Analysis of a finite volume-finite element method for Darcy-Brinkman two-phase flows in porous media

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Abstract In this paper, we are interested in the displacement of two incompressible phases in a Darcy-Brinkman flow in a porous media. The equations are obtained by the conservation of the mass and by considering the Brinkman regularization velocity of each phase. This model is treated in its general form with the whole non-linear terms. This paper deals with construction and convergence analysis of a combined finite volume- non conforming finite element scheme together with a phase-by-phase upstream according to the total velocity.

1 The Darcy-Brinkman model

Different empiric laws are used to describe the filtration of a fluid through porous media. Darcy law is the most popular one, due to its simplicity. It states that the filtration velocity of the fluid is proportional to the pressure gradient. Darcy law cannot sustain the no-slip condition on an impermeable wall or a transmission condition on the contact with free flow [18, 22]. That motivated H. Brinkmann in 1947 to modify the Darcy law in order to be able to impose the no-slip boundary condition on an obstacle submerged in porous medium [7, 17, 18]. He assumed large permeability to compare his law with experimental data and assumed that the second viscosity μ equals to the physical viscosity of the fluid in the case of monophasic flow. Brinkman law could formally be obtained from the Stokes system describing the microscopic flow, by adding the resistance to the flow [2].
Let $T > 0$, fixed and let Ω be a bounded set of $\mathbb{R}^d (d \geq 1)$. We set $Q_T = (0, T) \times \Omega$, $\Sigma_T = (0, T) \times \partial \Omega$, and η is the outward normal to the boundary $\partial \Omega$. The mass conservation of two incompressible phases with Darcy-Brinkman velocity of each phase can be written in the following system, (see [11, 16] for more details),

$$\begin{cases} \phi \partial_t s - \mu \phi \Delta \partial_t s - \operatorname{div}\left(f(s)\lambda(s)\Lambda \nabla p\right) - \operatorname{div}\left(\alpha(s)\Lambda \nabla s\right) = 0, & \text{in } Q_T \\ -\operatorname{div}\left(\lambda(s)\Lambda \nabla p\right) = 0, & \text{in } Q_T \\ \int_\Omega p(t,x)\,\mathrm{d}x = 0, & \text{in } (0,T) \\ \lambda(s)\Lambda \nabla p(t,x) \cdot \eta = \pi(t,x), & \text{on } \Sigma_T \\ \mu \phi \nabla \partial_t s \cdot \eta + f(s)\pi + \alpha(s)\Lambda \nabla s \cdot \eta = h(t,x), & \text{on } \Sigma_T \\ s(0,x) = s_0(x), & \text{in } \Omega. \end{cases} \tag{1}$$

In the above model, the first saturation phase (volume fraction) is represented by $s = s(t,x)$ and the other saturation phase is stated to be $1 - s$ and by $p = p(t,x)$ the global pressure. Next, $\lambda(s) = \frac{\lambda_1(s)}{\mu_1} + \frac{\lambda_2(s)}{\mu_2}$ is the total mobility with λ_i and μ_i are the relative permeability and the viscosity of the phase $i = 1, 2$; where Λ is the permeability tensor of the porous medium and $f(s) = \frac{\lambda_1(s)}{\mu_1 \lambda(s)}$ is the fractional flow. Furthermore, the function

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$\alpha(s) = \frac{\lambda_1(s)\lambda_2(s)}{\mu_1\mu_2\lambda(s)} p'_c(s)$ is the capillary pressure term. The viscosity $\mu > 0$ is said the Brinkman viscosity.

We give the main assumptions made about the system:

(H1) Λ is a symmetric matrix where $\Lambda_{ij} \in L^\infty(\Omega)$, and there exists a constant $C_\Lambda > 0$ such that
       $\Lambda\xi \cdot \xi \geq C_\Lambda |\xi|^2, \forall \xi \in \mathbb{R}^d$.

(H2) The fractional flow f is a continuous function satisfies $f(0) = 0$, $f(1) = 1$ and $0 \leq f(u) \leq 1$.

(H3) The total mobility λ is a continuous function and there exists a positive constant $\lambda_\star$, such that $0 < \lambda_\star \leq \lambda(\cdot)$.

(H4) the capillary pressure term is a continuous function with $\alpha(0) = \alpha(1) = 0$.

(H5) The function $\pi : \Sigma_T \longrightarrow \mathbb{R}$ is a bounded function in $L^\infty(0,T;L^2(\partial\Omega))$ with $\int_{\partial\Omega} \pi\, d\sigma = 0$.

(H6) The initial function $s_0 \in H^1(\Omega)$, and the function $h \in L^\infty(0,T;L^2(\partial\Omega))$.

In the sequel, we use the Lipschitz continuous nondecreasing function $\beta : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ defined by

$$\beta(s) := \int_0^s \alpha(\zeta)\, d\zeta, \qquad \forall s \in \mathbb{R}^+.$$

We now give the definition of a weak solution of the problem (1).

Definition 1 *A weak solution of system* (1) *is a pair of function* (s,p), $s,p : Q_T \to \mathbb{R}$, *such that* $\forall T > 0$, $s \in W^{1,2}(0,T;H^1(\Omega))$, $p \in L^2(0,T;H^1_M(\Omega))$; *and for all* $\varphi \in L^2(0,T;H^1(\Omega))$ *we have :*

$$\int_0^T \int_\Omega \partial_t s\, \varphi\, dx\, dt + \mu \int_0^T \int_\Omega \nabla\partial_t s \cdot \nabla\varphi\, dx\, dt + \int_0^T \int_\Omega \Lambda f(s)\lambda(s)\nabla p \cdot \nabla\varphi\, dx\, dt$$

$$+ \int_0^T \int_\Omega \Lambda\alpha(s)\nabla s \cdot \nabla\varphi\, dx\, dt - \int_0^T \int_{\partial\Omega} h\varphi\, dx\, dt = 0, \tag{2}$$

$$\int_0^T \int_\Omega \Lambda\lambda(s)\nabla p \cdot \nabla\varphi\, dx\, dt - \int_0^T \int_{\partial\Omega} \pi\varphi\, dx\, dt = 0, \tag{3}$$

where, $H^1_M(\Omega) = \{u \in H^1(\Omega), \int_\Omega u = 0\}$.

Numerical discretization and simulation of multi-phase flow in porous media with Darcy velocity have been the object of several studies during the past decades. There is an extensive literature on the approximation of incompressible two-phase flows with Darcy's velocities for each phase. For instance the finite difference method can be found in the book of Aziz and Settari [23]. The method of finite element methods (mixed or hybrid) is also studied in the past years see Chavent et al. [8] and Chen et al. [9]. For the incompressible immiscible model; let us mention the work of Eymard, Herbin and Michel [15] where a coupled scheme, consisting in a finite volume method together with a phase-by-phase upstream weighting scheme, is analyzed on orthogonal admissible mesh. In [5], the authors propose to explore the limit of a finite volume scheme of two-phase flow model with discontinuous capillary pressure. For that, they proposed to study the limit of the upstream finite volume scheme for the incompressible immiscible model. In [19], the authors suggest a second order accurate finite volume scheme for the Richards equation. Otherwise, the numerical analysis of finite volume and a finite volume-finite element schemes for a compressible two phases flow in porous media for Darcy flow have been studied in the works [3, 21].

At our knowledge, the numerical analysis of combined finite volume-finite element scheme of the problem (1) is not studied on triangular mesh. Nevertheless, we mention the pioneer work [11] on the convergence analysis of the finite difference scheme of the Brinkman regularization of two incompressible phase flows in porous media. Recently, in [20], we have been studied the finite volume scheme of Darcy-Brinkman model.

Our aim is to propose a finite volume-finite element scheme based on upstream approach of fractional flow with respect to the gradient of the global pressure.

The rest of this paper is organized as follows. In section 2, we define a primal triangular mesh and its corresponding dual mesh, next, we introduce the nonlinear finite volume-finite element scheme and specify the discretization of the degenerate diffusion and convection terms. In Section 3, we prove the existence of a discrete solution to the finite volume-finite element scheme based on the establishment of a priori estimates on the

discrete solution. In Section 4, we give estimates on differences of time and space translates for the approximate solutions. Finally, in Section 5, using the Kolmogorov relative compactness criterion, we prove the convergence of a subsequent of discrete solutions to the weak solution in the sense of definition 1.

2 Combined Finite volume-Finite element scheme for system (1)

This section is devoted to the formulation of a combined scheme for the anisotropic Darcy-Brinkman model (1). First, we will describe the space and time discretization, define the approximation spaces and then we will introduce the combined scheme.

2.1 Space and time discretizations

In order to discretize problem (1), we perform a finite element triangulation $\{\mathcal{T}_h\}$ of the polygonal domain Ω, consisting of open bounded triangles such that $\bar{\Omega} = \cup_{K \in \mathcal{T}_h} \bar{K}$ and such that for all $K, L \in \mathcal{T}_h$, $K \neq L$, then $\bar{K} \cap \bar{L}$ is either an empty set or a common vertex or edge of K and L. We denote by $\mathcal{E}_h$ the set of all edges, by $\mathcal{E}_h^{int}$ the set of interior edges, by $\mathcal{E}_h^{ext}$ the set of all exterior edges and by $\mathcal{E}_K$ the set of all edges of an element K. We define $h := \max\{\text{diam}(K), K \in \mathcal{T}_h\}$ and make the following shape regularity assumption on the family of triangulation $\{\mathcal{T}_h\}_h$:

$$\min_{K \in \mathcal{T}_h} \frac{|K|}{(\text{diam}(K))^d} \geq k_\mathcal{T} \quad \forall h > 0. \tag{4}$$

Assumption (4) is equivalent to the more common requirement of the existence of a constant $\theta_\mathcal{T} > 0$ such that

$$\max_{K \in \mathcal{T}_h} \frac{\text{diam}(K)}{\rho_k} \geq \theta_\mathcal{T} \quad \forall h > 0,$$

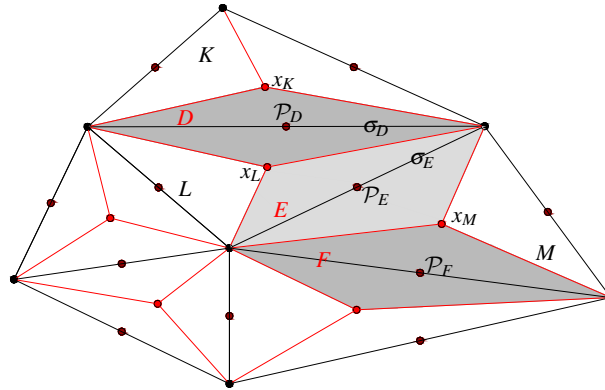where ρ_k is the diameter of the largest ball inscribed in the simplex K.

Fig. 1 Triangles $K, L \in \mathcal{T}_h$ and dual volumes $D, E \in \mathcal{D}_h$ associated with edges $\sigma_D, \sigma_E \in \mathcal{E}_h$.

We also define a dual mesh $\mathcal{D}_h$ generated by the triangulation mesh $\mathcal{T}_h$ such that $\bar{\Omega} = \cup_{D \in \mathcal{D}_h} \bar{D}$. There is one dual element D associated with each side $\sigma_D = \sigma_{K,L} \in \mathcal{E}_h$ a diamond. We construct it by connecting the barycenters of every $K \in \mathcal{T}_h$ that contains σ_D through the vertices of σ_D. As for the primal mesh, we define $\mathcal{F}_h$, $\mathcal{F}_h^{int}$ and $\mathcal{F}_h^{ext}$, respectively, as the set of dual, interior, and exterior mesh edges. For $\sigma_D \in \mathcal{F}_h^{ext}$, the contour of D is completed

by the edge σ_D itself which corresponds to a half diamond. We refer to the Fig. 1 for the 2D case.
We use the following notations :

- $|D| = \text{mes}(D) = d$-dimensional Lebesgue measure of D and $|\sigma|$ is the $(d-1)$-dimensional measure of σ.
- $\mathcal{P}_D$ is the barycenter of the edge σ_D.
- $\mathcal{N}(D)$ the set of neighbors of the diamond D.
- $d_{D,E} := |\mathcal{P}_E - \mathcal{P}_D|$ the distance between the center $\mathcal{P}_D$ and $\mathcal{P}_E$.
- $\sigma_{D,E}$ the interface between a dual volume D and $E \in \mathcal{N}(D)$.
- $\eta_{D,E}$ the unit normal vector to $\sigma_{D,E}$ outward to D.
- $K_{D,E} = \{K \in \mathcal{T}_h; \sigma_{D,E} \subset K\}$.
- $\mathcal{D}_h^{int}$ and $\mathcal{D}_h^{ext}$ are, respectively, the set of all interior and boundary dual volumes.

