A non-commutative algorithm for multiplying 5 x 5 matrices using 99 multiplications

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Abstract

We present a non-commutative algorithm for multiplying 5×5 matrices using 99 multiplications. This algorithm is a minor modification of Makarov's algorithm which exhibit the previous best known bound with 100 multiplications.

1 Introduction

In his seminal work [15], V. Strassen introduced a non-commutative algorithm for multiplication of two 2×2 matrices using only 7 coefficient multiplications. Since, several algorithms where proposed for the product of small matrices during the last 40 years (e.g. [7, 11, 8]). In 1987, O.M. Makarov shown in [9] that the product of two 5×5 matrices can be done using 100 coefficient multiplications.

In this note, we present a non-commutative $M_{5\times5\times5}$ performing the matrix product problem $AB = C$ expressed with the following generic 5×5 matrices:

$$M_{5\times5\times5} : \begin{pmatrix} a_{11} & \cdots & a_{15} \\ \vdots & & \vdots \\ a_{51} & \cdots & a_{55} \end{pmatrix} \begin{pmatrix} b_{11} & \cdots & b_{15} \\ \vdots & & \vdots \\ b_{51} & \cdots & b_{55} \end{pmatrix} = \begin{pmatrix} c_{11} & \cdots & c_{15} \\ \vdots & & \vdots \\ c_{51} & \cdots & c_{55} \end{pmatrix}; \quad (1)$$

this algorithms requires 99 coefficient multiplications. Furthermore, we explain briefly how this result is derived from Makarov's original algorithm and we conclude by some implications of this new upper bound.

2 An algorithm for multiplying 5×5 matrices

The algorithm presented below was *accidentally* obtained while implementing Makarov's algorithm in a computer algebra package devoted to matrix multiplication algorithms seen as geometric objects represented by tensors. In his original paper [9], O.M. Makarov decomposed the 5×5 matrix multiplication algorithm as 7 others matrix multiplication algorithms using a variant of Strassen's algorithm [15] and obtain an algorithm requiring 101 multiplications. Then, Makarov uses the particular form of one of the used sub-algorithms to obtain his algorithm that we are not be able to improved. But by explicitly implementing the intermediary algorithm requiring 101 multiplications, the simplification procedure of our computer algebra package produces automatically the following evaluation scheme that requires only 99 coefficient multiplications:

$$m_1 = (a_{51} + a_{35} + a_{45} - a_{55})\, b_{54}, \tag{2}$$
$$m_2 = (-a_{12} + a_{14})\, b_{21}, \tag{3}$$
$$m_3 = -a_{32}\, b_{23}, \tag{4}$$
$$m_4 = -a_{34}\, b_{43}, \tag{5}$$
$$m_5 = a_{14}\, (b_{21} + b_{41}), \tag{6}$$
$$m_6 = (-a_{32} + a_{42})\, (b_{14} - b_{24} + b_{34} - b_{44}), \tag{7}$$
$$m_7 = (a_{12} + a_{32} - a_{14})\, (b_{21} + b_{41} + b_{43}), \tag{8}$$
$$m_8 = a_{53}\, b_{35}\, \ell, \tag{9}$$
$$m_9 = a_{51}\, b_{15}\, \ell, \tag{10}$$
$$m_{10} = (-a_{43} + a_{34} - a_{44})\, (b_{21} - b_{41} + b_{12} - b_{22} - b_{32} + b_{42}), \tag{11}$$
$$m_{11} = (a_{21} - a_{12} + a_{22} + a_{23} - a_{14} + a_{24})\, (-b_{43} - b_{34} + b_{44}), \tag{12}$$
$$m_{12} = (a_{32} - a_{42} - a_{14} + a_{24})\, (b_{12} - b_{22} + b_{14} - b_{24} + b_{34} - b_{44}), \tag{13}$$
$$m_{13} = (-a_{21} + a_{12} - a_{22})\, (-b_{23} + b_{43} - b_{14} + b_{24} + b_{34} - b_{44}), \tag{14}$$
$$m_{14} = a_{45}\, (b_{32} - b_{42} + b_{52}), \tag{15}$$
$$m_{15} = (a_{12} - a_{21} - a_{22} - a_{43} + a_{34} - a_{44}) \begin{pmatrix} b_{22} - b_{21} - b_{12} \\ +b_{44} - b_{43} - b_{34} \end{pmatrix}, \tag{16}$$
$$m_{16} = (-a_{41} + a_{32} - a_{42} - a_{43} + a_{34} - a_{44})\, (-b_{21} - b_{12} + b_{22}), \tag{17}$$
$$m_{17} = (-a_{23} - a_{25})\, (-b_{53} + b_{54}), \tag{18}$$
$$m_{18} = (a_{43} + a_{44})\, (b_{21} - b_{41} - b_{22} + b_{42}), \tag{19}$$
$$m_{19} = (-a_{21} - a_{22} - a_{23} - a_{24})\, (-b_{43} + b_{44}), \tag{20}$$
$$m_{20} = (a_{14} - a_{24})\, (b_{12} - b_{22} + b_{32} - b_{42} + b_{14} - b_{24} + b_{34} - b_{44}), \tag{21}$$
$$m_{21} = (a_{21} + a_{22})\, (-b_{23} + b_{43} + b_{24} - b_{44}), \tag{22}$$
$$m_{22} = (-a_{12} - a_{32} + a_{14} + a_{34})\, (b_{41} + b_{43}), \tag{23}$$
$$m_{23} = (a_{12} + a_{32})\, (b_{21} + b_{41} + b_{23} + b_{43}), \tag{24}$$
$$m_{24} = (a_{12} - a_{22} - a_{32} + a_{42} - a_{25})\, (b_{12} - b_{22}), \tag{25}$$
$$m_{25} = (-a_{31} + a_{41} - a_{32} + a_{42})\, (b_{14} + b_{34}), \tag{26}$$
$$m_{26} = (-a_{41} - a_{43})\, (-b_{13} + b_{23} + b_{14} - b_{24}), \tag{27}$$
$$m_{27} = a_{35}\, (b_{32} + b_{52}), \tag{28}$$
$$m_{28} = (-a_{54} - a_{45})\, (b_{32} - b_{42} + b_{52} + b_{54}), \tag{29}$$
$$m_{29} = (a_{21} + a_{22} + a_{43} + a_{44})\, (-b_{21} + b_{22} - b_{43} + b_{44}), \tag{30}$$

