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ON A SERIES OF RAMANUJAN

Olivier Oloa¹

Abstract. We uncover a remarkable evaluation of a family of infinite series involving the logarithmic derivative of the Gamma function. We are then led to a new evaluation of an integral involving Riemann's zeta function on the critical line.

1. INTRODUCTION

Recently, B. C. Berndt and A. Dixit [5] offered a clear proof of an interesting identity coming from some manuscripts of Ramanujan involving an infinite series with the logarithmic derivative of the Gamma function and an integral with the Riemann zeta function. They proved

Theorem 1.1. *If α and β are positive numbers such that $\alpha\beta = 1$, then*

$$\begin{aligned} \sqrt{\alpha} \left\{ \frac{\gamma - \log(2\pi\alpha)}{2\alpha} + \sum_{n=1}^{\infty} \left(\psi(n\alpha) - \log(n\alpha) + \frac{1}{2n\alpha} \right) \right\} \\ = \sqrt{\beta} \left\{ \frac{\gamma - \log(2\pi\beta)}{2\beta} + \sum_{n=1}^{\infty} \left(\psi(n\beta) - \log(n\beta) + \frac{1}{2n\beta} \right) \right\} \\ = -\frac{1}{\pi^{3/2}} \int_0^{\infty} \left| \xi\left(\frac{1}{2} + \frac{it}{2}\right) \Gamma\left(\frac{-1+it}{4}\right) \right|^2 \frac{\cos\left(\frac{t}{2} \log \alpha\right)}{1+t^2} dt, \end{aligned} \quad (1.1)$$

where γ denotes Euler's constant, ψ denotes the logarithmic derivative of the Γ function, called the Digamma function,

$$\psi(x) := \frac{\Gamma'(x)}{\Gamma(x)} = -\gamma - \sum_{k=0}^{\infty} \left(\frac{1}{k+x} - \frac{1}{k+1} \right), \quad \operatorname{Re} x > 0, \quad (1.2)$$

and $\xi(x)$ is defined by

$$\xi(s) := \frac{s(s-1)}{2} \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s),$$

with Riemann's zeta function ζ .

In the present note we uncover a new evaluation of a family of infinite series including the one involved in Ramanujan's Theorem. Our main result is

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Theorem 1.2. *If α and s are complex numbers such that $\operatorname{Re} \alpha > 0$ and $\operatorname{Re}(\alpha + s) > 0$, then*

$$\begin{aligned} & \sum_{n=1}^{\infty} \left(\psi(\alpha n + s) - \log(\alpha n + s) + \frac{1}{2(\alpha n + s)} \right) \\ &= \log \Gamma(s/\alpha + 1) - (s/\alpha + 1/2) \psi(s/\alpha + 1) + s/\alpha + 1/2 - \frac{1}{2} \log(2\pi) \\ & \quad - \int_0^1 \left(\frac{1}{\alpha(1-u^{1/\alpha})} - \frac{1}{1-u} + \frac{1}{2} - \frac{1}{2\alpha} \right) \frac{u^{s/\alpha}}{1-u} du. \end{aligned} \quad (1.3)$$

The preceding result yields different consequences. First, the right hand side of (1.3) shows clearly that the infinite series on the left hand side is expressible in finite terms of standard functions, whenever α and s are positive rational numbers.

Next, combining Theorem 1.1 and Theorem 1.2 gives new informations for the non elementary integrals involving Riemann's ζ function.

Hence, when α tends to $+\infty$, one may deduce in particular

$$\int_0^{\infty} \left| \xi \left(\frac{1}{2} + \frac{it}{2} \right) \Gamma \left(\frac{-1+it}{4} \right) \right|^2 \frac{\cos(\frac{1}{2}t \log \alpha)}{1+t^2} dt \sim \frac{\pi^{3/2} \log \alpha}{2 \sqrt{\alpha}}, \quad (1.4)$$

moreover, one has the asymptotic expansion,

$$\begin{aligned} & \frac{1}{\pi^{3/2}} \int_0^{\infty} \left| \xi \left(\frac{1}{2} + \frac{it}{2} \right) \Gamma \left(\frac{-1+it}{4} \right) \right|^2 \frac{\cos(\frac{1}{2}t \log \alpha)}{1+t^2} dt \\ &= \frac{1}{2} \frac{\log \alpha}{\sqrt{\alpha}} - \frac{1}{2} (\log(2\pi) - \gamma) \frac{1}{\sqrt{\alpha}} + \frac{\pi^2}{72} \frac{1}{\alpha \sqrt{\alpha}} - \frac{\pi^4}{10 \cdot 800} \frac{1}{\alpha^3 \sqrt{\alpha}} + O\left(\frac{1}{\alpha^5 \sqrt{\alpha}}\right). \end{aligned} \quad (1.5)$$