Next, we define the following finite-dimensional space :

$$X_h := \{\varphi_h \in L^2(\Omega); \varphi_h|_K \text{ is linear } \forall K \in \mathcal{T}_h, \varphi_h \text{ is continuous at the points } \mathcal{P}_D, D \in \mathcal{D}_h^{int}\}.$$

The basis of X_h is spanned by the shape functions $\varphi_D, D \in \mathcal{D}_h$ such that $\varphi_D(\mathcal{P}_E) = \delta_{DE}, E \in \mathcal{D}_h$ and δ being the Kronecker symbol. We recall that the approximations in these spaces are nonconforming as $X_h \not\subset H^1(\Omega)$. We equip X_h with the semi-norm,

$$||S_h||^2_{X_h} := \sum_{K \in \mathcal{T}_h} \int_K |\nabla S_h|^2 dx.$$

For the time discretization, it's might be performed with a variable time step, to simplify the notation, we consider a constant time step $\Delta t \in [0,T]$. A discretization of $[0,T]$ is given by $N \in \mathbb{N}^*$ such that $t_n = n\Delta t$, for $n \in \{0,..,N+1\}$ with $t_{N+1} = T$. For a given value $W_D, D \in \mathcal{D}_h$ (resp. $W_D^n, D \in \mathcal{D}_h, n \in [0,N]$), we define a constant piecewise function as : $w(x) = W_D$ for $x \in D$ (resp. $w(t,x) = W_D^n$ for $x \in D, t \in]t_n, t_{n+1}]$).

2.2 Combined scheme

This subsection is committed to discretize the Darcy-Brinkman system (1) with anisotropic tensors on general meshes where we loose the orthogonality condition then the classical approximation of the normal diffusive flux used in the typical finite volume scheme VF4 (see [14] for more details) and therefore the consistency of the scheme. To discretize (1), we use the implicit Euler scheme in time and we consider the piecewise linear nonconforming finite element method for the discretization of the diffusion term in space. The convection terms are discretized by means of a finite volume scheme on the dual mesh.
The approximation of the flux $K\nabla p \cdot \eta_{D,E}$ on the interface $\sigma_{D,E}$ is denoted by $\delta P_{D,E}$. Now, we have to approximate $Kf(s)\lambda(s)\nabla p \cdot \eta_{D,E}$ by means of the values S_D, S_E, and $\delta P_{D,E}$ that are available in the neighborhood of the interface $\sigma_{D,E}$. To do this, we use a numerical flux function $G_s(S_D, S_E, \delta P_{D,E})$. Numerical convection flux functions G_s of arguments $(a,b,c) \in \mathbb{R}^3$ are required to satisfy the properties:

- $G_s(.,b,c)$ is nondecreasing for all $b,c \in \mathbb{R}$, and $G_s(a,.,c)$ is nonincreasing for all $a,c \in \mathbb{R}$;
- $G_s(a,b,c) = G_s(b,a,c)$ for all $a,b,c \in \mathbb{R}$; hence the flux is conservative.
- $G_s(a,a,c) = -f(a)\lambda(a)c$ for all $a,c \in \mathbb{R}$; hence the flux is consistent.

One possibility to construct the numerical flux corresponding to $g(s)c$ is by split $g(s)$ to a nondecreasing part $(g(.))_\uparrow$ and a nonincreasing part $(g(.))_\downarrow$ such that

$$g_\uparrow(z) := \int_0^z (g'(s))^+ ds, \quad g_\downarrow(z) := -\int_0^z (g'(s))^- ds,$$

herein $(g'(s))^+ = \max(g'(s)),0)$ and $(g'(s))^- = \max(-g'(s)),0)$. Then we take,

$$G_s(a,b,c) = c^+\big(g_\uparrow(a)+g_\downarrow(b)\big) - c^-\big(g_\uparrow(b)+g_\downarrow(a)\big).$$

In our case $g(.) = -f(.)\lambda(.)$ and since $f\lambda$ is nondecreasing with $f(0) = 0$ then,

$$G_s(a,b,c) = -f(b)\lambda(b)c^+ - (-f(a)\lambda(a))c^-. \tag{5}$$

Next, for all $S_h = \sum_{D\in\mathcal{D}_h} S_D \varphi_D$, we define a discrete function $\beta_h(S_h)$ as

$$\beta_h(S_h) = \sum_{D\in\mathcal{D}_h} \beta(S_D)\varphi_D.$$

Finally, a combined finite volume-nonconforming finite element scheme for the discretization of the problem (1) is given by the following set of equations: For all $D \in D_h$,

$$S_D^0 = \frac{1}{|D|}\int_D s_0(x)\,dx, \tag{6}$$

and for all $D \in \mathcal{D}_h$, $n \in \{0,1,...,N\}$,

$$|D|\frac{S_D^{n+1}-S_D^n}{\Delta t} - \mu \sum_{E\in\mathcal{N}(D)} \mathcal{N}_{D,E}\Big(\frac{S_E^{n+1}-S_E^n}{\Delta t} - \frac{S_D^{n+1}-S_D^n}{\Delta t}\Big) - \sum_{E\in\mathcal{N}(D)} \mathcal{M}_{E,D}\big(\beta(S_E^{n+1})-\beta(S_D^{n+1})\big)$$
$$+ \sum_{E\in\mathcal{N}(D)} G_s(S_D^{n+1},S_E^{n+1};\delta P_{D,E}^{n+1}) = |\partial D\cap\partial\Omega|h_D^{n+1}, \tag{7}$$

$$- \sum_{E\in\mathcal{N}(D)} G_p(S_D^{n+1},S_E^{n+1};\delta P_{D,E}^{n+1}) = |\partial D\cap\partial\Omega|\pi_D^{n+1}, \tag{8}$$

$$\sum_{D\in\mathcal{D}_h} P_D^{n+1}\varphi_D = 0. \tag{9}$$

$$h_D^{n+1} = \frac{1}{\Delta t|\partial D\cap\partial\Omega|}\int_{\partial D\cap\partial\Omega}\int_{t_n}^{t_{n+1}} h(t,x)dx, \quad \pi_D^{n+1} = \frac{1}{\Delta t|\partial D\cap\partial\Omega|}\int_{\partial D\cap\partial\Omega}\int_{t_n}^{t_{n+1}} \pi(t,x)dx.$$

The coefficient $\mathcal{N}_{D,E}$ (resp. $\mathcal{M}_{D,E}$) for $D,E \in \mathcal{D}_h^{int}$ is the stiffness coefficient of the nonconforming finite element method. So that:

$$\mathcal{N}_{D,E} = -\sum_{K\in\mathcal{T}_h} (\nabla\varphi_E,\nabla\varphi_D)_{L^2(K)} \quad \text{and} \quad \mathcal{M}_{D,E} = -\sum_{K\in\mathcal{T}_h} (\Lambda\nabla\varphi_E,\nabla\varphi_D)_{L^2(K)}, \tag{10}$$

$\delta P_{D,E}^{n+1}$ denotes the approximation of $K\nabla p$ on the interface $\sigma_{D,E}$:

$$\delta P_{D,E}^{n+1} = \mathcal{M}_{D,E}\big(P_E^{n+1}-P_D^{n+1}\big), \tag{11}$$

and we define G_p for all $a,b,c \in \mathbb{R}$ as

$$G_p(a,b,c) = \lambda(b)c^+ - \lambda(a)c^-. \tag{12}$$

Notice that the source terms are, for $n \in \{0,...,N\}$

Remark 1 *A rigorous justification for the construction of the schema for nonlinear convection-diffusion problems can be found in [1, 13] and we only put the formal construction of the schema in this paper.*

Remark 2 *We can rewrite (7) as follow :*

$$|D|\frac{S_D^{n+1}-S_D^n}{\Delta t} - \mu \sum_{E\in\mathcal{D}_h} \mathcal{N}_{D,E}\big(\frac{S_E^{n+1}-S_E^n}{\Delta t}\big) - \sum_{E\in\mathcal{D}_h} \mathcal{M}_{E,D}\big(\beta(S_E^{n+1})\big)$$
$$+ \sum_{E\in\mathcal{N}(D)} G_s(S_D^{n+1},S_E^{n+1};\delta P_{D,E}^{n+1}) = |\partial D\cap\partial\Omega|h_D^{n+1}. \quad (13)$$

Proof. The proof of this assertion is given in Lemma 5 and by equation (21)-(22).

Definition 2 *Using the values of ((S_D^{n+1},P_D^{n+1}), $\forall D\in\mathcal{D}_h$ and $n\in[0,..,N]$, we will define two approximate solutions of discrete problem (7)-(9) in the sense of the combined finite volume-nonconforming finite element scheme : i) A nonconforming finite element solution $(S_{h,\Delta t},P_{h,\Delta t})$ as a function piecewise linear and continuous at the barycenters of the interior sides in space and piecewise constant in time, such that:*

$$S_{h,\Delta t}(x,0) = S_h^0(x) \text{ for } x\in\Omega,$$

$$(S_{h,\Delta t}(x,t),P_{h,\Delta t}(x,t)) = (S_h^{n+1}(x),P_h^{n+1}(x)) \text{ for } x\in\Omega, t\in]t_n,t_{n+1}],$$

$$\text{where } S_h^{n+1} = \sum_{D\in\mathcal{D}_h} S_D^{n+1}\varphi_D \quad \text{and} \quad P_h^{n+1} = \sum_{D\in\mathcal{D}_h} P_D^{n+1}\varphi_D.$$

ii) A finite volume solution $(\tilde{S}_{h,\Delta t},\tilde{P}_{h,\Delta t})$ defined as piecewise constant on the dual volumes in space and piecewise constant in time, such that:

$$\tilde{S}_{h,\Delta t}(x,0) = S_D^0 \text{ for } x\in D, D\in\mathcal{D}_h,$$

$$(\tilde{S}_{h,\Delta t}(x,t),\tilde{P}_{h,\Delta t}(x,t)) = (S_D^{n+1},P_D^{n+1}) \text{ for } x\in D, D\in\mathcal{D}_h, t\in]t_n,t_{n+1}].$$

Furthermore, we define a piecewise linear function in time given by

$$f_S(t) = f(S(t,x)) = \frac{S(t_{n+1},x)-S(t_n,x)}{\Delta t}(t-t_n) + S(t_n,x) \text{ for } x\in\Omega, t\in]t_n,t_{n+1}]. \quad (14)$$

Now, we state the existence theorem under the assumption that all transmissibilities coefficients are positive:

$$\mathcal{N}_{D,E}\geq 0 \text{ and } \mathcal{M}_{D,E}\geq 0, \forall D\in\mathcal{D}_h, E\in\mathcal{N}(D). \quad (15)$$

Remark 3 *Assumption (15) is satisfied when the diffusion term reduces to a scalar function and when the magnitude of all interior angles smaller or equal to $\pi/2$ in two space dimension.*

Theorem 1. *Assume $(H1)-(H6)$ hold. Under assumption (15), the sequence $(S_{h,\Delta t},P_{h,\Delta t})$ converges to a solution (s,p) of system (1) in the sense of Definition 1 .*

The proof of this theorem is splitted in several Lemmas and Propositions in the following section.

3 Existence and discrete properties

In this section, we present some technical Lemmas that show the conservativity of the scheme, the coercivity, and the continuity of the diffusion term. We show also a priori estimate on the gradient of the solution which we shall need later in the proof of the existence of a discrete solution of (7)-(9) and in the proof of the convergence.

3.1 Preliminary results

This subsection is devoted to known results. For the sake of clarity, we summarize the following lemmas :

Lemma 1 (Discrete Gronwalls Inequality). *Given $\gamma_1 \geq 0$ and $\gamma_2 \geq 0$, assume that for all sequence $\{y_n\}_n$ satisfies $0 \leq y_{n+1} \leq y_n + \gamma_1 \Delta t + \gamma_2 \Delta t \, y_{n+1}$ for all $n \in \mathbb{N}$. Given a fixed time-step $\Delta t_0 \leq \frac{1}{\gamma_2}$ and a fixed time $T > 0$, we have for all $\Delta t \leq \Delta t_0$,*

$$\forall n \in N, \quad n\Delta t \leq T \Rightarrow y_n \leq \left(y_0 + \frac{\gamma_1}{\gamma_2}\right) \exp\left(\frac{\gamma_2 T}{1 - \gamma_2 \Delta t_0}\right)$$

Proof. the proof of this Lemma can be found in [12].

Lemma 2. *Let $\Omega \subset \mathbb{R}^d$ ($d = 2$ or $d = 3$) be an open bounded polygonal with a boundary $\partial\Omega$. Let $\mathcal{T}_h$ an admissible mesh. Let U_K the value of u on the control volume K. Then, there exists a positive constant C depend on Ω such that :*

$$\|u\|_{L^2(\Omega)}^2 \leq C\|u\|_{X_h}^2 + \frac{4}{|\Omega|}\left(\int_\Omega u(x)dx\right)^2.$$

Proof. The proof of this Lemma is given in [24, theorem 8.1]

Lemma 3 (Discrete trace lemma). *Let $\Omega \subset \mathbb{R}^d$ ($d = 2$ or $d = 3$) be an open bounded polygonal with a boundary $\partial\Omega$. Let $\mathcal{T}_h$ an admissible mesh. Let S_K the value of s_h on the control volume K. We define $\gamma(s_h)$ by $\gamma(s_h) = S_K$ piecewise on $\sigma_K = \partial K \cap \partial\Omega$ for all $K \in \mathcal{T}_h$. Then, there exists a positive constant C_1 depend on Ω such that :*

$$\|\gamma(s_h)\|_{L^2(\partial\Omega)} \leq C_1(\|s_h\|_{X_h} + \|s_h\|_{L^2(\Omega)}).$$

Proof. By applying the trace theorem on the reference simplex and then using a homogeneity/scaling argument (see [4]).

