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Yves Achdou, Nicoletta Tchou

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Abstract

We consider a family of optimal control problems in the plane with dynamics and running costs possibly discontinuous across a two-scale oscillatory interface. Typically, the amplitude of the oscillations is of the order of ε while the period is of the order of ε^2. As $\varepsilon \to 0$, the interfaces tend to a straight line Γ. We study the asymptotic behavior of the value function as $\varepsilon \to 0$. We prove that the value function tends to the solution of Hamilton-Jacobi equations in the two half-planes limited by Γ, with an effective transmission condition on Γ keeping track of the oscillations.

1 Introduction

The main motivation of this paper is to study the asymptotic behavior as $\varepsilon \to 0$ of the value function of an optimal control problem in $\mathbb{R}^2$ in which the running cost and dynamics may jump across a periodic oscillatory interface $\Gamma_{\varepsilon,\varepsilon}$, when the oscillations of $\Gamma_{\varepsilon,\varepsilon}$ have an amplitude of the order of ε and a period of the order of ε^2, (see Figure 1 below, which actually describes a more general case). The respective roles of the two indices in $\Gamma_{\varepsilon,\varepsilon}$ will be explained in § 1.1 below. The interface $\Gamma_{\varepsilon,\varepsilon}$ separates two unbounded regions of $\mathbb{R}^2$, $\Omega^L_{\varepsilon,\varepsilon}$ and $\Omega^R_{\varepsilon,\varepsilon}$. The present work is a natural continuation of a previous one, [3], in which both the amplitude and the period of the oscillations were of the order of ε. In [3], it was possible to make a change of variables in order to map the interface onto a flat one. Here, it is no longer possible, and the route leading to the homogenization result becomes more complex.

To characterize the optimal control problem, one has to specify the admissible dynamics at a point $x \in \Gamma_{\varepsilon,\varepsilon}$: in our setting, no mixture is allowed at the interface, i.e. the admissible dynamics are the ones corresponding to the subdomain $\Omega^L_{\varepsilon,\varepsilon}$ **and** entering $\Omega^L_{\varepsilon,\varepsilon}$, or corresponding to the subdomain $\Omega^R_{\varepsilon,\varepsilon}$ **and** entering $\Omega^R_{\varepsilon,\varepsilon}$. Hence the situation differs from those studied in the articles of G. Barles, A. Briani and E. Chasseigne [8, 9] and of G. Barles, A. Briani, E. Chasseigne and N. Tchou [11], in which mixing is allowed at the interface. The optimal control problem under consideration has been first studied in [25]: the value function is characterized as the viscosity solution of a Hamilton-Jacobi equation with special transmission conditions on $\Gamma_{\varepsilon,\varepsilon}$; a comparison principle for this problem is proved in [25] with arguments from the theory of optimal control similar to those introduced in [8, 9]. In parallel to [25], Imbert and Monneau have studied

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similar problems from the viewpoint of PDEs, see [18], and have obtained comparison results for quasi-convex Hamiltonians. There has been a very active research effort on finding simpler and more general/powerful proofs of the above-mentioned comparison results, see [10] and the very recent work of P-L. Lions and P. Souganidis [24].

In particular, [18] contains a characterization of the viscosity solution of the transmission problem with a reduced set of test-functions; this characterization will be used in the present work. Note that [25, 18] can be seen as extensions of articles devoted to the analysis of Hamilton-Jacobi equations on networks, see [1, 20, 2, 19, 23], because the notion of interface used there can be seen as a generalization of the notion of vertex (or junction) for a network.

We will see that as ε tends to 0, the value function converges to the solution of an effective problem related to a flat interface Γ, with Hamilton-Jacobi equations in the half-planes limited by Γ and a transmission condition on Γ. Whereas the partial differential equation far from the interface is unchanged, the main difficulty consists in finding the effective transmission condition on Γ. Naturally, the latter depends on the dynamics and running costs but also keeps memory of the vanishing oscillations. The present work is strongly related to [3], but also to two articles, [4] and [17], about singularly perturbed problems leading to effective Hamilton-Jacobi equations on networks. In [4], the authors of the present paper study a family of star-shaped planar domains D^ε made of N non intersecting semi-infinite strips of thickness ε and of a central region whose diameter is proportional to ε. As $\varepsilon \to 0$, the domains D^ε tend to a network $\mathcal{G}$ made of N half-lines sharing an endpoint O, named the vertex or junction point. For infinite horizon optimal control problems in which the state is constrained to remain in the closure of D^ε, the value function tends to the solution of a Hamilton-Jacobi equation on $\mathcal{G}$, with an effective transmission condition at O. The related effective Hamiltonian, which corresponds to trajectories staying close to the junction point, was obtained in [4] as the limit of a sequence of ergodic constants corresponding to larger and larger bounded subdomains. Note that the same problem and the question of the correctors in unbounded domains were also discussed by P-L. Lions in his lectures at Collège de France respectively in January 2017, and in January and February 2014, see [21]. The same kind of construction was then used in [17], in which Galise, Imbert and Monneau study a family of time dependent Hamilton-Jacobi equations in a simple network composed of two half-lines with a perturbation of the Hamiltonian localized in a small region close to the junction. In [17], a key point was the use of a single test-function at the vertex which was first proposed in [19, 18]. This idea will be also used in the present work. Note that similar techniques were used in the recent works of Forcadel et al, [14, 16, 15], which deal with applications to traffic flows. Finally, multiscale homogenization and singular perturbation problems with first and second order Hamilton Jacobi equations (without discontinuities) have been addressed in [5, 6].

Note that slight modifications of the techniques used below yield the asymptotic behavior of the transmission problems with oscillatory interfaces of amplitude ε and period ε^{1+q} with $q \geq 0$, (see § 2 and [3] for $q = 0$ and Remark 4.2 below for $q > 0$). Also, even if we focus on a two-dimensional problem, all the results below hold in the case when $\mathbb{R}^N$ is divided into two subregions, separated by a smooth and periodic $N-1$ dimensional oscillatory interface with two scales. Finally, we wish to stress the fact that an important possible application of our work is the homogenization of a transmission problem in geometrical optics, with two media separated by a two-scale interface.

The paper is organized as follows: in the remaining part of § 1, we set the problem. We will see in particular that it is convenient to consider a more general setting than the one described above, with two small parameters η and ε instead of one: more precisely, the amplitude of the oscillations will be of the order of η whereas the period will of the order of $\eta\varepsilon$. In § 2, we keep

η fixed while ε tends to 0: the region where the two media are mixed is a strip whose width is of the order of η: in this region, an effective Hamiltonian is found by classical homogenization techniques, see [22]; the main difficulty is to obtain the effective transmission conditions on the boundaries of the strip (two parallel straight lines) and to prove the convergence. The techniques will be reminiscent of [4, 17, 3], because only one parameter tends to 0. The effective transmission conditions keeps track of the geometry of the interface at the scale ε.
In § 3, we take the latter effective problem which depends on η, and have η tend to 0: we obtain a new effective transmission condition on a single flat interface, and prove the convergence result. Note that this passage to the limit is an intermediate step in order to study the two-scale homogenization problem described in the beginning of the introduction, but that it has also an interest for itself.
In § 4, we take $\eta = \varepsilon$, i.e. we consider the interface $\Gamma_{\varepsilon,\varepsilon}$ described at the beginning of the introduction, and let ε tend to 0: at the limit, we obtain the same effective problem as the one found in § 3, by letting first ε then η tend to 0.
Sections 2, 3 and 4 are organized in the same way: the main result is stated first, then proved in the remaining part of the section. For the conciseness of § 2, some technical proofs will be given in an appendix.

1.1 The geometry

Let (e_1, e_2) be an orthonormal basis of $\mathbb{R}^2$. For two real numbers a, b such that $0 < a < b < 1$, consider the set $\mathbb{S} = \{a, b\} + \mathbb{Z}$. Let $g : \mathbb{R} \to \mathbb{R}$ be a continuous function, periodic with period 1, such that

1. g is $\mathcal{C}^2$ in $\mathbb{R}\backslash\mathbb{S}$
2. $g(a) = g(b) = 0$
3. $\lim_{t\to a^-} g'(t) = \lim_{t\to a^+} g'(t) = +\infty$ and $\lim_{t\to a^-} \frac{g''(t)}{g'(t)} = \lim_{t\to a^+} \frac{g''(t)}{g'(t)} = 0$
4. $\lim_{t\to b^-} g'(t) = \lim_{t\to b^+} g'(t) = -\infty$ and $\lim_{t\to b^-} \frac{g''(t)}{g'(t)} = \lim_{t\to b^+} \frac{g''(t)}{g'(t)} = 0$

Let G be the multivalued Heavyside step function, periodic with period 1, such that

1. $G(a) = G(b) = [-1, 1]$
2. $G(t) = \{1\}$ if $t \in (a, b)$
3. $G(t) = \{-1\}$ if $t \in [0, a) \cup (b, 1]$

Let η and ε be two positive parameters: consider the $\mathcal{C}^2$ curve $\Gamma_{\eta,\varepsilon}$ defined as the graph of the multivalued function $g_{\eta,\varepsilon} : x_2 \mapsto \eta G(\frac{x_2}{\varepsilon\eta}) + \eta\varepsilon g(\frac{x_2}{\varepsilon\eta})$. We also define the domain $\Omega^R_{\eta,\varepsilon}$ (resp. $\Omega^L_{\eta,\varepsilon}$) as the epigraph (resp. hypograph) of $g_{\eta,\varepsilon}$:

$$\Omega^R_{\eta,\varepsilon} = \{x \in \mathbb{R}^2 : x_1 > g_{\eta,\varepsilon}(x_2)\}, \tag{1.1}$$
$$\Omega^L_{\eta,\varepsilon} = \{x \in \mathbb{R}^2 : x_1 < g_{\eta,\varepsilon}(x_2)\}. \tag{1.2}$$

The unit normal vector $n_{\eta,\varepsilon}(x)$ at $x \in \Gamma_{\eta,\varepsilon}$ is defined as follows: setting $y_2 = \frac{x_2}{\eta\varepsilon}$,

$$n_{\eta,\varepsilon}(x) = \begin{cases} \left(1 + \left(g'(y_2)\right)^2\right)^{-1/2} (e_1 - g'(y_2)e_2) & \text{if} \quad y_2 \notin \mathbb{S} \\ -e_2 & \text{if} \quad y_2 = a \mod 1 \\ e_2 & \text{if} \quad y_2 = b \mod 1. \end{cases}$$

Note that $n_{\eta,\varepsilon}(x)$ is oriented from $\Omega^L_{\eta,\varepsilon}$ to $\Omega^R_{\eta,\varepsilon}$.

