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▶ To cite this version:

Térence Bayen, Alain Rapaport. About the minimal time crisis problem and applications. IFAC Word Congress 2017, Jul 2017, Toulouse, France. pp.512-517, 10.1016/j.ifacol.2017.08.112 . hal-01562523

HAL Id: hal-01562523
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Submitted on 15 Jul 2017

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Preprints of the 20th World Congress
The International Federation of Automatic Control
Toulouse, France, July 9-14, 2017

About the minimal time crisis problem and applications ⋆

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Abstract We study the optimal control problem where the cost functional to be minimized represents the so-called time of crisis, i.e. the time spent by a trajectory solution of a control system outside a given set K. Such a problematic finds applications in population dynamics, such as in prey-predator models, which require to find a control strategy that may leave and enter the crisis domain K a number of time that increases with the time interval. One important feature of the time crisis function is that it can be expressed using the characteristic function of K that is discontinuous preventing the use of the standard Maximum Principle. We provide an approximation scheme of the problem based on the Moreau-Yosida approximation of the indicator function of K and prove the convergence of an optimal sequence for the approximated problem to an optimal solution of the original problem when the regularization parameter goes to zero. We illustrate this approach on a simple example and then on the Lotka-Volterra prey-predator model.

Keywords: Optimal control, Hybrid Maximum Principle, Viability, Prey-predator model.

1. THE MINIMAL TIME CRISIS PROBLEM

We consider a controlled dynamics in $\mathsf{X} \subset \mathbb{R}^n$

$$\dot{x}(t) = f(x(t), u(t)) \quad \text{a.e. } t \in [0, T], \tag{1}$$

a set of admissible controls

$$\mathcal{U} := \{u : [0, T] \to U \ ; \ u \text{ meas.}\},$$

where U is a compact convex set in $\mathbb{R}^m$. Let $K \subset \mathsf{X}$ be a closed set with non empty interior. The system is said to be in a *crisis* when $x(t)$ does not belong to the set K. We shall also consider the usual hypotheses, that we recall later, which guarantee the following assumption to be fulfilled.

Assumption 1. Given $t_0 \in (-\infty, T]$, $x_0 \in \mathsf{X}$ and $u(\cdot) \in \mathcal{U}$, we denote by $x_u(\cdot)$ the unique absolutely continuous solution of (1) such that $x(t_0) = x_0$ and defined over $[t_0, T]$.

We recall the following definitions from the Viability theory [1].
a) Viability kernel:

$$Viab(K) := \{x_0 \in K \ ; \ \exists u \in \mathcal{U}, \ x_u(t) \in K, \ \forall t \geq 0\}$$

b) Finite horizon viability kernel:

$$Viab_{[0,T]}(K) := \{x_0 \in K \ ; \ \exists u \in \mathcal{U}, \ x_u(t) \in K, \ \forall t \in [0, T]\}.$$

When the constraints set K is not viable (i.e. $Viab(K) \neq K$), and $x_0 \notin Viab_{[0,T]}(K)$ the trajectory spends *some time* outside K. One may then consider the *minimal time crisis* problem:

$$(\mathcal{P}) : \ J^T(u) := \int_{t_0}^{T} \mathbb{1}_{K^c}(x_u(t)) \, \mathrm{d}t \quad \to \inf_{u \in \mathcal{U}} J^T(u) \tag{2}$$

$$\text{where } \mathbb{1}_{K^c}(x) := \begin{cases} 0 & x \in K, \\ 1 & x \notin K. \end{cases}$$

Remark that trajectories may enter and leave K several times (possibly periodically). Let us recall some previous works related to this problem :

1. In [4, 5], linear parabolic equations (related to steel continuous casting model) are considered with the criterion

$$\sup\{t \ ; x_u(t) \in K\} - \inf\{t \ ; \ x_u(t) \notin K\}$$

A regularization method is proposed but considering only one crossing time from K to K^c.

2. In [8], the minimal time crisis problem is considered over an infinite horizon, and the value function is characterized as the smallest positive lower semi-continuous viscosity super-solution of

$$\begin{aligned} &H(x, \nabla V(x)) = 0, \ x \in \mathsf{X}, \\ &V(x) = 0, \quad x \in \partial Viab(K). \end{aligned}$$

Then, an approximation scheme of the epigraph of the value function with the discrete viability kernel algorithm (considering an augmented dynamics) is used. No necessary conditions are given for this problem.

Our objectives in the present work are

(1) to consider finite horizon,
(2) to do not fix any a priori numbers of entry and exit times, and to provide necessary optimal conditions,
(3) to setup a numerical scheme that could be used to approximate optimal trajectories on concrete problems.

⋆ The authors thank the "FMJH Program Gaspard Monge in optimization and operation research, the ECOS-SUD project C13E03", and the Matham-SUD project SOCDE.