Lemma 4. *For all $W_h = \sum_{D \in \mathcal{D}_h} W_D \varphi_D \in X_h$, we have*

$$\sum_{\sigma_{D,E} \in \mathcal{D}_h^{int}} diam(K_{D,E})^{d-2}(W_E - W_D)^2 \leq \frac{d+1}{2dk_{\mathcal{T}}}\|W_h\|_{X_h}^2, \tag{16}$$

$$\sum_{\sigma_{D,E} \in \mathcal{D}_h^{int}} \frac{|\sigma_{D,E}|}{d_{D,E}}(W_E - W_D)^2 \leq \frac{d+1}{2(d-1)k_{\mathcal{T}}}\|W_h\|_{X_h}^2. \tag{17}$$

Proof. This Lemma is proved in [13], but for the sake of completeness we reproduce the proof. We notice that :

$$d_{D,E} \leq \frac{diam(K_{D,E})}{d}, \; |\sigma_{D,E}| \leq \frac{\left(diam(K_{D,E})\right)^{d-1}}{d-1}. \tag{18}$$

then,

$$\sum_{\sigma_{D,E} \in \mathcal{F}_h^{int}} \left(diam(K_{D,E})\right)^{d-2}(W_E - W_D)^2 \leq \sum_{\sigma_{D,E} \in \mathcal{F}_h^{int}} \left(diam(K_{D,E})\right)^{d-2}\big|\nabla W_h\big|_{K_{D,E}}\big|^2 d_{D,E}^2$$

$$\leq \frac{d+1}{2d} \sum_{K \in \mathcal{T}_h} \left(diam(K)\right)^d \big|\nabla W_h|_K\big|^2 \leq \frac{d+1}{2dk_{\mathcal{T}}} \sum_{K \in \mathcal{T}_h} |K|\big|\nabla W_h|_K\big|^2 = \frac{d+1}{2dk_{\mathcal{T}}}\|W_h\|_{X_h}^2.$$

Using the fact that the gradient of W_h is a piecewise constant on $\mathcal{T}_h$, the inequality (18) and each $K \in \mathcal{T}_h$ contains exactly $\binom{d+1}{2} = \frac{d(d+1)}{2}$ dual sides with assumption (4), we manage to show (16). similarly,

$$\sum_{\sigma_{D,E} \in \mathcal{F}_h^{int}} \frac{|\sigma_{D,E}|}{d_{D,E}}(W_E - W_D)^2 \leq \sum_{\sigma_{D,E} \in \mathcal{F}_h^{int}} \big|\nabla W_h|_{K_{D,E}}\big|^2 d_{D,E}|\sigma_{D,E}| \leq \frac{d+1}{2(d-1)k_{\mathcal{T}}}\|W_h\|_{X_h}^2.$$

Lemma 5. *For all $D \in \mathcal{D}_h$:*

$$\mathcal{N}_{D,D} = - \sum_{E \in \mathcal{N}(D)} \mathcal{N}_{D,E} \quad and \quad \mathcal{M}_{D,D} = - \sum_{E \in \mathcal{N}(D)} \mathcal{M}_{D,E}. \tag{19}$$

Proof. We can find the proof in [13]. for reader convenience we reproduce this proof. We fix a dual volume $D \in \mathcal{D}_h$. By using the structure of φ_D, the sum (10) is reduced just for two triangles K and L which have σ_D as a common interface,

$$\mathcal{M}_{D,D} = -(\Lambda \nabla \varphi_D, \nabla \varphi_D)_{L^2(L)} - (\Lambda \nabla \varphi_D, \nabla \varphi_D)_{L^2(K)}. \tag{20}$$

We denote by E_1 and E_2 the two dual volumes associated of the two other side of element L. Since we have

$$\varphi_D + \varphi_{E_1} + \varphi_{E_2} = 1 \text{ on } L,$$

then

$$-(\Lambda \nabla \varphi_D, \nabla \varphi_D)_{L^2(L)} = (\Lambda \nabla \varphi_{E_1}, \nabla \varphi_D)_{L^2(L)} + (\Lambda \nabla \varphi_{E_2}, \nabla \varphi_D)_{L^2(L)}.$$

By using a similar eventual contribution for the element K and by replacing that in the equation (20), this yield to prove the first assertion of Lemma 5. The same guidelines are used to prove the second assertion of this Lemma.

Using the fact that $\mathcal{N}_{D,E} = 0$ and $\mathcal{M}_{D,E} = 0$ unless if $E \in \mathcal{N}(D)$ or if $E = D$, we deduce from (19) :

$$\sum_{E \in \mathcal{D}_h} \mathcal{M}_{D,E} \beta(M_E^{n+1}) = \sum_{E \in \mathcal{N}(D)} \mathcal{M}_{D,E} \beta(S_E^{n+1}) + \mathcal{M}_{D,D} \beta(S_D^{n+1})$$

$$= \sum_{E \in \mathcal{N}(D)} \mathcal{M}_{D,E} \left(\beta(S_E^{n+1}) - \beta(S_D^{n+1}) \right), \tag{21}$$

$$\sum_{E \in \mathcal{D}_h} \mathcal{N}_{D,E} S_E^{n+1} = \sum_{E \in \mathcal{N}(D)} \mathcal{N}_{D,E} (S_E^{n+1} - S_D^{n+1}). \tag{22}$$

Let us take two fixed neighboring dual volumes D and $E \in \mathcal{D}_h$. Using the symmetry of the tensor Λ, we remark that $\mathcal{M}_{D,E} = \mathcal{M}_{E,D}$, which yields to an equality up to the sign between two discrete diffusive flux, from D to E and from E to D. In other terms,

$$\mathcal{M}_{D,E}(S_E^{n+1} - S_D^{n+1}) = -\mathcal{M}_{E,D}(S_D^{n+1} - S_E^{n+1}).$$

Lemma 6 (Coercivity). *For all $\beta_h(S_h) = \sum_{D \in \mathcal{D}_h} \beta(S_D) \varphi_D$ and $W_h = \sum_{D \in \mathcal{D}_h} W_D \varphi_D \in X_h$, then, the discrete degenerate diffusion operator is continuous coercive and we have*

$$\sum_{D \in \mathcal{D}_h} \sum_{E \in \mathcal{N}(D)} \mathcal{M}_{D,E} \left(\beta(S_E) - \beta(S_D) \right)^2 \geq C_\Lambda \|\beta_h(S_h)\|_{X_h}^2, \quad \sum_{D \in \mathcal{D}_h} \sum_{E \in \mathcal{N}(D)} \mathcal{N}_{D,E} \left(S_E - S_D \right)^2 = \|W_h\|_{X_h}^2, \tag{23}$$

and

$$\left| \sum_{D \in \mathcal{D}_h} \sum_{E \in \mathcal{N}(D)} \mathcal{M}_{D,E} \left(\beta(S_E) - \beta(S_D) \right)^2 \right| \leq C_\Lambda' \|\beta_h(S_h)\|_{X_h}^2. \tag{24}$$

Proof. We have

$$\sum_{D \in \mathcal{D}_h} \sum_{E \in \mathcal{N}(D)} \mathcal{M}_{D,E} \left(\beta(S_E) - \beta(S_D) \right)^2 = -2 \sum_{D \in \mathcal{D}_h} \beta(S_D) \sum_{E \in \mathcal{N}(D)} \mathcal{M}_{D,E} \left(\beta(S_E) - \beta(S_D) \right)$$

$$= -2 \sum_{D \in \mathcal{D}_h} \beta(S_D) \sum_{E \in \mathcal{D}_h} \mathcal{M}_{D,E} \beta(S_E).$$

Then, by using the definition of $\beta_h(S_h)$ and (10) we get:

$$-\sum_{D\in\mathcal{D}_h}\beta(S_D)\sum_{E\in\mathcal{D}_h}\mathcal{M}_{D,E}\beta(S_E)=\sum_{K\in\mathcal{T}_h}\left(\Lambda\nabla\beta_h(S_h),\nabla\beta_h(S_h)\right)_{L^2(K)}.$$

Now thanks to the coercivity of the tensor Λ given by (H2), we can deduce that

$$\sum_{D\in\mathcal{D}_h}\sum_{E\in\mathcal{N}(D)}\mathcal{M}_{D,E}\left(\beta(S_E)-\beta(S_D)\right)^2\geq C_\Lambda\sum_{K\in\mathcal{T}_h}\left(\nabla\beta(S_h),\nabla\beta(S_h)\right)_{L^2(K)},$$

and therefore (24) is a straightforward consequence of the boundedness of the tensor Λ given in (H2).

3.2 A priori estimates

Proposition 1 *Let* $\left(S_D^{n+1},P_D^{n+1}\right)_{D\in\mathcal{D}_h,\,n\in\{0,\ldots,N\}}$ *be a solution of the scheme (7)–(9). Then, there exists a positive constant* $C(\Omega)$ *depends only on* Ω *such that :*

$$\sum_{D\in\mathcal{D}_h}\sum_{E\in\mathcal{N}(D)}\mathcal{M}_{E,D}\left(P_D^{n+1}-P_E^{n+1}\right)^2\leq C(\Omega)||\pi||^2_{L^\infty(0,T;L^2(\partial\Omega))},\qquad(25)$$

and consequently

$$||P_h^{n+1}||^2_{X_h}\leq C(\Omega)||\pi||^2_{L^\infty(0,T;L^2(\partial\Omega))}.$$

Proof. We multiply (8) by P_D^{n+1}, we sum for all $D\in\mathcal{D}_h$ and after a discrete integrating by parts we get :

$$\sum_{D\in\mathcal{D}_h}\sum_{E\in\mathcal{N}(D)}G_p(S_D^{n+1},S_E^{n+1},\delta P_{D,E}^{n+1})\left(P_E^{n+1}-P_D^{n+1}\right)=2\sum_{K\in\mathcal{T}_h}|\partial D\cap\partial\Omega|\pi_D^{n+1}P_D^{n+1}$$

but the numerical flux satisfies $G_p(a,b,c)c\geq\lambda_*|c|^2$, for all $(a.,b,c)\in\mathbb{R}^3$, and under the assumption that the transmissibilities coefficients are positive one gets

$$\lambda_*\sum_{D\in\mathcal{D}_h}\sum_{E\in\mathcal{N}(D)}\mathcal{M}_{E,D}\left(P_E^{n+1}-P_D^{n+1}\right)^2\leq 2\sum_{D\in\mathcal{D}_h}|\partial D\cap\partial\Omega|\pi_D^{n+1}P_D^{n+1}$$

$$\leq\frac{2}{\Delta t}\sum_{D\in\mathcal{D}_h}\left(\frac{1}{\beta}\int_{t_n}^{t_{n+1}}\int_{\partial D\cap\partial\Omega}\pi^2dtdx+4\beta\Delta t\int_{\partial D\cap\partial\Omega}(P_D^{n+1})^2dx\right).$$

Furthermore, by using the discrete trace lemma and the fact that $\pi\in L^\infty(0,T;L^2(\partial\Omega))$ we obtain :

$$\lambda_*\sum_{D\in\mathcal{D}_h}\sum_{E\in\mathcal{N}(D)}\mathcal{M}_{E,D}\left(P_E^{n+1}-P_D^{n+1}\right)^2\leq\frac{2}{\beta}||\pi||_{L^\infty(0,T;L^2(\partial\Omega))}+\beta C_1\left(||P_h^{n+1}||_{X_h}+||P_h^{n+1}||_{L^2(\Omega)}\right).$$

Applying Lemma 2 to the sequence $\left(P_h^{n+1}\right)_h$ and using the equation (9), we have

$$||P_h^{n+1}||_{L^2(\Omega)}\leq C||P_h^{n+1}||_{X_h},$$

then from above inequality, we get

$$\lambda_*\sum_{D\in\mathcal{D}_h}\sum_{E\in\mathcal{N}(D)}\mathcal{M}_{E,D}\left(P_E^{n+1}-P_D^{n+1}\right)^2\leq\frac{2}{\beta}||\pi||_{L^\infty(0,T;L^2(\partial\Omega))}+\beta C_1(1+C)||P_h^{n+1}||_{X_h}.$$

Finally, The result follow from the use of the Lemmas 6 and after choosing appropriate parameter β.

