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Rush-Larsen time-stepping methods of high order for stiff problems in cardiac electrophysiology

Yves Coudière *,1,2, Charlie Douanla-Lontsi †1 and Charles Pierre ‡3

1 INRIA Bordeaux Sud Ouest, Université de Bordeaux, France.
2 Institut de Mathématiques de Bordeaux, UMR CNRS 5241.
3 Laboratoire de Mathématiques et de leurs Applications, UMR CNRS 5142, Université de Pau et des Pays de l’Adour, France.

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Abstract

Stability and accuracy of numerical methods for reaction-diffusion equations still need improvements, which prompts for the development of high order and stable time-stepping methods. This is particularly true in the context of cardiac electrophysiology, where reaction-diffusion equations are coupled with stiff systems of ordinary differential equations. So as to address these issues, much research on implicit-explicit methods and exponential integrators has been carried out during the past 15 years. In 2009, Perego and Veneziani [25] proposed an innovative time-stepping scheme of order 2. In this paper we present an extension of this scheme to the orders 3 and 4, that we call Rush-Larsen schemes of order \( k \). These new schemes are explicit multistep methods, which belong to the classical class of exponential integrators. Their general formulation is simple and easy to implement. We prove that they are stable under perturbation and convergent of order \( k \). We analyze their Dahlquist stability, and show that they have a very large stability domain, provided that the stabilizer associated with the method captures well enough the stiff modes of the problem. We study their application to a system of equations that models the action potential in cardiac electrophysiology.

Keywords: stiff equations, explicit high-order multistep methods, exponential integrators, stability and convergence, Dahlquist stability

Subject classification: 65L04, 65L06, 65L20, 65L99

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*yves.coudiere@inria.fr
†charlie.douanla-lontsi@inria.fr
‡charles.pierre@univ-pau.fr
Introduction

This article concerns the problem of time integration of stiff reaction-diffusion equations, in particular when they are coupled to a system of ordinary differential equations (ODE). As developed below, for such problems, the matters of stability and accuracy are of first importance. As a systemic example of these questions, we will consider the monodomain model in cardiac electrophysiology [3, 4, 5]. Given the heart domain Ω and the time interval $[0, T]$, it has the general form

$$\frac{\partial v}{\partial t} = Av + f_1(v, \zeta) + s(x, t), \quad \frac{d\zeta}{dt} = f_2(v, \zeta),$$

where $A$ is a diffusion operator. The unknown function $v : \Omega \times [0, T] \to \mathbb{R}$ is the transmembrane potential. The unknown function $\zeta : \Omega \times [0, T] \to \mathbb{R}^{p+q}$ gathers $p+q$ variables describing the state of the cell membrane. It incorporates $p$ gating variables and $q$ ionic concentrations. The source term $s(x, t)$ is an applied stimulation current. The reaction terms $f_1$ and $f_2$ model ionic currents across the cell membrane, and are called ionic models. Ionic models have originally been developed by Hodgkin and Huxley [19] in 1952. Highly detailed ionic models specific to cardiac cells have been designed since the 1960’s, such as the Beeler and Reuter (BR) model [1] or the ten Tusscher, Noble, Noble and Panfilov (TNNP) model [29]. A comprehensive review is available in [28].

There are two major difficulties for numerical simulations in cardiac electrophysiology. First, the non-linear functions $f_1$ and $f_2$ in equation (1) induce expensive computations of the mappings $(v, \zeta) \to f_i(v, \zeta)$. For example, the TNNP model [29] involves the computation of 50 scalar exponentials, that have to be performed for each mesh node to approximate solutions of the partial differential equation (1). They represent the predominant computational load during numerical simulations, and their total amount needs to be maintained as low as possible. Fully implicit time-stepping methods, which require a non-linear solver, are therefore avoided. Second, the equations (1) are stiff, but since implicit methods are not affordable, numerical instabilities are challenging to manage. More precisely, the stiffness is caused by the presence of different space and time scales. The solutions of equation (1) display sharp wavefronts. Typically, the scaling factor between the fast and slow variables ranges from 100 to 1000. This is commonly coped with by resorting to very fine space and time discretization grids, associated with high computational costs.