$$m_{30} = (a_{41} + a_{42} + a_{43} + a_{44})(-b_{21} + b_{22}), \tag{31}$$
$$m_{31} = -a_{52}\ (b_{25} + b_{45}), \tag{32}$$
$$m_{32} = \left(\ell a_{31} + a_{51} - \ell^2 a_{35}\right)\left(b_{11} + \frac{b_{51}}{\ell} - \ell b_{15}\right), \tag{33}$$
$$m_{33} = (a_{23} + a_{25} + a_{45})(b_{53} - b_{54} - b_{35} + b_{45}), \tag{34}$$
$$m_{34} = \left(\ell a_{13} + a_{53} - \ell^2 a_{15}\right)\left(b_{31} + \frac{b_{51}}{\ell} - \ell b_{35}\right), \tag{35}$$
$$m_{35} = (a_{52} - a_{54})\, b_{45}, \tag{36}$$
$$m_{36} = (a_{14} - a_{23} + a_{43} - a_{24} - a_{34} + a_{44})\begin{pmatrix} b_{42} - b_{41} - b_{32} \\ -b_{43} - b_{34} + b_{44} \end{pmatrix}, \tag{37}$$
$$m_{37} = -a_{52}\ (b_{14} - b_{24} + b_{54}), \tag{38}$$
$$m_{38} = (a_{21} + a_{41} a_{43})(b_{31} - b_{41} - b_{32} + b_{42} - b_{13} + b_{23} + b_{14} - b_{24}), \tag{39}$$
$$m_{39} = (a_{53} + a_{54} - a_{35})(b_{32} + b_{52} + b_{54}), \tag{40}$$
$$m_{40} = \left(-\frac{a_{11}}{\ell} - \frac{a_{51}}{\ell^2} + a_{15}\right)\left(b_{13} + \frac{b_{53}}{\ell} - \ell b_{15}\right), \tag{41}$$
$$m_{41} = (a_{21} - a_{41} - a_{12} + a_{22} + a_{32} - a_{42})\begin{pmatrix} b_{24} - b_{21} - b_{12} \\ +b_{22} - b_{23} - b_{14} \end{pmatrix}, \tag{42}$$
$$m_{42} = a_{25}\ (b_{12} - b_{22} + b_{52}), \tag{43}$$
$$m_{43} = -a_{54}\ (b_{32} - b_{42} + b_{52} + b_{34} - b_{44} + b_{54}), \tag{44}$$
$$m_{44} = (a_{31} - a_{41} + a_{32} - a_{42} - a_{13} + a_{23} - a_{14} + a_{24})(b_{12} + b_{14} + b_{34}), \tag{45}$$
$$m_{45} = (-a_{52} - a_{25})(b_{12} - b_{22} - b_{54}), \tag{46}$$
$$m_{46} = \left(\ell a_{33} + a_{53} - \ell^2 a_{35}\right)\left(b_{33} + \frac{b_{53}}{\ell} - \ell b_{35}\right), \tag{47}$$
$$m_{47} = (a_{52} + a_{14})(b_{21} - b_{45}), \tag{48}$$
$$m_{48} = \left(\frac{a_{31}}{\ell} + \frac{a_{51}}{\ell^2}\right)\left(b_{11} + \frac{b_{51}}{\ell}\right), \tag{49}$$
$$m_{49} = a_{45}\ (-b_{53} + b_{54} - b_{15} + b_{25} + b_{55}), \tag{50}$$
$$m_{50} = \left(-\ell a_{31} + \ell^2 a_{35}\right)(b_{11} - \ell b_{15}), \tag{51}$$
$$m_{51} = (a_{21} - a_{45})(-b_{51} + b_{52} + b_{15} - b_{25}), \tag{52}$$
$$m_{52} = -a_{21}\ (b_{11} - b_{21} - b_{51} - b_{12} + b_{22} + b_{52} - b_{13} + b_{23} + b_{14} - b_{24}), \tag{53}$$
$$m_{53} = (a_{21} + a_{25})(b_{51} - b_{52}), \tag{54}$$
$$m_{54} = \left(\frac{a_{13}}{\ell} + \frac{a_{53}}{\ell^2}\right)\left(b_{31} + \frac{b_{51}}{\ell}\right), \tag{55}$$
$$m_{55} = (a_{52} - a_{34} - a_{54})(b_{23} + b_{25} + b_{45}), \tag{56}$$
$$m_{56} = \left(-\ell a_{13} + \ell^2 a_{15}\right)(b_{31} - \ell b_{35}), \tag{57}$$
$$m_{57} = (a_{23} + a_{43} + a_{25} + a_{45})(b_{35} - b_{45}), \tag{58}$$
$$m_{58} = (a_{23} - a_{43} + a_{24} - a_{44})(-b_{41} + b_{42} - b_{43} + b_{44}), \tag{59}$$

$$m_{59} = \left(-a_{41} - a_{45}\right)\left(-b_{15} + b_{25}\right), \tag{60}$$

$$m_{60} = \left(a_{13} - a_{23} + a_{14} - a_{24}\right)\left(b_{12} + b_{32} + b_{14} + b_{34}\right), \tag{61}$$

$$m_{61} = \left(a_{53} + a_{54}\right)\left(b_{32} + b_{52} + b_{34} + b_{54}\right), \tag{62}$$

$$m_{62} = \left(a_{21} + a_{41} + a_{23} + a_{43}\right)\left(b_{31} - b_{41} - b_{32} + b_{42}\right), \tag{63}$$