Proposition 2 *Let* $\left(S_D^{n+1}, P_D^{n+1}\right)_{D \in \mathcal{D}_h, n \in \{0,...,N\}}$ *be a solution of the scheme* (7)-(9). *then, there exists a positive constant* $C_1(\Omega)$ *depends only on* Ω, *where for all* $n \in \mathbb{N}$, $0 < \Delta t \leq \Delta t_0$ *and* $n\Delta t \leq T$ *with* $\Delta t_0 \leq \dfrac{\mu}{C_1(\Omega)}$ *we have:*

$$\sum_{D \in \mathcal{D}_h} |D| \left|S_D^n\right|^2 + \sum_{D \in \mathcal{D}_h} \sum_{E \in \mathcal{N}(D)} \mathcal{N}_{E,D} \left|S_E^n - S_L^n\right|^2 \leq g(s_0, h, \pi, \Delta t_0), \tag{26}$$

where

$$g = \left(\sum_{D \in \mathcal{D}_h} |D| \left|S_D^0\right|^2 + \sum_{D \in \mathcal{D}_h} \sum_{E \in \mathcal{N}(D)} \mathcal{N}_{E,D} \left|S_E^0 - S_D^0\right|^2 + \|h\|_{L^\infty(0,T;L^2(\partial\Omega))}^2 + \|\pi\|_{L^\infty(0,T;L^2(\partial\Omega))}^2\right) \exp\left(\frac{C_1(\Omega)T}{\mu - C_1(\Omega)\Delta t_0}\right).$$

Proof. We multiply (7) by S_D^{n+1} for all $D \in \mathcal{D}_h$, that gives :

$$\sum_{D \in \mathcal{D}_h} |D| \frac{S_D^{n+1} - S_D^n}{\Delta t} S_D^{n+1} - \mu \sum_{D \in \mathcal{D}_h} \sum_{E \in \mathcal{N}(D)} \mathcal{N}_{E,D} \left(\frac{S_E^{n+1} - S_E^n}{\Delta t} - \frac{S_D^{n+1} - S_D^n}{\Delta t}\right) S_D^{n+1}$$

$$+ \sum_{D \in \mathcal{D}_h} \sum_{E \in \mathcal{N}(D)} G_s(S_D^{n+1}, S_E^{n+1}, \delta P_{D,E}^{n+1}) S_D^{n+1} - \sum_{D \in \mathcal{D}_h} \sum_{E \in \mathcal{N}(D)} \mathcal{M}_{E,D} \left(\beta(S_E^{n+1}) - \beta(S_D^{n+1})\right) S_D^{n+1}$$

$$= \sum_{D \in \mathcal{D}_h} |\partial D \cap \partial \Omega| h_D^{n+1} S_D^{n+1}.$$

Making an integration by parts and using the inequality $\left[(a-b)a \geq \frac{1}{2}\left(a^2 - b^2\right)\right]$, and using the conservative property of the function G_s we obtain

$$\frac{1}{2} \sum_{D \in \mathcal{D}_h} |D| \frac{\left|S_D^{n+1}\right|^2 - \left|S_D^n\right|^2}{\Delta t} + \frac{\mu}{4\Delta t} \sum_{D \in \mathcal{D}_h} \sum_{E \in \mathcal{N}(D)} \mathcal{N}_{E,D} \left(\left|S_E^{n+1} - S_D^{n+1}\right|^2 - \left|S_E^n - S_D^n\right|^2\right)$$

$$+ \frac{1}{2} \sum_{D \in \mathcal{D}_h} \sum_{E \in \mathcal{N}(D)} \mathcal{M}_{E,D} \left(\beta(S_E^{n+1}) - \beta(S_D^{n+1})\right) \left(S_E^{n+1} - S_D^{n+1}\right) \leq$$

$$\frac{1}{2} \sum_{D \in \mathcal{D}_h} \sum_{E \in \mathcal{N}(D)} G_s(S_D^{n+1}, S_E^{n+1}, \delta P_{E,D}^{n+1}) \left(S_E^{n+1} - S_D^{n+1}\right) + \sum_{D \in \mathcal{D}_h} |\partial D \cap \partial \Omega| h_D^{n+1} S_D^{n+1}. \tag{27}$$

The function β is a nondecreasing and since $\mathcal{M}_{E,D} \geq 0$, then the diffusion term leads

$$\frac{1}{2} \sum_{D \in \mathcal{D}_h} \sum_{E \in \mathcal{N}(D)} \mathcal{M}_{E,D} \left(\beta(S_E^{n+1}) - \beta(S_D^{n+1})\right) \left(S_E^{n+1} - S_D^{n+1}\right) \geq 0.$$

Notice that from (5), we have $\left|G_s(S_D^{n+1}, S_E^{n+1}, \delta P_{D,E}^{n+1})\right| \leq C_1 \left|\delta P_{D,E}^{n+1}\right|$, then the first term in the right hand side of (27), is estimated as

$$\frac{1}{2} \sum_{D \in \mathcal{D}_h} \sum_{E \in \mathcal{N}(D)} G_s(S_D^{n+1}, S_E^{n+1}, \delta P_{D,E}^{n+1}) \left(S_E^{n+1} - S_D^{n+1}\right) \leq$$

$$C_1 \sum_{D \in \mathcal{D}_h} \sum_{E \in \mathcal{N}(D)} \mathcal{M}_{E,D} \left(P_E^{n+1} - P_D^{n+1}\right)^2 + C_1 \sum_{D \in \mathcal{D}_h} \sum_{E \in \mathcal{N}(D)} \mathcal{M}_{E,D} \left(S_E^{n+1} - S_D^{n+1}\right)^2.$$

But, according to Lemma 6 if we replace β by identity function, we obtain

$$\sum_{D\in\mathcal{D}_h}\sum_{E\in\mathcal{N}(D)}\mathcal{M}_{E,D}\big(S_E^{n+1}-S_D^{n+1}\big)^2 \leq C_\Lambda||S_h||_{X_h} = C_\Lambda \sum_{D\in\mathcal{D}_h}\sum_{E\in\mathcal{N}(D)}\mathcal{N}_{E,D}\big(S_E^{n+1}-S_D^{n+1}\big)^2,$$

then

$$\frac{1}{2}\sum_{D\in\mathcal{D}_h}\sum_{E\in\mathcal{N}(D)}G_s(S_D^{n+1},S_E^{n+1},\delta P_{D,E}^{n+1})\big(S_E^{n+1}-S_D^{n+1}\big) \leq$$

$$C_1 C(\Omega)||\pi||_{L^\infty(0,T;L^2(\partial\Omega))} + C_\Lambda \sum_{D\in\mathcal{D}_h}\sum_{E\in\mathcal{N}(D)}\mathcal{N}_{E,D}\big(S_E^{n+1}-S_D^{n+1}\big)^2.$$

Using the Cauchy-Schwarz inequality with the discrete trace lemma 3 give us

$$\sum_{D\in\mathcal{D}_h}|\partial D \cap \partial\Omega|h_D^{n+1}S_D^{n+1} \leq ||h||_{L^\infty(0,T;L^2(\partial\Omega))} + C_2\left(\sum_{D\in\mathcal{D}_h}|D|(|S_D^{n+1}|^2 + \sum_{D\in\mathcal{D}_h}\sum_{E\in\mathcal{N}(D)}\mathcal{N}_{E,D}|S_E^{n+1}-S_D^{n+1}|^2\right),$$

where C_2 is a positive constant depend only on Ω. This leads

$$\sum_{D\in\mathcal{D}_h}|D|(|S_D^{n+1}|^2 - |S_D^n|^2) + \sum_{D\in\mathcal{D}_h}\sum_{E\in\mathcal{N}(D)}\mathcal{N}_{E,D}(|S_E^{n+1}-S_D^{n+1}|^2 - |S_E^n-S_D^n|^2)$$

$$\leq \frac{4\Delta t}{\mu}\left(C_2\sum_{D\in\mathcal{D}_h}|D|\big(S_D^{n+1}\big)^2 + (C_2+C_\Lambda)\sum_{D\in\mathcal{D}_h}\sum_{E\in\mathcal{N}(D)}\mathcal{N}_{E,D}\big(S_E^{n+1}-S_D^{n+1}\big)^2\right.$$

$$\left.+ ||h||_{L^\infty(0,T;L^2(\partial\Omega))}^2 + C_1 C(\Omega)||\pi||_{L^\infty(0,T;L^2(\partial\Omega))}\right),$$

to end this proof, we take $C_1(\Omega) = \frac{1}{4}\max\{C_2, C_2+C_\Lambda, C_1 C(\Omega), 1\}$ and applying the discrete Gronwall lemma 1 with $y_{n+1} = \sum_{D\in\mathcal{D}_h}|D||S_D^{n+1}|^2 + \sum_{D\in\mathcal{D}_h}\sum_{E\in\mathcal{N}(D)}\mathcal{N}_{E,D}|S_E^{n+1}-S_D^{n+1}|^2$, $\gamma_2 = \frac{C_1(\Omega)}{\mu}$ and $\gamma_1 = \frac{C_1(\Omega)}{\mu}\left(||h||_{L^\infty(0,T;L^2(\partial\Omega))}^2 + ||\pi||_{L^\infty(0,T;L^2(\partial\Omega))}\right)$ to get (26).

Proposition 3 *Let* $\big(S_D^{n+1}, P_D^{n+1}\big)_{D\in\mathcal{D}_h, n\in\{0,\dots,N\}}$ *be a solution of the scheme* (7)-(9). *Then, there exist a positive constants* C_2 *depend only on* μ *such that :*

$$\sum_{D\in\mathcal{D}_h}|D|\big(\frac{S_D^{n+1}-S_D^n}{\Delta t}\big)^2 + \sum_{D\in\mathcal{D}_h}\sum_{E\in\mathcal{N}(D)}\mathcal{N}_{E,D}\big(\frac{S_E^{n+1}-S_E^n}{\Delta t} - \frac{S_D^{n+1}-S_D^n}{\Delta t}\big)^2 \leq C_2(\mu)g(s_0,h,\pi,\Delta t_0). \qquad (28)$$

In addition, we have

$$\sum_{n=0}^{N}\Delta t||\beta_h(S_h^{n+1})||_{X_h}^2 \leq C \qquad (29)$$

Proof. We multiply (7) by $\frac{S_D^{n+1}-S_D^n}{\Delta t}$, $D\in\mathcal{D}_h$, sum over all and $D\in\mathcal{D}_h$, and after integrating by parts we obtain $J_1 + J_2 + J_3 = J_4$ where

$$J_1 = \sum_{D \in \mathcal{D}_h} |D| \left(\frac{S_D^{n+1} - S_D^n}{\Delta t} \right)^2 + \frac{\mu}{2} \sum_{D \in \mathcal{D}_h} \sum_{E \in \mathcal{N}(D)} \mathcal{N}_{E,D} \left(\frac{S_E^{n+1} - S_E^n}{\Delta t} - \frac{S_D^{n+1} - S_D^n}{\Delta t} \right)^2,$$

$$J_2 = -\frac{1}{2} \sum_{D \in \mathcal{D}_h} \sum_{E \in \mathcal{N}(D)} G_s(S_D^{n+1}, S_E^{n+1}, \delta P_{D,E}^{n+1}) \left(\frac{S_E^{n+1} - S_E^n}{\Delta t} - \frac{S_D^{n+1} - S_D^n}{\Delta t} \right),$$

$$J_3 = \frac{1}{2} \sum_{D \in \mathcal{D}_h} \sum_{E \in \mathcal{N}(D)} \mathcal{M}_{E,D} \left(\beta(S_E^{n+1}) - \beta(S_D^{n+1}) \right) \left(\frac{S_E^{n+1} - S_E^n}{\Delta t} - \frac{S_D^{n+1} - S_D^n}{\Delta t} \right),$$

$$J_4 = \sum_{D \in \mathcal{D}_h} |\partial D \cap \partial \Omega| h_D^{n+1} \frac{S_D^{n+1} - S_D^n}{\Delta t}.$$

First, to estimate the second term J_2, let us recall that there exist a positive constant C, such that $\left| G_s(S_D^{n+1}, S_E^{n+1}, \delta P_{D,E}^{n+1}) \right| \le C \left| \delta P_{D,E}^{n+1} \right|$, and by using Cauchy-Schwartz inequality we obtain, for all $\gamma > 0$

$$J_2 \le \frac{C}{\gamma} \sum_{D \in \mathcal{D}_h} \sum_{E \in \mathcal{N}(D)} \mathcal{M}_{E,D} (P_E^{n+1} - P_E^{n+1})^2 + \gamma \sum_{D \in \mathcal{D}_h} \sum_{E \in \mathcal{N}(D)} \mathcal{M}_{E,D} \left(\frac{S_E^{n+1} - S_E^n}{\Delta t} - \frac{S_D^{n+1} - S_D^n}{\Delta t} \right)^2$$

from the estimate (25) and Lemma6, one gets

$$J_2 \le \frac{C}{\gamma} ||\pi||_{L^\infty(0,T;L^2(\partial \Omega))}^2 + C_1 \gamma \sum_{D \in \mathcal{D}_h} \sum_{E \in \mathcal{N}(D)} \mathcal{N}_{E,D} \left(\frac{S_E^{n+1} - S_E^n}{\Delta t} - \frac{S_D^{n+1} - S_D^n}{\Delta t} \right)^2.$$

