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# Computing a front-facing ellipse that subtends the same solid angle as an arbitrarily oriented ellipse 

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#### Abstract

A front-facing ellipse, with respect to an observation point, is an ellipse that is perpendicular to the line that goes through the origin and the center of the ellipse. Front-facing ellipses are more convenient than arbitrarily oriented ellipses for solid-angle computations.

In this report, we present an algebraic method for computing a front-facing ellipse that covers the same solid angle as an arbitrarily oriented ellipse. A previous approach was described by Conway [Con10] but it is involved, in terms of trigonometric derivations and computations. Our formulation shows that this problem is equivalent to computing the eigenvectors of a $3 \times 3$ matrix, which is simpler to derive and compute.




Figure 1: We compute a front-facing ellipse that covers the same solid angle as an arbitrarily oriented ellipse.

## References

[Con10] John T. Conway. Analytical solution for the solid angle subtended at any point by an ellipse via a point source radiation vector potential. Nuclear Instruments and Methods in Physics Research Section A: Accelerators, Spectrometers, Detectors and Associated Equipment, 614(1):17-27, 2010. (document)

## 1 Arbitrarily oriented ellipse

We define the observation point as the origin: $O=(0,0,0)$. We consider an arbitrarily oriented ellipse of center $C$ and of orthonormal axes of normalized directions $V_{x}$ and $V_{y}$ and of respective lengths $l_{x}$ and $l_{y}$. The normal of the ellipse is $V_{z}=V_{x} \times V_{y}$ such that $\left(V_{x}, V_{y}, V_{z}\right)$ is an orthonormal basis. In this basis, the coordinates of the ellipse center are

$$
\begin{align*}
x_{c} & =C \cdot V_{x},  \tag{1}\\
y_{c} & =C \cdot V_{y},  \tag{2}\\
z_{c} & =C \cdot V_{z} . \tag{3}
\end{align*}
$$



Figure 2: Arbitrarily oriented ellipse.

## 2 Planar Quadric

In the plane of the ellipse $z=z_{c}$, the equation of the ellipse is

$$
\begin{equation*}
\left(\frac{x-x_{c}}{l_{x}}\right)^{2}+\left(\frac{y-y_{c}}{l_{y}}\right)^{2} \leq 1 \tag{4}
\end{equation*}
$$

## 3 Spherical Quadric

If $(x, y, z)$ is a point on the sphere, the equation of the spherical quadric is

$$
\begin{equation*}
\left(\frac{\frac{x z_{c}}{z}-x_{c}}{l_{x}}\right)^{2}+\left(\frac{\frac{y z_{c}}{z}-y_{c}}{l_{y}}\right)^{2} \leq 1 \tag{5}
\end{equation*}
$$

which can be rewritten

$$
\left[\begin{array}{lll}
x & y & z
\end{array}\right] Q\left[\begin{array}{l}
x  \tag{6}\\
y \\
z
\end{array}\right]=0
$$

with

$$
Q=\left[\begin{array}{ccc}
\frac{z_{c}^{2}}{l_{x}^{2}} & 0 & -\frac{z_{c}}{l_{x}^{2}} x_{c}  \tag{7}\\
0 & \frac{z_{c}^{2}}{l_{y}^{2}} & -\frac{z_{c}}{l_{y}^{2}} y_{c} \\
-\frac{z_{c}}{l_{x}^{2}} x_{c} & -\frac{z_{c}}{l_{y}^{2}} y & \frac{x_{c}^{2}}{l_{x}^{2}}+\frac{y_{c}^{2}}{l_{y}^{2}}-1
\end{array}\right] .
$$

This quadratic equation defines an elliptic cone and the ellipse is a slice of this cone.

## 4 Eigendecomposition

Diagonalizing $Q$ yields

$$
Q=\left[\begin{array}{lll}
V_{1}^{+} & V_{2}^{+} & V^{-}
\end{array}\right]\left[\begin{array}{ccc}
e_{1}^{+} & 0 & 0  \tag{8}\\
0 & e_{2}^{+} & 0 \\
0 & 0 & e^{-}
\end{array}\right]\left[\begin{array}{lll}
V_{1}^{+} & V_{2}^{+} & V^{-}
\end{array}\right]^{T}
$$

where

- $V_{1}^{+}, V_{2}^{+}, V^{-}$are the normalized eigenvectors,
- $e_{1}^{+}, e_{2}^{+}, e^{-}$are the eigenvalues,
- $e_{1}^{+}$and $e_{2}^{+}$are two positive eigenvalues,
- $e^{-}$is the single negative eigenvalue.


## 5 Front-facing ellipses

A front-facing tangent ellipse with the same solid angle can be obtained from

- center $\lambda V^{-}$(or $-\lambda V^{-}$because the quadratic equation is symmetric), and
- main axes of normalized directions $\left(V_{1}^{+}, V_{2}^{+}\right)$and respective lengths $\lambda \sqrt{\frac{-e^{-}}{e_{1}^{+}}}$and $\lambda \sqrt{\frac{-e^{-}}{e_{2}^{+}}}$,
where $\lambda>0$ is an arbitrary scaling factor.


Figure 3: Front-facing ellipse.

