v_z \\ \rho c v_y & \rho v_y v_x & \rho v_y v_y & \rho v_y v_z \\ \rho c v_z & \rho v_z v_x & \rho v_z v_y & \rho v_z v_z \end{bmatrix} \quad (131)$$

Based on the definition of the stress tensor, (cf. equation 128), the stress energy tensor at low speed can be written as following:

$$T_{\mu v} = \begin{bmatrix} \frac{mc^2}{V} & \rho c v_x & \rho c v_y & \rho c v_z \\ \rho c v_x & \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \rho c v_y & \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \rho c v_z & \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix} \quad (132)$$

The Einstein equation build a link in 4 dimensions (space time) with the curvature tensor $G_{\mu v}$(dimension 1/m²) and the stress energy tensor $T_{\mu v}$ (dimension energy/m³) that is itself a generalization in 4 dimensions of the stress tensor of the continuum mechanics.

2.4) Conclusion of this second part

The table 2 below makes a synthesis of the results obtained. We can see that the general relativity is a generalization of the elastic theory in 4 dimensions of the space time.

Theory considered	Number of dimensions	Formulation between energy and curvature	Example of application (pure bending) or expression at low speed
Beam on two simple supports in elasticity	**1**	$\frac{1}{R} = \frac{\frac{d^2y}{dx^2}}{\left(\sqrt{1+\left(\frac{dy}{dx}\right)^2}\right)^3}$ $U = \frac{1}{2}\int_0^L \frac{M_{(x)}^2}{EI} dx \qquad \frac{d^2y}{dx^2} = -\frac{M_{(x)}}{EI} = \frac{1}{R}$	$\frac{1}{R^2} = \frac{2}{EI}\left(\frac{U}{L}\right)$
Thin plate	**2**	$\left[\left(-\frac{1}{R_x} - \frac{1}{R_y}\right)^2 - 2(1-v)\left[\frac{1}{R_x}\frac{1}{R_y} - \left(\frac{1}{R_{xy}}\right)^2\right]\right] = \left(\frac{2}{D}\right)\frac{\Delta U}{\Delta x \Delta y}$	$\frac{1}{R^2} = \frac{12(1-v)}{Eh^3} \times \frac{U}{A}$
General relativity	**4**	$R_{\mu v} - \frac{1}{2} g_{\mu v} R + \Lambda g_{\mu v} = \frac{8\pi G}{c^4} T_{\mu v}$	$G_{\mu v} = \kappa T_{\mu v}$ $T_{\mu v} = \begin{bmatrix} \frac{mc^2}{V} & \rho c v_x & \rho c v_y & \rho c v_z \\ \rho c v_x & \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \rho c v_y & \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \rho c v_z & \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix}$

Tableau 2 – Comparison of the different relations between energy and curvature in function of the number of dimensions considered –

Conclusions

We have showed on several examples that the strength of materials allows to understand at our scale certain fundamental principles of the quantum mechanics:

- The eigen mode of a beam on two simple supports correspond at the different shape of the wave function connected with the jump energy of a particle in a quantum well,
- The natural circular frequencies of a beam on two simple supports correspond at the quantification of the energy of a particle in a potential well.

We have showed that the general relativity is a generalization in 4 dimensions (space time) of the relation curvature/energy equally present in strength of materials for the beam and for the thin plates

Finally, we have showed the stress energy tensor written at low speed give the stress tensor of the elasticity theory.

<u>Extension of this article, next steps:</u>

We have equally shown that in strength of material 2 main equations are necessary. One in static (curvature = K energy density) and one in dynamic (natural frequencies and eigen modes). So if the analogy between the beam (or plate) in strength of material and the space time is total; two equations should be also necessary at large scale. The first is the Einstein Equation connecting the curvature and the energy. The second remain to find.

If the analogy with the strength of the materials is exact we have some clues about this second equation.

This second equation would allow to quantify the space time (analogy with the natural frequency and eigen mode of the beam that represent also the energy quantification and shape of the wave function ψ in quantum mechanics).see table 3 in annex A.

Notice that this equation should be in 4th derivative of the space metric and 2th derivative of the time metric, and of course tensorial written to be valid in all the referential (covariant derivative).