In this context, our strategy for solving problem (1) is to use high order methods, so as to have accurate simulations with coarser space and time discretization grids. A high order time-stepping method that fulfills the two following conditions is required: it must have strong stability properties, and has to be explicit for the reaction terms. To this aim, we will focus on the time integration of stiff ODE systems of the form

$$\frac{dy}{dt} = f(t, y), \quad y(0) = y_0,$$

in which the nonlinear function $f : [0, +\infty[ \times \mathbb{R}^N \to \mathbb{R}^N$ (e.g. $N = p + q$ for the ionic models presented above) may be split as $f(t, y) = a(t, y)y + b(t, y)$. This leads to a
formulation more suited to our needs,

\[
\frac{dy}{dt} = a(t, y)y + b(t, y), \quad y(0) = y_0.
\]  

(3)

It involves the non-linear term \( b(t, y) \) and the operator \( y \in \mathbb{R}^N \mapsto a(t, y)y \in \mathbb{R}^N \), which can be easily linearized as e.g. \( a(t, y_n)y \). This term \( a(t, y) \) will be inserted into the numerical scheme in order to stabilize the computations. It will be called the stabilizer in the sequel. In practice, the term \( a(t, y) \) may be related to the Jacobian of the system \( \partial_y f(t, y) \).

However, no a priori definition of the stabilizer is made (such as \( a(t, y) = \partial_y f(t, y) \)), because we plan to analyze the formulation in (3) in general. This will allow us for instance to define the stabilizer as an approximation of the Jacobian, for technical reasons detailed below. This approach is relevant in cardiac electrophysiology, where the fastest variables are gating variables that are given by the \( p \) first equations of the ODE system \( \frac{d\zeta}{dt} = f_2(v, \zeta) \) in (1). They have the general form

\[
\frac{d\zeta_i}{dt} = \frac{\zeta_i,\infty(v) - \zeta_i}{\tau_i(v)},
\]

(see Section 4.1) that motivates the reformulation (3) with the diagonal stabilizer \( a = \text{diag}(-1/\tau_i) \).

Exponential integrators are well suited in this framework, we refer to [23, 16, 12] for general reviews. They have been widely studied for the semilinear equation \( \partial_t y = Ay + b(t, y), \) see e.g. [14, 7, 15, 18, 30, 21]. Exponential integrators commonly define a time iteration based on the exact solution of an equation of the form \( \partial_t y = Ay + p(t) \) where \( p(t) \) is a polynomial. It is usually defined with the functions \( (\varphi_k)_{k\geq0} \)

\[
\varphi_0(z) = e^z, \quad \varphi_{j+1}(z) = \frac{\varphi_j(z) - 1/j!}{z},
\]

(4)

introduced by Nørsett [24]. In general, it requires to compute a matrix exponential applied to a vector, like \( e^{At} y \). This is the supplementary cost associated with exponential integrators. A gain in stability is expected when \( A \) is the predominant stiff part of the equation.

The target equation (3) incorporates a non-constant linear part \( a(t, y) \), exponential integrators have been less studied in that case. Exponential integrators of Adams type for a non-constant linear part have been first considered by Lee and Preiser [20] in 1978, and by Chu [2] in 1983. Recently, Ostermann et al., [17] developed and analyzed the linearized exponential Adams method. In general, the original equation (2) is formulated after each time step as \( \frac{dy}{dt} = J_n y + c_n(t, y) \), involving the Jacobian matrix \( J_n = \partial_y f(t_n, y_n) \), and the correction function \( c_n(t, y) = f(t, y) - J_n y \). This has several drawbacks. It requires the computation of matrix exponential applied to a vector with a different matrix at each time step. Moreover, stabilization can be performed on the fast variables only, in case they are known in advance, e.g. because of modeling assumptions, or of our physical understanding of the problem. In this case, using the full Jacobian as the stabilizer will
cause unnecessary computational efforts. As an alternative, the stabilizer can be set to a part or an approximation of the Jacobian. This had already been proposed by Nørsett [24] in 1969 and has been analyzed in [31], [26], and [6] for exponential Rosenbrock, exponential Runge-Kutta and exponential Adams methods, respectively. For exponential Adams methods, equation (3) is reformulated after each time step as \( \frac{dy}{dt} = a_n y + c_n(t, y) \), with \( a_n = a(t_n, y_n) \), and \( c_n(t, y) = f(t, y) - a_n y \). The resulting scheme with a time-step \( h > 0 \) is (see the details in [17, 6])

\[
y_{n+1} = y_n + h \left( \varphi_1(a_n h)(a_n y_n + \gamma_1) + \varphi_2(a_n h)\gamma_2 + \ldots \varphi_k(a_n h)\gamma_k \right),
\]

where the numbers \( \gamma_i \) are the coefficients of the Lagrange interpolation polynomial of \( c_n(t, y) \) (in a classical \( k \)-step setting), and the functions \( \varphi_j \) are given by (4).

