A new four parameter extension of Burr-XII Distribution: its properties and applications

Laba Handique, Subrata Chakraborty

A new four parameter extension of Burr-XII Distribution: its properties and applications

[1] Laba Handique, Subrata Chakraborty [2*]

Department of Statistics, Dibrugarh University, Dibrugarh-786004, India
[1] Email: handiquelaba@gmail.com and [2]Email: subrata_stats@dibru.ac.in
* Corresponding Author

(**11th June 2017**)

Abstract

A new four parameter flexible extension of the Burr-XII distribution is proposed. A genesis for this distribution is presented. Some well known distributions are shown as special and related cases. Liner expansions for density and cumulative density functions, quantile function, moment generating function, ordinary moments, incomplete moments, order statistics, power weighted moments, Renyi entropy, relative entropy, stochastic orderings and stress-strength reliability are investigated. The proposed distribution is compared with its sub models and some existing generalizations of Bur XII taking five real life data sets for fitting using maximum likelihood method for parameter estimation. In all the five examples of applications the proposed model is found to be the best one in terms of different goodness of fit tests as well as model selection criteria.

Keywords: *Exponentiated family, Burr-XII distribution, Maximum likelihood, AIC, K-S test, LR test.*

1 Introduction

The Burr-XII (BXII) distribution was first introduced way back in 1942 by Burr (1942) as a two-parameter family. The cumulative distribution function (cdf) and probability density function (pdf) (for $t > 0$) of the BXII distribution are respectively given by $F(t) = 1 - (1 + t^{\lambda})^{-\beta}$ and $f(t) = \lambda \beta t^{\lambda-1} (1 + t^{\lambda})^{-\beta-1}$, where $\lambda > 0$ and $\beta > 0$ are the shape parameters.

A number of new extensions of the Burr-XII distribution are introduced to achieve extra flexibility in modelling data from variety of applications. Following are the notable ones:

Preprint submitted for publication

- Five parameter beta Burr XII [$\text{FBBXII}(a,b,c,k,s)$]distribution (Paranaiba *et al.*, 2011) with cdf and pdf

  $$F(t) = I_{1-\{1+(t/s)^c\}^{-k}} \text{ and } f(t) = \frac{c\,k\,t^{c-1}}{s^c\, B(a,b)}[1+(t/s)^c]^{-(bk-1)}[1-\{1+(t/s)^c\}^{-k}]^{(a-1)},$$

  where $t > 0,\ a,b > 0,\ c > 0,\ k > 0$ and $s > 0$

- Five parameter Kumaraswamy Burr-XII [$\text{FKwBXII}(a,b,c,k,s)$] distribution (Paranaiba *et al.*, 2013) with cdf and pdf

  $F(t) = 1 - [1-[1-\{1+(t/s)^c\}^{-k}]^a]^b$ and

  $$f(t) = a\,b\,c\,k\,s^{-c}\ t^{c-1}[1+(t/s)^c]^{-(k+1)}[1-\{1+(t/s)^c\}^{-k}]^{a-1}[1-[1-\{1+(t/s)^c\}^{-k}]^a]^{b-1},$$

  where $t > 0,\ a,b > 0,\ c > 0,\ k > 0$ and $s > 0$

- the Marshall-Olkin Extended Burr XII [$\text{MOBXII}(\lambda,\beta,\alpha)$] distribution (Arwa Y. Al-Saiari *et al.*, 2014) with cdf and pdf

  $$F(t) = \frac{1-(1+t^{\lambda})^{-\beta}}{1-\overline{\alpha}\,(1+t^{\lambda})^{-\beta}} \text{ and } f(t) = \frac{\alpha\,\lambda\,\beta\,t^{\lambda-1}(1+t^{\lambda})^{-\beta-1}}{[1-\overline{\alpha}\,(1+t^{\lambda})^{-\beta}]^2},$$

  where $-\infty < t < \infty,\ 0 < \alpha < \infty,\ \lambda > 0$ and $\beta > 0$

- A new generalization of Burr XII distribution (Mead, 2014) [in short $\text{BEBXII}(a,b,c,k,\beta)$] with pdf and cdf

  $$f(t) = \frac{\beta\,k\,c}{B(a,b)} t^{c-1}(1+t^c)^{-k-1}\{1-(1+t^c)^{-k}\}^{a\beta-1}[1-\{1-(1+t^c)^{-k}\}^{\beta}]^{b-1}$$

  and $F(t) = I_{\{1-(1+t^c)^{-k}\}^{\beta}}$, where $t > 0,\ a,b > 0,\ c > 0,\ k > 0$ and $\beta > 0$

- the new extended Burr-XII distribution [in short $\text{EBXII}(\lambda,\beta)$ (Ghosh and Bourguignon, 2017) with cdf and pdf

  $$F(t) = 1 - \frac{2}{1+(1+t^{\lambda})^{\beta}} \text{ and } f(t) = \frac{2\lambda\,\beta\,t^{\lambda-1}(1+t^{\lambda})^{\beta-1}}{[1+(1+t^{\lambda})^{\beta}]^2},$$ where $t > 0,\ \lambda > 0$ and $\beta > 0$

- and the Top-Leone Burr-XII [$\text{TLBXII}(\lambda,\beta,\alpha)$] distribution (Hesham and Soha, 2017) with cdf and pdf

  $$F(t) = [1-(1+t^{\lambda})^{-2\beta}]^{\alpha} \text{ and } f(t) = 2\alpha\,\lambda\,\beta\ t^{\lambda-1}(1+t^{\lambda})^{-(2\beta+1)}[1-(1+t^{\lambda})^{-2\beta}]^{\alpha-1},$$

  where $t > 0,\ \lambda > 0,\ \beta > 0$ and $\alpha > 0$.

In this paper we introduce a new extension of BXII distribution having four parameters λ,β,α and θ by considering the BXII as the baseline distribution in the generalized Marshall-Olkin-G family of distribution studied by Jayakumar and Mathew (2008). The cdf, pdf, survival function (sf) and hazard rate function (hrf) of this proposed distribution are respectively given by:

$$F(t;\lambda,\beta,\alpha,\theta)=1-\left[\frac{\alpha(1+t^{\lambda})^{-\beta}}{[1-\overline{\alpha}\,(1+t^{\lambda})^{-\beta}]}\right]^{\theta}=1-\left[\frac{\alpha}{[(1+t^{\lambda})^{\beta}-\overline{\alpha}\,]}\right]^{\theta} \quad (1)$$

$$f(t;\lambda,\beta,\alpha,\theta)=\frac{\theta\,\alpha^{\theta}\,\lambda\,\beta\,t^{\lambda-1}(1+t^{\lambda})^{-\beta\theta-1}}{[1-\overline{\alpha}\,(1+t^{\lambda})^{-\beta}]^{\theta+1}},\, t>0,\ \lambda>0,\beta>0,\alpha>0,\theta>0 \quad (2)$$

$$\overline{F}(t;\lambda,\beta,\alpha,\theta)=\left[\frac{\alpha(1+t^{\lambda})^{-\beta}}{[1-\overline{\alpha}\,(1+t^{\lambda})^{-\beta}]}\right]^{\theta} \quad (3)$$

and $h(t;\lambda,\beta,\alpha,\theta)=\{\theta\,\lambda\,\beta\,t^{\lambda-1}(1+t^{\lambda})^{-1}\}\big/[1-\overline{\alpha}\,(1+t^{\lambda})^{-\beta}]$, where $\overline{\alpha}=1-\alpha$.

We refer to this distribution as the Generalized Marshall-Olkin Extended Burr-XII, in short as GMOBXII$(\lambda,\beta,\alpha,\theta)$. One of the main attractions of the proposed distribution is that it includes many known one as a special and related case as narrated in table 1(a), 1(b)).

Table 1 (a): Some special case of GMOBXII$(\lambda,\beta,\alpha,\theta)$

λ	β	α	θ	**Reduced distribution**
---	---	---	1	MOBXII(λ,β,α)
---	---	1	---	BXII$(\lambda,\beta\theta)$
---	---	1	1	BXII(λ,β)
---	---	2	1	EBXII(λ,β)

Table 1 (b): Some related distributions of

T ~ GMOBXII$(\lambda,\beta,\alpha,\theta)$, for $s>0$

λ	β	α	θ	**Related distribution**
1	---	---	---	sT ~ GMOLomax(s,β,α,θ)
1	---	---	1	sT ~ MOLomax(s,β,α)
1	---	1	---	sT ~ Lomax$(s,\beta\theta)$
1	---	---	---	T/s ~ GMOLogLogistic(s,β,α,θ)
1	---	---	1	T/s ~ MOLogLogistic(s,β,α)
1	---	1	---	T/s ~ LogLogistic$(s,\beta\theta)$

From the plots of the pdf and hrf of the GMOBXII$(\lambda,\beta,\alpha,\theta)$ in figures 1 and 2 for different parameter values it can be seen that the distribution is very flexible and can offer many different types of shapes of density from right skewed to left skewed and symmetric and increasing as well as decreasing hazard rate functions. With increase in λ the skewness shifts from right to left, increase in β increases the kurtosis, increase in α (θ) results in the decrease of kurtosis when rest of the parameters are kept fixed. Thus the additional parameters α and θ in the proposed distribution accounts for peaked ness. The hazard function is increasing with α and θ and decreasing with λ and β.

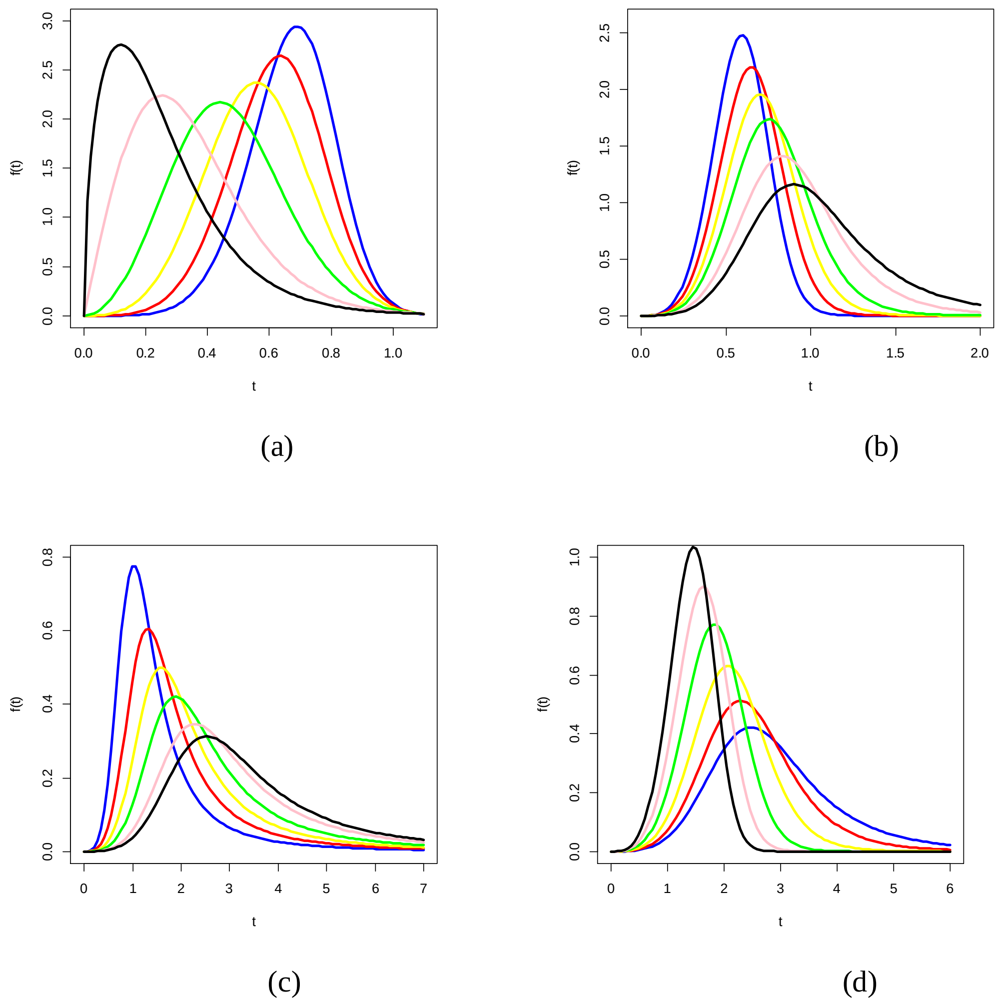
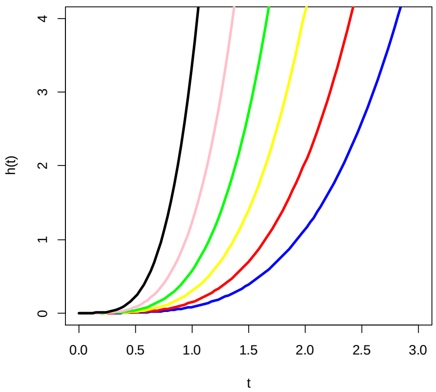
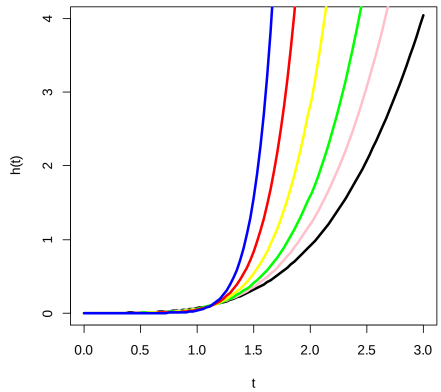