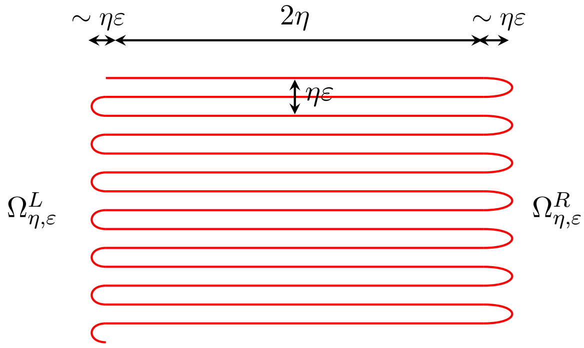

Figure 1: The oscillatory interface $\Gamma_{\eta,\varepsilon}$ separates $\Omega^L_{\eta,\varepsilon}$ and $\Omega^R_{\eta,\varepsilon}$. It has two scales: its amplitude η and period $\eta\varepsilon$

In § 2, we will let ε tend to zero and keep η fixed. In § 4, we will focus on the case when $\eta = \varepsilon$ and let ε tend to 0.

1.2 The optimal control problem in $\Omega^L_{\eta,\varepsilon} \cup \Omega^R_{\eta,\varepsilon} \cup \Gamma_{\eta,\varepsilon}$

We consider infinite-horizon optimal control problems which have different dynamics and running costs in the regions $\Omega^i_{\eta,\varepsilon}$, $i = L, R$. The sets of controls associated to the index $i = L, R$ will be called A^i; similarly, the notations f^i and ℓ^i will be used for the dynamics and running costs. The following assumptions will be made in all the present work.

1.2.1 Standing Assumptions

[H0] A is a metric space (one can take $A = \mathbb{R}^m$). For $i = L, R$, A^i is a non empty compact subset of A and $f^i : A^i \to \mathbb{R}^2$ is a continuous function. The sets A^i are disjoint. Define $M_f = \max_{i=L,R} \sup_{a \in A^i} |f^i(a)|$. The notation F^i will be used for the set $F^i = \{f^i(a), a \in A^i\}$.

[H1] For $i = L, R$, the function $\ell^i : A^i \to \mathbb{R}$ is continuous and bounded. Define $M_\ell = \max_{i=L,R} \sup_{a \in A^i} |\ell^i(a)|$.

[H2] For any $i = L, R$, the non empty set $\mathrm{FL}^i = \{(f^i(a), \ell^i(a)), a \in A^i\}$ is closed and convex.

[H3] There is a real number $\delta_0 > 0$ such that for $i = L, R$, $B(0, \delta_0) \subset F^i$.

We stress the fact that all the results below hold provided the latter assumptions are satisfied, although, in order to avoid tedious repetitions, we will not mention them explicitly in the statements.

Remark 1.1. *We have assumed that the dynamics f^i and running costs ℓ^i, $i = L, R$, do not depend on x. This assumption is made only for simplicity. With further classical assumptions, it would be possible to generalize all the results contained in this paper to the case when f^i and ℓ^i depend on x: typical such assumptions are*

1. *the Lipschitz continuity of f^i with respect to x uniformly in $a \in A^i$: there exists L_f such that for $i = L, R$, $\forall a \in A^i$, $x, y \in \mathbb{R}^2$, $|f^i(x, a) - f^i(y, a)| \le L_f |x - y|$*

2. the existence of a modulus of continuity ω_ℓ such that for any $i = L, R$, $x, y \in \mathbb{R}^2$ and $a \in A^i$,

$$|\ell^i(x,a) - \ell^i(y,a)| \le \omega_\ell(|x-y|).$$

Even if these assumptions are standard, keeping track of a possible slow dependency of the Hamiltonian with respect to x in the homogenization process below would have led us to tackle several technical questions and to significantly increase the length of the paper. It would have also made the essential ideas more difficult to grasp.
Moreover, it is clear that if f^i and ℓ^i do depend on x except in a strip containing the oscillatory interface, for example the strip $\{x : |x_1| < 1\}$ for ε and η small enough, then all what follows holds and does not require any further technicality.

1.2.2 The optimal control problem

Let the closed set $\mathcal{M}_{\eta,\varepsilon}$ be defined as follows:

$$\mathcal{M}_{\eta,\varepsilon} = \left\{(x,a);\ x \in \mathbb{R}^2, \quad a \in A^i \text{ if } x \in \Omega^i_{\eta,\varepsilon},\ i = L, R, \ \text{ and } a \in A^L \cup A^R \text{ if } x \in \Gamma_{\eta,\varepsilon}\right\}. \tag{1.3}$$

The dynamics $f_{\eta,\varepsilon}$ is a function defined in $\mathcal{M}_{\eta,\varepsilon}$ with values in $\mathbb{R}^2$:

$$\forall (x,a) \in \mathcal{M}_{\eta,\varepsilon}, \qquad f_{\eta,\varepsilon}(x,a) = f^i(a) \quad \text{if } x \in \Omega^i_{\eta,\varepsilon} \text{ or } (x \in \Gamma_{\eta,\varepsilon} \text{ and } a \in A^i).$$

The function $f_{\eta,\varepsilon}$ is continuous on $\mathcal{M}_{\eta,\varepsilon}$ because the sets A^i are disjoint. Similarly, let the running cost $\ell_{\eta,\varepsilon} : \mathcal{M}_{\eta,\varepsilon} \to \mathbb{R}$ be given by

$$\forall (x,a) \in \mathcal{M}_{\eta,\varepsilon}, \qquad \ell_{\eta,\varepsilon}(x,a) = \ell^i(a). \quad \text{if } x \in \Omega^i_{\eta,\varepsilon} \text{ or } (x \in \Gamma_{\eta,\varepsilon} \text{ and } a \in A^i).$$

For $x \in \mathbb{R}^2$, the set of admissible trajectories starting from x is

$$\mathcal{T}_{x,\eta,\varepsilon} = \left\{ \begin{array}{ll} (y_x, a) \in L^\infty_{\text{loc}}(\mathbb{R}^+; \mathcal{M}_{\eta,\varepsilon}) : & y_x \in \text{Lip}(\mathbb{R}^+; \mathbb{R}^2), \\ & y_x(t) = x + \displaystyle\int_0^t f_{\eta,\varepsilon}(y_x(s), a(s))ds \quad \forall t \in \mathbb{R}^+ \end{array} \right\}. \tag{1.4}$$

The cost associated to the trajectory $(y_x, a) \in \mathcal{T}_{x,\eta,\varepsilon}$ is

$$\mathcal{J}_{\eta,\varepsilon}(x; (y_x, a)) = \int_0^\infty \ell_{\eta,\varepsilon}(y_x(t), a(t)) e^{-\lambda t} dt, \tag{1.5}$$

with $\lambda > 0$. The value function of the infinite horizon optimal control problem is

$$v_{\eta,\varepsilon}(x) = \inf_{(y_x,a) \in \mathcal{T}_{x,\eta,\varepsilon}} \mathcal{J}_{\eta,\varepsilon}(x; (y_x, a)). \tag{1.6}$$

Proposition 1.2. *The value function $v_{\eta,\varepsilon}$ is bounded uniformly in η and ε and continuous in $\mathbb{R}^2$.*

Proof. This result is classical and can be proved with the same arguments as in [7]. □

1.3 The Hamilton-Jacobi equation

Similar optimal control problems have recently been studied in [2, 19, 25, 18]. It turns out that $v_{\eta,\varepsilon}$ can be characterized as the viscosity solution of a Hamilton-Jacobi equation with a discontinuous Hamiltonian, (once the notion of viscosity solution has been specially tailored to cope with the above mentioned discontinuity). We briefly recall the definitions used e.g. in [25].