Independently, Perego and Veneziani [25] presented in 2009 an innovative exponential integrator of order 2, of a different nature. They proposed a scheme of the form

\[
y_{n+1} = y_n + h\varphi_1(a_n h)(a_n y_n + \beta_n),
\]

involving two coefficients \( \alpha_n \) and \( \beta_n \) to be computed at each time step. The resulting scheme has a very simple definition, and is in particular simpler than the exponential Adams integrators (5). The essential difference with the previous approaches is that \( \alpha_n \neq a(t_n, y_n) \), but instead is fixed for the scheme to be consistent of order 2. Specifically, the coefficients \( \alpha_n \) and \( \beta_n \) are given by \( \alpha_n = \frac{3}{2}a_n - \frac{1}{2}a_{n-1} \) and \( \beta_n = \frac{3}{2}b_n - \frac{1}{2}b_{n-1} \) with \( a_j = a(t_j, y_j) \) and \( b_j = b(t_j, y_j) \). Perego and Veneziani presented their scheme as a “generalization of the Rush-Larsen method” in reference to the Rush-Larsen scheme [27] commonly used in electrophysiology.

This scheme resembles the Magnus integrator introduced by Hochbruck et al. in [13] for the time dependent Schrödinger equation \( iy' = H(t)y \), and extended by González et al. in [9] to parabolic equations with time-dependent linear part \( y' = a(t)y + b(t) \). The second-order Magnus integrator also formulates as (6), but with \( \alpha_n = a(t_{n+1/2}) \) and \( b_n = b(t_{n+1/2}) \).

The scheme of Perego and Veneziani generalizes the second-order Magnus integrator to the case where \( a = a(t, y) \) and \( b = b(t, y) \): it presents an approximation of the unknown terms \( a \left( t_{n+1/2}, y(\tau_{n+1/2}) \right) \) and \( b \left( t_{n+1/2}, y(\tau_{n+1/2}) \right) \) using a two-points interpolation.

In this paper we will study schemes under the form (6). We will show that they also exist at the orders 3 and 4, and will exhibit explicit definitions of the two coefficients \( \alpha_n \) and \( \beta_n \). The schemes will be referred to as as Rush-Larsen schemes of order \( k \) (shortly denoted by RL\(_k\)), in the continuation of the denomination used in [25]. They will be shown to be stable under perturbation and convergent of order \( k \). We also present the Dahlquist stability analysis for the RL\(_k\) schemes. It is a practical tool that allows one to scale the time step \( h \) with respect to the variations of the function \( f(t, y) \) in problem (2), see e.g. [11]. The splitting \( f(t, y) = a(t, y)y + b(t, y) \) may be arbitrary, but obviously the choice of the stabilizer term \( a(t, y) \) is critical for the stability of the method. When considering time-dependent stabilizers, the stability domain depends on this splitting. We compute stability domains numerically, and show that they are much larger if \( a(t, y) \) captures the variations of \( f(t, y) \), than in absence of stabilization (i.e., when \( a(t, y) = 0 \)).
We finally evaluate the performances of the RL\(_k\) methods as applied to the membrane equation in cardiac electrophysiology. They are compared to the exponential Adams integrators (5). The two methods have a very similar robustness with respect to stiffness, allowing stable computations with large time steps. For the considered test case, the RL\(_3\) and RL\(_4\) schemes are more accurate for large time steps.

The paper is organized as follows. The RL\(_k\) schemes are derived in Section 1, and their numerical analysis is made in Sections 1 and 2. The Dahlquist stability analysis is completed in Section 3. The numerical results are presented in Section 4. The paper ends with a conclusion in Section 5.

In the sequel \(h\) denotes the time step, and \(t_n = nh\) are the associated time instants, starting at \(t_0 = 0\).

1 Definition of RL\(_k\) schemes and consistency

Let us consider a solution \(y(t)\) of equation (3) on a time interval \([0, T]\). It is recalled that the scheme (6) is consistent of order \(k\) if, given a time step \(h\), a time instant \(kh \leq t_n \leq T - h\), and the numerical approximation \(y_{n+1}\) in (6) computed with \(y_{n-j} = y(t_{n-j})\) for \(j = 0, \ldots, k - 1\), we have \(|y_{n+1} - y(t_n + h)| \leq Ch^{k+1}\), for a constant \(C\) only depending on the data \(a, b, y_0\) and \(T\) of the problem (3).