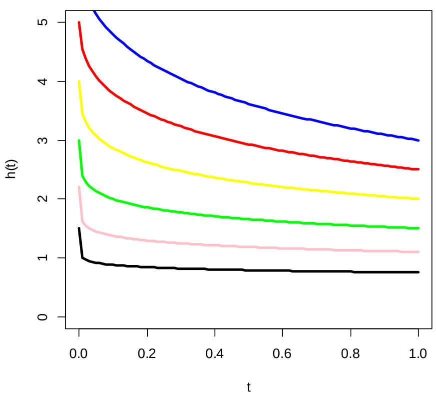
Fig. 1: Pdf plots for (a) $\beta = 15, \alpha = 1, \theta = 0.5$ with varying $\lambda = 1.5, 2, 3, 4, 5, 6$ (left to right) (b) $\lambda = 4.4, \alpha = 1, \theta = 0.5$ with varying $\beta = 15, 10, 7, 5, 3, 2$ (left to right) (c) $\lambda = 4.4, \beta = 1, \theta = 0.5$ with varying $\alpha = 1, 3, 7, 15, 35, 55$ (left to right) (d) $\lambda = 4.4, \beta = 1, \alpha = 85.5$ with varying $\theta = 12.5, 7.5, 4.5, 2.5, 1.5, 1$ (left to right)



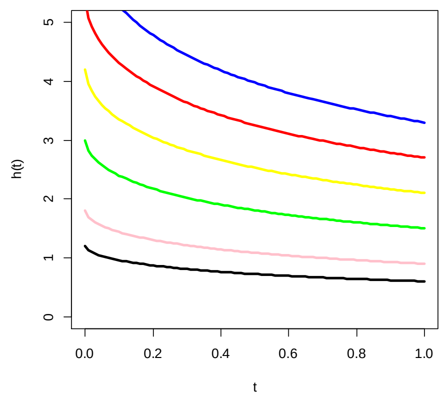
(a)



(b)



(c)



(d)

Fig. 2: hrf plot for (a) $\lambda = 1, \beta = 1, \alpha = 5$ with varying $\theta = 20.5, 7.5, 3.5, 1.8, 0.9, 0.5$ (left to right) (b) $\lambda = 1, \beta = 1, \theta = 0.4$ with varying $\alpha = 10.5, 8.5, 7, 6, 5.5, 5$ (left to right) (c) $\beta = 10, \alpha = 1, \theta = 1$ with varying $\lambda = 0.6, 0.5, 0.4, 0.3, 0.22, 0.15$ (top to down) (d) $\lambda = 0.6, \alpha = 1, \theta = 1$ with varying $\beta = 11, 9, 7, 5, 3, 2$ (top to down)

The rest of this article is organized in six more sections. A genesis of the proposed distribution in section 2. In section 3 we discuss some general results of the proposed family, while entropy is studied in section 4. Stochastic ordering and reliability properties are discussed in section 5 and 6. In section 7 maximum likelihood methods of estimation of parameters is presented. The data fitting applications are presented in section 8. The article ends with a conclusion in section 9.

2. A Genesis

Theorem1. For $i=1,2,...,\theta$ where $\theta>1$ is an integer, if $\{T_{\mathrm{i1}},T_{\mathrm{i2}},...,T_{\mathrm{iN}}\}$ be a sequence of *i.i.d.* Bur-XII random variables then (i) if N has a geometric distribution with parameter $\alpha\ (0<\alpha\le 1)$ independent of T_{ij}'s, then $\min\limits_{1\le i\le\theta}\{\min(T_{\mathrm{i1}},T_{\mathrm{i2}},...,T_{\mathrm{iN}})\}$ is distributed as $\mathrm{GMOBXII}(\lambda,\beta,\alpha,\theta)$ or (ii) if N has a geometric distribution with parameter $1/\alpha\,(\alpha>1)$ independent of T_{ij}'s, then $\min\limits_{1\le i\le\theta}\{\max(T_{\mathrm{i1}},T_{\mathrm{i2}},...,T_{\mathrm{iN}})\}$ is distributed as $\mathrm{GMOBXII}(\lambda,\beta,\alpha,\theta)$.

Proof: Let $T_{\mathrm{i1}},T_{\mathrm{i2}},...,T_{\mathrm{iN}}$ be a sequence of *i.i.d.* random variables from Bur-XII distribution with survival function $(1+t^{\lambda})^{-\beta}$, and suppose N has a geometric distribution with parameter p independent of T_{ij} 's. Then $W_{\mathrm{i}}=\min(T_{\mathrm{i1}},T_{\mathrm{i2}},...,T_{\mathrm{iN}})$ and $V_{\mathrm{i}}=\max(T_{\mathrm{i1}},T_{\mathrm{i2}},...,T_{\mathrm{iN}})$ are distributed as $\mathrm{MOBXII}(p,\lambda,\beta)$ and $\mathrm{MOBXII}(p^{-1},\lambda,\beta)$ respectively. Now

(i) For $0<\alpha\le 1$, if N has a geometric distribution with parameter α ,then

$$\mathrm{P}[\min\{W_1,W_2,...,W_\theta\}>t]=\mathrm{P}[W_1>t]\,\mathrm{P}[W_2>t]...\mathrm{P}[W_\theta>t]$$

$$=\prod_{\mathrm{i}=1}^{\theta}\mathrm{P}[W_{\mathrm{i}}>t]=[\overline{F}^{\mathrm{MOBXII}}(t;\alpha;\lambda,\beta)]^{\theta}=[\alpha\,(1+t^{\lambda})^{-\beta}\big/\{1-\overline{\alpha}\,(1+t^{\lambda})^{-\beta}\}]^{\theta}$$

This is the sf of $\mathrm{GMOBXII}(\lambda,\beta,\alpha,\theta)$.

(ii) For $\alpha>1$, if N has a geometric distribution with parameter α^{-1}, then

$$\mathrm{P}[\min\{V_1,\ V_2,...,V_\theta\}>t]=\mathrm{P}[V_1>t]\,\mathrm{P}[V_2>t]...\mathrm{P}[V_\theta>t]$$

$$=\prod_{i=1}^{\theta}P[V_i>t]=[\overline{F}^{\mathrm{MOBXII}}(t;\alpha;\lambda,\beta)]^{\theta}=[\alpha\,(1+t^{\lambda})^{-\beta}\big/\{1-\overline{\alpha}\,(1+t^{\lambda})^{-\beta}\}]^{\theta}\ \ [\because(\alpha^{-1})^{-1}=\alpha\,]$$

3 General Properties

3.1 Expansions of the survival and density functions as infinite linear mixture

Here we express the sf and pdf of the GMOBXII as infinite linear mixture of the corresponding functions of BXII distribution.

Consider the series representation $(1-z)^{-k}=\sum\limits_{j=0}^{\infty}\dfrac{\Gamma(k+j)}{\Gamma(k)\,j!}z^{j}$. (4)

This is valid for $|z|<1$ and $k>0$, where $\Gamma(.)$ is the gamma function.

Using equation (4) in equation (3), for $\alpha\in(0,1)$ we obtain

$$\overline{F}^{\mathrm{GMOBXII}}(t;\lambda,\beta,\alpha,\theta)=\alpha^{\theta}\{\overline{F}^{BXII}(t;\lambda,\beta)\}^{\theta}\sum_{j=0}^{\infty}\frac{(j+\theta-1)!}{(\theta-1)!\,j!}(1-\alpha)^{j}\{\overline{F}^{BXII}(t;\lambda,\beta)\}^{j}$$

$$= \sum_{j=0}^{\infty} A'_j [\overline{F}^{BXII}(t;\lambda,\beta)]^{j+\theta} = \sum_{j=0}^{\infty} A'_j \, \overline{F}^{BXII}(t;\lambda,\beta(j+\theta)) \quad (5)$$

Differentiating in equation (5) with respect to 't' we get

$$f^{\text{GMOBXII}}(t;\lambda,\beta,\alpha,\theta) = f^{\text{BXII}}(t;\lambda,\beta) \sum_{j=0}^{\infty} A_j [\overline{F}^{\text{BXII}}(t;\lambda,\beta)]^{j+\theta-1} \quad (6)$$

$$= -\sum_{j=0}^{\infty} A'_j \frac{d}{dt} [\overline{F}^{\text{BXII}}(t;\lambda,\beta)]^{j+\theta} \quad (7)$$

$$= \sum_{j=0}^{\infty} A'_j \, f^{\text{BXII}}(t;\lambda,\beta(j+\theta)) \quad (8)$$

Where $A'_j = A'_j(\alpha) = \binom{j+\theta-1}{j} (1-\alpha)^j \alpha^{\theta}$ and $A_j = A_j(\alpha) = (j+\theta) A'_j$.

3.2 Moment generating function

The moment generating function of $\text{GMOBXII}(t;\lambda,\beta,\alpha,\theta)$ family can be easily expressed in terms of those of the exponentiated B-XII distribution using the results of section 3.1. For example using equation (7) it can be seen that

$$M_T(s) = E[e^{sT}] = -\int_0^{\infty} e^{st} \sum_{j=0}^{\infty} A'_j \frac{d}{dt} [\overline{F}^{\text{BXII}}(t;\lambda,\beta)]^{j+\theta} \, dt$$

$$= \sum_{j=0}^{\infty} A'_j \int_0^{\infty} e^{st} \frac{d}{dt} \left\{ \overline{F}^{\text{BXII}}(t;\lambda,\beta) \right\}^{j+\theta} dt = \sum_{j=0}^{\infty} A_j M_X(s) \quad (9)$$

Where $M_X(s)$ is the mgf of a B-XII $[(\lambda,\beta(j+\theta))]$ distribution.