Observe that the above expansion may be read in terms of Fourier cosine integral and thus may give, via the inverse transform, a possible path to estimate

$$\left| \xi \left(\frac{1}{2} + \frac{it}{2} \right) \Gamma \left(\frac{-1+it}{4} \right) \right|^2 \frac{1}{1+t^2}, \quad (1.6)$$

or equivalently

$$\left| \zeta \left(\frac{1}{2} + it \right) \right|.$$

In section 2 we give a proof of our main result Theorem 1.2, then we display certain closed forms in section 3 and we establish the asymptotic expansion (1.5) in section 4.

2. THE PROOF

In this section we establish our main result, Theorem 1.2.

Recall Binet's formula ([13], p. 248), $\operatorname{Re} z > 0$,

$$\log \Gamma(z) = \left(z - \frac{1}{2}\right) \log z - z + \frac{1}{2} \log(2\pi) + \int_0^\infty \left(\frac{1}{2} - \frac{1}{x} + \frac{1}{e^x - 1}\right) \frac{e^{-zx}}{x} dx, \quad (2.1)$$

which, upon making $x = -\log v$, can be written as

$$\log \Gamma(z) = \left(z - \frac{1}{2}\right) \log z - z + \frac{1}{2} \log(2\pi) - \int_0^1 \left(\frac{1}{\log v} + \frac{1}{1-v} - \frac{1}{2}\right) \frac{v^{z-1}}{\log v} dv. \quad (2.2)$$

If one differentiates (2.2) with respect to z , one may obtain

$$-\left(\psi(z) - \log z + \frac{1}{2z}\right) = \int_0^1 \left(\frac{1}{\log v} + \frac{1}{1-v} - \frac{1}{2}\right) v^{z-1} dv, \quad (2.3)$$

or equivalently, Gauss' formula ([13], p. 249)

$$-(\psi(z) - \log z) = \int_0^1 \left(\frac{1}{\log v} + \frac{1}{1-v}\right) v^{z-1} dv. \quad (2.4)$$

Let $\alpha > 0$ and $s > 0$. From (2.3), we deduce

$$\begin{aligned} & -\sum_{n=1}^{\infty} \left(\psi(\alpha n + s) - \log(\alpha n + s) + \frac{1}{2(\alpha n + s)} \right) \\ &= \int_0^1 \left(\frac{1}{\log v} + \frac{1}{1-v} - \frac{1}{2} \right) \sum_{n=1}^{\infty} v^{\alpha n + s - 1} dv, \\ &= \int_0^1 \left(\frac{1}{\log v} + \frac{1}{1-v} - \frac{1}{2} \right) \frac{v^{\alpha + s - 1}}{1 - v^\alpha} dv, \end{aligned} \quad (2.5)$$

where we have interchanged the integral and sum, which may be justified by considering the finite sum and estimating the remainder. Substitute $u = v^\alpha$ in the latter integral to obtain

$$\int_0^1 \left(\frac{1}{\log u} + \frac{1}{\alpha(1 - u^{1/\alpha})} - \frac{1}{2\alpha} \right) \frac{u^{s/\alpha}}{1 - u} du. \quad (2.6)$$

We then split (2.6) in two integrals

$$\int_0^1 \left(\frac{1}{\log u} + \frac{1}{1-u} - \frac{1}{2} \right) \frac{u^{s/\alpha}}{1-u} du + \int_0^1 \left(\frac{1}{\alpha(1 - u^{1/\alpha})} - \frac{1}{1-u} + \frac{1}{2} - \frac{1}{2\alpha} \right) \frac{u^{s/\alpha}}{1-u} du. \quad (2.7)$$

Thus to prove Theorem 1.2 it is sufficient to evaluate

$$\int_0^1 \left(\frac{1}{\log u} + \frac{1}{1-u} - \frac{1}{2} \right) \frac{u^{s/\alpha}}{1-u} du. \quad (2.8)$$