Annex A - Synthesis of the results –

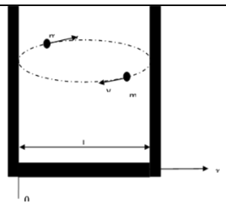
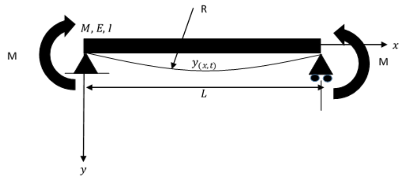
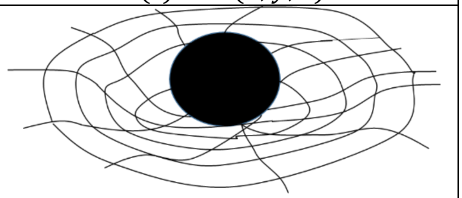
Principle	**Quantum mechanic** **Space (1-3) dimensions**	**Strength of material** **Space (1-3 dimensions) or time**	**General relativity** **Space-time (4 dimensions)**
What is distorted	Wave function $\psi_{(x)}$	Deflection of the beam $y_{(x,t)}$	Metric $g_{\mu\nu}$ with$\mu\nu =$ $0(t)\ to\ 3(x,y,z)$
Simplified /symbolic drawing of the case studied			
Curvature = K energy	Curvature of the wave function Example in 1 dimension $$\boldsymbol{\frac{d^2\Psi_{(x)}}{dx^2} = -\left(\frac{2m}{\hbar^2}\Psi_{(x)}\right)E_m}$$	Curvature and energy density Example in 1 dimension $$\frac{d^2y}{dx^2} = -\frac{M_{(x)}}{EI} = \frac{1}{R}$$ $$U = \frac{1}{2}\int_0^L \frac{M_{(x)}^2}{EI}dx$$ $$\boldsymbol{\int_0^L \frac{1}{R^2}dx = \frac{2}{EI}U}$$ If R constant => $\frac{1}{R^2} = \frac{2}{EI}\left(\frac{U}{L}\right)$	Curvature and energy density Example in 4 dimensions $$\boldsymbol{R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu}}$$
Eigen value (natural frequency) and eigen mode	Example of a particle in a well potential in 1 dimension $$\omega = \frac{n^2\pi^2}{L^2}\left(\frac{\hbar}{2m}\right)$$ $$E = \frac{n^2\pi^2}{L^2}\left(\frac{\hbar^2}{2m}\right)$$ $$\Psi_{n(x)} = \sqrt{\frac{2}{L}}sin\left(\frac{n\pi x}{L}\right)$$ $$\frac{d^2\Psi_{(x)}}{dx^2} + \left(\frac{2mE_m}{\hbar^2}\right)\Psi_{(x)} = 0$$	Example of a beam simply supported in 1 dimension (natural frequencies) $$\boldsymbol{\frac{\partial^4 y_{(x,t)}}{\partial x^4} + \left(\frac{m}{EI}\right)\frac{\partial^2 y_{(x,t)}}{\partial t^2} = 0}$$ $$y_{(x,t)} = q_{(t)}\phi_{(x)}$$ $$\boldsymbol{y_{n(x,t)} = a\sqrt{\frac{2}{L}}sin\left(\frac{n\pi x}{L}\right)cos(\omega t)}$$ $$\omega = \frac{n^2\pi^2}{L^2}\left(\sqrt{\frac{EI}{\rho S}}\right)$$ $$\frac{\phi^{IV}}{\phi} = -\frac{\rho S}{EI}\times\frac{\ddot{q}}{q} = \frac{\alpha^4}{L^4}$$ $$\phi_{(x)}^{IV} - \frac{\alpha^4}{L^4}\phi_{(x)} = 0$$ In space : $$\frac{d^4\phi_{(x)}}{dx^4} - \frac{\alpha^4}{L^4}\phi_{(x)} = 0$$ $$\phi_{n(x)} = \sqrt{\frac{2}{L}}sin\left(\frac{n\pi x}{L}\right)$$ In time : $$\frac{d^2q_{(t)}}{dt^2} + \left(\frac{\alpha^4 EI}{L^4\rho S}\right)q_{(t)} = 0$$	To be developed… In 4 dimensions In 4th derivative of the space metric and 2th derivative of the time metric, and of course tensorial written to be valid in all the referential (covariant derivative)?

Table 3 – Basic evidence of one equation is missing at the space time scale by analogy with the 2 main equations in strength of material –

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