Paranaiba *et al.* (2011) provided the mgf of BXII as

$$M_{BXII}(s) = E[e^{sX}] = \lambda\beta \int_0^{\infty} e^{sx} x^{\lambda-1} (1+x^{\lambda})^{-(\beta+1)} \, dx$$

$$= m\, l'(-s, (m/\beta)-1, (m/\beta), -\beta-1) \quad (10)$$

Where for m and β positive integers, $\mu > -1$ and $p > 0$ (Prudnikov *et al.*,1992)

$$l'(p,\mu,\frac{m}{\beta},\nu) = \int_0^{\infty} e^{-pt} t^{\mu} (1+t^{\frac{m}{\beta}})^{\nu} \, dt$$

$$= \frac{\beta^{-\nu} m^{\mu+\frac{1}{2}}}{(2\pi)^{\frac{m-1}{2}} \Gamma(-\nu) \, p^{\mu+1}} G^{\beta,\beta+m}_{\beta+m,\beta} \left(\frac{m^m}{p^m} \middle| \begin{matrix} \Delta(m,-\mu), \ \Delta(\beta,\nu+1) \\ \Delta(\beta,0) \end{matrix} \right)$$

Where $\Delta\,(\beta,a)=\frac{a}{\beta},\frac{a+1}{\beta},...,\frac{a+\beta}{\beta};\lambda=\frac{m}{\beta}$; and the Meijer G-function defined by

$$G_{p,q}^{m,n}\left(x\left|\begin{matrix}a_1,...,a_p\\ b_1,...,b_q\end{matrix}\right.\right)=\frac{1}{2\pi i}\int_L\frac{\prod_{j=1}^{m}\Gamma(b_j+t)\prod_{j=1}^{n}\Gamma(1-a_j-t)}{\prod_{j=n+1}^{p}\Gamma(a_j+t)\prod_{j=m+1}^{p}\Gamma(1-b_j-t)}x^{-t}dt,$$

where $i=\sqrt{-1}$ is the complex unit and L denotes an integration path (see section 9.3 in Gradshteyn and Ryzhik, (2000) for a description of this path).

Hence from equation (9), the mgf of $\text{GMOBXII}(\lambda,\beta,\alpha,\theta)$ distribution for $s<0$ is given by

$$M_{GMOBXII}(s)=m\sum_{j=0}^{\infty}A_j\,l'\left(-s,\frac{m}{\beta(j+\theta)}-1,\ \frac{m}{\beta(j+\theta)},-\beta(j+\theta)-1\right) \qquad (11)$$

The Characteristic function (chf) is simply $\phi(s)=M(is)$, which holds for $s<1$. For special cases $\lambda=1$ and $\lambda=2$, we can obtain simple expressions for $M_{BXII}(s)$ and consequently for $M_{GMOBXII}(s)$ using equation (1) (on page 16) and (2) (on page 20) of the book by Prudnikov *et al.* (1992). For $\lambda=1$ and $s<0$, we have the mgf of BXII distribution is defined by

$M_{BXII}(s)=\beta\,(-s)^{\beta}\exp(-s)\Gamma(-\beta,-s)$, where $\Gamma(v,x)=\int_x^{\infty}t^{v-1}\exp(-t)\,dt$ is the complementary incomplete gamma function. For $\lambda=2$ and $t<0$, we obtain

$$M_{BXII}(s)={}_1F_2(1;\frac{1}{2};1-\beta;\frac{s^2}{4})+\frac{s}{2}B(2,\beta-\frac{1}{2})\,{}_1F_2(1;\frac{3}{2};\beta+\frac{7}{2};\frac{-s^2}{4})+\frac{\Gamma(-2\beta)}{(-s)^{-2\beta}}$$

where ${}_1F_2\,(a;b;c;x)=\sum_{i=0}^{\infty}\frac{(a)_i}{(b)_i(c)_\beta}\frac{x^i}{i!}$ is a generalized hyper geometric function and

$(a)_i=a(a+1)...(a+i-1)$ denotes the ascending factorial.

Using (9) we can express the moments of GMOBXII in terms of those of B-XII as

$$\mu'_r(\text{GMOBXII})=\sum_{j=0}^{\infty}A_j\mu'_r(\text{BXII}).$$

Where $\mu'_r(\text{BXII})$ is the r^{th} moment of B-XII [$(\lambda,\beta(j+\theta))$] given by

$\beta(j+\theta)B(\beta(j+\theta)-r/\lambda,1+r/\lambda)$ provided $\beta(j+\theta)\lambda>r\ (\beta\,\theta\,\lambda>r)$.

Alternatively, we can derive the ordinary moments [say μ'_r for $\alpha\in(0,1)$] directly as follows

$$\mu'_r = \mathrm{E}(X^r) = \theta\,\alpha^{\theta}\,\lambda\,\beta\int_0^{\infty} t^{r+\lambda-1}(1+t^{\lambda})^{-(\beta\theta+1)}[1-\overline{\alpha}\,(1+t^{\lambda})^{-\beta}]^{-(\theta+1)}\,dt$$

By using equation (4) in the last term of above integrand, we get

$$\mu'_r = \theta\,\alpha^{\theta}\,\lambda\,\beta\sum_{j=0}^{\infty}\frac{(j+\theta-1)!}{(\theta-1)!\,j!}(1-\alpha)^{j}\int_0^{\infty} t^{r+\lambda-1}(1+t^{\lambda})^{-(\beta(j+\theta)+1)}\,dt$$

Let $z=(1+t^{\lambda})^{-1}$ in the above equation, so we have

$$\mu'_r = \theta\,\alpha^{\theta}\,\beta\sum_{j=0}^{\infty}\binom{j+\theta-1}{j}(1-\alpha)^{j}\int_0^{1} z^{\beta(j+\theta)-(r/\lambda)-1}(1-z)^{r/\lambda}\,dz$$

$$=\theta\,\alpha^{\theta}\beta\sum_{j=0}^{\infty}\binom{j+\theta-1}{j}(1-\alpha)^{j}\,B[\beta(j+\theta)-(r/\lambda),\,(r/\lambda)+1] \qquad (12)$$

Where $B[.,.]$ is the beta function.

Substituting $r=1,2$ in equation (12), then we get the mean and variance respectively as follows.

$$\mu'_1 = \theta\,\alpha^{\theta}\,\beta\sum_{j=0}^{\infty}\binom{j+\theta-1}{j}(1-\alpha)^{j}\,B[\beta(j+\theta)-(1/\lambda),\,(1/\lambda)+1] \text{ and}$$

$$\mathrm{Var}(T) = \theta\,\alpha^{\theta}\,\beta\left[\sum_{j=0}^{\infty}\binom{j+\theta-1}{j}(1-\alpha)^{j}\,B[\beta(j+\theta)-(2/\lambda),\,(2/\lambda)+1]\right.$$
$$\left.-\theta\,\alpha^{\theta}\,\beta\left\{\sum_{j=0}^{\infty}\binom{j+\theta-1}{j}(1-\alpha)^{j}\,B[\beta(j+\theta)-(1/\lambda),\,(1/\lambda)+1]\right\}^{2}\right]$$

3.4 Incomplete moments

Suppose T is a random variable having the GMOBXII$(\lambda,\beta,\alpha,\theta)$ distribution, then for $\alpha\in(0,1)$ the r^{th} incomplete moments denoted as $m_r(z)$ can be obtained as follows:

$$m_r(z)=\int_0^{z} t^{r} f(t)\,dt = \theta\,\alpha^{\theta}\,\lambda\,\beta\sum_{j=0}^{\infty}\binom{j+\theta-1}{j}(1-\alpha)^{j}\int_0^{z} t^{r+\lambda-1}(1+t^{\lambda})^{-(\beta(j+\theta)+1)}\,dt$$

Substituting $y=(1+t^{\lambda})^{-1}$ and then using the binomial expansion we get

$$m_r(z)=\theta\,\alpha^{\theta}\,\lambda\,\beta\sum_{j=0}^{\infty}\sum_{k=o}^{r/\lambda}\binom{j+\theta-1}{j}\binom{r/\lambda}{k}(1-\alpha)^{j}(-1)^{k}\int_{(1+t^{\lambda})^{-1}}^{1} y^{\beta(j+\theta)+k-(r/\lambda)-1}\,dy$$

$$=\theta\,\alpha^{\theta}\,\lambda\,\beta\sum_{j=0}^{\infty}\sum_{k=o}^{r/\lambda}\binom{j+\theta-1}{j}\binom{r/\lambda}{k}(1-\alpha)^{j}(-1)^{k}\,\frac{[1-(1+t^{\lambda})^{(r/\lambda)-[\beta(j+\theta)+k]}]}{\beta(j+\theta)+k-(r/\lambda)}$$

3.5 Quantile function and median

A random number T from GMOBXII distribution can be easily generated by using inversion method by inverting the cdf or the survival function as

$$\Rightarrow T=\left(\left(\frac{[1-U]^{1/\theta}}{\alpha+\bar{\alpha}[1-U]^{1/\theta}}\right)^{-1/\beta}-1\right)^{1/\lambda} \text{ where } U\sim Uniform(0,1) \qquad (13)$$

From (13) the p^{th} quantile t_p and the median for GMOBXII$(\lambda,\beta,\alpha,\theta)$ distribution can be respectively obtained as

$$\left(\left(\frac{[1-p]^{1/\theta}}{\alpha+\bar{\alpha}[1-p]^{1/\theta}}\right)^{-1/\beta}-1\right)^{1/\lambda} \text{ and } \left(\left(\frac{[0.5]^{1/\theta}}{\alpha+\bar{\alpha}[0.5]^{1/\theta}}\right)^{-1/\beta}-1\right)^{1/\lambda}$$

3.6 Distribution of order statistics

Suppose $T_1,T_2,...,T_n$ is a random sample from the GMOBXII$(\lambda,\beta,\alpha,\theta)$ distribution and $T_{i:n}$ denote the i^{th} order statistics. Then the pdf of $T_{i:n}$ can be expressed as

$$f_{i:n}(t)=\frac{n!}{(i-1)!\,(n-i)!}f(t)\sum_{l=0}^{i-1}(-1)^l\binom{i-1}{l}\bar{F}(t)^{n+l-i}$$

Now using the general expansion of the pdf and sf of the GMOBXII$(\lambda,\beta,\alpha,\theta)$ distribution we get the pdf of the i^{th} order statistics as

$$f_{i:n}(t)=\frac{n!}{(i-1)!\,(n-i)!}\left\{f^{BXII}(t;\lambda,\beta)\sum_{j=0}^{\infty}A_j[\bar{F}^{\text{BXII}}(t;\lambda,\beta)]^{j+\theta-1}\right\}$$

$$\sum_{l=0}^{i-1}(-1)^{l+1}\binom{i-1}{l}\left\{\sum_{p=0}^{\infty}A'_p[\bar{F}^{\text{BXII}}(t;\lambda,\beta)]^{p+\theta(n+l-i)}\right\}$$

where A_j and A'_p are defined earlier

$$=\frac{n!}{(i-1)!\,(n-i)!}\sum_{l=o}^{i-1}\binom{i-1}{l}(-1)^{l+1}f^{\text{BXII}}(t;\lambda,\beta)\sum_{j=0}^{\infty}\sum_{p=0}^{\infty}A_j\,A'_p[\bar{F}^{\text{BXII}}(t;\lambda,\beta)]^{j+p+\theta(n+l-i+1)-1}$$

$$=f^{\text{BXII}}(t;\lambda,\beta)\sum_{j,p=0}^{\infty}M_{j,p}[\bar{F}^{\text{BXII}}(t;\lambda,\beta)]^{j+p+\theta(n+l-i+1)-1} \qquad (14)$$

$$=-\sum_{j,p=0}^{\infty}M'_{j,p}\frac{d}{dt}[\bar{F}^{\text{BXII}}(t;\lambda,\beta)]^{j+p+\theta(n+l-i+1)}$$

$$=\sum_{j,p=0}^{\infty}M'_{j,p}\,f^{\text{BXII}}(t;\lambda,\beta(j+p+\theta(n+l-i+1)))], \text{ where}$$

$$M_{j,p} = n A_j A'_p \binom{n-1}{i-1} \sum_{l=0}^{i-1} \binom{i-1}{l} (-1)^{l+1} \quad \text{and} \quad M'_{j,p} = M_{j,p} / (j + p + \theta(n+l-i+1))$$

3.7 Power moments

The probability weighted moments (PWMs), first proposed by Greenwood *et al.* (1979), are expectations of certain functions of a random variable whose mean exists. The $(p,q,r)^{th}$ PWM of T is defined by $\Gamma_{p,q,r} = \int_{-\infty}^{\infty} t^p [F(t)]^q [1-F(t)]^r f(t)\,dt$.