$$m_{63} = \left(-a_{21} + a_{41} - a_{22} + a_{42}\right)\left(-b_{21} + b_{22} - b_{23} + b_{24}\right), \tag{64}$$

$$m_{64} = \left(\frac{a_{11}}{\ell} + \frac{a_{51}}{\ell^2}\right)\left(b_{13} + \frac{b_{53}}{\ell}\right), \tag{65}$$

$$m_{65} = \left(-a_{32} + a_{42} + a_{34} - a_{44} + a_{54}\right)\left(b_{34} - b_{44}\right), \tag{66}$$

$$m_{66} = \left(-\ell a_{11} + \ell^2 a_{15}\right)\left(b_{13} - \ell b_{15}\right), \tag{67}$$

$$m_{67} = a_{15}\,\left(b_{12} + b_{52}\right), \tag{68}$$

$$m_{68} = \left(a_{51} + a_{52}\right)\left(b_{14} + b_{54}\right), \tag{69}$$

$$m_{69} = \left(-\ell a_{33} + \ell^2 a_{35}\right)\left(b_{33} - \ell b_{35}\right), \tag{70}$$

$$m_{70} = \left(-a_{53} - a_{15} - a_{25} + a_{55}\right)\left(b_{52} + b_{54}\right), \tag{71}$$

$$m_{71} = \left(\frac{a_{33}}{\ell} + \frac{a_{53}}{\ell^2}\right)\left(b_{33} + \frac{b_{53}}{\ell}\right), \tag{72}$$

$$m_{72} = \left(-a_{14} + a_{24} + a_{34} - a_{44} - a_{45}\right)\left(b_{32} - b_{42}\right), \tag{73}$$

$$m_{73} = \left(a_{11} - a_{21} - a_{31} + a_{41} + a_{12} - a_{22} - a_{32} + a_{42} - a_{15}\right) b_{12}, \tag{74}$$

$$m_{74} = \left(a_{12} - a_{22} + a_{52} - a_{14} + a_{24}\right)\left(b_{12} - b_{22} + b_{14} - b_{24}\right), \tag{75}$$

$$m_{75} = \left(a_{51} + a_{52} - a_{15}\right)\left(b_{12} - b_{54}\right), \tag{76}$$

$$m_{76} = \left(a_{21} + a_{41}\right)\left(\begin{array}{c} b_{11} - b_{21} - b_{31} + b_{41} - b_{12} \\ +b_{22} + b_{32} - b_{42} - b_{15} + b_{25} \end{array}\right), \tag{77}$$

$$m_{77} = a_{33}\, b_{31}, \tag{78}$$

$$m_{78} = a_{11}\, b_{11}, \tag{79}$$

$$m_{79} = \left(-a_{13} + a_{23} + a_{33} - a_{43} - a_{14} + a_{24} + a_{34} - a_{44} - a_{35}\right) b_{32}, \tag{80}$$

$$m_{80} = \left(a_{14} + a_{54}\right)\left(b_{41} + b_{45}\right), \tag{81}$$

$$m_{81} = \left(-a_{31} - \frac{a_{51}}{\ell} - a_{13} - \frac{a_{53}}{\ell} + \ell a_{15} + \ell a_{35} + a_{55}\right) b_{51}, \tag{82}$$

$$m_{82} = a_{23}\,\left(-b_{31} + b_{41} + b_{32} - b_{42} + b_{33} - b_{43} - b_{53} - b_{34} + b_{44} + b_{54}\right), \tag{83}$$

$$m_{83} = a_{13}\, b_{33}, \tag{84}$$

$$m_{84} = \left(\begin{array}{c} a_{11} - a_{21} - a_{51} + a_{12} - a_{22} \\ -a_{52} - a_{13} + a_{23} - a_{14} + a_{24} \end{array}\right)\left(b_{12} + b_{14}\right), \tag{85}$$

$$m_{85} = \left(a_{34} + a_{54}\right)\left(b_{23} + b_{43} + b_{25} + b_{45}\right), \tag{86}$$