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The paper is organized as follows. In the second section, we recall the existence of an optimal control for the time crisis problem and we apply the hybrid maximum principle provided that an optimal trajectory satisfies a transverse condition at every crossing time of K. The third section is devoted to the study of a regularization scheme of the time crisis problem and to the convergence of an optimal solution of the regularized problem to a solution of the time crisis problem. In particular, no transverse condition is required when studying the approximated optimal control problem. Finally, we consider in the last section two examples where the time crisis is of particular interest. In the first one, we show that a so-called *myopic* strategy is optimal : it consists in minimizing the time in K^c and in maximizing the time in K. In this example, the viability kernel is empty. The second example is based on the Lotka-Volterra model and involves a viability kernel. We first compute this set and provide an optimal synthesis for the minimum time problem to reach the viability kernel. Finally, we show that there exist initial conditions for which the time crisis is strictly less than the minimum time to reach the viability kernel.

2. EXISTENCE RESULT AND HYBRID MAXIMUM PRINCIPLE

Let us first state the results about the existence of optimal solutions, under the following hypotheses.

Hypothesis 2. f, U and K fulfill the following properties:

(1) The set U is a non-empty compact set of $\mathbb{R}^m$.
(2) The dynamics f is continuous w.r.t. (x, u), locally Lipschitz w.r.t. x and satisfies the linear growth condition: $\exists\, c > 0$ s.t. $\forall\, (x, u) \in \mathbb{R}^n \times U$, one has:
$$\|f(x, u)\| \leq c(1 + \|x\|).$$
(3) For any $x \in \mathbb{R}^n$, the set $F(x) := \{f(x, u) \ ; \ u \in U\}$ is a non-empty convex set.
(4) The set K is a compact set in $\mathbb{R}^n$ with non-empty interior.

Proposition 3. For any $t_0 \in (-\infty, T]$ and $x_0 \in \mathbb{R}^n$, there exists an optimal control of problem $(\mathcal{P})$.

Proof. As a sketch of proof, one can consider the *extended* set-valued map G from $\mathbb{R}^{n+1}$ into the subsets of $\mathbb{R}^{n+1}$:
$$G(z) := \left| \begin{array}{ll} f(x, U) \times \{0\} & \text{if } x \in \mathrm{Int}(K), \\ f(x, U) \times [0, 1] & \text{if } x \in \partial K, \\ f(x, U) \times \{1\} & \text{if } x \in K^c, \end{array} \right.$$
where $z = (x, y)$, and use classical compactness arguments.

Define now the Hamiltonian associated to the optimal control problem:
$$H(x, p, p_0, u) = p \cdot f(x, u) - p_0 \mathbb{1}_{K^c}(x)$$
and notice that it is discontinuous w.r.t. to x preventing the use of the standard Maximum Principle. Instead, one may consider the Hybrid Maximum Principle (HMP). For this, we recall the definition of crossing times.

Definition 4. We say that a time $t_c \in [t_0, T]$ is a *regular crossing time* for a given trajectory $x(\cdot)$ if:

(1) The point $x(t_c)$ is s.t. $x(t_c) \in \partial K$, and there exists $\eta > 0$ such that for any $t \in [t_c - \eta, t_c)$, resp. $t \in (t_c, t_c + \eta]$, one has $x(t) \in K$, resp. $x(t) \in K^c$.
(2) The control u associated to the solution x is left- and right-continuous at t_c.
(3) The trajectory is transverse to K at $x(t_c)$, i.e. for any $h^\star \in N_K(x(t_c))$ such that there exists $h \in T_K(x(t_c)) \backslash R_K(x(t_c))$ with $h^\star \cdot h = 0$, then one has:
$$h^\star \cdot f(x(t_c), u(t_c)) \neq 0.$$

Hypothesis 5. An optimal trajectory $(x(\cdot), u(\cdot))$ has no $(m = 0)$ or a finite number $m \geq 1$ of regular crossing times $\{t_1, \cdots, t_m\}$ over $[t_0, T]$.

Proposition 6. Suppose that Hypotheses 2 and 5 are fulfilled and let $\mathcal{T}_c := \{t_1, ..., t_m\}$. Then, one has:

(1) $\exists\, p_0 \leq 0,\ p : [t_0, T] \to \mathbb{R}^n$ s.t. $(p_0, p(\cdot)) \neq 0$ and
$$\dot{p}(t) = -\partial_x H(x(t), p(t), p_0, u(t)) \quad a.e.\ t \notin \mathcal{T}_c.$$
(2) $u(t) \in \arg\max_{v \in U} H(x(t), p(t), p_0, v) \quad$ a.e. $t \in [t_0, T]$
(3) Transversality condition : $p(T) = 0$.
(4) The Hamiltonian is constant along the trajectory.
(5) At any crossing time t_c, we have :
$$p(t_c^+) - p(t_c^-) \in N_K(x(t_c)).$$

Moreover $\exists\, h \in N_K(x(t_c))$ with $\|h\| = 1$ s.t. :
$$p(t_c^+) = p(t_c^-) + \frac{p(t_c^-) \cdot (f(x(t_c), u(t_c^-)) - f(x(t_c), u(t_c^+))) + \sigma p_0}{h.f(x(t_c), u(t_c^+))} h$$
where $\sigma = -1$ (inner) or 1 (outer).