From equation (6) the s^{th} moment of T can be written either as

$$E(T^s) = \int_{-\infty}^{+\infty} t^s f^{\text{BXII}}(t;\lambda,\beta) \sum_{j=0}^{\infty} A_j [\bar{F}^{\text{BXII}}(t;\lambda,\beta)]^{j+\theta-1} dt$$

$$= \sum_{j=0}^{\infty} A_j \int_{-\infty}^{+\infty} t^s [\bar{F}^{\text{BXII}}(t;\lambda,\beta)]^{j+\theta-1} f^{\text{BXII}}(t;\lambda,\beta)\,dt = \sum_{j=0}^{\infty} A_j \Gamma^{\text{BXII}}_{s,0,\,j+\theta-1}$$

Where $\Gamma^{\text{BXII}}_{p,q,r} = \int_{-\infty}^{\infty} t^p \{F^{\text{BXII}}(t;\lambda,\beta)\}^q \{\bar{F}^{\text{BXII}}(t;\lambda,\beta)\}^r [f^{\text{BXII}}(t;\lambda,\beta)]dt$ is the PWM of $\text{B-XII}(\lambda,\beta)$ distribution.

Proceeding as above we can derive s^{th} moment of the i^{th} order statistic $T_{i:n}$, on using equations (14) as $E(T^s{}_{i,n}) = \sum_{j,p=0}^{\infty} M_{j,p} \Gamma^{\text{BXII}}_{s,0,\,j+p+\theta(n+l-i+1)-1}$. Where A_j and $M_{j,p}$ defined in section 3.1 and 3.6.

4 Entropy

In this subsection we present results on Rényi and relative entropy.

4.1 Rényi entropy

The entropy of a random variable is a measure of uncertainty variation and has been used in various situations in science and engineering. The Rényi entropy is defined by

$$I_R(\delta) = (1-\delta)^{-1} \log\left(\int_{-\infty}^{\infty} f(t)^{\delta} dt \right)$$

where $\delta > 0$ and $\delta \neq 1$. For furthers details, see Song (2001).

Using expansion (4), in (2) we can write

$$f(t;\alpha,\theta,\lambda,\beta)^{\delta} = [\theta\, \alpha^{\theta} \lambda \beta t^{\lambda-1} (1+t^{\lambda})^{-\beta\theta-1}]^{\delta} [1-\bar{\alpha}\,(1+t^{\lambda})^{-\beta}]^{-\delta(\theta+1)}$$

$$= \frac{\theta^{\delta} \alpha^{\delta\theta} [\lambda \beta t^{\lambda-1} (1+t^{\lambda})^{-\beta\theta-1}]^{\delta}}{\Gamma[\delta(\theta+1)]} \sum_{j=0}^{\infty} (1-\alpha)^j\, \Gamma[\delta(\theta+1)+j]\, \frac{[(1+t^{\lambda})^{-\beta}]^j}{j!}$$

Thus the Rényi entropy of $\text{GMOBXII}(\lambda,\beta,\alpha,\theta)$ distribution can be obtained as

$$I_R(\delta) = (1-\delta)^{-1} \log\left(\sum_{j=0}^{\infty} \mu_j \int_{-\infty}^{\infty} [\lambda \beta t^{\lambda-1} (1+t^{\lambda})^{-\beta\theta-1}]^{\delta} [(1+t^{\lambda})^{-\beta}]^{j} \, dt \right)$$

Where $\mu_j = \mu_j(\alpha) = \{\theta^{\delta} \alpha^{\delta\theta} (1-\alpha)^{j} \Gamma[\delta(\theta+1)+j]\}/\{\Gamma[\delta(\theta+1)] \; j!\}$

4.2 Relative entropy

The relative entropy (R.E.) between two distributions with pdfs $f_1(t)$ and $f_2(t)$ is define as

$$R.E(f_1, f_2) = E_{f_1}\left(\log\{f_1(t)/f_2(t)\}\right) = \int \left(\log\{f_1(t)/f_2(t)\}\right) f_1(t) \, dt \, .$$

This is also known as Kullback-Leibler divergence (distance) (Kullback, 1959). It is a measure of distance between $f_1(t)$ and $f_2(t)$ and can be seen as inefficiency of assuming f_2 for modeling when f_1 is the true distribution. We have presented the estimated $R.E.(GMOB-XII, \, f_2)$ where f_2 is a

For example taking, $f_1 = f^{\text{GMOBXII}}(t; \lambda, \beta, \alpha, \theta) = \dfrac{\theta \, \alpha^{\theta} \, \lambda \beta t^{\lambda-1} (1+t^{\lambda})^{-\beta\theta-1}}{[1-\bar{\alpha} \, (1+t^{\lambda})^{-\beta}]^{\theta+1}}$ and

$f_2 = f^{\text{MOBXII}}(t; \lambda, \beta, \alpha) = \dfrac{\alpha \lambda \beta t^{\lambda-1} (1+t^{\lambda})^{-\beta-1}}{[1-\bar{\alpha} (1+t^{\lambda})^{-\beta}]^{2}}$ then the relative entropy (R.E.) between two distributions with pdfs $\text{GMOBXII}(t; \lambda, \beta, \alpha, \theta)$ and $\text{MOBXII}(t; \lambda, \beta, \alpha)$ is defined by

$$R.E(\text{GMOBXII, MOBXII}) = E_{f_1}\left(\log\{f^{\text{GMOBXII}}(t; \lambda, \beta, \alpha, \theta) / f^{\text{MOBXII}}(t; \lambda, \beta, \alpha)\}\right)$$

$$= \int_0^{\infty} \log\left(\frac{\theta \, \alpha^{\theta-1} \, (1+t^{\lambda})^{-\beta(\theta-1)}}{[1-\bar{\alpha} \, (1+t^{\lambda})^{-\beta}]^{\theta-1}} \right) \frac{\theta \, \alpha^{\theta} \, \lambda \beta t^{\lambda-1} (1+t^{\lambda})^{-\beta\theta-1}}{[1-\bar{\alpha} \, (1+t^{\lambda})^{-\beta}]^{\theta+1}} \, dt$$

5 Stochastic orderings

Here we use the concept of stochastic ordering (Shaked and Shanthikumar, 2007) to derive different ordering conditions on parameters for random variables following $\text{GMOBXII}(\lambda, \beta, \alpha, \theta)$ distributions.

Let X and Y be two random variables with cfds F and G, respectively, survival functions $\bar{F} = 1 - F$ and $\bar{G} = 1 - G$, and corresponding pdf's f and g. Then X is said to be smaller than Y in the likelihood ratio order ($X \leq_{lr} Y$) if $f(t)/g(t)$ is decreasing in $t \geq 0$; stochastic order ($X \leq_{st} Y$) if $\bar{F}(t) \leq \bar{G}(t)$ for all $t \geq 0$; hazard rate order ($X \leq_{hr} Y$) if $\bar{F}(t)/\bar{G}(t)$ is decreasing in $t \geq 0$; reversed hazard rate order ($X \leq_{rhr} Y$) if $F(t)/G(t)$ is decreasing in $t \geq 0$. These four stochastic orders are related to each other, as $X \leq_{rhr} Y \Leftarrow X \leq_{lr} Y \Rightarrow X \leq_{hr} Y \Rightarrow X \leq_{st} Y$

Theorem 2: Let $X \sim \text{GMOBXII}(\lambda, \beta, \alpha_1, \theta)$ and $Y \sim \text{GMOBXII}(\lambda, \beta, \alpha_2, \theta)$. If $\alpha_1 < \alpha_2$, then $X \leq_{lr} Y$

Proof: $f(t)/g(t) = (\alpha_1/\alpha_2)^{\theta}[\{1-\bar{\alpha}_2[1+t^{\lambda}]^{-\beta}\}/\{1-\bar{\alpha}_1[1+t^{\lambda}]^{-\beta}\}]^{\theta+1}$

$$\frac{d}{dt}(f(t)/g(t)) = (\theta+1)(\alpha_1/\alpha_2)^{\theta}(\alpha_1-\alpha_2)\frac{\{1-\bar{\alpha}_2(1+t^{\lambda})^{-\beta}\}^{\theta}\lambda\beta(1+t^{\lambda})^{-\beta-1}t^{\lambda-1}}{\{1-\bar{\alpha}_1(1+t^{\lambda})^{-\beta}\}^{\theta+2}}$$

Now this is always less than 0 since $\alpha_1 < \alpha_2$. Hence, $f(t)/g(t)$ is decreasing in *t*. That is $X \le_{lr} Y$.

6 Reliability

In reliability, a component which has a random strength X_1 and exposed to a random stress X_2 fails at the instant when the stress exceeds the strength that is $X_2 > X_1$, and keeps performing satisfactorily whenever $X_1 > X_2$. Here we derive the reliability $R = \Pr(X_2 < X_1)$ when the strength X_1 and the stress X_2 have independent GMOBXII$(\lambda,\beta_1,\alpha_1,\theta)$ and GMOBXII$(\lambda,\beta_2,\alpha_2,\theta)$ distributions. First the cdf of X_2 and pdf of X_1 are written using the equations (5) and (8) as

$F_2(x) = \sum_{j=0}^{\infty} A'_j(\alpha_2) F^{BXII}(t;\lambda,\beta_2(j+\theta))$ and $f_1(x) = \sum_{k=0}^{\infty} A'_k(\alpha_1) f^{\mathrm{BXII}}(t;\lambda,\beta_1(k+\theta))$ respectively,

where $A'_k(\alpha_1)$ and $A'_j(\alpha_2)$ are described in section 3.

Hence $R = \Pr(X_2 < X_1) = \int_0^{\infty} f_1(t) F_2(t)\, dt$

$$= \sum_{j,k=0}^{\infty} A'_k(\alpha_1) A'_j(\alpha_2) \int_0^{\infty} f^{BXII}(t;\lambda,\beta_1(k+\theta)) F^{BXII}(t;\lambda,\beta_2(j+\theta))\, dt$$

$$= \sum_{j,k=0}^{\infty} A'_k(\alpha_1) A'_j(\alpha_2) \lambda\beta_1(k+\theta) \int_0^{\infty} t^{\lambda-1}(1+t^{\lambda})^{-\beta_1(k+\theta)-1}[1-(1+t^{\lambda})^{-\beta_2(j+\theta)}]\, dt$$

$$= \sum_{j,k=0}^{\infty} A'_k(\alpha_1) A'_j(\alpha_2) \beta_1(k+\theta)\eta'(\lambda,\beta_1,\beta_2,j,k,\theta).$$

Where $\eta'(\lambda,\beta_1,\beta_2,j,k,\theta) = \int_0^{\infty} t^{\lambda-1}(1+t^{\lambda})^{-\beta_1(k+\theta)-1}[1-(1+t^{\lambda})^{-\beta_2(j+\theta)}]\, dt$

$= \dfrac{(j+\theta)\beta_2}{(k+\theta)\beta_1[(k+\theta)\beta_1 + (j+\theta)\beta_2]}$ can be shown by substituting $u = (1+t^{\lambda})$

7 Maximum Likelihood Estimation

In this section, parameters estimation of the GMOBXII$(\lambda,\beta,\alpha,\theta)$ distribution is conducted using the maximum likelihood method.

Let $t=(t_1,t_2,...,t_r)^T$ be a random sample of size r from the GMOBXII$(\lambda,\beta,\alpha,\theta)$ distribution with parameter vector $\vartheta=(\lambda,\beta,\alpha,\theta)^T$, then the log-likelihood function for ϑ is given by

$$\ell=\ell(\vartheta)=r\log\theta+r\log\lambda+r\log\beta+r\,\theta\log\alpha+(\lambda-1)\sum_{i=0}^{r}\log t_i-(\beta\theta+1)\sum_{i=0}^{r}\log(1+t_i{}^{\lambda})$$

$$-(\theta+1)\sum_{i=0}^{r}\log[1-\overline{\alpha}\,(1+t_i{}^{\lambda})^{-\beta}]$$

This log-likelihood function can not be solved analytically because of its complex form but it can be maximized numerically by employing global optimization methods available with software's like R, SAS, Mathematica.