**Lemma 1.** Assume that the functions \(a(t, y)\) and \(b(t, y)\) are \(C^k\) regular on \([0, T] \times \mathbb{R}^N\). Moreover, assume that \(a(t, y)\) is diagonal \((a(t, y) = \text{diag}(a_i(t, y)))\) or constant. Then the scheme in (6) is consistent of order \(k\) for \(k = 2, 3, 4\) if

- for \(k = 2\), we have
  \[
  \alpha_n = a_n + \frac{1}{2} a_n' h + O(h^2), \quad \text{and} \quad \beta_n = b_n + \frac{1}{2} b_n' h + O(h^2);
  \]

- for \(k = 3\), we have
  \[
  \alpha_n = a_n + \frac{1}{2} a_n' h + \frac{1}{6} a_n'' h^2 + O(h^3),
  \]
  \[
  \beta_n = b_n + \frac{1}{2} b_n' h + \frac{1}{12} (a_n' b_n - a_n b_n') h^2 + O(h^3);
  \]

- for \(k = 4\), we have
  \[
  \alpha_n = a_n + \frac{1}{2} a_n' h + \frac{1}{6} a_n'' h^2 + \frac{1}{24} a_n''' h^3 + O(h^4),
  \]
  \[
  \beta_n = b_n + \frac{1}{2} b_n' h + \frac{1}{12} (a_n' b_n - a_n b_n') h^2 + \frac{1}{24} (b_n'' + a_n'' b_n - a_n b_n'') h^3 + O(h^4);
  \]
where \(a'_n, a''_n, a'''_n\) and \(b'_n, b''_n, b'''_n\) denote the successive derivatives at time \(t_n\) of the functions \(t \mapsto a(t, y(t))\) and \(t \mapsto b(t, y(t))\).

**Remark 1.** The assumption “\(a(t, y)\) is diagonal or constant” in Lemma 1 has the following origin. To analyze the consistency of the scheme, we will compute a Taylor expansion in \(h\) of the scheme in (6). This expansion is derived from Taylor expansions of \(\alpha_n\) and \(\beta_n\). For the sake of simplicity, assume the simple form \(\alpha_n = \alpha_{n,0} + h\alpha_{n,1}\). We need to expand \(\varphi_1(\alpha_n h)\) as a power series in \(h\), where the function \(\varphi_1\) is analytic on \(C\). However, in the matrix case, the equality, \(\varphi_1(M + N) = \varphi_1(M) + \varphi_1'(M)N + \ldots + \varphi_1^{(i)}(M)N^i/i! + \ldots\) holds if \(M\) and \(N\) are commutative matrices. Therefore one cannot expand \(\varphi_1(\alpha_n h)\) without the assumptions that \(\alpha_{n,0}\) and \(\alpha_{n,1}\) commute. This difficulty vanishes if \(a(t, y)\) is constant or a varying diagonal matrix.

**Proof.** Consider equation (3) on the closed time interval \([0, T]\), and its solution, the function \(y\). Since the functions \(a\) and \(b\) are \(C^k\) regular on \([0, T] \times \mathbb{R}^N\), the solution \(y(t)\) is \(C^{k+1}\) regular on \([0, T]\). Its derivatives up to order \(k + 1\) are bounded by constants only depending on the data of problem (3), and on \(T\). The Taylor expansion of \(y\) at time instant \(t_n\) is

\[
y(t_n + h) = y(t_n) + \sum_{j=1}^{k} \frac{s_j}{j!} h^j + O(h^{k+1}),
\]

with \(s_j = y^{(j)}(t_n)\). Using that \(y' = ay + b\) we get that

\[
s_1 = a_n y_n + b_n, \\
s_2 = (a'_n + a''_n) y_n + a_n b_n + b'_n, \\
s_3 = (a''_n + 3a'_n a'_n + a'''_n) y_n + b''_n + a_n b'_n + 2a'_n b_n + a_2 b_n, \\
s_4 = (a'''_n + 4a''_n a'_n + 3a''_n a'_n + 6a'_n a''_n + a''_n) y_n \\
+ b'''_n + b''_n a_n + 3a''_n b_n + 5a'_n a_n b_n + 3a'_n b'_n + a^3_n b_n + a^2_n b'_n.
\]