Proof. It is based on application of [9, 10, 7].

Recall that a statement of the HMP without the Hypothesis 5 is an open problem.

Another approach to encounter the difficulty due to the discontinuity of the Hamiltonian is to consider a regularization of the criterion.

3. REGULARIZATION OF THE PROBLEM AND CONVERGENCE RESULTS

In this Section we shall assume that K is a convex set (although extensions to prox-regular sets could be achieved and be the matter of a future work). For convex sets, it is natural to consider the Moreau-Yosida approximation (see [2] and references herein for more details on the Moreau envelope). The characteristic function of K^c can be written:
$$\mathbb{1}_{K^c}(x) = \gamma(\chi_K(x)),$$
where $\gamma(v) = 1 - e^{-v}$ and χ_K is the indicator of K :
$$\chi_K(x) := \left| \begin{array}{ll} 0 & \text{if } x \in K, \\ +\infty & \text{if } x \notin K. \end{array} \right.$$
Then we recall the following results:
1. When K is convex, the Moreau envelope of K is of class $C^{1,1}$:
$$x \longmapsto e_\varepsilon(x) := \frac{1}{2\varepsilon} d(x, K)^2$$
2. One has $\gamma(e_\varepsilon(x)) \to \mathbb{1}_{K^c}(x)$ when $\varepsilon \downarrow 0$ for any $x \in \mathbb{R}^n$.

Thus we consider the following regularized problem
$$(\mathcal{P}_\varepsilon): \quad \inf_{u \in \mathcal{U}} J_\varepsilon^T(u) \quad \text{with} \quad J_\varepsilon^T(u) := \int_{t_0}^T \gamma(e_\varepsilon(x_u(t)))\, \mathrm{d}t.$$

Proposition 7. Let $\varepsilon_n \downarrow 0$ and $x_n(\cdot)$ be a sequence of optimal trajectories for $(\mathcal{P}_{\varepsilon_n})$. Then up to a sub-sequence:

(1) $x_n(\cdot)$ uniformly converges to an absolutely continuous (a.c.) function $x^\star(\cdot)$ and $\dot{x}_n(\cdot)$ weakly converges in $L^2(0,T)$ to $\dot{x}^\star(\cdot)$.
(2) $x^\star(\cdot)$ is an optimal solution of $(\mathcal{P})$.

Proof. See [2].

To apply the Maximum Principle on the regularized problem, it is convenient to transform the problem into a Mayer problem:

$$\begin{cases} \dot{x}_\varepsilon &= f(x_\varepsilon, u) \\ \dot{y}_\varepsilon &= \gamma\left(\dfrac{1}{2\varepsilon}\|x_\varepsilon - v\|^2\right) \end{cases} \quad \rightarrow \quad \inf_{(u(\cdot),v(\cdot))} y_\varepsilon(T)$$

where $v : [0,T] \to K$ is measurable and $u(\cdot) \in \mathcal{U}$. Let H_ε be the Hamiltonian:

$$H_\varepsilon(x,p,u,v) := p \cdot f(x,u) - \gamma\left(\frac{1}{2\varepsilon}\|x - v\|^2\right),$$

Let $(x_\varepsilon, u_\varepsilon)$ be an optimal pair. Then, Pontryagin's Principle implies:

(1) There exists an a.c. function $p_\varepsilon : [0,T] \to \mathbb{R}^n$ s.t.:
$$\begin{cases} \dot{p}_\varepsilon(t) = -\partial_x H_\varepsilon(x_\varepsilon(t), p_\varepsilon(t), u_\varepsilon(t), v_\varepsilon(t)), \text{ a.e.} \\ p_\varepsilon(T) = 0. \end{cases}$$
(2) One has the maximization condition:
$$(u_\varepsilon(t), v_\varepsilon(t)) \in \arg \max_{(\alpha,\beta)\in U\times K} H_\varepsilon(x_\varepsilon(t), p_\varepsilon(t), \alpha, \beta),$$
for a.e. $t \in [0,T]$.
(3) H_ε is constant along any extremal trajectory.

Our main result is the following.