By taking the partial derivatives of the log-likelihood function with respect to λ,β,α and θ we obtain the components of the score vector $U_{\vartheta}=(U_{\lambda},U_{\beta},U_{\alpha},U_{\theta})^{\mathrm{T}}$

$$U_{\lambda}=\frac{\partial\ell(\vartheta)}{\partial\lambda}=\frac{r}{\lambda}+\sum_{i=0}^{r}\log t_i-(\beta\theta+1)\sum_{i=0}^{r}\frac{t_i{}^{\lambda}\log t_i}{1+t_i{}^{\lambda}}-(\theta+1)\sum_{i=0}^{r}\frac{\overline{\alpha}\beta(1+t_i{}^{\lambda})^{-\beta-1}t_i{}^{\lambda}\log t_i}{1-\overline{\alpha}\,(1+t_i{}^{\lambda})^{-\beta}}$$

$$U_{\beta}=\frac{\partial\ell(\vartheta)}{\partial\beta}=\frac{r}{\beta}-\theta\sum_{i=0}^{r}\log(1+t_i{}^{\lambda})-(\theta+1)\sum_{i=0}^{r}\frac{\overline{\alpha}\,(1+t_i{}^{\lambda})^{-\beta}\log(1+t_i{}^{\lambda})}{1-\overline{\alpha}\,(1+t_i{}^{\lambda})^{-\beta}}$$

$$U_{\alpha}=\frac{\partial\ell(\vartheta)}{\partial\alpha}=\frac{r\theta}{\alpha}-(\theta+1)\sum_{\mathrm{i}=0}^{\mathrm{r}}\frac{(1+t_i{}^{\lambda})^{-\beta}}{1-\overline{\alpha}\,(1+t_i{}^{\lambda})^{-\beta}}$$

$$U_{\theta}=\frac{\partial\ell(\vartheta)}{\partial\theta}=\frac{r}{\theta}+r\log\alpha-\beta\sum_{i=0}^{r}\log(1+t_i{}^{\lambda})-\sum_{i=0}^{r}\log[1-\overline{\alpha}\,(1+t_i{}^{\lambda})^{-\beta}]$$

Setting these equations to zero $U_{\vartheta}=(U_{\lambda},U_{\beta},U_{\alpha},U_{\theta})^{\mathrm{T}}=0$ and solving them simultaneously yields the maximum likelihood estimate (MLE) $\hat{\vartheta}=(\hat{\lambda},\hat{\beta},\hat{\alpha},\hat{\theta})^T$ of $\vartheta=(\lambda,\beta,\alpha,\theta)^T$.

7.1 Asymptotic standard error for the mles

The asymptotic variance-covariance matrix of the MLEs of parameters can obtained by inverting the Fisher information matrix $\mathrm{I}(\vartheta)$ which can be derived using the second partial derivatives of the log-likelihood function with respect to each parameter. The $i\,j^{th}$ elements of $\mathrm{I}_n(\vartheta)$ are given by $\mathrm{I}_{ij}=-E[\partial^2 l(\vartheta)/\partial\vartheta_i\partial\vartheta_j]$, $\quad i,\ j=1,2,\cdots,3+q$.

The exact evaluation of the above expectations may be cumbersome. In practice one can estimate $\mathrm{I}_n(\vartheta)$ by the observed Fisher's information matrix $\hat{\mathrm{I}}_n(\hat{\vartheta})=(\hat{\mathrm{I}}_{ij})$ is defined as:

$$\hat{\mathrm{I}}_{ij}\approx\left(-\partial^2 l(\vartheta)/\partial\vartheta_i\partial\vartheta_j\right)_{\vartheta=\hat{\vartheta}},\qquad i,\ j=1,2,\cdots,3+q$$

Using the general theory of MLEs under some regularity conditions on the parameters as $n \to \infty$ the asymptotic distribution of $\sqrt{n}\,(\hat{\vartheta} - \vartheta)$ is $N_k(0, V_n)$ where $V_n = (v_{jj}) = \mathrm{I}_n^{-1}(\vartheta)$. The asymptotic behaviour remains valid if V_n is replaced by $\hat{V}_n = \hat{\mathrm{I}}^{-1}(\hat{\vartheta})$. This result can be used to provide large sample standard errors for the model parameters. Thus an approximate standard error for the MLE of j^{th} parameter ϑ_j is given by $\sqrt{\hat{v}_{jj}}$.

8 Applications

We consider modelling of the following five real life data sets with different shapes to illustrate the suitability of the $\mathrm{GMOBXII}(\lambda, \beta, \alpha, \theta)$ distribution in comparison to some existing distributions by estimating the parameters by numerical maximization of log-likelihood functions.

Data Set I: This data set about 346 nicotine measurements made from several brands of cigarettes in 1998. The data have been collected by the Federal Trade Commission which is an independent agency of the US government, whose main mission is the promotion of consumer protection**.** [http: //www.ftc.gov/ reports/tobacco or http: // pw1.netcom.com/ rdavis2/ smoke. html.]

Data Set II: This data set is obtained from Smith and Naylor (1987). The data consists of 63 observations of the strengths of 1.5 cm glass fibres, measured at the National Physical Laboratory, England.

Data Set III: In this application, we work with the survival times (in days) of 72 guinea pigs infected with virulent tubercle bacilli, observed and reported by Bjerkedal (1960).

Data Set IV: This data have been obtained from Aarset (1987), and widely reported in many literatures. It represents the lifetimes of 50 devices.

Data Set V: This data set consists of 153 observations, of which 85 are classified as failed windshields, and the remaining 68 are service times of windshields that had not failed at the time of observation. The unit for measurement is 1000 h. This data is previously analyzed by Ramos *et al.* (2013) and Tahir *et al.* (2015).

We have presented the descriptive statistics of all the data sets in table 1.

Table 2: Descriptive Statistics for the data set I, II, III, IV and V

Data Sets	Minimum	Mean	Median	s.d.	Skewness	Kurtosis	1st Qu.	3rd Qu.	Maximum
I	0.100	0.853	0.900	0.334	0.171	0.296	0.600	1.100	2.000
II	0.550	1.507	1.590	0.324	-0.879	0.800	1.375	1.685	2.240
III	0.100	1.851	1.560	1.201	1.788	4.158	1.080	2.303	7.000
IV	0.100	45.690	48.500	32.835	-0.134	-1.642	13.500	81.250	86.000
V	0.040	2.563	2.385	1.113	0.085	-0.689	1.866	3.376	4.663

Upon fitting the best model is chosen as the one having lowest AIC (Akaike Information Criterion), BIC (Bayesian Information Criterion), CAIC (Consistent Akaike Information Criterion) and HQIC (Hannan-Quinn Information Criterion). It may be noted that $AIC = 2k - 2l$; $BIC = k\log(n) - 2l$; $CAIC = AIC + (2k(k+1))/(n-k-1)$; and $HQIC = 2k\log[\log(n)] - 2l$, where k is the number of parameters in the statistical model, n the sample size and l is the maximized value of the log-likelihood function under the considered model. The Kolmogorov-Smirnov (K-S) statistics is also used to compare the fitted models. For visual comparison all the fitted densities and the fitted cdf's are plotted with the corresponding observed histograms and ogives.

First we have compared the $\text{GMOBXII}(\lambda,\beta,\alpha,\theta)$ distribution with some of its sub models namely MOB-XII, B-XII and EB-XII models and also compared the proposed $\text{GMOBXII}(\lambda,\beta,\alpha,\theta)$ distribution with another recently introduced models TLB-XII, KwB-XII and BetaB-XII distributions for all five data sets.

We have also carried out Likelihood Ratio Test for nested models. The $\text{GMOBXII}(\lambda,\beta,\alpha,\theta)$ distribution reduces to $\text{MOB-XII}(\lambda,\beta,\alpha)$ when $\theta = 1$, to $\text{BXII}(\lambda,\beta)$ if $\alpha = \theta = 1$ to $\text{EBXII}(\lambda,\beta)$ if $\alpha = 2, \theta = 1$. So here we have employed likelihood ratio test the following null hypothesis:

(i) $H_0: \theta = 1$, that is the sample is from $\text{MOB-XII}(\lambda,\beta,\alpha)$
$H_1: \theta \neq 1$, that is the sample is from $\text{GMOBXII}(\lambda,\beta,\alpha,\theta)$.

(ii) $H_0: \alpha = \theta = 1$, that is the sample is from $\text{BXII}(\lambda,\beta)$
$H_1: \alpha \neq \theta \neq 1$, that is the sample is from $\text{GMOBXII}(\lambda,\beta,\alpha,\theta)$.

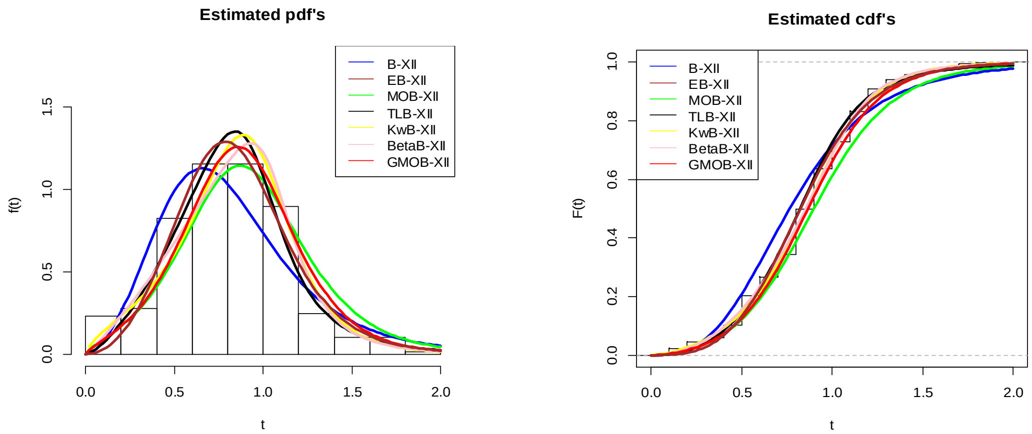
(iii) $H_0: \alpha = 2, \theta = 1$, that is the sample is from $\text{EBXII}(\lambda,\beta)$
$H_1: \alpha \neq 2, \theta \neq 1$, that is the sample is from $\text{GMOBXII}(\lambda,\beta,\alpha,\theta)$.

Writing $\vartheta = (\lambda,\beta,\alpha,\theta)$ the likelihood ratio test statistic is given by LR $= -2\ln(L(\hat{\vartheta}^*; x)/L(\hat{\vartheta}, x))$, where $\hat{\vartheta}^*$ is the restricted ML estimates under the null hypothesis H_0 and $\hat{\vartheta}$ is the unrestricted ML estimates under the alternative hypothesis H_1. Under the null hypothesis H_0 the LR criterion follows Chi-square distribution with degrees of freedom (df) $(df_{alt} - df_{null})$. The null hypothesis is rejected for *p*-value less than 0.05.

Visual comparison fitted densities and the fitted cdf's are is presented in the form of a Histograms and Ogives of the data in Figures 3, 4, 5, 6 and 7. These plots indicate that the proposed distributions provide a good fit to these data.