One may check by a direct calculation that

$$\begin{aligned} \left(\frac{1}{\log u} + \frac{1}{1-u} - \frac{1}{2} \right) \frac{u^{s/\alpha}}{1-u} &= u^{s/\alpha} \frac{d}{du} \left\{ u \left(\frac{1}{\log u} + \frac{1}{1-u} \right) \right\} + \\ &\quad \left(\frac{1}{\log u} + \frac{1}{1-u} - \frac{1}{2} \right) \frac{u^{s/\alpha}}{\log u} - \frac{1}{2} \left(\frac{1}{\log u} + \frac{1}{1-u} \right) u^{s/\alpha}. \end{aligned} \quad (2.9)$$

Hence the integral in (2.8) is the sum of three integrals. The first,

$$\begin{aligned} I_1 &= \int_0^1 u^{s/\alpha} \frac{d}{du} \left\{ u \left(\frac{1}{\log u} + \frac{1}{1-u} \right) \right\} du \\ &= \left(\frac{1}{\log u} + \frac{1}{1-u} \right) u^{s/\alpha+1} \Big|_0^1 - \frac{s}{\alpha} \int_0^1 \left(\frac{1}{\log u} + \frac{1}{1-u} \right) u^{s/\alpha} du, \end{aligned} \quad (2.10)$$

and, using (2.4),

$$I_1 = \int_0^1 u^{s/\alpha} \frac{d}{du} \left\{ u \left(\frac{1}{\log u} + \frac{1}{1-u} \right) \right\} du = \frac{1}{2} + \frac{s}{\alpha} (\psi(s/\alpha + 1) - \log(s/\alpha + 1)). \quad (2.11)$$

The second, applying (2.2),

$$\begin{aligned} I_2 &= \int_0^1 \left(\frac{1}{\log u} + \frac{1}{1-u} - \frac{1}{2} \right) \frac{u^{s/\alpha}}{\log u} du \\ &= -\log \Gamma(s/\alpha + 1) + (s/\alpha + 1/2) \log(s/\alpha + 1) - s/\alpha - 1 + \frac{1}{2} \log(2\pi). \end{aligned} \quad (2.12)$$

The third, using (2.4),

$$I_3 = -\frac{1}{2} \int_0^1 \left(\frac{1}{\log u} + \frac{1}{1-u} \right) u^{s/\alpha} du = \frac{1}{2} (\psi(s/\alpha + 1) - \log(s/\alpha + 1)). \quad (2.13)$$

Consequently,

$$\begin{aligned} \int_0^1 \left(\frac{1}{\log u} + \frac{1}{1-u} - \frac{1}{2} \right) \frac{u^{s/\alpha}}{1-u} du &= I_1 + I_2 + I_3 \\ &= -\log \Gamma(s/\alpha + 1) + (s/\alpha + 1/2) \psi(s/\alpha + 1) - s/\alpha - 1/2 + \frac{1}{2} \log(2\pi). \end{aligned} \quad (2.14)$$

Using (2.5), (2.6), (2.7), (2.14), and analytic continuation, gives Theorem 1.2.

3. CLOSED FORMS

Theorem 1.2 yields different closed forms.

For example, the substitution $u = v^{12}$ in the following integrand leads us to integrate a rational function and produces

$$\int_0^1 \left(\frac{1}{4(1-u^{1/4})} - \frac{1}{1-u} - \frac{3}{2} \right) \frac{u^{1/6}}{1-u} du = \frac{\sqrt{3}}{16} \log \left(\frac{2-\sqrt{3}}{2+\sqrt{3}} \right) + \frac{\pi}{8} + \frac{4}{9} \sqrt{3} \pi - \frac{21}{8}, \quad (3.1)$$

yielding, with (1.3),

$$\begin{aligned} \sum_{n=1}^{\infty} \left(\psi(4n + 2/3) - \log(4n + 2/3) + \frac{1}{2(4n + 2/3)} \right) &= \\ \log \Gamma(7/6) - \frac{2}{3} \psi(7/6) - \frac{1}{2} \log(2\pi) - \frac{\sqrt{3}}{16} \log \left(\frac{2-\sqrt{3}}{2+\sqrt{3}} \right) - \frac{\pi}{8} - \frac{4}{9} \sqrt{3} \pi + \frac{89}{8}. \end{aligned} \quad (3.2)$$

