Computing a front-facing ellipse that subtends the same solid angle as an arbitrarily oriented ellipse

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Abstract

A front-facing ellipse, with respect to an observation point, is an ellipse that is perpendicular to the line that goes through the origin and the center of the ellipse. Front-facing ellipses are more convenient than arbitrarily oriented ellipses for solid-angle computations.

In this report, we present an algebraic method for computing a front-facing ellipse that covers the same solid angle as an arbitrarily oriented ellipse. A previous approach was described by Conway [Con10] but it is involved, in terms of trigonometric derivations and computations. Our formulation shows that this problem is equivalent to computing the eigenvectors of a 3×3 matrix, which is simpler to derive and compute.

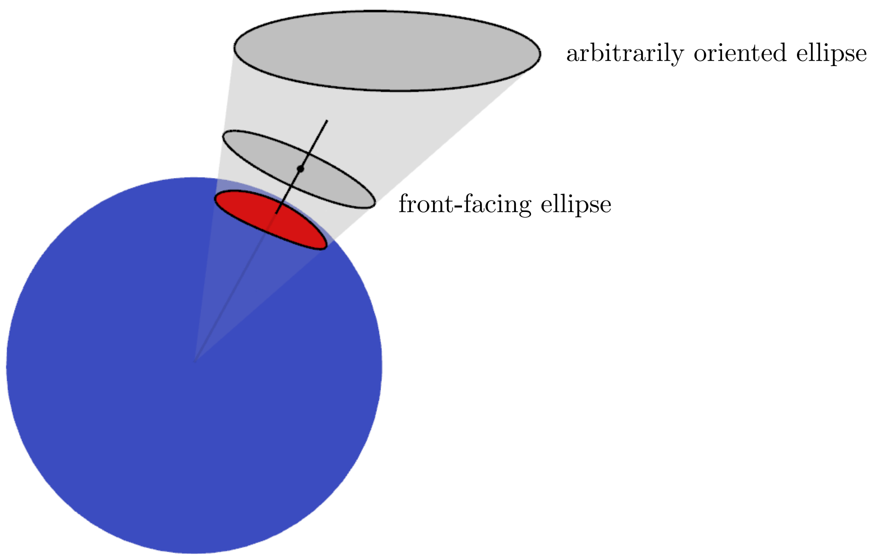

Figure 1: We compute a front-facing ellipse that covers the same solid angle as an arbitrarily oriented ellipse.

References

[Con10] John T. Conway. Analytical solution for the solid angle subtended at any point by an ellipse via a point source radiation vector potential. *Nuclear Instruments and Methods in Physics Research Section A: Accelerators, Spectrometers, Detectors and Associated Equipment*, 614(1):17 – 27, 2010. (document)

1 Arbitrarily oriented ellipse

We define the observation point as the origin: $O = (0, 0, 0)$. We consider an arbitrarily oriented ellipse of center C and of orthonormal axes of normalized directions V_x and V_y and of respective lengths l_x and l_y. The normal of the ellipse is $V_z = V_x \times V_y$ such that (V_x, V_y, V_z) is an orthonormal basis. In this basis, the coordinates of the ellipse center are

$$x_c = C \cdot V_x, \tag{1}$$
$$y_c = C \cdot V_y, \tag{2}$$
$$z_c = C \cdot V_z. \tag{3}$$

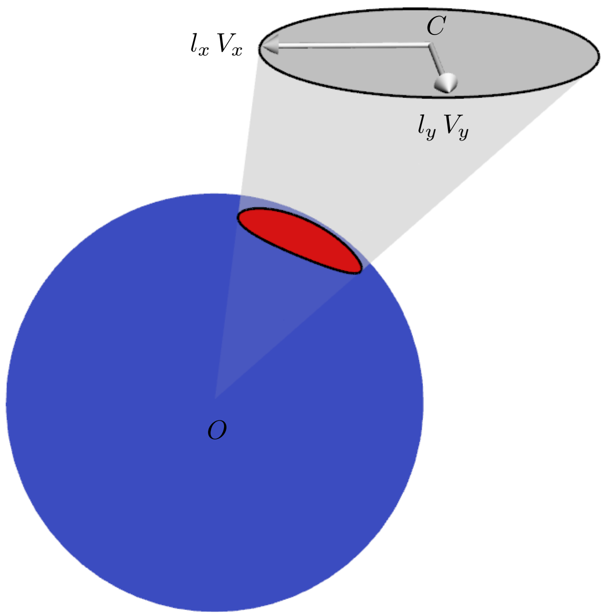

Figure 2: Arbitrarily oriented ellipse.