Table 3: MLEs, standard errors, confidence interval (in parentheses) with AIC, BIC, CAIC, HQIC, K-S (*p*-value) and LR (*p*-value) values for the data set I

Models	$\hat{\lambda}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\theta}$	AIC	BIC	CAIC	HQIC	K-S (*p*-value)	L-R (*p*-value)
B-XII (λ,β)	2.846 (0.154) (2.54, 3.15)	1.808 (0.097) (1.62, 1.99)	---	---	258.58	266.27	258.61	261.65	0.25 (0.14)	40.24 (0.0002)
EB-XII (λ,β)	3.060 (0.139) (2.79, 3.33)	2.529 (0.113) (2.31, 2.75)	---	---	240.58	248.28	240.61	243.66	0.27 (0.09)	22.24 (0.009)
MOB-XII (λ,β,α)	1.733 (0.298) (1.14, 2.32)	4.953 (0.096) (4.76, 5.14)	19.137 (12.990) (0, 44.59)	---	226.98	238.51	227.05	231.57	0.26 (0.12)	6.64 (0.009)
TLB-XII (α,λ,β)	9.213 (2.369) (4.57, 13.85)	0.262 (0.036) (0.19,0.33)	0.271 (0.018) (0.24, 0.31)	---	232.70	244.23	232.77	237.29	0.29 (0.05)	---
KwB-XII $(\alpha,\theta,\lambda,\beta)$	6.201 (0.892) (4.45, 7.95)	4.948 (2.011) (1.01,8.89)	0.271 (0.056) (0.16, 0.38)	0.245 (0.071) (0.11,0.38)	224.94	240.32	225.05	231.06	0.28 (0.07)	---
BetaB-XII $(\alpha,\theta,\lambda,\beta)$	12.778 (0.021) (12.73,12.82)	2.022 (0.596) (0.85,3.19)	0.170 (0.012) (0.15,0.19)	0.249 (0.057) (0.14,0.36)	224.12	239.50	224.24	230.24	0.29 (0.06)	---
GMOBXII $(\lambda,\beta,\alpha,\theta)$	1.812 (0.303) (1.22, 2.41)	4.315 (1.103) (2.15,6.48)	22.128 (14.349) (0, 50.25)	1.856 (0.770) (0.35,3.37)	**222.34**	**237.72**	**222.45**	**228.47**	0.21 (**0.18**)	---

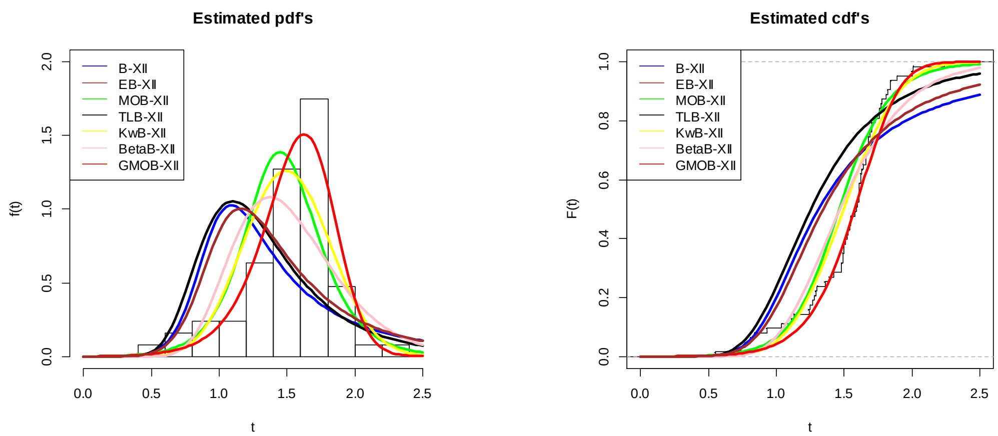


(a) (b)

Fig 3: Plots of the (a) observed histogram and estimated pdf's and (b) observed ogive and estimated cdf's for the B - XII, EB-XII, MOB - XII ,TLB-XII, KwB-XII, BetaB-XII and GMOBXII for data set I.

Table 4: MLEs, standard errors, confidence interval (in parentheses) with AIC, BIC, CAIC, HQIC, K-S (*p*-value) and LR (*p*-value) values for the data set II

Models	$\hat{\lambda}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\theta}$	AIC	BIC	CAIC	HQIC	K-S (*p*-value)	L-R (*p*-value)
B-XII	7.483	0.321	---	---	101.44	105.72	101.64	103.12	0.18	74.72
(λ,β)	(1.285)	(0.064)							(0.09)	(0.002)
	(4.96, 10.00)	(0.19, 0.45)								
EB-XII	7.155	0 .489	---	---	84.34	88.62	84.54	86.02	0.24	57.62
(λ,β)	(1.253)	(0.094)							(0.06)	(0.017)
	(4.69,9.61)	(0.30,0.67)								
MOB-XII	3.064	3.453	143.422	---	40.70	47.12	41.11	43.22	0.15	11.98
(λ,β,α)	(1.212)	(1.880)	(89.056)						(0.17)	(0.0005)
	(0.69, 5.44)	(0, 7.14)	(0, 317.97)							
TLB-XII	2.643	0.941	7.505	---	76.70	83.52	77.10	79.62	0.31	---
(α,λ,β)	(0.796)	(0.369)	(4.548)						(0.05)	
	(1.08, 4.20)	(0, 1.66)	(0, 16.42)							
KwB-XII	0.367	4.485	188.968	299.578	46.88	55.44	47.56	50.24	0.10	---
$(\alpha,\theta,\lambda,\beta)$	(0.155)	(0.983)	(130.407)	(134.773)					(0.21)	
	(0.06, 0.67)	(2.56, 6.411)	(0, 444.56)	(35.62, 563.73)						
BetaB-XII	0.418	0.966	184.248	164.788	64.22	72.78	64.90	67.58	0.14	---
$(\alpha,\theta,\lambda,\beta)$	(0.036)	(0.196)	(71.239)	(51.405)					(0.25)	
	(0.35, 0.49)	(0.58, 1.35)	(44.61, 323.88)	(64.03, 265.54)						
GMOBXII	2.913	3.128	838.36	4.710	**30.72**	**39.29**	**31.41**	**34.08**	0.13	---
$(\lambda,\beta,\alpha,\theta)$	(1.700)	(2.399)	(138.16)	(4.446)					(**0.34**)	
	(0, 6.25)	(0, 7.83)	(567.56,1109.15)	(0, 13.42)						

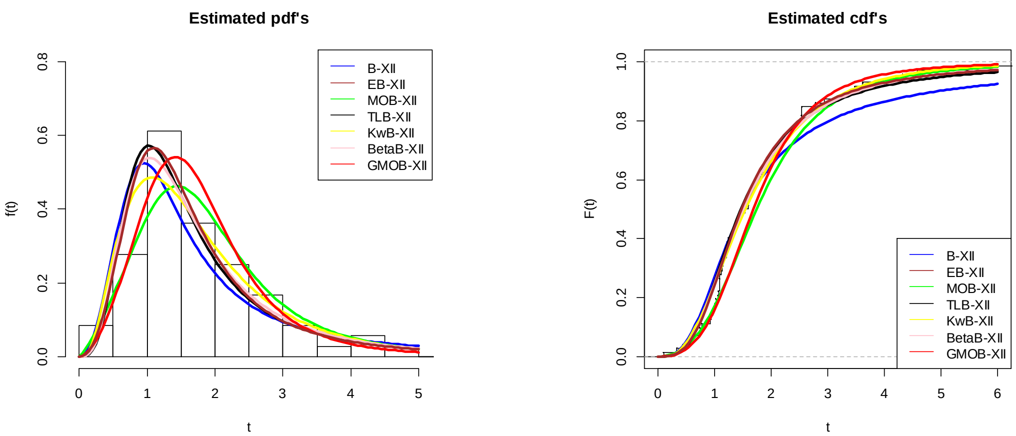


(a) (b)

Fig 4: Plots of the (a) observed histogram and estimated pdf's and (b) observed ogive and estimated cdf's for the B - XII , EB-XII, MOB - XII ,TLB-XII, KwB-XII, BetaB-XII and GMOBXII for data set II.

Table 5: MLEs, standard errors, confidence interval (in parentheses) with AIC, BIC, CAIC, HQIC, K-S (*p*-value) and LR (*p*-value) values for the data set III

Models	$\hat{\lambda}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\theta}$	AIC	BIC	CAIC	HQIC	K-S (*p*-value)	L-R (*p*-value)
B-XII (λ,β)	3.102 (0.538) (2.05, 4.16)	0.465 (0.077) (0.31, 0.62)	---	---	209.60	214.15	209.77	211.40	0.11 (0.35)	10.46 (0.005)
EB-XII (λ,β)	3.263 (0.491) (2.30,4.23)	0 .729 (0.117) (0.49,0.96)	---	---	205.80	210.36	205.97	207.60	0.08 (0.73)	6.66 (0.04)
MOB-XII (λ,β,α)	2.259 (0.864) (0.57, 3.95)	1.533 (0.907) (0, 3.31)	6.760 (4.587) (0, 15.75)	---	209.74	216.56	210.09	212.44	0.10 (0.43)	8.60 (0.003)
TLB-XII (α,λ,β)	2.393 (0.907) (0.62, 4.17)	0.458 (0.244) (0, 0.94)	1.796 (0.915) (0.002, 3.59)	---	211.80	218.63	212.15	214.52	0.09 (0.61)	---
KwB-XII $(\alpha,\theta,\lambda,\beta)$	0.525 (0.279) (0, 1.07)	2.274 (0.990) (0.33, 4.21)	14.105 (10.805) (0, 35.28)	7.424 (11.850) (0, 30.65)	208.76	217.86	209.36	212.38	0.08 (0.62)	---
BetaB-XII $(\alpha,\theta,\lambda,\beta)$	1.800 (0.955) (0, 3.67)	0 .294 (0.466) (0, 1.21)	2.555 (1.859) (0, 6.28)	6.058 (10.391) (0, 26.42)	210.44	219.54	211.03	214.06	0.08 (0.66)	---
GMOBXII $(\lambda,\beta,\alpha,\theta)$	2.481 (0.848) (0.82, 4.14)	1.113 (1.002) (0, 3.08)	6.032 (5.098) (0, 16.02)	1.369 (1.238) (0, 3.79)	**203.14**	**212.22**	**203.73**	**206.75**	0.05 (**0.86**)	---

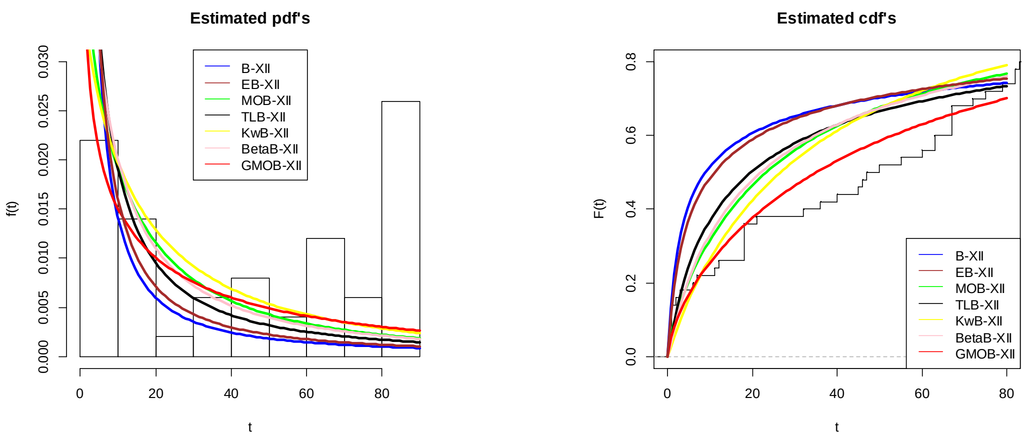


(a) (b)

Fig 5: Plots of the (a) observed histogram and estimated pdf's and (b) observed ogive and estimated cdf's for the B - XII, EB-XII, MOB - XII ,TLB-XII, KwB-XII, BetaB-XII and GMOBXII for data set III.