Next, we use again Cauchy-Schwartz inequality with the proprieties of β to obtain for all $\delta > 0$:

$$J_3 \le \frac{C_2}{2\delta} \sum_{D \in \mathcal{D}_h} \sum_{E \in \mathcal{N}(D)} \mathcal{M}_{E,D} (S_E^{n+1} - S_D^{n+1})^2 + \frac{\delta}{2} \sum_{D \in \mathcal{D}_h} \sum_{E \in \mathcal{N}(D)} \mathcal{M}_{E,D} \left(\frac{S_E^{n+1} - S_E^n}{\Delta t} - \frac{S_D^{n+1} - S_D^n}{\Delta t} \right)^2.$$

this yield us by Proposition 2 and Lemma 6 to have

$$J_3 \le \frac{C_1 C_2}{2\delta} g(s_0, h, \pi, \Delta t_0) + \frac{C_1 \delta}{2} \sum_{D \in \mathcal{D}_h} \sum_{E \in \mathcal{N}(D)} \mathcal{N}_{E,D} \left(\frac{S_E^{n+1} - S_E^n}{\Delta t} - \frac{S_D^{n+1} - S_D^n}{\Delta t} \right)^2.$$

Similarly, we have for all $\zeta > 0$

$$J_4 \le \frac{C_3}{\zeta} ||h||_{L^\infty(0,T;L^2(\partial \Omega))}^2 + \zeta C \left[\sum_{D \in \mathcal{D}_h} |D| \left(\frac{S_D^{n+1} - S_D^n}{\Delta t} \right)^2 + \sum_{D \in \mathcal{D}_h} \sum_{E \in \mathcal{N}(D)} \mathcal{N}_{E,D} \left(\frac{S_E^{n+1} - S_E^n}{\Delta t} - \frac{S_D^{n+1} - S_D^n}{\Delta t} \right)^2 \right].$$

Finally, if we chose γ, δ and ζ small enough, we obtain

$$\sum_{D \in \mathcal{D}_h} |D| \left(\frac{S_D^{n+1} - S_D^n}{\Delta t} \right)^2 + \sum_{D \in \mathcal{D}_h} \sum_{E \in \mathcal{N}(D)} \mathcal{N}_{E,D} \left(\frac{S_E^{n+1} - S_E^n}{\Delta t} - \frac{S_D^{n+1} - S_D^n}{\Delta t} \right)^2 \le C_4 \mu g(s_0, h, \pi, \Delta t_0).$$

Now, we are concerned with estimate (29), for that we multiply (7) by $\Delta t \beta(S_D^{n+1})$ and we sum for all $D \in \mathcal{D}_h$ and $n = \{0, ..., N\}$ to obtain $E_1 + E_2 + E_3 + E_4 = E_5$, where $(E_i)_{i=1..5}$ are explicitly given and estimate in what follow. According to the convexity of the function $\Upsilon(s) = \int_0^s \beta(z)dz$, since $\Upsilon''(s) = \alpha(s) \ge 0$, we shall have the following inequality : $(a-b)\beta(a) \ge \Upsilon(a) - \Upsilon(b)$. Then,

$$E_1 = \sum_{n=0}^{N} \sum_{D \in \mathcal{D}_h} |D| \left(S_D^{n+1} - S_D^n\right) \beta(S_D^{n+1}) \geq \sum_{n=0}^{N} \sum_{D \in \mathcal{D}_h} |D| \left(\Upsilon(S_D^{n+1}) - \Upsilon(S_D^n)\right) = \sum_{D \in \mathcal{D}_h} |D| \left(\Upsilon(S_D^N) - \Upsilon(S_D^0)\right).$$

Using of Young inequality, an integrating by parts and the positivity of transmissibilities coefficient $\mathcal{M}_{E,D}$ we obtain for all $\delta > 0$

$$E_2 = -\mu \sum_{n=0}^{N} \Delta t \sum_{D \in \mathcal{D}_h} \sum_{E \in \mathcal{N}(D)} \mathcal{M}_{E,D} \left(\frac{S_E^{n+1} - S_E^n}{\Delta t} - \frac{S_D^{n+1} - S_D^n}{\Delta t}\right) \beta(S_D^{n+1})$$

$$\leq \frac{\mu}{\delta} \sum_{n=0}^{N} \Delta t \sum_{D \in \mathcal{D}_h} \sum_{E \in \mathcal{N}(D)} \mathcal{M}_{E,D} \left(\frac{S_E^{n+1} - S_E^n}{\Delta t} - \frac{S_D^{n+1} - S_D^n}{\Delta t}\right)^2 + \delta\mu \sum_{n=0}^{N} \Delta t \sum_{D \in \mathcal{D}_h} \sum_{E \in \mathcal{N}(D)} \mathcal{M}_{E,D} \left(\beta(S_D^{n+1}) - \beta(S_E^{n+1})\right)^2,$$

this yield to

$$E_2 = \leq \frac{C\mu}{\delta} g(s_0, h, \pi, \Delta t_0) + \delta\mu \sum_{n=0}^{N} \Delta t \sum_{D \in \mathcal{D}_h} \sum_{E \in \mathcal{N}(D)} \mathcal{M}_{E,D} \left(\beta(S_D^{n+1}) - \beta(S_E^{n+1})\right)^2.$$

According to equation (23), we have

$$E_3 = -\sum_{n=0}^{N} \Delta t \sum_{D \in \mathcal{D}_h} \sum_{E \in \mathcal{N}(D)} \mathcal{M}_{E,D} \left(\beta(S_E^{n+1}) - \beta(S_D^{n+1})\right) \beta(S_D^{n+1})$$

$$= \frac{1}{2} \sum_{n=0}^{N} \Delta t \sum_{D \in \mathcal{D}_h} \sum_{E \in \mathcal{N}(D)} \mathcal{M}_{E,D} \left(\beta(S_E^{n+1}) - \beta(S_D^{n+1})\right)^2 \geq C_\Lambda \sum_{n=0}^{N} \Delta t \|\beta_h(S_h^{n+1})\|_{X_h}^2.$$

Next, we have

$$E_4 = \sum_{n=0}^{N-1} \Delta t \sum_{D \in \mathcal{D}_h} \sum_{E \in \mathcal{N}(D)} G_s(S_D^{n+1}, S_E^{n+1}, \delta P_{D,E}^{n+1}) \beta(S_D^{n+1}).$$

Integrating by part, using $\left|G_s(S_D^{n+1}, S_E^{n+1}, \delta P_{D,E}^{n+1})\right| \leq C \left|\delta P_{D,E}^{n+1}\right|$ and the Cauchy-Schwarz inequality, give us

$$E_4 \leq \frac{1}{\delta} \sum_{n=0}^{N} \Delta t \sum_{D \in \mathcal{D}_h} \sum_{E \in \mathcal{N}(D)} \mathcal{M}_{E,D} \left(P_E^{n+1} - P_D^{n+1}\right)^2 + \delta \sum_{n=0}^{N} \Delta t \sum_{D \in \mathcal{D}_h} \sum_{E \in \mathcal{N}(D)} \mathcal{M}_{E,D} \left(\beta(S_E^{n+1}) - \beta(S_D^{n+1})\right)^2,$$

for any $\delta > 0$. The estimate (25) and Lemma 6 lead to

$$E_4 \leq \frac{C}{\delta} \|\pi\|_{L^\infty(0,T;L^2(\partial\Omega))}^2 + \delta \sum_{n=0}^{N} \Delta t \|\beta_h(S_h)\|_{X_h}^2.$$

Finally, we use the continuity of β and lemma 3 to obtain for all $\delta > 0$

$$E_5 = \sum_{n=0}^{N} \Delta t \sum_{D \in \mathcal{D}_h} |\partial D \cap \partial\Omega| h_D^{n+1} \beta(S_D^{n+1})$$

$$\leq \frac{C}{\delta} \|h(t, \cdot)\|_{L^\infty(0,T;L^2(\partial\Omega))}^2 + \delta C_1 \left(\sum_{n=0}^{N} \Delta t \sum_{D \in \mathcal{D}_h} |D| |S_D^n|^2 + \sum_{n=0}^{N} \Delta t \|\beta_h(S_h)\|_{X_h}^2\right)$$

$$\leq C' g(s_0, h, \pi, \Delta t_0) + \delta C_1 \sum_{n=0}^{N} \Delta t \|\beta_h(S_h)\|_{X_h}^2.$$

In summarize, from the equation $E_1 + E_2 + E_3 + E_4 = E_5$ and by choosing an appropriate δ we have

$$\sum_{n=0}^{N} \Delta t ||\beta_h(S_h^{n+1})||_{X_h}^2 \leq C'' g(s_0, h, \pi, \Delta t_0) + \sum_{D \in \mathcal{D}_h} |D| \Upsilon(S_D^0) \leq C,$$

where C'' and C are positives constants.

3.3 Existence of discrete solution

The existence of a discrete solution for the combined scheme is given in the following proposition.

Proposition 4 *There exists a least one solution of problem* (7)-(9).

Proof. First, we will use the following notation :

$$\mathcal{M} := Card(\mathcal{D}_h), \quad s_{\mathcal{M}} := \{s_D^{n+1}\}_{D \in \mathcal{D}_h} \in \mathbb{R}^{\mathcal{M}}, \quad p_{\mathcal{M}} := \{p_D^{n+1}\}_{D \in \mathcal{D}_h} \in \mathbb{R}^{\mathcal{M}}.$$

We define the application $\quad \mathcal{P}_h : \mathbb{R}^{\mathcal{M}} \times \mathbb{R}^{\mathcal{M}} \mapsto \mathbb{R}^{\mathcal{M}} \times \mathbb{R}^{\mathcal{M}}$

$$\mathcal{P}_h(s_{\mathcal{M}}, p_{\mathcal{M}}) = (\{\mathcal{P}_{1,D}\}_{D \in \mathcal{D}_h}, \{\mathcal{P}_{2,D}\}_{D \in \mathcal{D}_h}), \tag{30}$$

with

$$\mathcal{P}_{1,K} = |D| \frac{S_D^{n+1} - S_D^n}{\Delta t} - \mu \sum_{D \in \mathcal{D}_h} \sum_{E \in \mathcal{N}(D)} \mathcal{N}_{D,E} \left(\frac{S_E^{n+1} - S_E^n}{\Delta t} - \frac{S_D^{n+1} - S_D^n}{\Delta t} \right)$$

$$- \sum_{D \in \mathcal{D}_h} \sum_{E \in \mathcal{N}(D)} \mathcal{M}_{E,D} \left(\beta(S_E^{n+1}) - \beta(S_D^{n+1}) \right) + \sum_{E \in \mathcal{N}(D)} G_s(S_D^{n+1}, S_E^{n+1}; \delta P_{D,E}^{n+1}) - |\partial D \cap \partial \Omega| h_D^{n+1}$$

$$\mathcal{P}_{2,K} = - \sum_{E \in \mathcal{N}(D)} G_p(S_D^{n+1}, S_E^{n+1}; \delta P_{D,E}^{n+1}) - |\partial D \cap \partial \Omega| \pi_D^{n+1}.$$

The scalar product of $(s_{\mathcal{M}}, p_{\mathcal{M}})$ by the equation (30), using the estimates (25)-(26) and Cauchy-Schwarz inequality permit to get

$$[\mathcal{P}_h(s_{\mathcal{M}}, p_{\mathcal{M}}), (s_{\mathcal{M}}, p_{\mathcal{M}})] \geq \frac{C_1}{\Delta t} \sum_{D \in \mathcal{D}_h} |D| |S_D^{n+1}|^2 + C_2 \sum_{D \in \mathcal{D}_h} |D| |P_D^{n+1}|^2 - \frac{1}{2\Delta t} \sum_{D \in \mathcal{D}_h} |D| |S_K^n|^2 - C_3 g(s_0, h, \pi, \Delta t_0)$$

where C_1, C_2 and C_3 are positives constants, this yields to

$$[\mathcal{P}_h(s_{\mathcal{M}}, p_{\mathcal{M}}), (s_{\mathcal{M}}, p_{\mathcal{M}})] > 0$$

For $|(s_{\mathcal{M}}, p_{\mathcal{M}})|_{\mathbb{R}^{\mathcal{M}}}$ large enough. And therefore we obtain the existence of $(s_{\mathcal{M}}, p_{\mathcal{M}})$ such that $\mathcal{P}_h(s_{\mathcal{M}}, p_{\mathcal{M}}) = 0$. Indeed, reasoning by contradiction. Assume that there is no $(s_{\mathcal{M}}, p_{\mathcal{M}})$ such that $\mathcal{P}_h(s_{\mathcal{M}}, p_{\mathcal{M}}) = 0$. In this case, one can define on a ball of origin center and radius k, the following application: $A : \bar{B}(0, k) \to \bar{B}(0, k)$ which give for each $(s_{\mathcal{M}}, p_{\mathcal{M}})$, $A((s_{\mathcal{M}}, p_{\mathcal{M}})) = -k \frac{\mathcal{P}_h(s_{\mathcal{M}}, p_{\mathcal{M}})}{|(s_{\mathcal{M}}, p_{\mathcal{M}})|}$, where the constant $k > 0$. The map A is continuous due to the continuity of $\mathcal{P}_h$, $|\mathcal{P}_h(s_{\mathcal{M}}, p_{\mathcal{M}})| \neq 0$ over $\bar{B}(0, k)$ and $\bar{B}(0, k)$ is closed convex and compact. The Brower fixed point theorem implies then that there exists $(s_{\mathcal{M}}, p_{\mathcal{M}})$ in $\bar{B}(0, k)$ such that $-k \frac{\mathcal{P}_h(s_{\mathcal{M}}, p_{\mathcal{M}})}{|(s_{\mathcal{M}}, p_{\mathcal{M}})|} = (s_{\mathcal{M}}, p_{\mathcal{M}})$, If we take the norm of both sides of this equation, we see that $|(s_{\mathcal{M}}, p_{\mathcal{M}})| = k > 0$ and if we take the scalar product of each side with $(s_{\mathcal{M}}, p_{\mathcal{M}})$, we find $[(s_{\mathcal{M}}, p_{\mathcal{M}}), (s_{\mathcal{M}}, p_{\mathcal{M}})] = |(s_{\mathcal{M}}, p_{\mathcal{M}})|^2 = -k \frac{[\mathcal{P}_h(s_{\mathcal{M}}, p_{\mathcal{M}}), (s_{\mathcal{M}}, p_{\mathcal{M}})]}{|(s_{\mathcal{M}}, p_{\mathcal{M}})|} \leq 0$, this yield to contradiction.