Theorem 8. Let $\varepsilon_n \downarrow 0$ and $(x_n(\cdot), p_n(\cdot), u_n(\cdot))$ be a sequence of extremals for $\mathcal{P}_{\varepsilon_n}$. Then, up to a sub-sequence:

(1) $x_n(\cdot)$ uniformly converges to an a.c. function $x^\star(\cdot)$ which is a solution of $(\mathcal{P})$.
(2) $\dot{x}_n(\cdot)$ weakly converges in $L^2(0,T)$ to $\dot{x}^\star(\cdot)$.
(3) When Hypothesis 5 is satisfied, let $\mathcal{T}_c := \{t_1, ..., t_m\}$. Then, one has:
   (a) $p_n(\cdot)$ is uniformly bounded on $[0,T]$,
   (b) $p_n(\cdot)$ uniformly converges to $p^\star(\cdot)$ on compacts of $[0,T] \setminus \mathcal{T}_C$, where $p^\star(\cdot)$ is a.c. on $[0,T] \setminus \mathcal{T}_C$ and $(x^\star(\cdot), p^\star(\cdot), u^\star(\cdot))$ satisfies the HMP.
   (c) $\dot{p}_n(\cdot)$ weakly converges to $\dot{p}^\star(\cdot)$ in $L^2(I)$ where I is any closed interval of $[0,T] \setminus \mathcal{T}_C$.

Proof. see [2].

4. AN ACADEMIC EXAMPLE

The aim of this section is to provide a simple example where the naive strategy which consists in minimizing the time spent in K^c and maximizing the time spent in K is optimal. We shall call this strategy *myopic* as the control acts independently in K and in K^c.

Consider the dynamics

$$\begin{cases} \dot{x}_1 &= -x_2(2+u), \\ \dot{x}_2 &= \quad x_1(2+u), \end{cases} \qquad u \in [-1,1],$$

and the set $K = \{x_2 \geq 0\}$. We prove in [2] the following optimality result.

Proposition 9. The *myopic* strategy defined by the autonomous feedback

$$u_m[x] := \left| \begin{array}{rl} 1 & x \notin K \\ -1 & x \in K \end{array} \right.$$

is optimal.

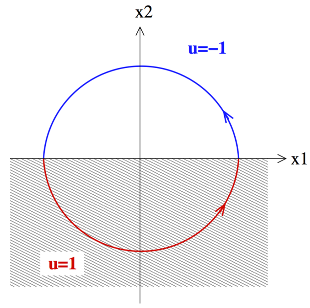

Figure 1. The myopic strategy.

The feedback u_m is non unique and it is depicted on Fig. 1. For the regularized problem defined by:

$$\inf_{u(\cdot)} J_\varepsilon^T(u) := \int_{t_0}^T \left(1 - e^{-\frac{\min(x_2(t),0)^2}{2\varepsilon}}\right) \mathrm{d}t,$$

we have the following result.

Proposition 10. Let $x_\varepsilon^\star(\cdot)$ be an optimal trajectory for the problem $\mathcal{P}_\varepsilon$.

(1) If $x_\varepsilon^\star(T) \in K$, then the myopic strategy is optimal.
(2) If $x_\varepsilon^\star(T) \notin K$, there exists $\theta_\varepsilon \in (0,\pi)$ s.t. the *delayed myopic* strategy is optimal.

Proof. see [2].

The *delayed myopic* strategy is depicted on Fig. 2.

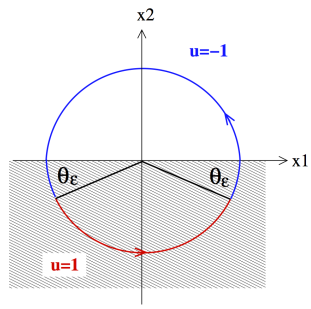

Figure 2. The delayed myopic strategy.

We also depict on Fig. 3 the convergence of the value function V_ε for the regularized problem to the value function V of the minimal time crisis problem (in line with Proposition 7). The exact expressions of V_ε and V can be found in [2].

5. APPLICATION TO LOTKA-VOLTERRA MODEL

We consider the classical prey-predator model

$$\begin{cases} \dot{x} &= rx - xy, \\ \dot{y} &= -my + xy - uy, \end{cases} \tag{3}$$

where x and y stands respectively for the prey and predator density. Here $r > 0$ represents the reproduction of species x and $m > 0$ the mortality of y. The control variable $u \in [0, \bar{u}]$ is a predator mortality (by removal or

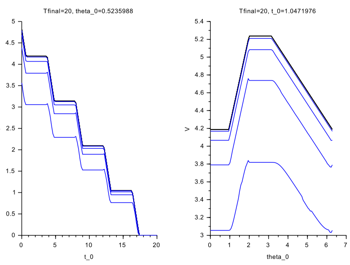

Figure 3. Convergence of the value function of the regularized problem (in blue) to the value function of the original problem (in black).

chemical actions). One consider that the prey are in danger when their density is under a threshold $\underline{x} > 0$. Therefore, we consider the set

$$K(\underline{x}) := \{(x, y) \in \mathcal{D} \,|\, x \geq \underline{x}\}.$$

Because of lack of space, all the proofs of the results given below are not provided but are available in [3].