In the same manner, from

$$\int_0^1 \left(\frac{1}{2(1-u^{1/2})} - \frac{1}{1-u} + \frac{1}{4} \right) \frac{u^{1/10}}{1-u} du = \frac{\sqrt{5}}{20} \log \left(\frac{9+4\sqrt{5}}{9-4\sqrt{5}} \right) + \frac{3}{5} \log 2 + \frac{3\pi}{50} \sqrt{10(5+\sqrt{5})} - \frac{11}{4}, \quad (3.3)$$

one deduces, with (1.3),

$$\begin{aligned} \sum_{n=1}^{\infty} \left(\psi(2n+1/5) - \log(2n+1/5) + \frac{1}{2(2n+1/5)} \right) = \\ \log \Gamma(11/10) - \frac{3}{5} \psi(11/10) - \frac{1}{2} \log(2\pi) - \frac{\sqrt{5}}{20} \log \left(\frac{9+4\sqrt{5}}{9-4\sqrt{5}} \right) - \frac{3}{5} \log 2 - \frac{3\pi}{50} \sqrt{10(5+\sqrt{5})} + \frac{67}{20}. \end{aligned} \quad (3.4)$$

One may observe the following particular family of unexpected closed forms.

Corollary 3.1. *If s is a complex number such that $\operatorname{Re} s > -1$, then*

$$\begin{aligned} \sum_{n=1}^{\infty} \left(\psi(n+s) - \log(n+s) + \frac{1}{2(n+s)} \right) \\ = \log \Gamma(s+1) - \left(s + \frac{1}{2} \right) \psi(s+1) + s + \frac{1}{2} - \frac{1}{2} \log(2\pi). \end{aligned} \quad (3.5)$$

Proof. Put $\alpha = 1$ in (1.3). □

Particular cases are,

$$\sum_{n=1}^{\infty} \left(\psi(n) - \log n + \frac{1}{2n} \right) = \frac{\gamma}{2} - \frac{1}{2} \log(2\pi) + \frac{1}{2}, \quad (3.6)$$

$$\sum_{n=1}^{\infty} \left(\psi(n+1/2) - \log(n+1/2) + \frac{1}{2(n+1/2)} \right) = \gamma + \frac{1}{2} \log 2 - 1, \quad (3.7)$$

$$\sum_{n=1}^{\infty} \left(\psi(n-1/2) - \log(n-1/2) + \frac{1}{2(n-1/2)} \right) = -\frac{1}{2} \log 2, \quad (3.8)$$

$$\begin{aligned} \sum_{n=1}^{\infty} \left(\psi(n+1/3) - \log(n+1/3) + \frac{1}{2(n+1/3)} \right) \\ = \log \Gamma(1/3) + \frac{1}{4} \log 3 + \frac{5}{6} \gamma - \log(2\pi) - \frac{5}{3} + \frac{5}{36} \pi \sqrt{3}. \end{aligned} \quad (3.9)$$

One may differentiate (3.5) several times with respect to s , obtaining

Proposition 3.2. *If s is a complex number such that $\operatorname{Re} s > -1$ and m is a natural number with $m \geq 2$ then,*

$$\sum_{n=1}^{\infty} \psi^{(m)}(n+s) = -m \psi^{(m-1)}(s+1) - s \psi^{(m)}(s+1). \quad (3.10)$$

For example, putting $m = 3$ and $s = 0$ in (3.10) gives

$$\sum_{n=1}^{\infty} \left(\zeta(4) - \sum_{k=1}^n \frac{1}{k^4} \right) = 2\zeta(3). \quad (3.11)$$

where we have used

$$\psi^{(m)}(s+1) = (-1)^m (m)! \sum_{n=1}^{\infty} \frac{1}{(n+s)^{m+1}}. \quad (3.12)$$

Another interesting particular case of Theorem 1.2 is given by

Corollary 3.3. *If α is a complex number such that $\operatorname{Re} \alpha > 0$, then*

$$\begin{aligned} \sum_{n=1}^{\infty} \left(\psi(\alpha n) - \log(\alpha n) + \frac{1}{2\alpha n} \right) &= \frac{\gamma}{2} - \frac{1}{2} \log(2\pi) + \frac{1}{2} \\ &\quad - \int_0^1 \left(\frac{1}{\alpha(1-u^{1/\alpha})} - \frac{1}{1-u} + \frac{1}{2} - \frac{1}{2\alpha} \right) \frac{du}{1-u}. \end{aligned} \quad (3.13)$$