Hamiltonians For $i = L, R$, let the Hamiltonians $H^i : \mathbb{R}^2 \to \mathbb{R}$ and $H_{\Gamma_{\eta,\varepsilon}} : \Gamma_{\eta,\varepsilon} \times \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$H^i(p) = \max_{a \in A^i}(-p \cdot f^i(a) - \ell^i(a)), \tag{1.7}$$
$$H_{\Gamma_{\eta,\varepsilon}}(x, p^L, p^R) = \max\{ H^{+,L}_{\Gamma_{\eta,\varepsilon}}(x, p^L), H^{-,R}_{\Gamma_{\eta,\varepsilon}}(x, p^R)\}, \tag{1.8}$$

where in (1.8), $p^L \in \mathbb{R}^2$ and $p^R \in \mathbb{R}^2$.

$$H^{-,i}_{\Gamma_{\eta,\varepsilon}}(x,p) = \max_{a \in A^i \text{ s.t. } f^i(a)\cdot n_{\eta,\varepsilon}(x) \geq 0} (-p \cdot f^i(a) - \ell^i(a)), \quad \forall x \in \Gamma_{\eta,\varepsilon}, \forall p \in \mathbb{R}^2, \tag{1.9}$$
$$H^{+,i}_{\Gamma_{\eta,\varepsilon}}(x,p) = \max_{a \in A^i \text{ s.t. } f^i(a)\cdot n_{\eta,\varepsilon}(x) \leq 0} (-p \cdot f^i(a) - \ell^i(a)), \quad \forall x \in \Gamma_{\eta,\varepsilon}, \forall p \in \mathbb{R}^2. \tag{1.10}$$

Test-functions For $\eta > 0$ and $\varepsilon > 0$, the function $\phi : \mathbb{R}^2 \to \mathbb{R}$ is an admissible test-function if ϕ is continuous in $\mathbb{R}^2$ and for any $i \in \{L, R\}$, $\phi|_{\overline{\Omega^i_{\eta,\varepsilon}}} \in \mathcal{C}^1(\overline{\Omega^i_{\eta,\varepsilon}})$.
The set of admissible test-functions is noted $\mathcal{R}_{\eta,\varepsilon}$. If $\phi \in \mathcal{R}_{\eta,\varepsilon}$, $x \in \Gamma_{\eta,\varepsilon}$ and $i \in \{L, R\}$, we set $D\phi^i(x) = \lim_{\substack{x' \to x \\ x' \in \Omega^i_{\eta,\varepsilon}}} D\phi(x')$.

Remark 1.3. *If $x \in \Gamma_{\eta,\varepsilon}$, ϕ is test-function and $p^L = D\phi^L(x)$, $p^R = D\phi^R(x)$, then $p^L - p^R$ is colinear to $n_{\eta,\varepsilon}(x)$ defined in § 1.1.*

Definition of viscosity solutions We are going to define viscosity solutions of the following transmission problem:

$$\lambda u(x) + H^L(Du(x)) = 0, \qquad \text{if } x \in \Omega^L_{\eta,\varepsilon}, \tag{1.11}$$
$$\lambda u(x) + H^R(Du(x)) = 0, \qquad \text{if } x \in \Omega^R_{\eta,\varepsilon}, \tag{1.12}$$
$$\lambda u(x) + H_{\Gamma_{\eta,\varepsilon}}(x, Du^L(x), Du^R(x)) = 0, \qquad \text{if } x \in \Gamma_{\eta,\varepsilon}, \tag{1.13}$$

where u^L (respectively u^R) stands for $u|_{\overline{\Omega^L_{\eta,\varepsilon}}}$ (respectively $u|_{\overline{\Omega^R_{\eta,\varepsilon}}}$). For brevity, we also note this problem

$$\lambda u + \mathcal{H}_{\eta,\varepsilon}(x, Du) = 0. \tag{1.14}$$

- An upper semi-continuous function $u : \mathbb{R}^2 \to \mathbb{R}$ is a subsolution of (1.14) if for any $x \in \mathbb{R}^2$, any $\phi \in \mathcal{R}_{\eta,\varepsilon}$ s.t. $u - \phi$ has a local maximum point at x, then

$$\lambda u(x) + H^i(D\phi^i(x)) \leq 0, \qquad \text{if } x \in \Omega^i_{\eta,\varepsilon}, \tag{1.15}$$
$$\lambda u(x) + H_{\Gamma_{\eta,\varepsilon}}(x, D\phi^L(x), D\phi^R(x)) \leq 0, \qquad \text{if } x \in \Gamma_{\eta,\varepsilon}, \tag{1.16}$$

  where, for $x \in \Gamma_{\eta,\varepsilon}$, the notation $D\phi^i(x)$ is introduced in the definition of the test-functions, see also Remark 1.3.

- A lower semi-continuous function $u : \mathbb{R}^2 \to \mathbb{R}$ is a supersolution of (1.14) if for any $x \in \mathbb{R}^2$, any $\phi \in \mathcal{R}_{\eta,\varepsilon}$ s.t. $u - \phi$ has a local minimum point at x, then

$$\lambda u(x) + H^i(D\phi^i(x)) \geq 0, \qquad \text{if } x \in \Omega^i_{\eta,\varepsilon}, \tag{1.17}$$
$$\lambda u(x) + H_{\Gamma_{\eta,\varepsilon}}(x, D\phi^L(x), D\phi^R(x)) \geq 0 \qquad \text{if } x \in \Gamma_{\eta,\varepsilon}. \tag{1.18}$$

- A continuous function $u : \mathbb{R}^2 \to \mathbb{R}$ is a viscosity solution of (1.14) if it is both a viscosity sub and supersolution of (1.14).

We skip the proof of the following theorem, see [25, 18].

Theorem 1.4. *The value function $v_{\eta,\varepsilon}$ defined in* (1.6) *is the unique bounded viscosity solution of* (1.14).

1.4 The main result and the general orientation

We set

$$\Omega^L = \{x \in \mathbb{R}^2, x_1 < 0\}, \quad \Omega^R = \{x \in \mathbb{R}^2, x_1 > 0\}, \quad \Gamma = \{x \in \mathbb{R}^2, x_1 = 0\}. \tag{1.19}$$

Informal statement of the main result Our main result, namely Theorem 4.1 below, is that, as $\varepsilon \to 0$, $v_{\varepsilon,\varepsilon}$ converges locally uniformly to v, the unique bounded viscosity solution of

$$\lambda v(z) + H^L(Dv(z)) = 0 \qquad \text{if } z \in \Omega^L, \tag{1.20}$$

$$\lambda v(z) + H^R(Dv(z)) = 0 \qquad \text{if } z \in \Omega^R, \tag{1.21}$$

$$\lambda v(z) + \max\left(E(\partial_{z_2} v(z)), H^{L,R}(Dv^L(z), Dv^R(z))\right) = 0 \qquad \text{if } z \in \Gamma. \tag{1.22}$$

The Hamiltonians H^L and H^R are defined in (1.7). In the effective transmission condition (1.22),

$$H^{L,R}(p^L, p^R) = \max\{\, H^{+,1,L}(p^L), H^{-,1,R}(p^R)\}, \tag{1.23}$$

for $p^L, p^R \in \mathbb{R}^2$. For $i = L, R$, $H^{+,1,i}(p)$ (respectively $H^{-,1,i}(p)$) is the nondecreasing (respectively nonincreasing) part of the Hamiltonian H^i with respect to p_1. In what follows, $p^L - p^R$ will be colinear to e_1. The effective flux-limiter $E : \mathbb{R} \to \mathbb{R}$ will be characterized in §3 below. For brevity, the problem in (1.20)-(1.22) will sometimes be noted

$$\lambda v(z) + \mathcal{H}(z, Dv(z)) = 0. \tag{1.24}$$

General orientation The proof of this result will be done by using Evans' method of perturbed test-functions, see [13]. Such a method requires to build a family of correctors depending on a single real variable p_2 (which stands for the derivative of v along Γ). The corrector, that will be noted $\xi_\varepsilon(p_2, \cdot)$, solves a cell problem, see (4.4) below, with a transmission condition on the interface $\Gamma_{1,\varepsilon}$, (note that the original geometry is dilated by a factor $1/\varepsilon$). The ergodic constant associated to the latter cell problem will be noted $E_\varepsilon(p_2)$ in § 4. The fact that the corrector and the ergodic constant still depend on ε is connected to the existence of two small scales in the problem.

The existence of the pairs $(\xi_\varepsilon(p_2), E_\varepsilon(p_2))$ and the asymptotic behavior of $\xi_\varepsilon(p_2)$ as $\varepsilon \to 0$ will be obtained essentially by using the arguments proposed in §2 below. In fact, for a reason that will soon become clear, in §2, we consider the interfaces $\Gamma_{\eta,\varepsilon}$ instead of $\Gamma_{1,\varepsilon}$, for a fixed arbitrary positive parameter η. Then, the region where the two media are mixed is a strip whose width is $\sim \eta$, see Figure 2: in this region, an effective Hamiltonian is found by classical homogenization techniques; the main achievement of §2 is to obtain the effective transmission conditions on the boundaries of the strip (two parallel straight lines) and to prove the convergence of the solutions of the transmission problems as $\varepsilon \to 0$.

Next, in § 3, we pass to the limit as $\eta \to 0$ in the effective problem that we have just obtained in §2. We obtain (1.24) at the limit $\eta \to 0$, whose solution is unique, (the well posedness of

(1.24) implies the convergence of the whole family of solutions as $\eta \to 0$) and a characterization of $E(p_2)$ in (1.22), see (3.1) below.

In the proof of the main result stated above and in Theorem 4.1, Evans' method requires to study the asymptotic behavior of $E_\varepsilon(p_2)$ and of $x \mapsto \varepsilon\xi_\varepsilon(p_2, x/\varepsilon)$ as $\varepsilon \to 0$. This is precisely what is done in § 4 below, where, in particular, we prove that $\lim_{\varepsilon\to 0} E_\varepsilon(p_2) = E(p_2)$ given by (3.1).