$$m_{86} = \left(-a_{11} - \frac{a_{51}}{\ell} - a_{33} - \frac{a_{53}}{\ell} + \ell a_{15} + \ell a_{35} + a_{55}\right) b_{53}, \tag{87}$$

$$m_{87} = a_{43}\,\left(-b_{13} + b_{23} + b_{33} - b_{43} + b_{14} - b_{24} - b_{34} + b_{44} - b_{35} + b_{45}\right), \tag{88}$$

$$m_{88} = a_{31}\, b_{13}, \tag{89}$$

$$m_{89} = a_{15}\,\left(-b_{31} - \frac{b_{51}}{\ell} - b_{13} - \frac{b_{53}}{\ell} + \ell b_{15} + \ell b_{35} + b_{55}\right), \tag{90}$$

$$m_{90} = (-a_{31} + a_{41} - a_{32} + a_{42} + a_{33} - a_{43} - a_{53} + a_{34} - a_{44} - a_{54})\, b_{34}, \quad (91)$$
$$m_{91} = a_{55}\, b_{55}, \quad (92)$$
$$m_{92} = (a_{43} + a_{44})\, b_{45}, \quad (93)$$
$$m_{93} = (a_{41} + a_{42})\, b_{25}, \quad (94)$$
$$m_{94} = (a_{32} + a_{52} - a_{34} - a_{54})\,(b_{23} + b_{25})\,, \quad (95)$$
$$m_{95} = -a_{35}\, \left(b_{11} + \frac{b_{51}}{\ell} + b_{33} + \frac{b_{53}}{\ell} - \ell b_{15} - \ell b_{35} - b_{55}\right), \quad (96)$$
$$m_{96} = (a_{23} + a_{24})\, b_{45}, \quad (97)$$
$$m_{97} = (a_{21} + a_{22})\, b_{25}, \quad (98)$$
$$m_{98} = (a_{25} + a_{45})\,(-b_{51} + b_{52} - b_{35} + b_{45} + b_{55})\,, \quad (99)$$
$$m_{99} = (a_{12} + a_{52})\,(b_{21} + b_{25})\,. \quad (100)$$

$$c_{11} = -\frac{m_8}{\ell} - m_2 + m_5 + m_{78} + m_{54}\,\ell - \frac{m_{56}}{\ell} - \frac{m_{34}}{\ell}, \quad (101)$$
$$c_{21} = m_{10} + m_{11} + m_{12} - m_2 + m_5 + m_6 - m_{52} + m_{53} + m_{62} + m_{36} - m_{38} + m_{42} + m_{15} + m_{20} + m_{24} - m_{26}, \quad (102)$$
$$c_{31} = m_7 - \frac{m_9}{\ell} + m_2 + m_4 + m_{77} - \frac{m_{50}}{\ell} - \frac{m_{32}}{\ell} + m_{48}\,\ell + m_{22}, \quad (103)$$
$$c_{41} = m_7 + m_{10} + m_{12} + m_{14} + m_2 + m_4 + m_6 + m_{72} + m_{76} + m_{51} + m_{52} + m_{59} + m_{38} + m_{16} + m_{20} + m_{22} + m_{26}, \quad (104)$$
$$c_{51} = m_8 + m_9 - m_5 + m_{80} + m_{81} + m_{50} + m_{56} + m_{32} + m_{34} + m_{35} + m_{47}, \quad (105)$$
$$c_{12} = m_{10} + m_{11} - m_2 + m_5 + m_{67} + m_{73} + m_{58} + m_{60} + m_{36} + m_{44} + m_{15} + m_{18} + m_{19} + m_{25} + m_{29}, \quad (106)$$
$$c_{22} = m_{10} + m_{11} + m_{12} - m_2 + m_5 + m_6 + m_{58} + m_{36} + m_{42} + m_{15} + m_{18} + m_{19} + m_{20} + m_{24} + m_{29}, \quad (107)$$
$$c_{32} = m_7 + m_{10} + m_2 + m_4 + m_{79} + m_{60} + m_{44} + m_{16} + m_{18} + m_{22} + m_{25} + m_{27} + m_{30}, \quad (108)$$
$$c_{42} = m_7 + m_{10} + m_{12} + m_{14} + m_2 + m_4 + m_6 + m_{72} + m_{16} + m_{18} + m_{20} + m_{22} + m_{30}, \quad (109)$$
$$c_{52} = m_{14} + m_1 + m_{67} + m_{70} + m_{75} + m_{39} + m_{42} + m_{45} + m_{27} + m_{28}, \quad (110)$$
$$c_{13} = -m_7 - \frac{m_9}{\ell} + m_3 - m_5 + m_{83} - \frac{m_{66}}{\ell} + m_{64}\,\ell + m_{40}\,\ell + m_{23}, \quad (111)$$
$$c_{23} = -m_7 - m_{11} - m_{12} + m_{13} + m_3 - m_5 - m_6 + m_{74} + m_{82} + m_{62} + m_{37} - m_{38} + m_{45} + m_{17} + m_{23} - m_{24} - m_{26}, \quad (112)$$
$$c_{33} = -\frac{m_8}{\ell} - m_3 - m_4 + m_{88} - \frac{m_{69}}{\ell} + m_{71}\,\ell - \frac{m_{46}}{\ell}, \quad (113)$$
$$c_{43} = m_{13} - m_{14} - m_3 - m_4 - m_6 + m_{87} + m_{57} + m_{65} + m_{33} + m_{41} + m_{43} + m_{15} - m_{16} - m_{17} + m_{26} - m_{28}, \quad (114)$$