Series expansions in \(h\) for \(\alpha_n\) and for \(\beta_n\) are introduced as

\[
\alpha_n = \alpha_{n,0} + \alpha_{n,1} h + \ldots + \alpha_{n,k-1} h^{k-1} + O(h^k), \\
\beta_n = \beta_{n,0} + \beta_{n,1} h + \ldots + \beta_{n,k-1} h^{k-1} + O(h^k).
\]

If the matrix \(a(t, y)\) is diagonal or constant (see Remark 1), the Taylor expansion of the numerical solution \(y_{n+1}\) in (6) can be performed

\[
y_{n+1} = y(t_n) + \sum_{j=1}^{k} \frac{r_j}{j!} h^j + O(h^{k+1}).
\]
A direct computation of the $r_j$ gives

$$r_1 = \alpha_{n,0} y_n + \beta_{n,0},$$

$$r_2 = (2\alpha_{n,1} + \alpha_{n,0}^2) y_n + 2\beta_{n,1} + \alpha_{n,0} \beta_{n,0},$$

$$r_3 = (6\alpha_{n,2} + \alpha_{n,0}^3 + 6\alpha_{n,0} \alpha_{n,1}) y_n + 3\alpha_{n,1} \beta_{n,0} + 6\beta_{n,2} + \alpha_{n,0}^2 \beta_{n,0} + 3\alpha_{n,0} \beta_{n,1},$$

$$r_4 = (24\alpha_{n,0} \alpha_{n,2} + 24\alpha_{n,3} + 12\alpha_{n,1} \alpha_{n,0}^2 + 12\alpha_{n,1}^2 + \alpha_{n,0}^4) y_n$$

$$+ 12\alpha_{n,2} \beta_{n,0} + 24\beta_{n,3} + 12\alpha_{n,0} \beta_{n,2} + 12\alpha_{n,1} \beta_{n,1} + 4\alpha_{n,0}^2 \beta_{n,1} + 8\alpha_{n,0} \alpha_{n,1} \beta_{n,0} + \alpha_{n,0}^3 \beta_{n,0},$$

where $y_n$ denotes $y(t_n)$. The condition to be consistent of order $k$ is: $r_i = s_i$ for $1 \leq i \leq k$. The consistency conditions in Lemma 1 are obtained by solving recursively these relations.

We then can state our main result, which includes the definition of the RL$_k$ schemes.

**Theorem 1.** Assume (as in Lemma 1) that the functions $a(t, y)$ and $b(t, y)$ are $C^k$ regular on $[0, T] \times \mathbb{R}^N$, and that $a(t, y)$ is diagonal or constant. Then, the three schemes defined for $k = 2, 3, 4$ by equation (6) and the coefficients,

- for $k = 2$,
  $$\alpha_n = \frac{3}{2} a_n - \frac{1}{2} a_{n-1}, \quad \beta_n = \frac{3}{2} b_n - \frac{1}{2} b_{n-1},$$

- for $k = 3$,
  $$\alpha_n = \frac{1}{12} (23a_n - 16a_{n-1} + 5a_{n-2}),$$
  $$\beta_n = \frac{1}{12} (23b_n - 16b_{n-1} - 5b_{n-2}) + \frac{h}{12} (b_{n+1} - b_n - b_{n-1} b_n),$$

- for $k = 4$,
  $$\alpha_n = \frac{1}{24} (55a_n - 59a_{n-1} + 37a_{n-2} - 9a_{n-3}),$$
  $$\beta_n = \frac{1}{24} (55b_n - 59b_{n-1} + 37b_{n-2} - 9b_{n-3})$$
  $$+ \frac{h}{12} (a_n (3b_{n+1} - b_{n-2} - (3a_{n-1} - a_{n-2}) b_n),$$

where $a_j = a(t_j, y_j)$ and $b_j = b(t_j, y_j)$, are consistent of order $k$.

The three methods stated above are called Rush-Larsen methods of order $k$, and denoted by RL$_k$. They are explicit and $k$-step methods.
Remark 2. If the matrix $a$ is a constant, $a(t, y) = A$, then we have $\alpha_n = A$ for all three methods. In this case, the expressions of the coefficients $\beta_n$ in Theorem 1 for $k = 3, 4$ simplify as follows:

RL3 case: $\beta_n = \frac{1}{12}(23b_n - 16b_{n-1} + 5b_{n-2}) - \frac{h}{12} A(b_n - b_{n-1})$.