Therefore, we will prove that the limit of $v_{\varepsilon,\varepsilon}$ as $\varepsilon \to 0$ can be obtained by considering the transmission problems with interfaces $\Gamma_{\eta,\varepsilon}$, letting ε tend to 0 first, then η tend to 0. This is coherent with the intuition that since the two scales ε^2 and ε are well separated, the asymptotic behavior can be obtained in two successive steps.

In what follows, a significant difficulty in the construction of the correctors is the unboundedness of the domains in which they should be defined. It is addressed by using the ideas proposed in [4, 17, 3].

2 The effective problem obtained by letting ε tend to 0

In § 2, η is a fixed positive number, whereas ε tends to 0.

2.1 Main result

Let the domains Ω_η^L, Ω_η^M and Ω_η^R and the straight lines $\Gamma_\eta^{L,M}$, $\Gamma_\eta^{M,R}$ be defined by

$$\Omega_\eta^L = \{x \in \mathbb{R}^2, x_1 < -\eta\}, \quad \Omega_\eta^M = \{x \in \mathbb{R}^2, |x_1| < \eta\}, \quad \Omega_\eta^R = \{x \in \mathbb{R}^2, x_1 > \eta\}, \tag{2.1}$$

$$\Gamma_\eta^{L,M} = \{x \in \mathbb{R}^2, x_1 = -\eta\}, \quad \Gamma_\eta^{M,R} = \{x \in \mathbb{R}^2, x_1 = \eta\}. \tag{2.2}$$

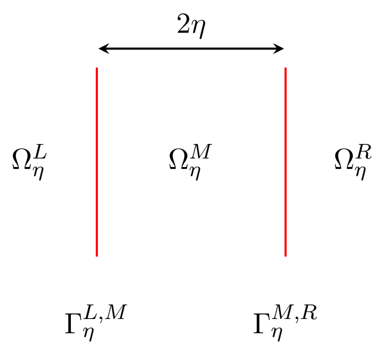

Figure 2: The geometry of the asymptotic problem when $\varepsilon \to 0$

Theorem 2.1. *As $\varepsilon \to 0$, $v_{\eta,\varepsilon}$ converges locally uniformly to v_η the unique bounded viscosity solution of*

$$\lambda v(z) + H^L(Dv(z)) = 0 \qquad \text{if } z \in \Omega_\eta^L, \tag{2.3}$$

$$\lambda v(z) + H^M(Dv(z)) = 0 \qquad \text{if } z \in \Omega_\eta^M, \tag{2.4}$$

$$\lambda v(z) + H^R(Dv(z)) = 0 \qquad \text{if } z \in \Omega_\eta^R, \tag{2.5}$$

$$\lambda v(z) + \max\left(E^{L,M}(\partial_{z_2} v(z)), H^{L,M}(Dv^L(z), Dv^M(z))\right) = 0 \qquad \text{if } z \in \Gamma_\eta^{L,M}, \tag{2.6}$$

$$\lambda v(z) + \max\left(E^{M,R}(\partial_{z_2} v(z)), H^{M,R}(Dv^M(z), Dv^R(z))\right) = 0 \qquad \text{if } z \in \Gamma_\eta^{M,R}, \tag{2.7}$$

where v^L (respectively v^M,v^R) stands for $v|_{\overline{\Omega_\eta^L}}$ (respectively $v|_{\overline{\Omega_\eta^M}}$, $v|_{\overline{\Omega_\eta^R}}$), and that we note for short

$$\lambda v(z) + \mathcal{H}_\eta(z, Dv(z)) = 0. \tag{2.8}$$

The Hamiltonians H^L and H^R are defined in (1.7). The effective Hamiltonian H^M will be defined in § 2.2 below (note that $p \mapsto H^M(p)$ is convex). In (2.6),

$$H^{L,M}(p^L, p^M) = \max\{ H^{+,1,L}(p^L), H^{-,1,M}(p^M)\}, \tag{2.9}$$

for $p^L \in \mathbb{R}^2$ and $p^M \in \mathbb{R}^2$ (in what follows, $p^L - p^M$ is colinear to e_1) and, for $i = L, M, R$, $p \mapsto H^{+,1,i}(p)$ (respectively $p \mapsto H^{-,1,i}(p)$) is the nondecreasing (respectively nonincreasing) part of the Hamiltonian H^i with respect to the first coordinate p_1 of p. In (2.7),

$$H^{M,R}(p^M, p^R) = \max\{ H^{+,1,M}(p^M), H^{-,1,R}(p^R)\}, \tag{2.10}$$

for $p^M \in \mathbb{R}^2$ and $p^R \in \mathbb{R}^2$.
The effective flux limiters $E^{L,M}$ and $E^{M,R}$ will be defined in § 2.3.

Let us list the notions which are needed by Theorem 2.1 and give a few comments:

1. Problem (2.8) is a transmission problem across the interfaces $\Gamma_\eta^{L,M}$ and $\Gamma_\eta^{M,R}$, with the respective effective transmission conditions (2.6) and (2.7). The notion of viscosity solutions of (2.8) is similar to the one defined for problem (1.14).

2. Note that the Hamilton-Jacobi equations in Ω_η^L and Ω_η^R are directly inherited from (1.15): this is quite natural, since the Hamilton-Jacobi equation at $x \in \Omega_\eta^L$ and $x \in \Omega_\eta^L$ does not depend on ε if ε is small enough.

3. The effective Hamiltonian $H^M(p)$ arising in Ω_η^M will be found by solving classical one dimensional cell-problems in the fast vertical variable $y_2 = x_2/\varepsilon$. The cell problems are one dimensional, since for any interval I such that $I \subset\subset (-\eta, \eta)$, $\Gamma_{\eta,\varepsilon} \cap (I \times \mathbb{R})$ is made of straight horizontal lines as soon as ε is small enough.

4. The Hamiltonian $H^{L,M}$ appearing in the effective transmission condition at the interface $\Gamma_\eta^{L,M}$ is built by considering only the effective dynamics related to $\Omega^{i,\eta}$ which point from $\Gamma_\eta^{L,M}$ toward $\Omega^{i,\eta}$, for $i = L, M$. The same remark holds for $H^{M,R}$ mutatis mutandis.

5. The effective flux limiters $E^{L,M}$ and $E^{M,R}$ are the only ingredients in the effective problem that keep track of the function g. They are constructed in § 2.3 and 2.4 below, see (2.27), as the limit of a sequence of ergodic constants related to larger and larger domains bounded in the horizontal direction. This is reminiscent of a construction first performed in [4] for singularly perturbed problems in optimal control leading to Hamilton-Jacobi equations posed on a network. Later, similar constructions were used in [17, 3].

6. For proving Theorem 2.1, the chosen strategy is reminiscent of [17], because it relies on the construction of a single corrector, whereas the method proposed in [4] requires the construction of an infinite family of correctors. This will be done in § 2.5 and the slopes at infinity of the correctors will be studied in § 2.4.2.

2.2 The effective Hamiltonian H^M

The first step in understanding the asymptotic behavior of the value function $v_{\eta,\varepsilon}$ as $\varepsilon \to 0$ is to look at what happens in Ω^M, i.e. in the region where $|x_1| < \eta$. For that, it is possible to rely on existing results, see [22] for the first work on the topic. In Ω^M, if the sequence of value functions $v_{\eta,\varepsilon}$ converges to v_η uniformly as $\varepsilon \to 0$, then v_η is a viscosity solution of a first order partial differential equation involving an effective Hamiltonian noted H^M in (2.4) and in the rest of the paper. The latter will be obtained by solving a one-dimensional periodic boundary value problem in the fast variable $y \in \mathbb{R}$, usually named *a cell problem*. Before stating the result, it is convenient to introduce the open sets $Y_\eta^L = (\eta a, \eta b) + \eta\mathbb{Z}$ and $Y_\eta^R = \mathbb{R} \setminus \overline{Y_\eta^L}$, and the discrete sets $\gamma_\eta^a = \{\eta a\} + \eta\mathbb{Z}$, $\gamma_\eta^b = \{\eta b\} + \eta\mathbb{Z}$. For $p \in \mathbb{R}^2$, $i = L, R$, we also define the Hamiltonians:

$$
\begin{aligned}
H^{-,2,i}(p) &= \max_{\alpha \in A^i, f^i(\alpha)\cdot e_2 \geq 0} (-p \cdot f^i(\alpha) - \ell^i(\alpha)), && (2.11)\\
H^{+,2,i}(p) &= \max_{\alpha \in A^i, f^i(\alpha)\cdot e_2 \leq 0} (-p \cdot f^i(\alpha) - \ell^i(\alpha)). && (2.12)
\end{aligned}
$$

Note that $H^{+,2,i}(p)$ (respectively $H^{-,2,i}(p)$) is the nondecreasing (respectively nonincreasing) part of the Hamiltonian $p \mapsto H^i(p)$ with respect to the second coordinate p_2 of p.

Proposition 2.2. *For any $p \in \mathbb{R}^2$ there exists a unique real number $H^M(p)$ such that the following one dimensional cell-problem has a Lipschitz continuous viscosity solution $\zeta(p,\cdot)$:*

$$
\begin{aligned}
H^R\left(p + \frac{d\zeta}{dy}(y)e_2\right) &= H^M(p), \quad \text{if } y \in Y_\eta^R, && (2.13)\\
H^L\left(p + \frac{d\zeta}{dy}(y)e_2\right) &= H^M(p), \quad \text{if } y \in Y_\eta^L, && (2.14)\\
\max\left(H^{+,2,R}\left(p + \frac{d\zeta}{dy}(y^-)e_2\right), H^{-,2,L}\left(p + \frac{d\zeta}{dy}(y^+)e_2\right)\right) &= H^M(p), \quad \text{if } y \in \gamma_\eta^a, && (2.15)\\
\max\left(H^{+,2,L}\left(p + \frac{d\zeta}{dy}(y^-)e_2\right), H^{-,2,R}\left(p + \frac{d\zeta}{dy}(y^+)e_2\right)\right) &= H^M(p), \quad \text{if } y \in \gamma_\eta^b, && (2.16)\\
\zeta \text{ is periodic in } y \text{ with period } \eta. & && (2.17)
\end{aligned}
$$

The following lemma contains information on H^M: we skip its proof because it is very much like the proof of [3, Lemma 4.16].