2 Planar Quadric

In the plane of the ellipse $z = z_c$, the equation of the ellipse is

$$\left(\frac{x - x_c}{l_x}\right)^2 + \left(\frac{y - y_c}{l_y}\right)^2 \leq 1. \tag{4}$$

3 Spherical Quadric

If (x, y, z) is a point on the sphere, the equation of the spherical quadric is

$$\left(\frac{\frac{x\,z_c}{z} - x_c}{l_x}\right)^2 + \left(\frac{\frac{y\,z_c}{z} - y_c}{l_y}\right)^2 \leq 1, \tag{5}$$

which can be rewritten

$$\begin{bmatrix} x & y & z \end{bmatrix} Q \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0, \tag{6}$$

with

$$Q = \begin{bmatrix} \frac{z_c^2}{l_x^2} & 0 & -\frac{z_c}{l_x^2}\,x_c \\ 0 & \frac{z_c^2}{l_y^2} & -\frac{z_c}{l_y^2}\,y_c \\ -\frac{z_c}{l_x^2}\,x_c & -\frac{z_c}{l_y^2}\,y & \frac{x_c^2}{l_x^2} + \frac{y_c^2}{l_y^2} - 1 \end{bmatrix}. \tag{7}$$

This quadratic equation defines an elliptic cone and the ellipse is a slice of this cone.

4 Eigendecomposition

Diagonalizing Q yields

$$Q = \begin{bmatrix} V_1^+ & V_2^+ & V^- \end{bmatrix} \begin{bmatrix} e_1^+ & 0 & 0 \\ 0 & e_2^+ & 0 \\ 0 & 0 & e^- \end{bmatrix} \begin{bmatrix} V_1^+ & V_2^+ & V^- \end{bmatrix}^T \tag{8}$$

where

- V_1^+, V_2^+, V^- are the normalized eigenvectors,
- e_1^+, e_2^+, e^- are the eigenvalues,
- e_1^+ and e_2^+ are two positive eigenvalues,
- e^- is the single negative eigenvalue.

5 Front-facing ellipses

A front-facing tangent ellipse with the same solid angle can be obtained from

- center λV^- (or $-\lambda V^-$ because the quadratic equation is symmetric), and
- main axes of normalized directions (V_1^+, V_2^+) and respective lengths $\lambda\sqrt{\frac{-e^-}{e_1^+}}$ and $\lambda\sqrt{\frac{-e^-}{e_2^+}}$,

where $\lambda > 0$ is an arbitrary scaling factor.

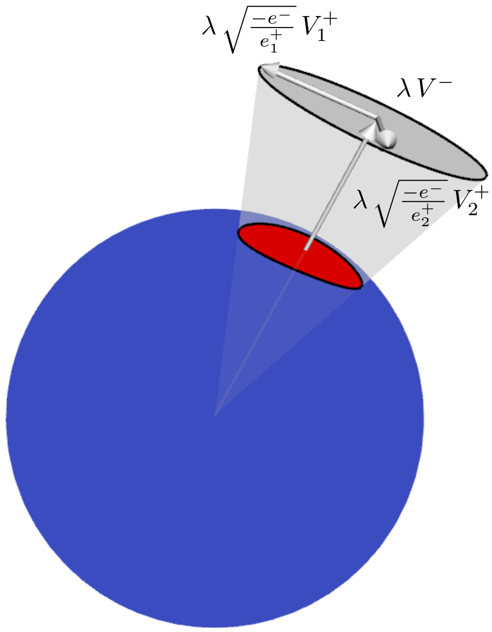

Figure 3: Front-facing ellipse.