RL4 case: $\beta_n = \frac{1}{24}(55b_n - 59b_{n-1} + 37b_{n-2} - 9b_{n-3}) - \frac{h}{12} A(2b_n - 3b_{n-1} + b_{n-2})$.

Proof. It is a direct consequence of backwards differentiation formulas, that we first recall. The derivatives of a real function $f$ at the time instant $t_n$ can be approximated as follows (with common notations):

- first derivative,
  \[
  f'_n = \frac{f_n - f_{n-1}}{h} + O(h) = \frac{1}{2h}(3f_n - 4f_{n-1} + f_{n-2}) + O(h^2) = \frac{1}{6h}(11f_n - 18f_{n-1} + 9f_{n-2} - 2f_{n-3}) + O(h^3);
  \]

- second derivative,
  \[
  f''_n = \frac{1}{h^2}(f_n - 2f_{n-1} + f_{n-2}) + O(h) = \frac{1}{h^2}(2f_n - 5f_{n-1} + 4f_{n-2} - f_{n-3}) + O(h^2);
  \]

- third derivative,
  \[
  f'''_n = \frac{1}{h^3}(f_n - 3f_{n-1} + 3f_{n-2} - f_{n-3}) + O(h).
  \]

With these formulas, the consistency condition at order 3 on the coefficient $\alpha_n$ becomes

\[
\alpha_n = a_n + \frac{1}{2}a'_n h + \frac{1}{6}a''_n h^2 + O(h^3) = a_n + \frac{1}{4}(3a_n - 4a_{n-1} + a_{n-2}) + \frac{1}{6}(a_n - 2a_{n-1} + a_{n-2}) + O(h^3) = \frac{1}{12}(23a_n - 16a_{n-1} + 5a_{n-2}) + O(h^3).
\]

We retrieve the definition of $\alpha_n$ for the RL3 scheme. The same proof holds for $\beta_n$, and extends to order 4. □
2 Stability under perturbation and convergence

We refer to [10, Ch. III-8] for the definitions of convergence and of stability under perturbation. For the analysis of time-stepping methods, it is commonly assumed that \( f \) in equation (2) is uniformly Lipschitz with respect to its second variable \( y \). This hypothesis will be replaced by assumptions based on the formulation (3). Precisely it will be assumed that

\[
a(t, y) \text{ is bounded, } a(t, y), b(t, y) \text{ are uniformly Lipschitz in } y.
\]  

The Lipschitz constants of \( a \) and \( b \) are denoted by \( L_a \) and \( L_b \), respectively. The upper bound on \( |a(t, y)| \) is denoted by \( M_a \).

**Proposition 1.** If the assumption (7) holds, then, the RL\(_k\) schemes are stable under perturbation, for \( k = 2, 3, 4 \). In addition, also for \( k = 2, 3, 4 \), if the consistency assumptions of Theorem 1 are satisfied (\( a(t, y) \) and \( b(t, y) \) are \( C^k \) regular and \( a(t, y) \) is diagonal or constant), then the RL\(_k\) scheme is convergent of order \( k \).

Stability under perturbation together with consistency implies (nonstiff) convergence, see e.g. [10], or [6], where the current setting has been detailed. Therefore the proof of the convergence statement in Proposition 1 is immediate, and will not be recalled here.

The following definitions are necessary to prove Proposition 1. Equation (2) is considered on \( E = \mathbb{R}^N \) with the max norm \( |\cdot| \). A final time \( T > 0 \) is considered. The space of \( N \times N \) matrices is equipped with the operator norm \( \| \cdot \| \) associated with \( |\cdot| \). The space \( E^k \) is equipped with the max norm \( |Y|_\infty = \max_{1\leq i\leq k} |y_i| \) with \( Y = (y_1, \ldots, y_k) \). The RL\(_k\) scheme is defined by the mapping

\[
s_{t,h} : Y = (y_1, \ldots, y_k) \in E^k \rightarrow s_{t,h}(Y) \in E,
\]

with

\[
s_{t,h}(Y) = y_k + h\varphi_1(\alpha_{t,h}(Y)h)(\alpha_{t,h}(Y)y_k + \beta_{t,h}(Y)),
\]

in such a way that the scheme in (6) reads \( y_{n+1} = s_{t_n,h}(y_{n-k+1}, \ldots, y_n) \). The functions \( \alpha_{t,h} \) and \( \beta_{t,h} \) map the vector \( Y \) of the \( k \) previous values to the values \( \alpha_n \) and \( \beta_n \) given in Theorem 1. For instance, the function \( \alpha_{t,h}(Y) \) for \( k = 3 \) (the RL\(_3\) scheme) reads

\[
\alpha_{t,h}(Y) = \frac{1}{12}(23a(t, y_3) - 16a(t - h, y_2) + 5a(t - 2h, y_1)).
\]