4 Convergence

4.1 Compactness Estimates on Discrete Solutions

In this subsection, we derive estimates on differences of space and time translates necessary to apply Kolmogorov's compactness theorem which will allow us to pass to the limit in the nonlinear second order terms.

Lemma 7 (Time translate estimate). *There exists a constant $C > 0$ depend of Ω, and T such that*

$$\iint_{\Omega \times [0,T-\tau]} \left(\tilde{S}_{h,\Delta t}(t+\tau,x) - \tilde{S}_{h,\Delta t}(t,x) \right)^2 dx\, dt \le C(\tau + \Delta t), \forall \tau \in [0,T].$$

Lemma 8 (space translate estimate). *There exists a constant $C > 0$ depend of Ω and T such that :*

$$\iint_{\Omega \times [0,T]} \left(\tilde{S}_{h,\Delta t}(t,x+\xi)) - \tilde{S}_{h,\Delta t}(t,x) \right)^2 dx\, dt \le C|\xi|(|\xi|+h), \forall \xi \in \mathbb{R}^d.$$

The estimates on the discrete global pressure (25) and on the saturation (26) are sufficient to prove the Lemma. The technics is being classical and widely used in the works of Eymard, Galouet, Herbin and their co-authors [3, 12, 13, 21].

4.2 Convergence of the combined scheme

This subsection is mainly devoted to the proof of the strong $L^2(Q_T)$ convergence of approximate saturation solutions, using the estimations proved in the subsection 3.2, 4.1 and Kolmogorovs compactness criterion (see [6]) for the convergence. Then, we prove that the limit is a weak solution to the continuous problem .

Lemma 9. *The sequence $\left(g_h(S_{h,\Delta t}) - g(\tilde{S}_{h,\Delta t}) \right)_{h,\Delta t}$ converges strongly to zero in $L^2(Q_T)$ when $h \to 0$ for $g = \beta$ and $g = Id$.*

Proof: Using the definition of the basis functions of the finite dimensional space X_h, and the reconstruction of discrete saturation in definition 2 give us

$$|\beta(\tilde{S}_{h,\Delta t}) - \beta_h(S_{h,\Delta t})|^2 = |\beta_h(S_{h,\Delta t})(\mathcal{P}_D) - \beta_h(S_{h,\Delta t})(x)|^2 = |\nabla \beta_h(S_{h,\Delta t}).(\mathcal{P}_D - x)|^2.$$

Integrating over Ω leads

$$\sum_{K \in \mathcal{T}_h} \sum_{\sigma_D \in \mathcal{E}_K} \int_{K \cap D} \left| \nabla \beta_h(S_{h,\Delta t}) \cdot (\mathcal{P}_D - x) \right|^2 dx \le \sum_{K \in \mathcal{T}_h} \sum_{\sigma_D \in \mathcal{E}_K} \left| \nabla \beta_h(S_{h,\Delta t})|_K \right|^2 (diam(D))^2 |K \cap D|$$

$$\le h^2 \sum_{K \in \mathcal{T}_h} \left| \nabla \beta_h(S_{h,\Delta t})|_K \right|^2 |K| \le h^2 ||\beta_h(S_{h,\Delta t})||_{X_h}^2.$$

Thus, the estimate (29) gives

$$||\beta_h(S_{h,\Delta t}) - \beta(\tilde{S}_{h,\Delta t})||_{L^2(Q_T)} \le h^2 \sum_{n=0}^{N} \Delta t ||\beta_h(S_{h,\Delta t})||_{X_h}^2 \le C_1 h^2 \overset{h,\Delta t \to 0}{\longrightarrow} 0.$$

Same proof for $g = Id$.

Theorem 2. *($\boldsymbol{Strong\ convergence\ in}$ $L^2(Q_T)$) There exists a subsequence of $(S_{h,\Delta t})_{h,\Delta t}$ which converges strongly to a function s in $L^2(Q_T)$. In addition, $(\beta_h(S_{h,\Delta t}))_{h,\Delta t}$ converges strongly in $L^2(Q_T)$ to the function $\Gamma = \beta(s)$.*

Proof. Proposition 2, and lemmas 7-8 implies that the sequence $\left(\tilde{S}_{h,\Delta t}\right)_{h,\Delta t}$ satisfies the assumptions of the Kolmogorov compactness criterion, and consequently the sequence $\left(\tilde{S}_{h,\Delta t}\right)_{h,\Delta t}$ is relatively compact in $L^2(Q_T)$. Thanks to lemma 9, $\left(S_{h,\Delta t}\right)_{h,\Delta t}$ converge strongly to a function $s \in L^2(Q_T)$.
Since β is well defined and continuous, thus, we extend it as follow :

$$\beta(s) = \begin{cases} 0, & \text{if } s \leq 0 \\ \int_0^s \alpha(z)dz, & \text{if } 0 \leq s \leq 1 \\ \int_0^1 \alpha(z)dz, & \text{if } s \geq 1. \end{cases}$$

applying L^∞ bound on $\tilde{S}_{h,\Delta t}$ and the dominated convergence theorem of Lebesgue to $\beta(S_h)$, there exists subsequence $\beta(S_{h,\Delta t})$ converge strongly in $L^2(Q_T)$ and a.e in Q_T to $\beta(s)$. Finally, by Lemma 9 we deduce that $\left(\beta_h(S_{h,\Delta t})\right)_{h,\Delta t}$ converges strongly in $L^2(Q_T)$ to $\Gamma = \beta(s)$.

Lemma 10. *There exists a subsequence of* $\left(f_{\tilde{S}_{h,\Delta t}}\right)_{h,\Delta t}$ *defined in* (14) *which is weakly convergent in* $W^{1,2}(0,T;L^2(\Omega))$ *to the function s defined in theorem 2.*

Proof. For the first part of convergence, it is useful to introduce the following inequality, for all $a,b \in \mathbb{R}$

$$\int_0^1 |\theta a + (1-\theta)b|d\theta \geq \frac{1}{2}(|a|+|b|)$$

Applying this inequality to $a = \tilde{S}_h^{n+1} - \tilde{S}_h^n$, $b = \tilde{S}_h^n - \tilde{S}_h^N$, from the definition of $f_{\tilde{S}_{h,\Delta t}}$ we deduce

$$\int_0^T \int_\Omega |f_{\tilde{S}_{h,\Delta t}}(t,x) - \tilde{S}_{h,\Delta t}(t,x)|dxdt \leq \int_0^{T-\Delta t} \int_\Omega |\tilde{S}_{h,\Delta t}(t+\Delta t,x) - \tilde{S}_{h,\Delta t}(t,x)|dxdt.$$

Since Δt tends to zero , the translate in time estimate in Lemma 7 implies that the right-hand side of the above inequality converges to zero as h and Δt tends to zero. Therefore, since we obtain a strongly convergence in $L^1(Q_T)$ and by construction of f_S we have that $f_{\tilde{S}_{h,\Delta t}}$ is bounded in $L^2(Q_T)$, this yield to deduce that $f_{\tilde{S}_{h,\Delta t}}$ weakly converge to s in $L^2(0,T;L^2(\Omega))$. Next, for $\psi \in \Theta = \{u \in C^{1,1}([0,T] \times \bar{\Omega}), u(T,.) = 0\}$, we denote by $\psi_D^n = \psi(t_n,x_D)$ and we define

$$T_T = -\sum_{n=0}^N \sum_{D \in \mathcal{D}_h} |D|S_D^{n+1}(\psi_D^n - \psi_D^{n+1}) - \sum_{D \in \mathcal{D}_h^{int}} |D|S_D^0(\psi_D^0)$$

$$= -\sum_{n=0}^N \sum_{D \in \mathcal{D}_h} \int_{t_n}^{t_{n+1}} \int_D S_D^{n+1} \frac{\partial \psi}{\partial t}(t,P_D)\,dxdt - \sum_{D \in \mathcal{D}_h} \int_D S_D^0 \psi(0,x_D)\,dx.$$

also we define

$$\tilde{T}_T = -\iint_{Q_T} \tilde{S}_{h,\Delta t} \frac{\partial \psi}{\partial t}dxdt - \int_\Omega \tilde{S}_{0,h}(0,x)\psi(x,0)\,dx$$

$$= -\sum_{n=0}^N \sum_{D \in \mathcal{D}_h} \int_{t_n}^{t_{n+1}} \int_D S_D^{n+1} \frac{\partial \psi}{\partial t}(t,x)\,dxdt - \sum_{D \in \mathcal{D}_h} \int_D S_D^0 \psi(0,x)\,dx.$$

We have, for all $x \in D$, $D \in \mathcal{D}_h$ and for all $h > 0$, $|\frac{\partial \psi}{\partial t}(t,P_D) - \frac{\partial \psi}{\partial t}(t,x)| \leq \varepsilon(h)$, where the function ε satisfies $\varepsilon(h) > 0$ and $\varepsilon(h) \xrightarrow{h\to 0} 0$. This follows by the fact that $\frac{\partial \psi}{\partial t} \in C^0(\bar{\Omega})$, then $\frac{\partial \psi}{\partial t}$ is uniformly continuous on $\bar{\Omega}$. Consequently, there exist a constant $C > 0$ such that

$$|T_T - \tilde{T}_T| \leq C\varepsilon(h) \xrightarrow{h,\Delta t \to 0} 0.$$

Note that $\tilde{S}_{h,\Delta t}$ converge also to s strongly in $L^2(Q_T)$, then we have but by theorem 2 and Lemma 9 we have,

$$\tilde{T}_T \rightarrow -\iint_{Q_T} s\frac{\partial \psi}{\partial t}dxdt - \int_{\Omega} s(0,x)\psi(x,0)\,dx$$

Notice that,

$$T_T = \sum_{n=0}^{N}\sum_{D\in\mathcal{D}_h}|D|\frac{S_D^{n+1}-S_D^n}{\Delta t}\Delta t\,\psi_D^{n+1} = \iint_{Q_T}\frac{\partial f_{\tilde{S}_{h,\Delta t}}}{\partial t}\psi_h \rightarrow -\iint_{Q_T} s\frac{\partial \psi}{\partial t}dxdt - \int_{\Omega} s(0,x)\psi(0,x)\,dx.$$