Lemma 2.3. *The function $p \mapsto H^M(p)$ is convex. There exists a constant C such that for any $p, p' \in \mathbb{R}^2$,*

$$
\begin{aligned}
&| H^M(p) - H^M(p') | \leq C|p - p'|, && (2.18)\\
&\delta_0|p| - C \leq H^M(p) \leq C|p| + C. && (2.19)
\end{aligned}
$$

As in [4, 17], we introduce three functions $E_0^i : \mathbb{R} \to \mathbb{R}$, $i = L, M, R$, and two functions $E_0^{L,M} : \mathbb{R} \to \mathbb{R}$ and $E_0^{M,R} : \mathbb{R} \to \mathbb{R}$:

$$
\begin{aligned}
E_0^i(p_2) &= \min\left\{H^i(p_2 e_2 + q e_1), \quad q \in \mathbb{R}\right\}, && (2.20)\\
E_0^{L,M}(p_2) &= \max\left\{E_0^L(p_2), E_0^M(p_2)\right\}, && (2.21)\\
E_0^{M,R}(p_2) &= \max\left\{E_0^M(p_2), E_0^R(p_2)\right\}. && (2.22)
\end{aligned}
$$

For $i = L, M, R$, $H^{+,1,i}(p)$ (respectively $H^{-,1,i}(p)$) is the nondecreasing (respectively nonincreasing) part of the Hamiltonian $p \mapsto H^i(p)$ with respect to the first coordinate p_1 of p. For $p_2 \in \mathbb{R}$, there exists a unique pair of real numbers $p_{1,0}^{-,i}(p_2) \leq p_{1,0}^{+,i}(p_2)$ such that

$$
\begin{aligned}
H^{-,1,i}(p_2 e_2 + p_1 e_1) &= \begin{cases} H^i(p_2 e_2 + p_1 e_1) & \text{if } p_1 \leq p_{1,0}^{-,i}(p_2), \\ E_0^i(p_2) & \text{if } p_1 > p_{1,0}^{-,i}(p_2), \end{cases} \\
H^{+,1,i}(p_2 e_2 + p_1 e_1) &= \begin{cases} E_0^i(p_2) & \text{if } p_1 \leq p_{1,0}^{+,i}(p_2), \\ H^i(p_2 e_2 + p_1 e_1) & \text{if } p_1 > p_{1,0}^{+,i}(p_2). \end{cases}
\end{aligned}
$$

2.3 Truncated cell problems for the construction of the flux limiters $E^{M,R}$ and $E^{L,M}$

In what follows, we focus on the construction of $E^{M,R}$ and on its properties, the construction of $E^{L,M}$ being completely symmetric.

2.3.1 Zooming near the line $\Gamma_\eta^{M,R}$

Asymptotically when $\varepsilon \to 0$, the two lines $\Gamma_\eta^{L,M}$ and $\Gamma_\eta^{M,R}$ appear very far from each other at the scale ε. This is why we are going to introduce another geometry obtained by first zooming near $\Gamma_\eta^{M,R}$ at a scale $1/\varepsilon$, then letting ε tend to 0.
Let $\widetilde{G}$ be the multivalued step function, periodic with period 1, such that

1. $\widetilde{G}(a) = \widetilde{G}(b) = [-\infty, 0]$
2. $\widetilde{G}(t) = \{0\}$ if $t \in (a, b)$
3. $\widetilde{G}(t) = \{-\infty\}$ if $t \in [0, a) \cup (b, 1]$

Consider the curve $\widetilde{\Gamma}_\eta$ defined as the graph of the multivalued function $\tilde{g}_\eta : x_2 \mapsto \eta \widetilde{G}(\frac{x_2}{\eta}) + \eta g(\frac{x_2}{\eta})$. We also define the domain $\widetilde{\Omega}_\eta^R$ (resp. $\widetilde{\Omega}_\eta^L$) as the epigraph (resp. hypograph) of $\tilde{g}_\eta$:

$$
\begin{aligned}
\widetilde{\Omega}_\eta^R &= \{x \in \mathbb{R}^2 : x_1 > \tilde{g}_\eta(x_2)\}, \\
\widetilde{\Omega}_\eta^L &= \{x \in \mathbb{R}^2 : x_1 < \tilde{g}_\eta(x_2)\}.
\end{aligned}
$$

The unit normal vector $\tilde{n}_\eta(x)$ at $x \in \widetilde{\Gamma}_\eta$ is defined as follows: setting $y_2 = \frac{x_2}{\eta}$,

$$
\tilde{n}_\eta(x) = \begin{cases} \left(1 + \left(g'(y_2)\right)^2\right)^{-1/2} \left(e_1 - g'(y_2) e_2\right) & \text{if} \quad y_2 \notin \mathbb{S} \\ -e_2 & \text{if} \quad y_2 = a \mod 1 \\ e_2 & \text{if} \quad y_2 = b \mod 1. \end{cases}
$$

Note that $\tilde{n}_\eta(x)$ is oriented from $\widetilde{\Omega}_\eta^L$ to $\widetilde{\Omega}_\eta^R$.

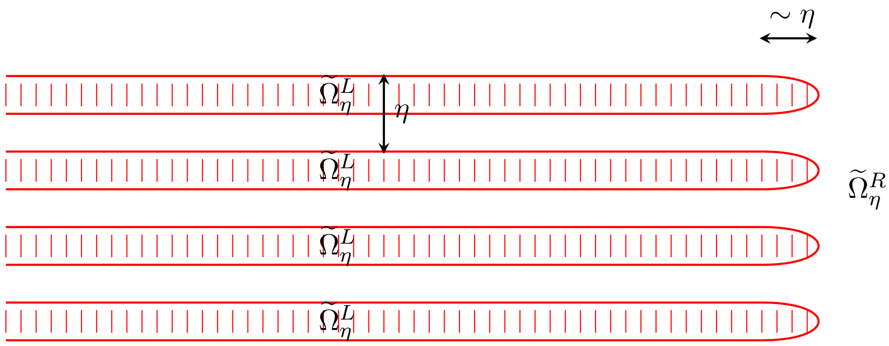

Figure 3: The interface $\widetilde{\Gamma}_\eta$ separates the disconnected domain $\widetilde{\Omega}^L_\eta$ and the connected domain $\widetilde{\Omega}^R_\eta$.

2.3.2 State-constrained problem in truncated domains

We introduce the following Hamiltonians:

$$H^{-,i}_{\widetilde{\Gamma}_\eta}(p,y) = \max_{a\in A^i \text{ s.t. } f^i(a)\cdot\tilde{n}_\eta(y)\geq 0} (-p\cdot f^i(a) - \ell^i(a)), \quad \forall y \in \widetilde{\Gamma}_\eta, \forall p \in \mathbb{R}^2, \tag{2.23}$$

$$H^{+,i}_{\widetilde{\Gamma}_\eta}(p,y) = \max_{a\in A^i \text{ s.t. } f^i(a)\cdot\tilde{n}_\eta(y)\leq 0} (-p\cdot f^i(a) - \ell^i(a)), \quad \forall y \in \widetilde{\Gamma}_\eta, \forall p \in \mathbb{R}^2, \tag{2.24}$$

with $\tilde{n}_\eta(y)$ defined in § 2.3.1, and, for $p^L, p^R \in \mathbb{R}^2$

$$H_{\widetilde{\Gamma}_\eta}(p^L,p^R,y) = \max\{\, H^{+,L}_{\widetilde{\Gamma}_\eta}(p^L,y), H^{-,R}_{\widetilde{\Gamma}_\eta}(p^R,y)\}. \tag{2.25}$$

In what follows, $p^L - p^R$ will always be colinear to $\tilde{n}_\eta(y)$. For $\rho > 0$, let us set $Y^\rho = \{y : |y_1| < \rho\}$. For ρ large enough such that $\widetilde{\Gamma}_\eta$ is strictly contained in $\{y : y_1 < \rho\}$, consider the *truncated cell problem*

$$\begin{cases} H^L(Du(y) + p_2e_2) & \leq \lambda_\rho(p_2) & \text{if } y \in \widetilde{\Omega}^L_\eta \cap Y^\rho, \\ H^L(Du(y) + p_2e_2) & \geq \lambda_\rho(p_2) & \text{if } y \in \widetilde{\Omega}^L_\eta \cap \overline{Y^\rho}, \\ H^R(Du(y) + p_2e_2) & \leq \lambda_\rho(p_2) & \text{if } y \in \widetilde{\Omega}^R_\eta \cap Y^\rho, \\ H^R(Du(y) + p_2e_2) & \geq \lambda_\rho(p_2) & \text{if } y \in \widetilde{\Omega}^R_\eta \cap \overline{Y^\rho}, \\ H_{\widetilde{\Gamma}_\eta}(Du^L(y) + p_2e_2, Du^R(y) + p_2e_2, y) & \leq \lambda_\rho(p_2) & \text{if } y \in \widetilde{\Gamma}_\eta \cap Y^\rho, \\ H_{\widetilde{\Gamma}_\eta}(Du^L(y) + p_2e_2, Du^R(y) + p_2e_2, y) & \geq \lambda_\rho(p_2) & \text{if } y \in \widetilde{\Gamma}_\eta \cap \overline{Y^\rho}, \\ u \text{ is 1-periodic w.r.t. } y_2/\eta, \end{cases} \tag{2.26}$$

where the inequations are understood in the sense of viscosity.