5.1 *Study of the viability kernel*

Given a non-empty subset A of $\mathbb{R}^2$, we will denote by $\mathrm{Int}(A)$ its interior. For a fixed $u \in [0, \bar{u}]$, we define a function $V_u : \mathcal{D} \to \mathbb{R}$ by:

$$V_u(x, y) := x - (m + u) \ln x + y - r \ln y, \quad (x, y) \in \mathcal{D},$$

together with the number $\underline{c}(u) \in \mathbb{R}$ defined by

$$\underline{c}(u) := V_u(m+u, r) = (m+u)(1-\ln(m+u))+r(1-\ln r),$$

and the equilibrium point $E^\star(u)$ for (3)

$$E^\star(u) := (x^\star(u), y^\star) = (m + u, r).$$

For a given number $c \geq \underline{c}(u)$, we denote by $L_u(c)$, resp. by $S_u(c)$, the level set, resp. the sub-level set of V_u defined by

$$L_u(c) := \{(x, y) \in \mathcal{D},\ V_u(x, y) = c\},$$

resp.

$$S_u(c) := \{(x, y) \in \mathcal{D},\ V_u(x, y) \leq c\}.$$

The following lemma is standard when dealing with Lotka-Volterra type systems.

Lemma 11. For a constant control u, a trajectory of (3) belongs to a level set $L_u(c)$ with $c \geq \underline{c}(u)$. The sets $L_u(c)$ are closed curves that surround the steady state $E^\star(u)$.

For $u \in [0, \bar{u}]$, we define two functions $\phi_u : \mathbb{R}_+ \to \mathbb{R}$ and $\psi : \mathbb{R}_+ \to \mathbb{R}$ by

$$\phi_u(x) := x - (m + u) \ln x,\ x \in \mathbb{R}_+,$$
$$\psi(y) := y - r \ln y,\ y \in \mathbb{R}.$$

Lemma 12. Given $u \in [0, \bar{u}]$, on has the following properties

- For any $x > \phi_u(m + u)$, there exists an unique $x_u^+(x) \in (m+u, +\infty)$ and an unique $x_u^-(x) \in (0, m+u)$ such that $\phi_u(x_u^+(x)) = \phi_u(x^-(x)) = x$.
- If $p > \psi(r)$, the equation $\psi(y) = p$ has exactly two roots $y^-(p)$, $y^+(p)$ that satisfy $y^-(p) < r < y^+(p)$.

For $c \in \mathbb{R}$, we consider the subsets of $\mathcal{D}$, $L_u^+(c)$, $L_u^-(c)$, $S_u^+(c)$ and $S_u^-(c)$ defined by:

$$L_u^+(c) := L_u(c) \cap \{y \geq r\}, \quad L_u^-(c) := L_u(c) \cap \{y \leq r\},$$

and

$$S_u^+(c) := S_u(c) \cap \{y \geq r\}, \quad S_u^-(c) := S_u(c) \cap \{y \leq r\},$$

and let $r_- \in (0, r]$ be defined by:

$$r_- := y^-(V_0(x_{\bar{u}}^+(\underline{x}), r) - \phi_0(\underline{x})).$$

The next proposition provides a description of the viability kernel, $Viab(\underline{x})$, of $K(\underline{x})$ for (3).

Proposition 13. One has the following characterization of the viability kernel:

- If $m + \bar{u} < \underline{x}$, the set $Viab(\underline{x})$ is empty.
- If $\bar{u} \geq \underline{x} - m$, the viability kernel is non empty and we have:

$$Viab(\underline{x}) = S_{\bar{u}}^+(V_{\bar{u}}(\underline{x}, r)) \bigcup \left(S_0^-(V_0(x_{\bar{u}}^+(\underline{x}), r)) \cap K(\underline{x})\right),$$

and its boundary is the union of the three curves

$$B^+(\underline{x}) := L_{\bar{u}}^+(V_{\bar{u}}(\underline{x}, r)),$$
$$B^-(\underline{x}) := L_0^-(V_0(x_{\bar{u}}^+(\underline{x}), r)) \cap \{x \geq \underline{x}\},$$
$$B^0(\underline{x}) := \{\underline{x}\} \times [r_-, r].$$

The viability kernel is depicted on Fig. 4 in case (ii) of Proposition 13 together with the three curves $B^+(\underline{x})$, $B^-(\underline{x})$,$B^0(\underline{x})$ that define its boundary. We admit here that it is a convex set (see the proof in [2]).

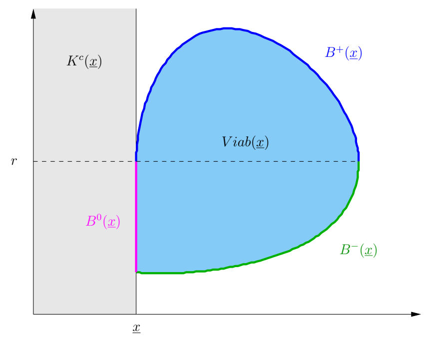

Figure 4. Viability kernel when $\bar{u} > \underline{x} - m$.