From estimate (28), the sequence $(f_{\tilde{S}})_{h,\Delta t}$ is bounded in $L^2(Q_T)$, then there a exist a function ζ such that $(f_{\tilde{S}})_{h,\Delta t}$ converge to ζ weakly in $L^2(Q_T)$, thus

$$\iint_{Q_T}\zeta\psi dtdx = -\iint_{Q_T} s\frac{\partial \psi}{\partial t}dxdt - \int_{\Omega} s(0,x)\psi(0,x)\,dx = \iint_{Q_T}\frac{\partial f_{\tilde{S}_{h,\Delta t}}}{\partial t}\psi dtdx, \quad \forall \psi \in \Theta,$$

therefor, we can deduce that $\zeta = \dfrac{\partial f_{\tilde{S}_{h,\Delta t}}}{\partial t}$ and $\iint_{Q_T}\dfrac{\partial f_{\tilde{S}_{h,\Delta t}}}{\partial t}\psi \rightarrow \iint_{Q_T}\dfrac{\partial s}{\partial t}\psi, \quad \forall \psi \in \Theta$. Furthermore, if we take $\psi = \phi - \phi(T,.)$ for all $\phi \in C^{1,1}([0,T]\times\bar{\Omega})$, we obtain

$$\iint_{Q_T}\frac{\partial f_{\tilde{S}_{h,\Delta t}}}{\partial t}\phi - \iint_{Q_T}\frac{\partial f_{\tilde{S}_{h,\Delta t}}}{\partial t}\phi(T,.) \rightarrow \iint_{Q_T}\frac{\partial s}{\partial t}\phi - \int_{Q_T}\frac{\partial s}{\partial t}\phi(T,.) \quad \forall \phi \in C^{1,1}([0,T]\times\bar{\Omega}).$$

then, it is enough to show

$$\iint_{Q_T}\frac{\partial f_{\tilde{S}_{h,\Delta t}}}{\partial t}\phi(T,.) \rightarrow \int_{Q_T}\frac{\partial s}{\partial t}\phi(T,.) \quad \forall \phi \in C^{1,1}([0,T]\times\bar{\Omega}),$$

to proof that

$$\iint_{Q_T}\frac{\partial f_{\tilde{S}_{h,\Delta t}}}{\partial t}\phi \rightarrow \iint_{Q_T}\frac{\partial s}{\partial t}\phi \quad \forall \phi \in C^{1,1}([0,T]\times\bar{\Omega}).$$

An integration by parts with respect to t gives us

$$\iint_{Q_T}\frac{\partial f_{\tilde{S}_{h,\Delta t}}}{\partial t}\phi(T,x)dxdt = \int_{\Omega} f_{\tilde{S}_{h,\Delta t}}(T,x)\phi(T,x)dx - \int_{\Omega} f_{\tilde{S}_{h,\Delta t}}(0,x)\phi(T,x)dx = \int_{\Omega}\tilde{S}_h^{N+1}\phi(T,x)dx - \int_{\Omega}\tilde{S}_h^0\phi(T,x)dx,$$

by proposition 2 and by applying Lemma 8 for $\tilde{S}_h^{N+1}, \tilde{S}_h^0$ respectively, we deduce by Kolmogorov's theorem that $\tilde{S}_h^{N+1}, \tilde{S}_h^0 \rightarrow s(T,.), s(0,.)$, strongly in $L^2(\Omega)$. Then

$$\iint_{Q_T}\frac{\partial f_{\tilde{S}_{h,\Delta t}}}{\partial t}\phi(T,.)dxdt \rightarrow \int_{\Omega} s(T,.)\phi(T,.) - \int_{\Omega} s(0,.)\phi(T,.) = \int_{Q_T}\frac{\partial s}{\partial t}\phi(T,.), \forall \phi \in C^{1,1}([0,T]\times\bar{\Omega}),$$

Finally, by using the density of $C^{1,1}([0,T]\times\bar{\Omega})$ in $L^2(Q_T)$ we obtain

$$\iint_{Q_T}\frac{\partial f_{\tilde{S}_{h,\Delta t}}}{\partial t}\phi \rightarrow \iint_{Q_T}\frac{\partial s}{\partial t}\psi \quad \forall \phi \in L^2(Q_T).$$

5 Proof of theorem 1

We will prove now that the limit couple (s,p) is a weak solution of the continuous problem. We take $\psi \in C^{1,1}([0,T] \times \bar{\Omega})$. We then multiply (7) by $\Delta t \psi(t_{n+1}, \mathcal{P}_D)$ and we sum the result over $D \in \mathcal{D}_h^{int}$ and for all $n \in [0,...,N]$, we have :

$$\mathcal{E}_1 + \mathcal{E}_2 + \mathcal{E}_3 + \mathcal{E}_4 = \mathcal{E}_5,$$

with,

$$\mathcal{E}_1 = \sum_{n=0}^{N} \Delta t \sum_{D \in \mathcal{D}_h} |D| \frac{S_D^{n+1} - S_D^n}{\Delta t} \psi(t_{n+1}, \mathcal{P}_D),$$

$$\mathcal{E}_2 = -\mu \sum_{n=0}^{N} \Delta t \sum_{D \in \mathcal{D}_h} \sum_{E \in \mathcal{N}(D)} \mathcal{N}_{D,E} \Big(\frac{S_E^{n+1} - S_E^n}{\Delta t} - \frac{S_D^{n+1} - S_D^n}{\Delta t} \Big) \psi(t_{n+1}, \mathcal{P}_D),$$

$$\mathcal{E}_3 = -\sum_{n=0}^{N} \Delta t \sum_{D \in \mathcal{D}_h} \sum_{E \in \mathcal{N}(D)} \mathcal{M}_{D,E} \big(\beta(S_E^{n+1}) - \beta(S_D^{n+1}) \big) \psi(t_{n+1}, \mathcal{P}_D),$$

$$\mathcal{E}_4 = \sum_{n=0}^{N} \Delta t \sum_{D \in \mathcal{D}_h} \sum_{E \in \mathcal{N}(D)} G_s(S_D^{n+1}, S_E^{n+1}; \delta P_{D,E}^{n+1}) \psi(t_{n+1}, \mathcal{P}_D),$$

$$\mathcal{E}_5 = \sum_{n=0}^{N} \sum_{D \in \mathcal{D}_h} \Delta t |\partial D \cap \partial \Omega| h_D^{n+1} \psi(t_{n+1}, \mathcal{P}_D).$$

We now show that each of the above terms converges to its continuous version as $h, \Delta t \to 0$. The first term is equal to

$$\mathcal{E}_1 = \sum_{n=0}^{N} \sum_{D \in \mathcal{D}_h} \int_{t_n}^{t_{n+1}} \int_D \frac{\partial f_{\tilde{S}_h}}{\partial t} \psi(t_{n+1}, \mathcal{P}_D) \, dx dt,$$

Define

$$\tilde{\mathcal{E}}_1 = \iint_{Q_T} \frac{\partial f_{\tilde{S}_h}}{\partial t} \psi dx dt = \sum_{n=0}^{N} \sum_{D \in \mathcal{D}_h} \int_{t_n}^{t_{n+1}} \int_D \frac{\partial f_{\tilde{S}_h}}{\partial t} \psi(t,x) \, dx dt.$$

We have, for all $x \in D$, $D \in \mathcal{D}_h$ and for all $h > 0$, $\Delta t > 0$, $|\psi(t_{n+1}, \mathcal{P}_D) - \psi(t,x)| \le \varepsilon(h, \Delta t)$, as we have seen before, so we have

$$|\mathcal{E}_1 - \tilde{\mathcal{E}}_1| \le C \varepsilon(h, \Delta t) \overset{h, \Delta t \to 0}{\longrightarrow} 0.$$

Lemma 10 ensures that $\dfrac{\partial f_{\tilde{S}_h}}{\partial t} \to \partial_t s$ weakly in $L^2(Q_T)$, then

$$\mathcal{E}_1 \longrightarrow \iint_{Q_T} \partial_t s \psi dx dt.$$

According to Remark 2, we can rewrite $\mathcal{E}_2$ as follow

$$\mathcal{E}_2 = -\mu \sum_{n=0}^{N} \Delta t \sum_{D \in \mathcal{D}_h} \sum_{E \in \mathcal{D}_h} \mathcal{N}_{D,E} \Big(\frac{S_E^{n+1} - S_E^n}{\Delta t}\Big) \psi(t_{n+1}, \mathcal{P}_D)$$

$$= \mu \sum_{n=0}^{N} \Delta t \sum_{K \in \mathcal{T}_h} \int_K \nabla \partial_t f_{S_h} \cdot \nabla \Big(\sum_{D \in \mathcal{D}_h} \psi(t_{n+1}, \mathcal{P}_D) \varphi_D(x) \Big) dx.$$

We set

$$I_\psi(t_{n+1}, x) := \sum_{D \in \mathcal{D}_h} \psi(t_{n+1}, \mathcal{P}_D) \varphi_D(x).$$

we will prove that

$$\mathcal{E}_{2,1} = \mu \sum_{n=0}^{N} \Delta t \sum_{K \in \mathcal{T}_h} \int_K \nabla \partial_t f_{S_h} \cdot \nabla \big(I_\psi(t_{n+1}, x) - \psi(t_{n+1}, x) \big) dx \xrightarrow{h, \to 0} 0.$$

Indeed, using Cauchy-Schwarz inequality, we estimate :

$$|\mathcal{E}_{2,1}| \le \mu c_I \sum_{n=0}^{N} \Delta t \|\partial_t f_{S_h^{n+1}}\|_{X_h} \Big(\sum_{K \in \mathcal{T}_h} \int_K |\nabla(I_\psi(t_{n+1}, x) - \psi(t_{n+1}, x))|^2 dx \Big)^{\frac{1}{2}}$$

$$\le \mu c_I \sum_{n=0}^{N} \Delta t \|\partial_t f_{S_h^{n+1}}\|_{X_h} \Big(C \theta_{\mathcal{T}} h \Big(\sum_{K \in \mathcal{T}_h} |\psi(t_{n+1}, x)|_{2,K}^2 \Big)^{\frac{1}{2}} \Big),$$

where $|.|_{2,K}$ denotes the H^2 seminorm and the constant C does not depend on h (nor on Δt) see for instance [10, Theorem 15.3]. Finally, using Cauchy-Schwarz inequality, we conclude that

$$|\mathcal{E}_{2,1}| \le \mu c_I C \theta_{\mathcal{T}} h C_\psi \Big(\sum_{n=0}^{N} \|\partial_t f_{S_h^{n+1}}\|_{X_h}^2 \Big)^{\frac{1}{2}} \Big(\sum_{n=0}^{N} \Delta t \Big)^{\frac{1}{2}} \xrightarrow{h, \Delta t \to 0} 0.$$

Then, we can proof that

$$\mathcal{E}_2 \longrightarrow \mu \iint_{Q_T} \nabla \partial_t s \cdot \nabla \psi \, dx \, dt.$$

For more details see the convergence of diffusion term in what follow.
To study the convergence of **diffusion term**, we use the same technic as in [24]. At first, according to remark 2 and if we replace $M_{D,E}$ by its value, we obtain :

$$\mathcal{E}_3 = \sum_{n=0}^{N} \sum_{K \in \mathcal{T}_h} \int_K \Lambda \nabla \beta_h(S_h^{n+1}(x)) \cdot \nabla \Big(\sum_{D \in \mathcal{D}_h} \psi(t_{n+1}, \mathcal{P}_D) \varphi_D(x) \Big) dx.$$

a) As a first step, we use the boundedness of the tensor Λ and same technic as before to prove that

$$\mathcal{E}_{3,1} = \sum_{n=0}^{N} \sum_{K \in \mathcal{T}_h} \int_K \Lambda \nabla \beta_h(S_h^{n+1}(x)) \cdot \nabla \big(I_\psi(t_{n+1}, x) - \psi(t_{n+1}, x) \big) dx \xrightarrow{h, \Delta t \to 0} 0.$$

b) We next show that:

$$\sum_{n=0}^{N} \Delta t \sum_{K \in \mathcal{T}_h} \int_K \Lambda \nabla \beta_h(S_h^{n+1}(x)) \cdot \nabla \psi(t_{n+1}, x) \, dx \, dt - \iint_{Q_T} \Lambda \nabla \beta(s) \cdot \nabla \psi \, dx \, dt \xrightarrow{h, \Delta t \to 0} 0.$$

For that, we add and subtract $\int_{Q_T} \Lambda \nabla \beta_h(S_h^{n+1}) \cdot \nabla \psi \, dxdt$ and consider

$$\mathcal{E}_{3,2} = \sum_{n=0}^{N} \int_{t_n}^{t_{n+1}} \sum_{K \in \mathcal{T}_h} \int_K \Lambda \nabla \beta_h(S_h^{n+1}(x)) \cdot \nabla \big(\psi(t_{n+1},x) - \psi(x,t) \big) \, dxdt$$

$$\mathcal{E}_{3,3} = \iint_{Q_T} \Lambda \big(\nabla \beta_h(S_h(x)) - \nabla \beta(s) \big) \cdot \nabla \psi(x,t) \, dxdt.$$

Since ψ is of class C^1 with respect to t, then $|\nabla \psi(t_{n+1},x) - \nabla \psi(x,t)| \le g(\Delta t) \xrightarrow{\Delta t \to 0} 0$ and

$$|\mathcal{E}_{3,2}| \le C_\Lambda g(\Delta t) \sum_{n=0}^{N} \sum_{K \in \mathcal{T}_h} \left| \nabla \beta_h(S_h^{n+1})_{|K} \right| |K|.$$

The Cauchy-Schwarz inequality and the estimate (29) give

$$|\mathcal{E}_{3,2}| \le C_\Lambda g(\Delta t) T^{\frac{1}{2}} \big(\sum_{n=0}^{N} \Delta t \| \beta_h(S_h^{n+1}) \|_{X_h}^2 \big)^{\frac{1}{2}} \le C C_\Lambda g(\Delta t) T^{\frac{1}{2}} |\Omega|^{\frac{1}{2}} \xrightarrow{\Delta t \to 0} 0.$$