Lemma 2.4. *There is a unique $\lambda_\rho(p_2) \in \mathbb{R}$ such that* (2.26) *admits a viscosity solution. For this choice of $\lambda_\rho(p_2)$, there exists a solution $\chi_\rho(p_2,\cdot)$ which is Lipschitz continuous with Lipschitz constant L depending on p_2 only (independent of ρ).*

Proof. We skip the proof of this lemma, since it is very much like that of [3, Lemma 4.6]. □

2.4 The effective flux limiter $E^{M,R}(p_2)$ and the global cell problem

As in [4, 3], using the optimal control interpretation of (2.26), it is easy to prove that for a positive K which may depend on p_2 but not on ρ, and for all $0 < \rho_1 \le \rho_2$,

$$\lambda_{\rho_1}(p_2) \le \lambda_{\rho_2}(p_2) \le K.$$

For $p_2 \in \mathbb{R}$, the effective tangential Hamiltonian $E^{M,R}(p_2)$ is defined by

$$E^{M,R}(p_2) = \lim_{\rho\to\infty} \lambda_\rho(p_2). \tag{2.27}$$

For a fixed $p_2 \in \mathbb{R}$, the *global cell-problem* reads

$$\begin{cases} H^L(Du(y) + p_2e_2) & = E^{M,R}(p_2) \quad \text{if } y \in \widetilde{\Omega}^L_\eta, \\ H^R(Du(y) + p_2e_2) & = E^{M,R}(p_2) \quad \text{if } y \in \widetilde{\Omega}^R_\eta, \\ H_{\widetilde{\Gamma}_\eta}(Du^L(y) + p_2e_2, Du^R(y) + p_2e_2, y) & = E^{M,R}(p_2) \quad \text{if } y \in \widetilde{\Gamma}_\eta, \\ u \text{ is 1-periodic w.r.t. } y_2/\eta. & \end{cases} \tag{2.28}$$

The following theorem is proved exactly as Theorem 4.8 in [3].

Theorem 2.5. *Let $\chi_\rho(p_2,\cdot)$ be a sequence of uniformly Lipschitz continuous solutions of the truncated cell-problem* (2.26) *which converges to $\chi(p_2,\cdot)$ locally uniformly in $\mathbb{R}^2$. Then $\chi(p_2,\cdot)$ is a Lipschitz continuous viscosity solution of the global cell-problem* (2.28)*. By subtracting $\chi(p_2,0)$ to $\chi_\rho(p_2,\cdot)$ and $\chi(p_2,\cdot)$, we may also assume that $\chi(p_2,0)=0$.*

2.4.1 Comparison between $E_0^{M,R}(p_2)$ and $E^{M,R}(p_2)$ respectively defined in (2.22) and (2.27)

For $\varepsilon > 0$, let us set $W_\varepsilon(p_2,y) = \varepsilon\chi(p_2, \frac{y-\eta e_1}{\varepsilon})$. The following result is reminiscent of [17, Theorem 4.6,iii]:

Lemma 2.6. *For any $p_2 \in \mathbb{R}$, there exists a sequence ε_n of positive numbers tending to 0 as $n \to +\infty$ such that $W_{\varepsilon_n}(p_2,\cdot)$ converges locally uniformly to a Lipschitz function $y \mapsto W(p_2,y)$ (the Lipschitz does not depend on η). This function is constant with respect to y_2 and satisfies $W(p_2,\eta e_1)=0$. It is a viscosity solution of*

$$\begin{array}{rclll} H^R(Du(y) + p_2e_2) & = & E^{M,R}(p_2), & \text{if} & y_1 > \eta, \\ H^M(Du(y) + p_2e_2) & = & E^{M,R}(p_2), & \text{if} & y_1 < \eta. \end{array} \tag{2.29}$$

Proof. It is clear that $y \mapsto W_\varepsilon(p_2,y)$ is a Lipschitz continuous function with a constant Λ independent of ε and that $W_\varepsilon(p_2,\eta e_1) = 0$. Thus, from Ascoli-Arzela's Theorem, we may assume that $y \mapsto W_\varepsilon(p_2,y)$ converges locally uniformly to some function $y \mapsto W(p_2,y)$, maybe after the extraction of a subsequence. The function $y \mapsto W(p_2,y)$ is Lipschitz continuous with constant Λ and $W(p_2,\eta e_1) = 0$. Moreover, since $W_\varepsilon(p_2,y)$ is periodic with respect to y_2 with period ε, $W(p_2,y)$ does not depend on y_2.
To prove that $W(p_2,\cdot)$ is a viscosity solution of (2.29), we focus on the more difficult case when $y_1 < \eta$; we also restrict ourselves to proving that $W(p_2,\cdot)$ is a viscosity subsolution of (2.29), because the proof that $W(p_2,\cdot)$ is a viscosity supersolution follows the same lines.
Consider $\bar{y} \in \mathbb{R}^2$ such that $\bar{y}_1 < \eta$, $\phi \in \mathcal{C}^1(\mathbb{R}^2)$ and $r_0 < 0$ such that $B(\bar{y}, r_0)$ is contained in $\{y_1 < \eta\}$ and that

$$W(p_2,y) - \phi(y) < W(p_2,\bar{y}) - \phi(\bar{y}) = 0 \text{ for } y \in B(\bar{y},r_0) \setminus \{\bar{y}\}.$$

We first observe that $y \mapsto W_\varepsilon(p_2, y)$ is a viscosity solution of

$$\begin{array}{rcll} H^i(Du(y) + p_2 e_2) &=& E^{M,R}(p_2), & \text{if } y \in \Omega^i_{\eta,\varepsilon} \cap B(\bar y, r_0),\ i = L, R, \\ H_{\Gamma_{\eta,\varepsilon}}(y, Du^L(y) + p_2 e_2, Du^R(y) + p_2 e_2) &=& E^{M,R}(p_2), & \text{if } y \in \Gamma_{\eta,\varepsilon} \cap B(\bar y, r_0). \end{array} \tag{2.30}$$

We wish to prove that $H^M(D\phi(\bar y) + p_2 e_2) \le E^{M,R}(p_2)$. Let us argue by contradiction and assume that there exists $\theta > 0$ such that

$$H^M(D\phi(\bar y) + p_2 e_2) = E^{M,R}(p_2) + \theta. \tag{2.31}$$

Take $\phi_\varepsilon(y) = \phi(y) + \varepsilon\zeta(D\phi(\bar y) + p_2 e_2, \frac{y_2}{\varepsilon}) - \delta$, where ζ is a one-dimensional periodic corrector constructed in Proposition 2.2 and $\delta > 0$ is a fixed positive number. We claim that for $r > 0$ small enough, ϕ_ε is a viscosity supersolution of

$$\begin{array}{rcll} H^i(Du(y) + p_2 e_2) &\ge& E^{M,R}(p_2) + \frac{\theta}{2}, & \text{if } y \in \Omega^i_{\eta,\varepsilon} \cap B(\bar y, r), \\ H_{\Gamma_{\eta,\varepsilon}}(y, Du^L(y) + p_2 e_2, Du^R(y) + p_2 e_2) &\ge& E^{M,R}(p_2) + \frac{\theta}{2}, & \text{if } y \in \Gamma_{\eta,\varepsilon} \cap B(\bar y, r). \end{array} \tag{2.32}$$

This comes from (2.31), the definition of $\zeta(D\phi(\bar y) + p_2 e_2, \frac{y_2}{\varepsilon})$, the $\mathcal{C}^1$ regularity of ϕ and the Lipschitz continuity of H^i and $H_{\Gamma_{\eta,\varepsilon}}$ with respect to the p variables.
Hence, $W_\varepsilon(p_2, \cdot)$ is a subsolution of (2.30) and ϕ_ε is a supersolution of (2.32) in $B(\bar y, r)$. Moreover for $r > 0$ small enough, $\max_{y \in \partial B(\bar y, r)} (W(p_2, y) - \phi(y)) < 0$. Hence, for $\delta > 0$ and $\varepsilon > 0$ small enough $\max_{y \in \partial B(\bar y, r)} (W_\varepsilon(p_2, y) - \phi_\varepsilon(y)) \le 0$.
Thanks to a standard comparison principle (which holds thanks to the fact that $\frac{\theta}{2} > 0$)

$$\max_{y \in B(\bar y, r)} (W_\varepsilon(p_2, y) - \phi_\varepsilon(y)) \le 0. \tag{2.33}$$

Letting $\varepsilon \to 0$ in (2.33), we deduce that $W(p_2, \bar y) \le \phi(\bar y) - \delta$, which is in contradiction with the assumptions. $\square$

Using Lemma 2.6, it is possible to compare $E_0^{M,R}(p_2)$ and $E^{M,R}(p_2)$ respectively defined in (2.22) and (2.27)