5.2 *Minimal time to reach the viability kernel*

In this section, we suppose that the condition

$$\bar{u} \geq \underline{x} - m,$$

is fulfilled, i.e. that $Viab(\underline{x}) \neq \emptyset$ (see Proposition 13). We recall from the Viability Theory (see e.g. [1]) that the viability kernel $Viab(\underline{x})$ can be reached from outside only at its boundary in common with the boundary of $K(\underline{x})$, that is, accordingly to Proposition 13 at the line segment (possibly reduced to a singleton when $\bar{u} = \underline{x} - m$) $B^0(\underline{x}) = \{\underline{x}\} \times [r_-, r]$. In order to show the attainability of the target set, it is convenient to introduce the feedback control:

$$\mathbf{u}[x, y] := \begin{cases} \bar{u} \text{ if } & y \geq r, \\ 0 \text{ if } & y < r. \end{cases} \qquad (4)$$

Given an initial condition $(x_0, y_0) \in (\mathbb{R}_+^* \times \mathbb{R}_+^*) \backslash Viab(\underline{x})$, we denote by $(x_m(\cdot), y_m(\cdot))$ the unique solution of (3) starting from (x_0, y_0) at time 0 and associated to the control $u_m(\cdot)$ defined by:

$$u_m(t) := \mathbf{u}[x_m(t), y_m(t)].$$

Proposition 14. The feedback control (4) steers (3) from any initial condition $(x_0, y_0) \in \mathcal{D} \backslash Viab(\underline{x})$ to the viability kernel $Viab(\underline{x})$.

The minimal time control problem to reach $Viab(\underline{x})$ is defined as follows:

$$v(x_0, y_0) := \inf_{u(\cdot)\in\mathcal{U}} T_u \quad \text{s.t. } (x(T_u), y(T_u)) \in Viab(\underline{x}), \quad (5)$$

where $(x_u(\cdot), y_u(\cdot))$ is the unique solution of (3) associated to the control u and starting at some point $(x_0, y_0) \in \mathcal{D}\backslash Viab(\underline{x})$, and $v(x_0, y_0) \in [0, +\infty]$ is the *value function* associated to the problem. Recall that $Viab(\underline{x})$ can be reached from $K(\underline{x})^c$ only through the line segment $B^0(\underline{x})$. From Proposition 14, the set $Viab(\underline{x})$ can be reached from any initial condition $(x_0, y_0) \in \mathcal{D}$ (i.e. v is finite everywhere in $\mathcal{D}$), thus the existence of an optimal control is straightforward using Fillipov's Theorem (see [6]). We are now in position to apply the Pontryagin Maximum Principle (PMP) to derive necessary optimality conditions on problem (5).

Recall that given a non-empty closed convex subset $K \subset \mathbb{R}^n$, $n \geq 1$, the normal cone to K at a point $x \in K$ is defined as $N_K(x) := \{p \in \mathbb{R}^n \ ; \ p \cdot (y - x) \leq 0, \ \forall y \in K\}$ where $a \cdot b$ denotes the standard scalar product of two vectors $a, b \in \mathbb{R}^n$. Let $H : \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be the Hamiltonian associated to (5) defined by:

$$H = H(x, y, p, q, p_0, u) = px(r - y) + qy(x - m - u) + p_0.$$

We now apply the Pontryagin Maximum Principle to (5). Let $u \in \mathcal{U}$ be an optimal control defined over a certain time interval $[0, T_u]$ with $T_u \geq 0$ and let $z_u := (x_u, y_u)$ be the associated solution. Then, there exists an absolutely continuous map $\lambda := (p, q) : [0, T_u] \to \mathbb{R}^2$ and $p_0 \leq 0$ such that the following conditions are satisfied:

- The pair $(\lambda(\cdot), p_0)$ is non-zero.
- The adjoint vector satisfies the adjoint equation $\dot{\lambda} = -\frac{\partial H}{\partial z}(z_u(t), \lambda(t), p_0, u(t))$ a.e. on $[0, T_u]$ that is:

$$\begin{cases} \dot{p} = p(y - r) - qy, \\ \dot{q} = px + q(u + m - x). \end{cases} \quad (6)$$

- As $Viab(\underline{x})$ is a non-empty compact convex subset of $\mathbb{R}^n$, the transversality condition can be expressed as $\lambda(T_u) \in -N_{Viab(\underline{x})}(z(T_u))$.
- The control u satisfies the maximization condition a.e. on $[0, T_u]$:

$$u(t) \in \arg\max_{0 \leq \omega \leq \bar{u}} H(z_u(t), \lambda(t), p_0, \omega). \quad (7)$$