$$c_{53} = m_8 + m_9 + m_4 + m_{85} + m_{86} + m_{66} + m_{69} + m_{55} - m_{40}\,\ell^2 + m_{46} + m_{31}, \tag{115}$$

$$c_{14} = -m_7 - m_{11} + m_{13} + m_3 - m_5 + m_{84} + m_{68} - m_{73} + m_{75} - m_{44} - m_{19} + m_{21} + m_{23} - m_{25}, \tag{116}$$

$$c_{24} = -m_7 - m_{11} - m_{12} + m_{13} + m_3 - m_5 - m_6 + m_{74} + m_{37} + m_{45} - m_{19} + m_{21} + m_{23} - m_{24}, \tag{117}$$

$$c_{34} = m_{13} - m_3 - m_4 + m_{90} + m_{61} + m_{63} - m_{39} + m_{41} + m_{15} - m_{16} + m_{21} - m_{25} - m_{27} + m_{29} - m_{30}, \tag{118}$$

$$c_{44} = m_{13} - m_{14} - m_3 - m_4 - m_6 + m_{63} + m_{65} + m_{41} + m_{43} + m_{15} - m_{16} + m_{21} - m_{28} + m_{29} - m_{30}, \tag{119}$$

$$c_{54} = -m_{14} - m_1 + m_{68} + m_{61} + m_{37} - m_{39} + m_{43} - m_{27} - m_{28}, \tag{120}$$

$$c_{15} = -\frac{m_8}{\ell^2} - \frac{m_9}{\ell^2} - \frac{m_{34}}{\ell^2} + m_2 + m_{89} + m_{99} + m_{54} + m_{64} + m_{40} - m_{47} + m_{31}, \tag{121}$$

$$c_{25} = m_{96} + m_{97} + m_{98} - m_{49} + m_{51} + m_{53} - m_{33} + m_{17}, \tag{122}$$

$$c_{35} = -\frac{m_8}{\ell^2} - \frac{m_9}{\ell^2} - \frac{m_{46}}{\ell^2} - \frac{m_{32}}{\ell^2} + m_3 + m_{94} + m_{95} + m_{71} - m_{55} + m_{35} + m_{48}, \tag{123}$$

$$c_{45} = m_{92} + m_{93} + m_{49} + m_{57} + m_{59} + m_{33} - m_{17}, \tag{124}$$

$$c_{55} = \frac{m_8}{\ell} + \frac{m_9}{\ell} + m_{91} - m_{35} - m_{31}. \tag{125}$$

Remark 1 — *The free parameter ℓ used in (2) – (125) came from the utilisation of Winograd variant of Strassen algorithms (see [1]) as presented in [13].*

In the following section, we present explicitly the difference between the original Makarov's algorithm and the improved version presented here.