Now, we still have to prove that

$$\mathcal{E}'_{3,3} = \iint_{Q_T} \big(\nabla \beta_h(S_h(x)) - \nabla \beta(s) \big) \cdot w(x,t) \, dxdt \xrightarrow{h, \Delta t \to 0} 0, \tag{31}$$

for all $w \in \big(C^\infty(Q_T) \big)^d$. Indeed,

$$\mathcal{E}'_{3,3} = \iint_{Q_T} \nabla \beta_h(S_h(x)) \cdot w(x,t) \, dxdt + \iint_{Q_T} \beta(s(x,t)) \nabla \cdot w(x,t) \, dxdt - \int_0^T \int_{\partial \Omega} \beta(s(x,t)) w(x,t) \cdot \eta \, d\gamma(x) dt$$
$$= B_1 + B_2 + B_3,$$

where we use the Green formula in B_2 because $\beta(s) \in L^2(0,T;H^1(\Omega))$ but in B_1 we apply the Green formula for each $K \in \mathcal{T}_h$ because $\beta_h(S_h^{n+1}) \notin H^1(\Omega)$. For that,

$$B_1 = - \int_0^T \sum_{K \in \mathcal{T}_h} \int_K \beta_h(S_h(x)) \nabla \cdot (w(x,t)) \, dxdt + \int_0^T \sum_{K \in \mathcal{T}_h} \int_{\partial K} \beta_h(S_h(x)) w(x,t) \cdot \eta \, d\gamma(x) dt.$$

By reordering the summation of the second term by sides, we have

$$B'_1 = \int_0^T \sum_{\sigma_{K,L} \in \mathcal{E}_h^{int}} \int_{\sigma_{K,L}} \left(\beta_h(S_h(x))|_K - \beta_h(S_h(x))|_L \right) w(x,t) \cdot \eta \, d\gamma(x) dt$$

$$+ \int_0^T \sum_{\sigma_K \in \mathcal{E}_h^{ext}} \int_{\sigma_K} \left(\beta_h(S_h(x))|_K \right) w(x,t) \cdot \eta \, d\gamma(x) dt.$$

Since the function w is smooth, we get

$$w \cdot \eta_{\sigma_D}(x) = w \cdot \eta_{\sigma_D}(\mathcal{P}_D) + f(\xi) |\mathcal{P}_D - x|, \ x \in \sigma_D, \ \xi \in [\mathcal{P}_D, x],$$

with $|f(\xi)| \le C_w$. Thus,

$$B_1' = \int_0^T \sum_{\sigma_{K,L} \in \mathcal{E}_h^{ext}} \int_{\sigma_{K,L}} \left(\beta_h(S_h(x))|_K - \beta_h(S_h(x))|_L \right) \left(w \cdot \eta_{\sigma_D}(\mathcal{P}_D) + f(\xi)|\mathcal{P}_D - x| \right) d\gamma(x) dt$$

$$+ \int_0^T \int_{\partial \Omega} \beta_h(S_h(x)) w(x,t) \cdot \eta \, d\gamma(x) dt$$

The function $\beta_h(S_h^{n+1}(x))|_K - \beta_h(S_h^{n+1}(x))|_L$ is a first-order polynomial, vanishing at the barycenter $\mathcal{P}_D$ of this side. Consequently

$$\int_{\sigma_{K,L}} \left(\beta_h(S_h^{n+1}(x))|_K - \beta_h(S_h^{n+1}(x))|_L \right) d\gamma(x) = 0.$$

In addition, we have for all $x \in \sigma_D = \sigma_{K,L}$,

$$\left| \beta_h(S_h^{n+1}(x))|_K - \beta_h(S_h^{n+1}(x))_L \right| \le \left| \nabla \beta_h(S_h^{n+1}(x))|_K \right| \frac{diam(\sigma_{K,L})}{4-d} + \left| \nabla \beta_h(S_h^{n+1}(x))|_L \right| \frac{diam(\sigma_{K,L})}{4-d}$$

here we use that $|x - \mathcal{P}_D| \le diam(\sigma_{K,L})/2$ for $d = 2$ and $|x - \mathcal{P}_D| \le diam(\sigma_{K,L})$ for $d = 3$. Thus, there exists a constant C depend on C_f and d such that

$$\sum_{\sigma_{K,L} \in \mathcal{E}_h^{int}} \int_{\sigma_{K,L}} \left(\beta_h(S_h^{n+1}(x))|_K - \beta_h(S_h^{n+1}(x))|_L \right) f(\xi)|\mathcal{P}_D - x| \, d\gamma(x)$$

$$\le C \sum_{\sigma_{K,L} \in \mathcal{E}_h^{int}} |\sigma_{K,L}| \left(\left| \nabla \beta_h(S_h^{n+1}(x))|_K \right| + \left| \nabla \beta_h(S_h^{n+1}(x))|_L \right| \right) \left(\frac{diam(\sigma_{K,L})}{4-d} \right)^2.$$

Finally, it's remains to show that

$$\int_0^T \int_{\partial \Omega} \beta_h(S_h(x)) w(x,t) \cdot \eta \, d\gamma(x) dt - \int_0^T \int_{\partial \Omega} \beta(s(x,t)) w(x,t) \cdot \eta \, d\gamma(x) dt \to 0,$$

By applying Lemma 3 on β_h and by using Theorem 2 we have the result. By reordering all estimates and using Cauchy-schwarz inequality, we can easily deduce that B_1' tends to $\int_0^T \int_{\partial \Omega} \beta(s(x,t)) w(x,t) \cdot \eta \, d\gamma(x)$ where h tends to zero.

According to the convergence of $\beta_h(S_h^{n+1})$ to $\beta(s)$ in $L^2(Q_T)$, we obtain that $\mathcal{E}_{3,3}'$ tend to 0 for all $w \in \left(C^\infty(Q_T) \right)^d$, consequently $\mathcal{E}_{3,3}'$ tends to 0 for all $w \in L^2(Q_T)$ by using the density argument.

Since $\mathcal{E}_{3,2}$ and $\tilde{\mathcal{E}}_{3,3}$ tends to 0 , then, the convergence of a step $b)$ is proved. By reordering the convergences of steps $a)$ and $b)$, we deduce:

$$\mathcal{E}_3 \xrightarrow{h,\Delta t \to 0} \iint_{Q_T} \Lambda \nabla \beta(s) \cdot \nabla \psi \, dx dt$$

we are concerned now with the **convective term:**

$$\mathcal{E}_4 = \sum_{n=0}^N \sum_{D \in \mathcal{D}_h} \sum_{E \in \mathcal{N}(D)} G_s(S_D^{n+1}, S_E^{n+1}; \delta P_{D,E}^{n+1}) \psi_D^{n+1}.$$

For each couple of neighbor volumes D and E we introduce

$$S_{D,E}^{n+1} = \min(S_D^{n+1}, S_E^{n+1}), \tag{32}$$

and we set

$$\mathcal{E}_4^* = -\sum_{n=0}^{N} \Delta t \sum_{D \in \mathcal{D}_h} \sum_{E \in \mathcal{N}(D)} f(S_{D,E}^{n+1}) \lambda(S_{D,E}^{n+1}) \delta P_{D,E}^{n+1} \psi_D^{n+1}.$$

The diamond constructed from the neighbor edge centers $\mathcal{P}_D$, $\mathcal{P}_E$ and the interface $\sigma_{D,E}$ of the dual mesh is denoted by $T_{D,E} \subset K_{D,E}$. Then, we introduce

$$\overline{S_h}\Big|_{]t_n,t_{n+1}] \times T_{D,E}} := \max(S_D^{n+1}, S_E^{n+1}), \quad \underline{S_h}\Big|_{]t_n,t_{n+1}] \times T_{D,E}} := \min(S_D^{n+1}, S_E^{n+1}),$$

and we rewrite

$$\mathcal{E}_4^* = \iint_{Q_T} \Lambda f(\underline{S_h}) \lambda(\underline{S_h}) \nabla P_h \cdot (\nabla \psi)_h \, dxdt.$$

By the monotonicity of β and thanks to the estimate (16), we have

$$\int_0^T \int_\Omega |\beta(\overline{S_h}) - \beta(\underline{S_h})|^2 \le \sum_{n=0}^{N} \Delta t \sum_{D \in \mathcal{D}_h} \sum_{E \in \mathcal{N}(D)} |T_{D,E}| |\beta(S_D^{n+1}) - \beta(S_E^{n+1})|^2$$

$$\le \sum_{n=0}^{N} \Delta t \sum_{D \in \mathcal{D}_h} \sum_{E \in \mathcal{N}(D)} (diam(K_{D,E}))^2 |\beta(S_D^{n+1}) - \beta(S_E^{n+1})|^2$$

$$\le Ch^{4-d} \overset{h,\Delta t \to 0}{\longrightarrow} 0,$$

with $d = 2$ or 3. Because β^{-1} is continuous, up to extraction of another subsequence, we deduce

$$|\overline{S_h} - \underline{S_h}| \overset{h,\Delta t \to 0}{\longrightarrow} 0 \text{ p.p. sur } Q_T.$$

In addition, $\underline{S_h} \le \tilde{S}_{h,\Delta t} \le \overline{S_h}$ and $\tilde{S}_{h,\Delta t} \overset{h,\Delta t \to 0}{\longrightarrow} s$ a.e on Q_T. Thus, we see that $f(\underline{S_h})\lambda(\underline{S_h}) \overset{h,\Delta t \to 0}{\longrightarrow} f(s)\lambda(s)$ a.e on Q_T. The dominated convergence theorem of Lebesgue implies that $\Lambda f(\underline{S_h})\lambda(\underline{S_h})(\nabla \psi)_h \longrightarrow \Lambda f(s)\lambda(s)\nabla \psi$ dans $(L^2(Q_T))^d$. Using the weak convergence of the gradient of the discrete solution P_h in $(L^2(Q_T))^d$ $\big($treated as for $\beta_h(S_h)$ in (31)$\big)$, we can deduce that

$$\mathcal{E}_4^* \overset{h,\Delta t \to 0}{\longrightarrow} \iint_{Q_T} \Lambda f(s)\lambda(s)\nabla p \cdot \nabla \psi \, dxdt.$$

It remains now to show that
$$\lim_{h \to 0} |\mathcal{E}_4 - \mathcal{E}_4^*| = 0. \tag{33}$$

By properties of G_s, we have

$$|G_s(S_D^{n+1}, S_E^{n+1}, \delta P_{D,E}^{n+1}) - f(S_{D,E}^{n+1})\lambda(S_{D,E}^{n+1})\delta P_{D,E}^{n+1}|$$
$$= |G_s(S_D^{n+1}, S_E^{n+1}, \delta P_{D,E}^{n+1}) - |G_s(S_{D,E}^{n+1}, S_{D,E}^{n+1}, \delta P_{D,E}^{n+1})|$$
$$\le C|\delta P_{D,E}^{n+1}||\overline{S_h} - \underline{S_h}|_{T_{D,E}}.$$

Therefore,

$$|\mathcal{E}_4 - \mathcal{E}_4^*| \le \frac{C}{2} \sum_{n=0}^{N} \Delta t \sum_{D \in \mathcal{D}_h} \sum_{E \in \mathcal{N}(D)} \mathcal{M}_{D,E} |P_E^{n+1} - P_D^{n+1}||\overline{S_h} - \underline{S_h}|_{T_{D,E}} |\psi_E^{n+1} - \psi_D^{n+1}|.$$

Using Youngs inequality, the estimate (25) and the Taylor formula for ψ, one can easily deduce (33). Similarly, for the last term $\mathcal{E}_5$ we have

$$\mathcal{E}_5 = \sum_{n=0}^{N} \sum_{D \in \mathcal{D}_h} |\partial D \cap \partial \Omega| h_D^{n+1} \psi(t_{n+1}, \mathcal{P}_D) = \sum_{n=0}^{N} \sum_{D \in \mathcal{D}_h} \int_{t_n}^{t_{n+1}} \int_D h(t,x) \psi(t_{n+1}, \mathcal{P}_D),$$

by continuity of ψ we have

$$\mathcal{E}_5 \xrightarrow{h, \Delta t \to 0} \int_0^T \int_{\partial \Omega} h \psi \, dx dt.$$

Otherwise, for the second equation of scheme (7)-(9) we use the same proof (as for convective term) to obtain

$$\int_0^T \int_\Omega \lambda(s) \Lambda \nabla p \cdot \nabla \varphi \, dx dt = \int_0^T \int_{\partial \Omega} \pi \varphi \, dx dt \quad \forall \psi \in C^{1,1}([0,T] \times \bar{\Omega}).$$

Finally, we conclude that the limit couple (s, p) is a weak solution of the continuous problem in the sense of definition 1 by using the fact that the space $C^{1,1}([0.T] \times \bar{\Omega})$ is dense in $L^2(0,T; H^1(\Omega))$.

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