Proof. Put $s = 0$ in (1.3). □

4. ASYMPTOTIC EXPANSION

Theorem 1.2, via Corollary 3.3, gives a tractable way to obtain an asymptotic expansion for the infinite series

$$\sum_{n=1}^{\infty} \left(\psi(\alpha n) - \log(\alpha n) + \frac{1}{2\alpha n} \right).$$

Inserting the asymptotic expansion of the previous series in Theorem 1.1 allows us to estimate, when α is great, the non elementary integrals involving Riemann's ζ function. Hence, we find

Theorem 4.1. *Let α tend to $+\infty$, then*

$$\begin{aligned} \frac{1}{\pi^{3/2}} \int_0^{\infty} \left| \xi \left(\frac{1}{2} + \frac{it}{2} \right) \Gamma \left(\frac{-1+it}{4} \right) \right|^2 \frac{\cos(\frac{1}{2}t \log \alpha)}{1+t^2} dt \\ = \frac{1}{2} \frac{\log \alpha}{\sqrt{\alpha}} - \frac{1}{2} (\log(2\pi) - \gamma) \frac{1}{\sqrt{\alpha}} + \frac{\pi^2}{72} \frac{1}{\alpha \sqrt{\alpha}} - \frac{\pi^4}{10 \cdot 800} \frac{1}{\alpha^3 \sqrt{\alpha}} + O\left(\frac{1}{\alpha^5 \sqrt{\alpha}}\right). \end{aligned} \quad (4.1)$$

Proof. Let $0 < u < 1$ and let α tend to $+\infty$. We readily have

$$\begin{aligned} \left(\frac{1}{\alpha(1-u^{1/\alpha})} - \frac{1}{1-u} + \frac{1}{2} - \frac{1}{2\alpha} \right) \frac{1}{1-u} \\ = - \left(\frac{1}{\log u} + \frac{1}{1-u} - \frac{1}{2} \right) \frac{1}{1-u} - \frac{\log u}{12(1-u)} \frac{1}{\alpha^2} + \frac{\log^3 u}{720(1-u)} \frac{1}{\alpha^4} + O\left(\frac{1}{\alpha^6}\right), \end{aligned} \quad (4.2)$$

from which we deduce

$$\begin{aligned} & \int_0^1 \left(\frac{1}{\alpha(1-u^{1/\alpha})} - \frac{1}{1-u} + \frac{1}{2} - \frac{1}{2\alpha} \right) \frac{du}{1-u} \\ &= - \int_0^1 \left(\frac{1}{\log u} + \frac{1}{1-u} - \frac{1}{2} \right) \frac{du}{1-u} - \frac{1}{12\alpha^2} \int_0^1 \frac{\log u}{1-u} du + \frac{1}{720\alpha^4} \int_0^1 \frac{\log^3 u}{1-u} du + O\left(\frac{1}{\alpha^6}\right). \end{aligned} \quad (4.3)$$

Now, using (2.14) with $s = 0$,

$$\int_0^1 \left(\frac{1}{\log u} + \frac{1}{1-u} - \frac{1}{2} \right) \frac{du}{1-u} = \frac{\gamma}{2} - \frac{1}{2} \log(2\pi) + \frac{1}{2} \quad (4.4)$$

and considering the well-known results

$$\int_0^1 \frac{\log u}{1-u} du = -\frac{\pi^2}{6}, \quad (4.5)$$

$$\int_0^1 \frac{\log^3 u}{1-u} du = -\frac{\pi^4}{15}, \quad (4.6)$$

we obtain

$$\begin{aligned} & \int_0^1 \left(\frac{1}{\alpha(1-u^{1/\alpha})} - \frac{1}{1-u} + \frac{1}{2} - \frac{1}{2\alpha} \right) \frac{du}{1-u} \\ &= \frac{\gamma}{2} - \frac{1}{2} \log(2\pi) + \frac{1}{2} + \frac{\pi^2}{72} \frac{1}{\alpha^2} - \frac{\pi^4}{10800} \frac{1}{\alpha^4} + O\left(\frac{1}{\alpha^6}\right). \end{aligned} \quad (4.7)$$

Finally inserting the expansion (4.7) in Corollary 3.3 and combining with Theorem 1.1 yields Theorem 4.1. $\square$

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