A first technique to prove the stability under perturbation consists in showing that the function \( s_{t,h} \) is globally Lipschitz in \( Y \). To this aim, the derivative \( \partial_Y s_{t,h} \) has to be analyzed. As developed in Remark 1, it implies restrictions on the function \( a(t, y) \): it has to be either diagonal or constant. A second technique consists in proving the following two stability conditions:

\[
|s_{t,h}(Y) - s_{t,h}(Z)| \leq |Y - Z|_\infty (1 + Ch(|Y|_\infty + 1)), \tag{8}
\]

\[
|s_{t,h}(Y)| \leq |Y|_\infty (1 + Ch) + Ch, \tag{9}
\]
for all $Y$ and $Z$ in $E^k$, and where the constant $C$ depends only on the data $a, b, y_0$ in equation (3), and on the final time $T$. These are sufficient conditions for the stability under perturbation, as proved in [6, Section 2]. We will use the conditions (8) and (9) here, because they are more general, and give rise to less computations. The core of the proof is the following property of the RL$_k$ scheme. For $Y = (y_1, \ldots, y_k) \in E^k$, we have

\[ s_{t,h}(Y) = z(t + h) \quad \text{for} \quad z' = \alpha_{t,h}(Y)z + \beta_{t,h}(Y), \quad z(t) = y_k. \tag{10} \]

It will be used together with the following Gronwall inequality (see [8, Lemma 196, p.150]).

Suppose that $z(t)$ is a $C^1$ function, and that there exists $M_1 > 0$ and $M_2 > 0$ such that $|z'(t)| \leq M_1|z(t)| + M_2$ for all $t \in [t_0, t_0 + h]$. Then

\[ \forall t \in [t_0, t_0 + h], \quad |z(t)| \leq e^{M_1(t-t_0)} (|z(t_0)| + M_2(t-t_0)). \tag{11} \]

**Proposition 1.** In this proof, we always assume that $0 \leq h, t \leq T$, and denote by $C_i$ a constant that depends only on the data $a, b$ and $T$ of problem (3). With the assumptions in (7), and the definitions of $\alpha_k$ ($k = 2, 3, 4$) in Theorem 1, the function $\alpha_{t,h}$ is uniformly Lipschitz with a Lipschitz constant equal to $L_a$. Moreover we have the uniform bound $||\alpha_{t,h}|| \leq M_a$. Since the function $b(t, y)$ is uniformly Lipschitz with respect to $y$, and since $0 \leq t \leq T$, we have

\[ |b(t, y)| \leq |b(t, 0)| + |b(t, y) - b(t, 0)| \leq K_b(1 + |y|), \tag{12} \]

with $K_b = \max(L_b, \sup_{0 \leq t \leq T} |b(t, 0)|)$. For the RL$_3$ scheme, we have

\[ |\beta_{t,h}(Y)| \leq \frac{11}{3} K_b(1 + |Y|) + \frac{h}{12} M_a 2 K_b (1 + |Y|) \leq C_1 (1 + |Y|). \]

The same inequality holds for the RL$_2$, and RL$_4$ schemes. Afterwards, we can apply these bounds to the differential equation in (10)

\[ |z'| = |\alpha_{t,h}(Y)z + \beta_{t,h}(Y)| \leq M_a |z| + C_1 (1 + |Y|). \]

The initial state is $|z(t)| = |y_k| \leq |Y|$. Finally, the Gronwall inequality (11) yields, for $t \leq \tau \leq t + h$,

\[ |z(\tau)| \leq e^{M_a h} (|Y| + h C_1 (1 + |Y|)) \leq e^{M_a h} (|Y| (1 + C_1 h) + C_1 h) \leq |Y| (1 + C_2 h) + C_2 h, \tag{13} \]

by bounding the exponential with an affine function for $0 \leq h \leq T$. This gives the stability condition (9) for $\tau = t + h$.