Table 6: MLEs, standard errors, confidence interval (in parentheses) with AIC, BIC, CAIC, HQIC, K-S (*p*-value) and LR (*p*-value) values for the data set IV

Models	$\hat{\lambda}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\theta}$	AIC	BIC	CAIC	HQIC	K-S (*p*-value)	L-R (*p*-value)
B-XII	1.259	0.246	---	---	548.72	552.54	548.97	550.16	0.29	55.80
(λ, β)	(0.321)	(0.069)							(0.007)	(0.011)
	(0.63, 1.89)	(0.11, 0.38)								
EB-XII	1.169	0 .384	---	---	536.40	540.22	536.66	537.84	0.61	43.48
(λ, β)	(0.298)	(0.105)							(0.09)	(0.0002)
	(0.58,1.75)	(0.18,0.59)								
MOB-XII	0.555	1.936	39.049	---	506.80	512.54	507.32	508.96	0.21	11.88
(λ, β, α)	(0.149)	(0.574)	(17.084)						(0.09)	(0.0005)
	(0.26, 0.85)	(0.81, 3.06)	(5.56, 72.53)							
TLB-XII	0.320	1.157	12.999	---	524.24	529.97	524.76	526.42	0.25	---
(α, λ, β)	(0.122)	(0.572)	(11.678)						(0.06)	
	(0.08, 0.56)	(0.04, 2.28)	(0, 35.89)							
KwB-XII	0.045	5.188	332.897	340.317	500.32	507.96	501.21	503.23	0.17	---
$(\alpha, \theta, \lambda, \beta)$	(0.013)	(1.498)	(110.30)	(171.36)					(0.19)	
	(0.02, 0.07)	(2.25, 8.12)	(116.71,549.08)	(4.55, 676.18)						
BetaB-XII	0.057	2.236	249.703	52.441	514.50	522.15	515.39	517.41	0.14	---
$(\alpha, \theta, \lambda, \beta)$	(0.012)	(1.023)	(117.55)	(10.391)					(0.25)	
	(0.03, 0.08)	(0.23, 4.24)	(19.31,480.10)	(32.07, 72.80)						
GMOBXII	0.715	0.945	485.85	31.282	**496.92**	**504.56**	**501.81**	**499.80**	0.28	---
$(\lambda, \beta, \alpha, \theta)$	(0.132)	(0.144)	(231.83)	(33.245)					(**0.32**)	
	(0.45, 0.97)	(0.66, 1.22)	(31.46, 940.24)	(0, 96.44)						

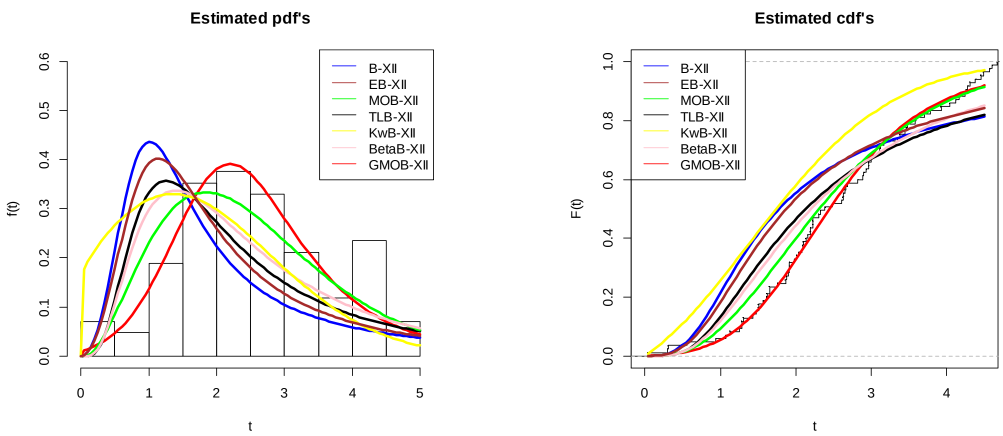


(a) (b)

Fig 6: Plots of the (a) observed histogram and estimated pdf's and (b) observed ogive and estimated cdf's for the B - XII, EB-XII, MOB - XII ,TLB-XII, KwB-XII, BetaB-XII and GMOBXII for data set IV.

Table 7: MLEs, standard errors, confidence interval (in parentheses) with AIC, BIC, CAIC, HQIC, K-S (*p*-value) and LR (*p*-value) values for the data set V

Models	$\hat{\lambda}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\theta}$	AIC	BIC	CAIC	HQIC	K-S (*p*-value)	L-R (*p*-value)
B-XII (λ,β)	3.179 (0.449) (2.29, 4.06)	0.350 (0.059) (0.23, 0.47)	---	---	347.36	352.24	347.50	349.32	0.28 (0.003)	87.16 (0.0008)
EB-XII (λ,β)	3.031 (0.437) (2.17,3.89)	0.537 (0.085) (0.37,0.70)	---	---	325.92	330.80	326.07	327.88	0.27 (0.09)	11.00 (0.004)
MOB-XII (λ,β,α)	1.138 (0.391) (0.37, 1.90)	4.403 (1.904) (0.67, 8.13)	343.36 (89.342) (168.25, 518.47)	---	271.26	278.59	271.55	274.21	0.08 (0.39)	9.06 (0.003)
TLB-XII (α,λ,β)	1.561 (0.341) (0.89, 2.23)	0.592 (0.173) (0.25, 0.93)	3.447 (1.132) (1.23, 5.67)	---	328.62	335.94	328.91	331.56	0.18 (0.09)	---
KwB-XII $(\alpha,\theta,\lambda,\beta)$	0.099 (0.016) (0.07, 0.13)	6.066 (0.598) (4.89, 7.24)	569.482 (226.98) (124.60, 1014.36)	515.413 (283.14) (0, 1070.36)	287.56	297.33	288.06	291.48	0.12 (0.15)	---
BetaB-XII $(\alpha,\theta,\lambda,\beta)$	0.449 (0.028) (0.39, 0.50)	0.753 (0.137) (0.48, 1.02)	25.632 (9.873) (6.28, 44.98)	26.983 (14.497) (0. 55.39)	318.92	328.68	319.42	322.85	0.17 (0.06)	---
GMOBXII $(\lambda,\beta,\alpha,\theta)$	1.058 (0.086) (0.89, 1.23)	2.088 (1.108) (0, 4.26)	764.99 (189.69) (393.19,1136.78)	70.389 (28.31) (14.90, 125.88)	**264.20**	**273.96**	**264.70**	**268.12**	0.07 (**0.58**)	---



(a) (b)

Fig 7: Plots of the (a) observed histogram and estimated pdf's and (b) observed ogive and estimated cdf's for the B - XII, EB-XII, MOB - XII ,TLB-XII, KwB-XII, BetaB-XII and GMOBXII for data set V.

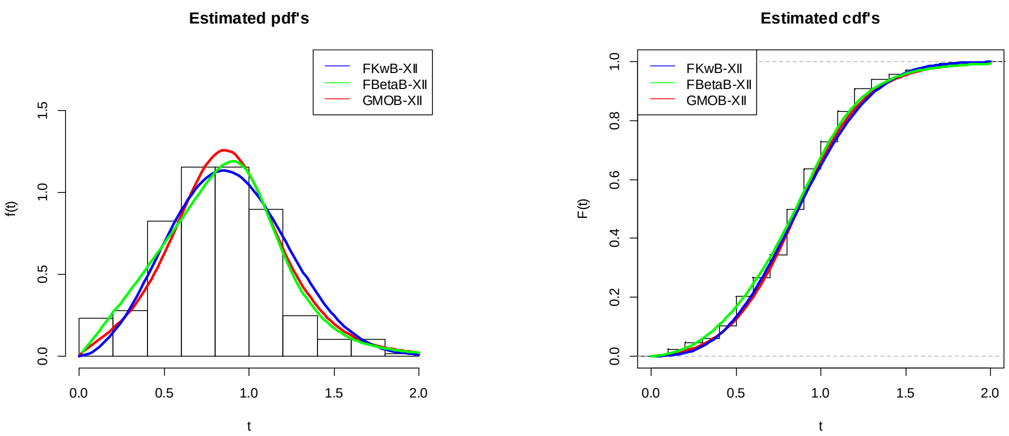
In the Tables 3, 4, 5, 6 and 7, the MLEs with standard errors of the parameters for all the fitted models along with their AIC, BIC, CAIC, HQIC, KS and LR statistic with *p*-value for the data sets I, II, III, IV and V are presented respectively. From table 3, 4, 5, 6 and 7, it is evident that for the all data sets, the GMOBXII distribution with lowest AIC, BIC, CAIC, HQIC and highest *p*-value of KS statistic. Hence it is a better model than all the sub models MOB - XII , B - XII, EB - XII and also the better than the recently introduced models namely TLB - XII, KwB - XII and BetaB - XII distributions.

As expected the LR test rejects the two sub models in favour of the GMOBXII distribution. These findings are further validated from the plots of fitted densities with histogram of the observed data and fitted cdfs with ogive of observed data in figure 3 (a), 4(a), 5(a), 6(a) and 7(a) and 3 (b), 4(b), 5(b), 6(b) and 7(b) for the data sets I, II, III, IV and V respectively. These plots indicate that the proposed distributions provide closest fit to all the observed data sets.

We now compare our proposed four parameter GMOBXII$(\lambda,\beta,\alpha,\theta)$ distribution with two recently introduced the FKwBXII(a,b,c,k,s)] and the FBBXII(a,b,c,k,s) distributions for all five data sets and present the finding in table 8 and figures 8 to 12.

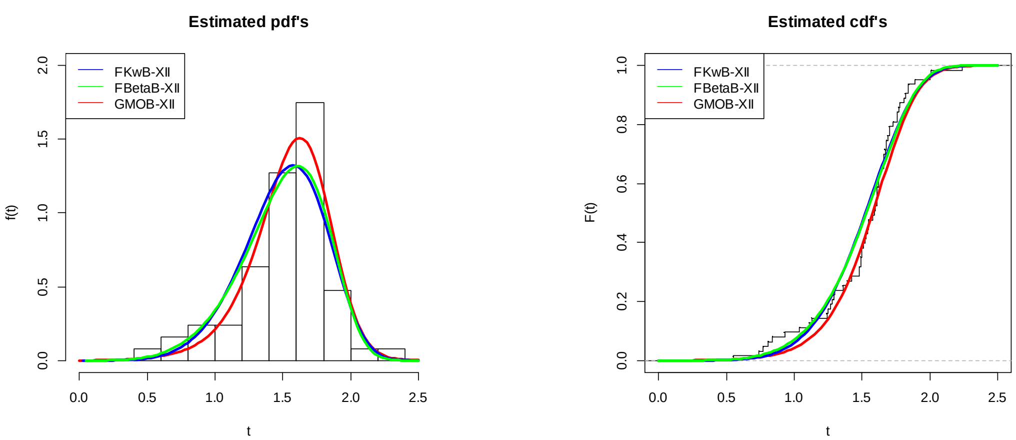
Table 8: MLEs, standard errors, confidence interval (in parentheses) with AIC, BIC, CAIC, HQIC and K-S (p-value) values for the data sets I, II, III, IV and V