An *extremal trajectory* is a triple $(z_u(\cdot), \lambda(\cdot), u(\cdot))$ satisfying (3)-(6)-(7). As the system is autonomous and T_u is free, the Hamiltonian is zero along any extremal trajectory. We say that the extremal is *normal* if $p_0 \neq 0$ and is *abnormal* if $p_0 = 0$. Whenever an extremal trajectory is normal, we can always suppose that $p_0 = -1$ (using that H and (6) are homogeneous). In view of (7), we define the *switching function* ϕ as

$$\phi := -qy,$$

and we obtain the following control law:

$$\begin{cases} \phi(t) > 0 \Rightarrow u(t) = \bar{u}, \\ \phi(t) < 0 \Rightarrow u(t) = 0, \\ \phi(t) = 0 \Rightarrow u(t) \in [0, \bar{u}]. \end{cases} \quad (8)$$

We call *switching time* (or switching point) a time t_c where the control is non-constant in any neighborhood of t_c. From (8), we deduce that any switching time satisfies $\phi(t_c) = 0$. A direct computation shows that we have:

$$\dot{\phi}(t) = -p(t)x(t)y(t) \quad \text{a.e. } t \in [0, T_u].$$

Let us now explicit the transversality condition. To do so, let $e_1 := (1, 0)$ and w the unit vector defined by $w := (\sin\psi, -\cos\psi)$ where $\psi \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ is defined by

$$\tan\psi := \frac{(\underline{x} - m)r_-}{(r - r_-)\underline{x}}.$$

Lemma 15. If $(x, y) \in B^0(\underline{x})$, we have:

$$N_{Viab(\underline{x})}(x, y) = \mathbb{R}_- \times \{0\},$$

whenever $y \in (r_-, r)$ and

$$N_{Viab(\underline{x})}(x, y) = \{\alpha(\beta w - [1 - \beta]e_1) \ ; \ (\alpha, \beta) \in \mathbb{R}_+ \times [0, 1]\}.$$

whenever $y = r_-$.

The Pontryagin Maximum Principle then implies the following properties.

Proposition 16. (i) An optimal control u is bang-bang i.e. it satisfies $u(t) \in \{0, \bar{u}\}$ for a.e. $t \in [0, T_u]$ and:

$$u(t) = \frac{\bar{u}}{2}(1 + \text{sign}(\phi(t))) \quad \text{a.e. } t \in [0, T_u]. \quad (9)$$

(ii) The transversality condition reads as follows: one has

$$(p(T_u), q(T_u)) \in \mathbb{R}_+ \times \{0\},$$

whenever $(x(T_u), y(T_u)) \in \{\underline{x}\} \times (r_-, r)$ and

$$(p(T_u), q(T_u)) \in \{\alpha([1 - \beta]e_1 - \beta w) \ ; \ (\alpha, \beta) \in \mathbb{R}_+ \times [0, 1]\},$$

whenever $(x(T_u), y(T_u)) = (\underline{x}, r_-)$.

(iii) Any extremal trajectory reaching the target at some point in $\{\underline{x}\} \times (r_-, r)$ is normal i.e. $p_0 \neq 0$.

(iv) Any switching point of an abnormal trajectory lies on the axis $\{y = r\}$.

Remark 17. From (6) and the fact that $(\lambda(\cdot), p_0)$ is non-zero, the mapping $t \longmapsto (p(t), q(t))$ is always non-zero. Following the proof of Proposition 16 (i), this argument allows to prove that the set of zeros of ϕ is isolated.

We first analyze the behavior of ϕ which is crucial in order to find an optimal control policy.

Lemma 18. A normal extremal trajectory $(z(\cdot), \lambda(\cdot), u(\cdot))$ defined over $[0, T_u]$ satisfies the following properties:

(i) The switching function satisfies the following ordinary differential equation (ODE) a.e. on $[0, T_u]$:

$$\dot{\phi}(t) = \frac{y(t)(m + u(t) - x(t))}{r - y(t)}\phi(t) - \frac{y(t)}{r - y(t)}. \quad (10)$$

(ii) At a time t_0 where $y(t_0) = r$, we have $\phi(t_0) \neq 0$ and:

$$\phi(t_0) = \frac{1}{u(t_0) + m - x(t_0)}. \quad (11)$$

The previous Lemma implies the following proposition.

Proposition 19. Let $(z(\cdot), \lambda(\cdot), u(\cdot))$ be a normal extremal trajectory defined over $[0, T_u]$. Then, the following properties are satisfied:

(i) If there exist two consecutive instants $t_2 > t_1 > 0$ such that $y(t_1) = y(t_2) = r$, then the control u has exactly one switching time $t_c \in (t_1, t_2)$.