Proposition 2.7. *For any $p_2 \in \mathbb{R}$,*

$$E^{M,R}(p_2) \ge E_0^{M,R}(p_2). \tag{2.34}$$

Proof. Thanks to Lemma 2.6, the function $y \mapsto W(p_2, y)$ is a viscosity solution $H^M(Du(y) + p_2 e_2) = E^{M,R}(p_2)$ in $\{y : y_1 < \eta\}$. Keeping in mind that $W(p_2, y)$ is independent of y_2, we see that for almost all $y_1 < \eta$, $E^{M,R}(p_2) = H^M(\partial_{y_1} W(p_2, y_1)e_1 + p_2 e_2) \ge E_0^M(p_2)$, from (2.20). Similarly, we show that $E^{M,R}(p_2) = H^R(\partial_{y_1} W(p_2, y_1)e_1 + p_2 e_2) \ge E_0^R(p_2)$ at almost $y_1 > \eta$, and we conclude using (2.22). $\square$

2.4.2 Asymptotic values of the slopes of χ as $y_1 \to \infty$

From Proposition 2.7 and the coercivity of the Hamiltonians H^i, $i = M, R$, the following numbers are well defined for all $p_2 \in \mathbb{R}$:

$$\overline{\Pi}^M(p_2) = \min\left\{q \in \mathbb{R} : H^M(p_2 e_2 + q e_1) = H^{-,1,M}(p_2 e_2 + q e_1) = E^{M,R}(p_2)\right\} \tag{2.35}$$

$$\widehat{\Pi}^M(p_2) = \max\left\{q \in \mathbb{R} : H^M(p_2 e_2 + q e_1) = H^{-,1,M}(p_2 e_2 + q e_1) = E^{M,R}(p_2)\right\} \tag{2.36}$$

$$\overline{\Pi}^R(p_2) = \min\left\{q \in \mathbb{R} : H^R(p_2 e_2 + q e_1) = H^{+,1,R}(p_2 e_2 + q e_1) = E^{M,R}(p_2)\right\} \tag{2.37}$$

$$\widehat{\Pi}^R(p_2) = \max\left\{q \in \mathbb{R} : H^R(p_2 e_2 + q e_1) = H^{+,1,R}(p_2 e_2 + q e_1) = E^{M,R}(p_2)\right\} \tag{2.38}$$

Remark 2.8. *From the convexity of the Hamiltonians H^i and $H^{\pm,1,i}$, we deduce that if for $i = M, R$, $E_0^i(p_2) < E^{M,R}(p_2)$, then $\overline{\Pi}^i(p_2) = \widehat{\Pi}^i(p_2)$. In this case, we will use the notation*

$$\Pi^i(p_2) = \overline{\Pi}^i(p_2) = \widehat{\Pi}^i(p_2). \tag{2.39}$$

Propositions 2.9 and 2.10 below, which will be proved in Appendix A, provide information on the growth of $y \mapsto \chi(p_2, y)$ as $|y_1| \to \infty$, where χ is obtained in Theorem 2.5 and is a solution of the cell problem (2.28):

Proposition 2.9. *With $\Pi^i(p_2) \in \mathbb{R}$ defined in* (2.39) *for $i = M, R$,*

1. *If $E^{M,R}(p_2) > E_0^R(p_2)$, then, there exist $\rho^* = \rho^*(p_2) > 0$ and $M^* = M^*(p_2) \in \mathbb{R}$ such that, for all $y \in [\rho^*, +\infty) \times \mathbb{R}$, $h_1 \geq 0$ and $h_2 \in \mathbb{R}$,*

$$\chi(p_2, y + h_1 e_1 + h_2 e_2) - \chi(p_2, y) \geq \Pi^R(p_2) h_1 - M^*. \tag{2.40}$$

2. *If $E^{M,R}(p_2) > E_0^M(p_2)$, then, there exist $\rho^* = \rho^*(p_2) > 0$ and $M^* = M^*(p_2) \in \mathbb{R}$ such that, for all $y \in (-\infty, -\rho^*] \times \mathbb{R}$, $h_1 \geq 0$ and $h_2 \in \mathbb{R}$,*

$$\chi(p_2, y - h_1 e_1 + h_2 e_2) - \chi(p_2, y) \geq -\Pi^M(p_2) h_1 - M^*. \tag{2.41}$$

Proposition 2.10. *For $p_2 \in \mathbb{R}$, $y \mapsto W(p_2, y)$ defined in Lemma 2.6 satisfies*

$$\overline{\Pi}^R(p_2) \leq \partial_{y_1} W(p_2, y) \leq \widehat{\Pi}^R(p_2) \qquad \text{for a.a. } y \in (\eta, +\infty) \times \mathbb{R}, \tag{2.42}$$

$$\overline{\Pi}^M(p_2) \leq \partial_{y_1} W(p_2, y) \leq \widehat{\Pi}^M(p_2) \qquad \text{for a.a. } y \in (-\infty, \eta) \times \mathbb{R}, \tag{2.43}$$

and for all y:

$$-\widehat{\Pi}^M(p_2)(y_1-\eta)^- + \overline{\Pi}^R(p_2)(y_1-\eta)^+ \leq W(p_2, y) \leq -\overline{\Pi}^M(p_2)(y_1-\eta)^- + \widehat{\Pi}^R(p_2)(y_1-\eta)^+. \tag{2.44}$$

2.5 Proof of Theorem 2.1

2.5.1 A reduced set of test-functions

From [19] and [18], we may use an equivalent definition for the viscosity solution of (2.8). We focus on the transmission condition at the interface $\Gamma_\eta^{M,R}$, because the same kind of arguments apply to the transmission at $\Gamma_\eta^{L,M}$. Theorem 2.12 below, which is reminiscent of [19, Theorem 2.7], will tell us that the transmission condition on $\Gamma_\eta^{M,R}$ can be tested with a reduced set of test-functions.

Definition 2.11. *Recall that $\overline{\Pi}^i$ and $\widehat{\Pi}^i$, $i = M, R$, have been introduced in (2.35)- (2.38). Let $\Pi : \Gamma_\eta^{M,R} \times \mathbb{R} \to \mathbb{R}^2$, $(z, p_2) \mapsto \big(\Pi^M(z, p_2), \Pi^R(z, p_2)\big)$ be such that, for all (z, p_2)*

$$\begin{aligned} \overline{\Pi}^M(p_2) \leq \Pi^M(z, p_2) \leq \widehat{\Pi}^M(p_2). \\ \overline{\Pi}^R(p_2) \leq \Pi^R(z, p_2) \leq \widehat{\Pi}^R(p_2). \end{aligned} \tag{2.45}$$

For $\bar{z} \in \Gamma_\eta^{M,R}$, the reduced set of test-functions $\mathcal{R}^\Pi(\bar{z})$ associated to the map Π is the set of the functions $\varphi \in \mathcal{C}^0(\mathbb{R}^2)$ such that there exists a $\mathcal{C}^1$ function $\psi : \Gamma_\eta^{M,R} \to \mathbb{R}$ with

$$\varphi(z + te_1) = \psi(z) + \big(\Pi^R(\bar{z}, \partial_{z_2}\psi(\bar{z}))\, 1_{t>0} + \Pi^M(\bar{z}, \partial_{z_2}\psi(\bar{z}))\, 1_{t<0}\big)\, t. \tag{2.46}$$

The following theorem is reminiscent of [19, Theorem 2.7].

Theorem 2.12. *Let $u : \mathbb{R}^2 \to \mathbb{R}$ be a subsolution (resp. supersolution) of (2.4) and (2.5). Consider a map $\Pi : \Gamma_\eta^{M,R} \times \mathbb{R} \to \mathbb{R}^2$, $(z, p_2) \mapsto \left(\Pi^M(z, p_2), \Pi^R(z, p_2)\right)$ such that (2.45) holds for all $(z, p_2) \in \Gamma_\eta^{M,R} \times \mathbb{R}$.*
We assume furthermore that u is Lipschitz continuous in $\Gamma_\eta^{M,R} + B(0, r)$ for some $r > 0$. The function u is a subsolution (resp. supersolution) of (2.7) if and only if for any $z \in \Gamma_\eta^{M,R}$ and for all $\varphi \in \mathcal{R}^\Pi(z)$ such that $u - \varphi$ has a local maximum (resp. local minimum) at z,

$$\lambda u(z) + \max\left(E^{M,R}(\partial_{z_2}\varphi(z)), H^{M,R}(D\varphi^M(z), D\varphi^R(z))\right) \le 0, \quad (\textit{resp. } \ge 0). \tag{2.47}$$

Proof. The proof follows the lines of that of [19, Theorem 2.7] and is also given in [3, Appendix C] $\square$

Remark 2.13. *In the statement of Theorem 2.12, we have chosen to restrict ourselves to functions that are Lipschitz continuous in $\Gamma_\eta^{M,R} + B(0, r)$ (this property makes the proof simpler); indeed, since the functions $v_{\eta,\varepsilon}$ are Lipschitz continuous with a Lipschitz constant Λ independent of ε (and also of η), the relaxed semi-limits of $v_{\eta,\varepsilon}$ as $\varepsilon \to 0$ are also Lipschitz continuous with the same Lipschitz constant Λ, see (2.48) below and [3, Remark 3.1].*
In fact, a more general version of Theorem 2.12 can be stated for any lower semi-continuous supersolution, and for the upper semi-continuous subsolutions u such that for all $z \in \Gamma_\eta^{M,R}$, $u(z) = \limsup_{z' \to z, z' \in \Omega_\eta^i} u(z')$, $\forall i = M, R$, as in [19, 18].