For the RL$_2$ scheme, the function $\beta_{t,h}$ is uniformly Lipschitz. For the RL$_3$ scheme, for $Y = (y_1, y_2, y_3)$ and $Z = (z_1, z_2, z_3)$ in $E^3$, we have

\[ |\beta_{t,h}(Y) - \beta_{t,h}(Z)| \leq \frac{11}{3} L_b |Y - Z| + \frac{h}{12} \left( |a(t, y_3)b(t - h, y_2) - a(t, z_3)b(t - h, z_2)| + |a(t - h, y_2)b(t, y_3) - a(t - h, z_2)b(t, z_3)| \right) \]
Let us bound the Lipschitz constant of a function of the type $F(Y) = a(\xi, y_2)b(\tau, y_3)$, for $0 \leq \tau, \xi \leq T$:

$$|F(Y) - F(Z)| = |a(\xi, y_2)(b(\tau, y_2) - b(\tau, z_2)) + (a(\xi, y_3) - a(\xi, z_3)) b(\tau, z_2)|$$

$$\leq M_aL_b|Y - Z|_\infty + L_a|Y - Z|_\infty|b(\tau, z_2)|.$$ 

With the inequality (12), this yields, for $0 \leq \tau, \xi \leq T$, and $Y, Z$ in $E^k$, $|F(Y) - F(Z)| \leq C_3|Y - Z|_\infty(1 + |Z|_\infty)$. As a result, we have

$$|\beta_{t,h}(Y) - \beta_{t,h}(Z)|_\infty \leq C_4|Y - Z|_\infty(1 + |Z|_\infty).$$

The same inequality holds for the $RL_4$ scheme.

Finally we consider $Y_1$ and $Y_2$ in $E^k$, and the notation $\alpha_i = \alpha_{t,h}(Y_i)$, and $\beta_i = \beta_{t,h}(Y_i)$. The property (10) shows that $s_{t,h}(Y_1) - s_{t,h}(Y_2) = (z_1 - z_2)(t + h)$, where $z_i$ is the solution to $z_i'(t) = Y_{i,k}$. On the first hand, with the inequality (13), we have $|z_2(\tau)| \leq C_5(1 + |Y_2|_\infty)$ for $t \leq \tau \leq t + h$. On the second hand, on $[t, t + h]$, we have

$$|z_2(\tau)| \leq M_a|z_1 - z_2| + L_a|Y_1 - Y_2|_\infty C_5(1 + |Y_2|_\infty) + C_4|Y_1 - Y_2|_\infty(1 + |Y_2|_\infty)$$

$$\leq M_a|z_1 - z_2| + C_6|Y_1 - Y_2|_\infty(1 + |Y_2|_\infty).$$

The initial condition yields $|(z_1 - z_2)(t)| = |Y_{1,k} - Y_{2,k}| \leq |Y_1 - Y_2|_\infty$. As a consequence, the Gronwall inequality (11) applied to these bounds shows that

$$|(z_1 - z_2)(t + h)| \leq e^{M_a h}(1 + C_6 h(1 + |Y_2|_\infty)).$$

This last inequality implies the stability condition (8), again by bounding the exponential with an affine function for $0 \leq h \leq T$.$\blacksquare$

## 3 Dahlquist stability

For the general ideas and definitions concerning the Dahlquist stability we refer to [11]. The background for the Dahlquist stability of exponential integrators with a general varying stabilizer $a(t, y)$ has been developed in [6], following the ideas of Perego and Veneziani [25]. The equation (2) is considered with the Dahlquist test function $f(t, y) = \lambda y$, which is split into $f(t, y) = a(t, y)y + b(t, y)$, in order to match the framework of equation (3), with

$$a(t, y) = \theta \lambda, \quad b(t, y) = \lambda(1 - \theta)y.$$

For $\theta = 1$, the methods are exact and thus $A$-stable. For $\theta \simeq 1$, the exact linear part of $f(t, y)$ in equation (2) is well approximated by $a(t, y)$. The stability domain depends on $\theta$, it is denoted by $D_\theta$. Given a value of $\theta$, the region $D_\theta$ is defined by the modulus of a stability function, with the same definition as for multistep methods, see e.g. [11]. This stability function has been numerically computed, pointwise on a grid inside the complex plane $\mathbb{C}$, for each of the three $RL_k$ schemes, $k = 2, 3, 4$. 

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