3 Where does the improvement come from?

Makarov's result is based on a divide-and-conquer strategy:

- The original problem is "divided" into 7 matrix multiplication subproblems using the Strassen's matrix multiplication algorithm [15] (see Drevet et all [3] or Sedoglavic [14] for a detailled description of similar—but inequivalent—decompositions).
- Each of these 7 subproblems could be handled by the more efficient known matrix multiplication algorithm adapted to its matrix sizes. These resolutions allow to "conquer" an algorithm solving the original problem more efficiently then the trivial approach.

Notation 2 — *In the sequel, we denote by $(a \times b \times c\,; d)$ a matrix multiplication algorithm computing the product of a matrix of size $a \times b$ by a matrix of size $b \times c$ using d coefficient multiplications.*

Makarov's algorithm $\mathcal{M}$ relies on the following subproblems:

- one $(2 \times 2 \times 2\,; 7)$ product M_9 done by Strassen algorithm $\mathcal{S}$ [15];
- one $(3 \times 3 \times 3\,; 23)$ product M_4 done by Laderman algorithm $\mathcal{L}$ [7];
- one $(2 \times 2 \times 3\,; 11)$ product M_6 done by Hopcroft-Kerr algorithm $\mathcal{K}$ [6, Th 3] (a.k.a. basic use of Strassen algorithm);
- and four $(3 \times 3 \times 2\,; 15)$ products M_3, M_5, M_1 and M_2 done by Hopcroft-Kerr algorithms $\mathcal{H}$ [6] (a.k.a. clearly not basic use of Strassen algorithm).

Above, the indices i in M_i refer directly to Makarov's numeration in [9] and the final algorithm (1) could be obtained in a trilinear form by the sum:

$$\langle \mathcal{M} | M_{5\times5\times5} \rangle = \langle \mathcal{S} | M_9 \rangle + \langle \mathcal{L} | M_4 \rangle + \langle \mathcal{K} | M_6 \rangle + \sum_{i \in \{1,2,3,5\}} \langle \mathcal{H} | M_i \rangle. \tag{126}$$

The interested reader could found in [13] a brief description of the framework (trilinear form, etc.) evoked above and in [12] the complete description of the used algorithms. The—tedious—complete presentation of these details is not necessary to expose the improvement done to Makarov's algorithm.

However, let us present with more details Makarov's subproblem [9, M_1] and [9, M_2] that are in our notations:

$$\begin{aligned} M_1 : &\begin{pmatrix} a_{11}+a_{12}-a_{21}-a_{22} & a_{13}+a_{14}-a_{23}-a_{24} & a_{15} \\ a_{31}+a_{32}-a_{41}-a_{42} & a_{33}+a_{34}-a_{43}-a_{44} & a_{35} \\ a_{51}+a_{52} & a_{53}+a_{54} & a_{55} \end{pmatrix} \begin{pmatrix} b_{12} & b_{14} \\ b_{32} & b_{34} \\ b_{52} & b_{54} \end{pmatrix} \\ &= \begin{pmatrix} n_{12} & n_{14} \\ n_{32} & n_{34} \\ \jmath c_{52} & \jmath c_{54} \end{pmatrix} \end{aligned} \tag{127a}$$

$$\begin{aligned} M_2 : &\begin{pmatrix} a_{12}-a_{22} & a_{14}-a_{24} & a_{25} \\ a_{32}-a_{42} & a_{34}-a_{44} & a_{45} \\ -a_{52} & -a_{54} & 0 \end{pmatrix} \begin{pmatrix} b_{12}-b_{22} & b_{14}-b_{24} \\ b_{32}-b_{42} & b_{34}-b_{44} \\ b_{52} & b_{54} \end{pmatrix} \\ &= \begin{pmatrix} n_{22} & n_{24} \\ n_{42} & n_{44} \\ (1-\jmath)c_{52} & (1-\jmath)c_{54} \end{pmatrix} \end{aligned} \tag{127b}$$

The quantities n_{ij} stand for intermediate variables allowing the computation of the wanted result C (similar to m_i in (2)–(125)) and $\jmath$ is a free parameters.