Models	$\hat{a}$	$\hat{b}$	$\hat{c}$	$\hat{k}$	$\hat{s}$	AIC	BIC	CAIC	HQIC	K-S (p-value)
Data Set I										
FKwB-XII (a,b,c,k,s)	1.029 (0.046) (0.94,1.12)	36.322 (12.132) (12.54,60.10)	2.735 (1.037) (0.70,4.77)	1.712 (0.087) (1.54,1.88)	4.146 (2.198) (0,8.45)	237.88	257.13	238.06	245.58	0.25 (0.13)
FBetaB-XII (a,b,c,k,s)	0.311 (0.066) (0.18,0.44)	0.192 (0.067) (0.06,0.32)	1.873 (1.264) (0, 4.35)	5.385 (1.923) (1.62,9.15)	0.995 (0.068) (0.86,1.13)	226.72	245.97	226.89	234.42	0.27 (0.09)
Data Set II										
FKwB-XII (a,b,c,k,s)	0.768 (0.681) (0, 2.10)	23.534 (1.046) (21.48,25.58)	7.431 (6.231) (0, 19.64)	4.116 (5.963) (0, 15.80)	3.315 (1.108) (1.14,5.49)	40.30	51.00	41.35	44.50	0.14 (0.26)
FBetaB-XII (a,b,c,k,s)	0.617 (0.234) (0.15,1.08)	3.584 (1.326) (0.98,6.18)	7.857 (1.898) (4.14,11.57)	14.031 (8.397) (0,30.48)	2.777 (1.234) (0.36,5.19)	38.98	49.68	40.03	43.18	0.13 (0.27)
Data Set III										
FKwB-XII (a,b,c,k,s)	0.558 (0.442) (0,1.42)	0.308 (0.314) (0, 0.92)	3.999 (2.082) (0, 8.07)	2.131 (1.833) (0, 5.72)	1.475 (0.361) (0.76, 2.18)	206.50	217.90	207.41	211.00	0.08 (0.65)
FBetaB-XII (a,b,c,k,s)	0.621 (0.541) (0, 1.68)	0.549 (1.011) (0, 2.53)	3.838 (2.785) (0, 9.29)	1.381 (2.312) (0, 5.91)	1.665 (0.436) (0.81, 4.48)	206.80	218.20	207.71	211.30	0.08 (0.64)
Data Set IV										
FKwB-XII (a,b,c,k,s)	2.952 (1.091) (0.81, 5.09)	11.139 (5.452) (0.45, 21.82)	0.409 (0.039) (0.33, 0.48)	0.977 (0.164) (0.66,1.29)	75.980 (0.361) (75.27,76.69)	504.98	514.53	506.34	508.48	0.21 (0.14)
FBetaB-XII (a,b,c,k,s)	3.446 (1.097) (1.29,5.59)	54.992 (72.592) (0, 197.27)	0.457 (0.038) (0.38, 0.53)	0.093 (0.120) (0, 0.32)	35.493 (16.954) (2.26,68.72)	512.54	522.09	513.90	516.14	0.22 (0.10)
Data Set V										
FKwB-XII (a,b,c,k,s)	0.362 (0.203) (0, 0.76)	1.183 (0.179) (0.83,1.53)	4.985 (0.930) (3.16,6.81)	0.517 (2.138) (0, 4.71)	5.100 (2.361) (0.47, 9.73)	269.12	281.32	269.41	274.02	0.09 (0.49)
FBetaB-XII (a,b,c,k,s)	0.222 (0.026) (0.17,0.27)	1.923 (0.078) (1.77,2.08)	7.833 (0.003) (7.82, 7.84)	0.505 (0.005) (0.49,0.51)	3.583 (0.049) (3.49, 3.68)	272.02	284.40	272.78	276.92	0.10 (0.33)



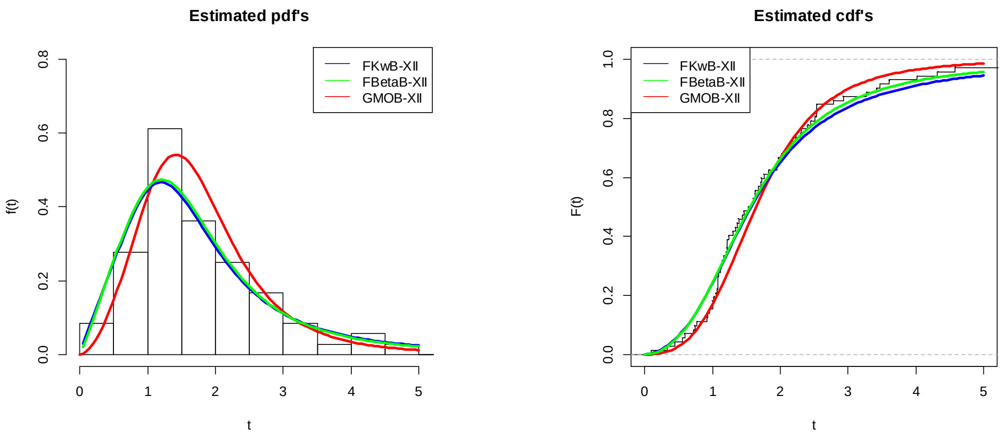
(a) (b)

Fig 8: Plots of the (a) observed histogram and estimated pdf's and (b) observed ogive and estimated cdf's for the FKwB-XII, FBetaB-XII and GMOBXII for data set I



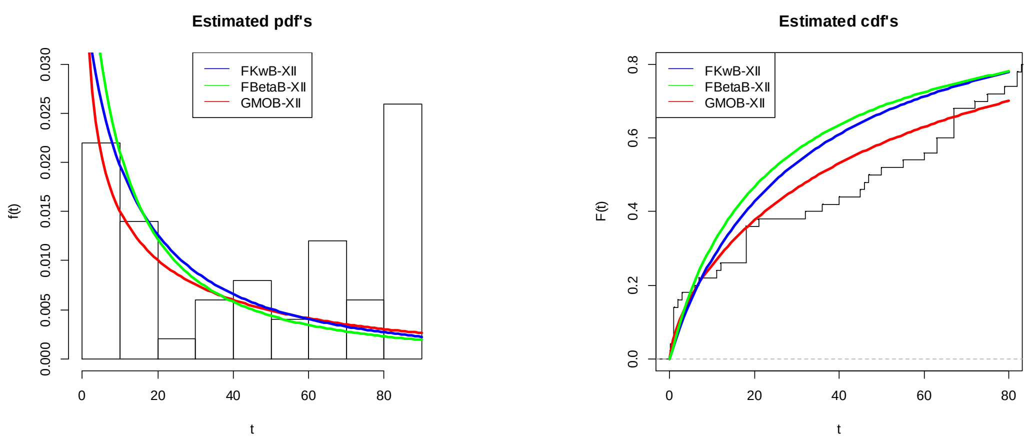
(a) (b)

Fig 9: Plots of the (a) observed histogram and estimated pdf's and (b) observed ogive and estimated cdf's for the FKwB-XII, FBetaB-XII and GMOBXII for data set II.



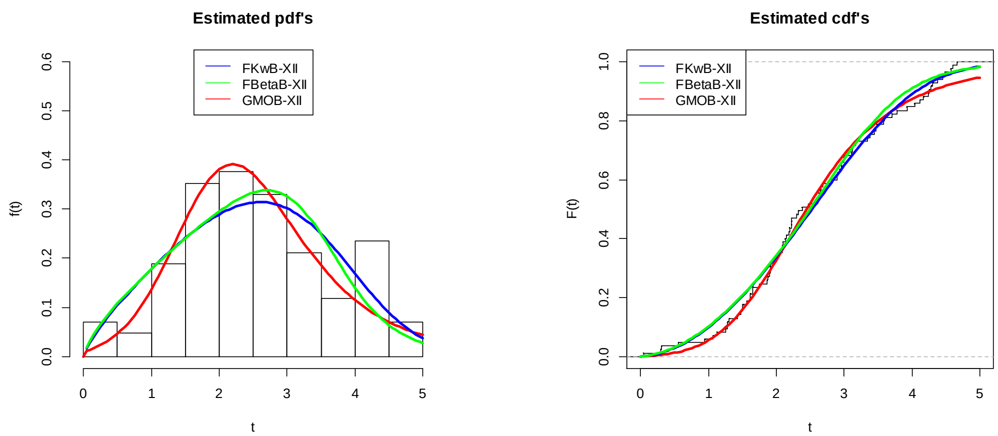
(a) (b)

Fig 10: Plots of the (a) observed histogram and estimated pdf’s and (b) observed ogive and estimated cdf’s for the FKwB-XII, FBetaB-XII and GMOBXII for data set III.



(a) (b)

Fig 11: Plots of the (a) observed histogram and estimated pdf’s and (b) observed ogive and estimated cdf’s for the FKwB-XII, FBetaB-XII and GMOBXII for data set IV.



(a) (b)

Fig 12: Plots of the (a) observed histogram and estimated pdf's and (b) observed ogives and estimated cdf's for the FKwB-XII, FBetaB-XII and GMOBXII for data set V.

Comparing the values of different criteria for GMOBXII from the last rows of table 3 to 7 with those in table 8 and inspecting figures 8 to12 it is clear that the proposed $\mathrm{GMOBXII}(\lambda,\beta,\alpha,\theta)$ distribution is better than both the $\mathrm{FKwBXII}(a,b,c,k,s)$ as well as the $\mathrm{FBBXII}(a,b,c,k,s)$ in all the five cases considered here.

9 Conclusion A new extension of the Burr-XII distribution which encompasses some important sub models is proposed along with its properties. Comparative evaluation findings from real life data modelling in terms of different model selection, goodness of fit criteria and test gave enough support to claim that the proposed distribution it as a better alternative than most of its sub models and the other extensions of the Burr-XII distribution including two five parameter distributions introduced recently. As such it is envisaged that the proposed extension of Burr-XII will be a useful addition to the existing knowledge.

Reference:

1. Aarset, M.V. (1987). How to identify a bathtub hazard rate, IEEE Transactions on Reliability, **36**, 106-108.
2. Arwa, Y. Al-Saiari, Lamya, A. Baharith and Salwa, A. Mousa (2014). Marshall-Olkin Extended Burr Type XII distribution. International Journal of Statistics and Probability, **3**, ISSN 1927-7032 E-ISSN, 1927-7040.
3. Bjerkedal, T. (1960). Acquisition of resistance in Guinea pigs infected with different doses of virulent tubercle bacilli. American Journal of Hygiene, **72**, 130–148.

4. Burr, W. (1942). Cumulative frequency functions, Annals of Mathematical Statistics, **13**, 215-232.
5. Gosh, I. and Bourguignon, M. (2017). A new extended Burr-XII distribution: properties and associated inference. Austrian journal of Statistics, **46**, 33-39, http://www.ajs.or.at/ doi: 10.17713/ajs.v46i1.139.
6. Gradshteyn, I.S. and Ryzhik, I.M. (2000). Table of Integrals, Series and Products, sixth ed.. Academic Press, San Diego.
7. Greenwood, J.A., Landwehr, J.M., Matalas, N.C. and Wallis, J.R. (1979). Probability weighted moments: definition and relation to parameters of several distributions expressible in inverse form. Water Resour Res, **15**, 1049-1054.
8. Hesham, M. Reyad and Soha, A. Othman (2017). The Top-Leone Burr-XII distribution: Properties and applications. British Journal of Mathematics and Computer Science, **21**, (Accepted paper).
9. Jayakumar, K., and Mathew, T. (2008). On a generalization to Marshall-Olkin scheme and its application to Burr type XII distribution. Stat Pap, **49**, 421-439.
10. Kullback, S. (1959). Information Theory and Statistics. John Wiley & Sons, New York.
11. Mead, M. (2014). A new generalization of Burr XII distribution. Journal of Statistics: Advances in Theory and Applications, **12**, 53–71.
12. Paranaíba, P.F., Ortega, E. M. M., Cordeiro, G.M. and Pescim, R.R. (2011). The beta Burr XII distribution with application to lifetime data, Computational Statistics and Data Analysis, **55**, 1118-1136.
13. Paranaíba, P.F., Ortega, E.M.M., Cordeiro, G.M. and de Pascoa, M.A. (2013). The Kumaraswamy Burr XII distribution: Theory and practice, Journal of Statistical Computation and Simulation, **83**, 2117-2143.
14. Prudnikov, A.P., Brychkov, Y.A., Marichev, O.I. (1986). Integrals and Series, vol. 1. Gordon and Breach Science Publishers, Amsterdam.
15. Prudnikov, A.P., Brychkov, Y.A., Marichev, O.I. (1992). Integrals and Series, vol. 4. Gordon and Breach Science Publishers, Amsterdam.
16. Ramos, M.W.A., Marinho, P.R.D., da Silva, R.V. and Cordeiro, G.M. (2013). The exponentiated Lomax Poisson distribution with an application to lifetime data. Advances and Applications in Statistics, **34**, 107–135.
17. Shaked, M. and Shanthikumar, J.G. (2007). Stochastic Orders, Springer, New York.
18. Smith, R.L. and Naylor, J.C. (1987). A comparison of maximum likelihood and Bayesian estimators for the three-parameter Weibull distribution. Applied Statistics, **36,** 358-369.

19. Song, K.S. (2001). Rényi information, log likelihood and an intrinsic distribution measure. EM J Stat Plan Infer, **93,** 51-69.
20. Tahir, M.H., Cordeiro, G.M., Mansoor, M. and Zubair, M. (2015). The Weibull-Lomax distribution: Properties and applications. Hacettepe Journal of Mathematics and Statistics, **44**, 461–480.