(ii) If in addition, $x(t_1) > x(t_2)$, resp. $x(t_1) < x(t_2)$, then an optimal control satisfies $u = 0$, resp. $u = 1$ on (t_1, t_c) and then $u = 1$, resp. $u = 0$ on (t_c, t_2).

We denote by γ the graph of the unique solution $(\tilde{x}(\cdot), \tilde{y}(\cdot))$ of (3) backward in time starting from the point $(\underline{x}, r)$

associated to the feedback control (4). Finally, let γ_1 be the restriction of $(\tilde{x}(\cdot), \tilde{y}(\cdot))$ to the interval $[\tau_1, \tau_2]$.

The optimal synthesis of the problem then reads as follows (see also Fig. 5).

Theorem 20. Let $(x_0, y_0) \in \mathcal{D} \backslash Viab(\underline{x})$.

(i) If $(x_0, y_0) \in \gamma$, then any extremal optimal trajectory steering (3) from (x_0, y_0) to the target set is abnormal. The corresponding control is given by u_m and switching points occur on the axis $\{y = r\}$.

(ii) If $(x_0, y_0) \notin \gamma$, then any extremal optimal trajectory steering (3) from (x_0, y_0) to the target set is normal. Moreover, if u denotes the optimal control, there exists $p \in \mathbb{N}^*$, $s \in \{0, 1\}$, and a sequence $(\tau_k)_{0 \leq k \leq p}$ such that:

- We have $\tau_0 = 0 < \tau_1 < \cdots < \tau_{p-1} < \tau_p = T_u$ and the time τ_k is a switching time of u for $1 \leq k \leq p$.
- The optimal control u is given by
$$u(t) = \frac{\bar{u}}{2}(1 + (-1)^{p-k-s}) \quad t \in (\tau_k, \tau_{k+1}), \tag{12}$$
where $0 \leq k \leq p - 1$.
- If $y(T_u) \in (r_-, r)$, resp. $y(T_u) = r_-$, then $s = 1$, resp. $s = 0$.

5.3 Discussion on the optimal synthesis

To highlight the optimal synthesis provided by Theorem 20, we provide the following remarks.

- The optimal control provided by Theorem 20 (ii) can be interpreted as a slight perturbation of the strategy (4): instead of switching on the axis $\{y = r\}$, switching times are delayed and the corresponding switching points occur after the last intersection between the corresponding trajectory and the axis $\{y = r\}$ (see the switching curves in red on Fig 5).
- Abnormal trajectories are contained in the curve γ and they are the only extremal trajectories for which switching points occur on the axis $\{y = r\}$.
- (iii) In Theorem 20, any normal trajectory reaching $B_0(\underline{x})$ in its interior satisfies:
$$p - (2j+1) \geq 0 \quad \text{and} \quad p - 2j \geq 0 \quad \Rightarrow$$
$$y(\tau_{p-2j+1}) > r \quad \text{and} \quad y(\tau_{p-2j}) < r.$$
- Any normal trajectory either reaches $B_0(\underline{x})$ with $u = \bar{u}$, or it reaches the point $(\underline{x}, r_-)$ with $u = 0$. In the latter case, an optimal trajectory switches from $u = 1$ to $u = 0$ on γ_1.

5.4 Study of the minimal time crisis problem

To compute an optimal feedback control for the minimal time crisis problem, we apply the hybrid maximum principle (see e.g. [7]). Here, we only provide a preliminary result obtained in [3].

Proposition 21. For any $(x_0, y_0) \in \gamma$ such that $(x_0, y_0) \notin Viab(\underline{x})$, the time crisis function $\theta(x_0, y_0)$ satisfies:
$$\theta(x_0, y_0) < v(x_0, y_0).$$

Let us now comment this result. First, we have seen that for initial conditions on γ, switching points for the minimum time problem occur on the axis $\{y = r\}$, and the corresponding optimal trajectory is abnormal. On the other hand, one can show that no abnormal trajectories exist for the time crisis problem [3]. Hence, no optimal trajectory can have a switching point on on the axis $\{y = r\}$. The proposition is then a consequence of the fact that the time crisis function for the viability kernel $Viab(\underline{x})$ (that is greater than the time crisis function for $K(\underline{x})$) is equal to the minimal time function v (this result is proved in [8]). In other words, the minimum time crisis function for such initial conditions (i.e. the time spent outside the set K) is strictly less than the minimal time needed to reach the viability kernel. Hence, it can be interesting to investigate optimal feedback controls for the time crisis problem from a practical point of view.

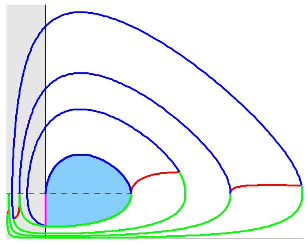

Figure 5. Optimal synthesis provided by Theorem 20: in red the switching curves, in green, resp. in blue the solutions of (3) with $u = 1$, resp. with $u = 0$.

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