As any other decomposition applied to this problem, the decomposition (126) shows directly the following statement:

Lemma 3 — *Strassen, Laderman and Hopcroft-Kerr algorithms allow to construct* $(5 \times 5 \times 5\,;101)$.

Remark 4 — *The second part of Makarov's paper use the fact that a coefficient in [9, M_2] is* 0 *in order to show that this last subproblem could be solved using* 14 *multiplications instead of* 15 *by avoiding a useless multiplication. Similarly, one can obtain* $(5 \times 5 \times 5\,;100)$ *by various decompositions (mainly based on Winograd variant of Strassen algorithm) not necessarily equivalent to Makarov's one.*

However, Makarov's decomposition (in contrary to others decompositions known by the author of this note) is the only one where two subproblems share—without any further manipulations like Pan's trilinear aggregations [10]—some common terms in such a way that the total complexity is reduced.

In fact, the part of the original problem corresponding to subproblems [9, M_1] and [9, M_2] could be computed using only 28 instead of expected 30 multiplications. To show that very briefly, let us present—in trilinear form—the concerned terms of Hopcroft-Kerr algorithm $\mathcal{H}$:

$$\begin{aligned}\langle \mathcal{H}, M_1\rangle = \cdots &\\ &+ \jmath\,(a_{55} - a_{51} - a_{52} - a_{35})\,b_{54}\,(c_{54} - c_{52}) \\ &+ \jmath\,(a_{55} - a_{53} - a_{54} - a_{15})\,(b_{52} + b_{54})\,c_{52},\end{aligned} \tag{128a}$$

$$\begin{aligned}\langle \mathcal{H}, M_2\rangle = \cdots &\\ &+ (1-\jmath)\,(a_{52} - a_{45})\,b_{54}\,(c_{54} - c_{52}) \\ &+ (1-\jmath)\,(a_{54} - a_{25})\,(b_{52} + b_{54})\,c_{52}.\end{aligned} \tag{128b}$$

These four last trilinear terms could be factorize as two trilinear terms:

$$\begin{aligned}\langle \mathcal{H}, M_1\rangle + \langle \mathcal{H}, M_2\rangle = \cdots &\\ + ((a_{55} - a_{51} - a_{35})\jmath + (1-2\jmath)\,a_{52} - (1-\jmath)\,a_{45})\,b_{54}\,(c_{54} - c_{52}) &\\ + ((a_{55} - a_{53} - a_{15})\jmath + (1-2\jmath)\,a_{54} - (1-\jmath)\,a_{25})\,(b_{52} + b_{54})\,c_{52}. &\end{aligned} \tag{129}$$

This simplification was automatically produced by our pilote computer algebra package and implies the new upper bound $(5 \times 5 \times 5\,;99)$. We unfortunately do not have any geometric interpretation of this simplification and thus, we do not know if it is possible to reproduce it on other matrix multiplication algorithm obtained by a divide-and-conquer process.

4 Concluding remarks

Remark 5 — *The algorithm presented in this note could be used to improve slightly other matrix multiplication algorithm's bounds like* $(10 \times 10 \times 10\,;693)$ *for example.*

Remark 6 — *It is shown in [4] that no group can realize* 5×5 *matrix multiplication better then Makarov's algorithm using the group-theoretic approach of Cohn and Umans [2]. Hence, the algorithm presented in this note shows that this approach does not produce better algorithms then* $(5 \times 5 \times 5\,;99)$*. The same assertion for* $(3 \times 3 \times 3\,;23)$ *and* $(4 \times 4 \times 4\,;49)$ *was proved in [5, Theorem 7.3].*

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