2.5.2 Proof of Theorem 2.1

Let us consider the relaxed semi-limits

$$\overline{v_\eta}(z) = \limsup_\varepsilon{}^* v_{\eta,\varepsilon}(z) = \limsup_{z' \to z, \varepsilon \to 0} v_{\eta,\varepsilon}(z') \quad \text{ and } \quad \underline{v_\eta}(z) = \liminf_\varepsilon{}_* v_{\eta,\varepsilon}(z) = \liminf_{z' \to z, \varepsilon \to 0} v_{\eta,\varepsilon}(z'). \tag{2.48}$$

Note that $\overline{v_\eta}$ and $\underline{v_\eta}$ are well defined, since $(v_{\eta,\varepsilon})_\varepsilon$ is uniformly bounded. We will prove that $\overline{v_\eta}$ and $\underline{v_\eta}$ are respectively a subsolution and a supersolution of (2.8). It is classical to check that the functions $\overline{v_\eta}(z)$ and $\underline{v_\eta}(z)$ are respectively a bounded subsolution and a bounded supersolution in Ω_η^i, $i = L, M, R$, of

$$\lambda u(z) + H^i(Du(z)) = 0. \tag{2.49}$$

From comparison theorems proved in [9, 18, 25], this will imply that $\overline{v_\eta} = \underline{v_\eta} = v_\eta = \lim_{\varepsilon \to 0} v_{\eta,\varepsilon}$. We just have to check the transmission conditions (2.6) and (2.7), and it is enough to focus on the latter, since the former is dealt with in a very same manner.
We focus on $\overline{v_\eta}$ since the proof for $\underline{v_\eta}$ is similar.

We are going to use Theorem 2.12 with the special choice for the map $\Pi : \mathbb{R} \to \mathbb{R}^2$: $\Pi(p_2) = \left(\widehat{\Pi}^M(p_2), \overline{\Pi}^R(p_2)\right)$. Note that Theorem 2.12 can indeed be applied, because, $\overline{v_\eta}$ is Lipschitz continuous, see Remark 2.13. Take $\bar{z} \in \Gamma_\eta^{M,R}$ and a test-function $\varphi \in \mathcal{R}^\Pi(\bar{z})$, i.e. of the form

$$\varphi(z + te_1) = \psi(z) + \left(\overline{\Pi}^R\left(\partial_{z_2}\psi(\bar{z})\right) 1_{t>0} + \widehat{\Pi}^M\left(\partial_{z_2}\psi(\bar{z})\right) 1_{t<0}\right) t, \quad \forall z \in \Gamma_\eta^{M,R}, t \in \mathbb{R}, \tag{2.50}$$

for a $\mathcal{C}^1$ function $\psi : \Gamma_\eta^{M,R} \to \mathbb{R}$, such that $\overline{v_\eta} - \varphi$ has a strict local maximum at $\bar{z}$ and that $\overline{v_\eta}(\bar{z}) = \varphi(\bar{z})$.
Let us argue by contradiction with (2.47) and assume that

$$\lambda\varphi(\bar{z}) + \max\left(E^{M,R}(\partial_{z_2}\varphi(\bar{z})), H^{M,R}(D\varphi^M(\bar{z}), D\varphi^R(\bar{z}))\right) = \theta > 0. \tag{2.51}$$

From (2.50), we see that $H^{M,R}(D\varphi^M(\bar z), D\varphi^R(\bar z)) \le E^{M,R}(\partial_{z_2}\varphi(\bar z))$ and (2.51) is equivalent to

$$\lambda\psi(\bar z) + E^{M,R}(\partial_{z_2}\psi(\bar z)) = \theta > 0. \tag{2.52}$$

Let $\chi(\partial_{z_2}\psi(\bar z),\cdot)$ be a solution of (2.28) such that $\chi(\partial_{z_2}\psi(\bar z),0) = 0$ (see Theorem 2.5), and $W(\partial_{z_2}\psi(\bar z), z_1) = \lim_{\varepsilon\to 0} \varepsilon\chi(\partial_{z_2}\psi(\bar z), \frac{z-\eta e_1}{\varepsilon})$.

Step 1 Hereafter, we will consider a small positive radius r such that $r < \eta/4$. Then for ε small enough, $\Omega^i_{\eta,\varepsilon} \cap B(\bar z, r) = \left(\eta e_1 + \varepsilon\widetilde\Omega^i_\eta\right) \cap B(\bar z, r)$ for $i = L, R$. We claim that for ε and r small enough, the function φ^ε:

$$\varphi^\varepsilon(z) = \psi(\eta e_1 + z_2 e_2) + \varepsilon\chi(\partial_{z_2}\psi(\bar z), \frac{z-\eta e_1}{\varepsilon})$$

is a viscosity supersolution of

$$\left\{\begin{array}{rcll} \lambda\varphi^\varepsilon(z) + H^i(D\varphi^\varepsilon(z)) & \ge & \frac{\theta}{2} & \text{if } z \in \Omega^i_{\eta,\varepsilon} \cap B(\bar z, r),\ i = L, R, \\ \lambda\varphi^\varepsilon(z) + H_{\Gamma_{\eta,\varepsilon}}(z, D\left(\varphi^\varepsilon\right)^L(z), D\left(\varphi^\varepsilon\right)^R(z)) & \ge & \frac{\theta}{2} & \text{if } z \in \Gamma_{\eta,\varepsilon} \cap B(\bar z, r), \end{array}\right. \tag{2.53}$$

where H^i and $H_{\Gamma_{\eta,\varepsilon}}$ are defined in (1.7)-(1.8).
Indeed, if ξ is a test-function in $\mathcal{R}_{\eta,\varepsilon}$ such that $\varphi^\varepsilon - \xi$ has a local minimum at $z^\star \in B(\bar z, r)$, then, from the definition of φ^ε, $y \mapsto \chi(\partial_{z_2}\psi(\bar z), y - \frac{\eta}{\varepsilon}e_1) - \frac{1}{\varepsilon}\left(\xi(\varepsilon y) - \psi(\eta e_1 + \varepsilon y_2 e_2)\right)$ has a local minimum at $\frac{z^\star}{\varepsilon}$.
If $\frac{z^\star - \eta e_1}{\varepsilon} \in \widetilde\Omega^i_\eta$, for $i = L$ or R, then $H^i(D\xi(z^\star) - \partial_{z_2}\psi(\eta e_1 + z_2^\star e_2)e_2 + \partial_{z_2}\psi(\bar z)e_2) \ge E^{M,R}(\partial_{z_2}\psi(\bar z))$.
From the regularity properties of H^i,

$$H^i(D\xi(z^\star) - \partial_{z_2}\psi(\eta e_1 + z_2^\star e_2)e_2 + \partial_{z_2}\psi(\bar z)e_2) = H^i(D\xi(z^\star)) + o_{r\to 0}(1),$$

thus

$$\lambda\varphi^\varepsilon(z^\star) + H^i(D\xi(z^\star)) \ge E^{M,R}(\partial_{z_2}\psi(\bar z)) + \lambda\left(\psi(\eta e_1 + z_2^\star e_2) + \varepsilon\chi(\partial_{z_2}\psi(\bar z), \frac{z^\star - \eta e_1}{\varepsilon})\right) + o_{r\to 0}(1).$$

From (2.52), this implies that

$$\lambda\varphi^\varepsilon(z^\star) + H^i(D\xi(z^\star)) \ge \theta + \lambda\varepsilon\chi(\partial_{z_2}\psi(\bar z), \frac{z^\star - \eta e_1}{\varepsilon}) + o_{r\to 0}(1).$$

Recall that the function $y \mapsto \varepsilon\chi(\partial_{z_2}\psi(\bar z), \frac{y-\eta e_1}{\varepsilon})$ converges locally uniformly to $y \mapsto W(\partial_{z_2}\psi(\bar z), y)$, which is a Lipschitz continuous function, independent of y_2 and such that $W(\partial_{z_2}\psi(\bar z), 0) = 0$. Therefore, for η and r small enough, $\lambda\varphi^\varepsilon(z^\star) + H^i(D\xi(z^\star)) \ge \frac{\theta}{2}$.
If $\frac{z^\star - \eta e_1}{\varepsilon} \in \widetilde\Gamma_\eta$, then, we have

$$H^{+,L}_{\widetilde\Gamma_\eta}(D\xi^L(z^\star) - \partial_{z_2}\psi(\eta e_1 + z_2^\star e_2)e_2 + \partial_{z_2}\psi(\bar z)e_2, \frac{z^\star - \eta e_1}{\varepsilon}) \ge E^{M,R}(\partial_{z_2}\psi(\bar z))$$

or

$$H^{-,R}_{\widetilde\Gamma_\eta}(D\xi^R(z^\star) - \partial_{z_2}\psi(\eta e_1 + z_2^\star e_2)e_2 + \partial_{z_2}\psi(\bar z)e_2, \frac{z^\star - \eta e_1}{\varepsilon}) \ge E^{M,R}(\partial_{z_2}\psi(\bar z)).$$

Since the Hamiltonians $H^{\pm,i}_{\widetilde\Gamma_\eta}$ enjoy the same regularity properties as H^i, it is possible to use the same arguments as in the case when $\frac{z^\star - \eta e_1}{\varepsilon} \in \Omega^i$. For r and ε small enough,

$$\lambda\varphi^\varepsilon(z^\star) + H_{\Gamma_{\eta,\varepsilon}}(z^\star, D\left(\varphi^\varepsilon\right)^L(z^\star), D\left(\varphi^\varepsilon\right)^R(z^\star)) \ge \frac{\theta}{2}.$$

The claim that φ^ε is a supersolution of (2.53) is proved.

Step 2 Let us prove that there exist some positive constants $K_r > 0$ and $\varepsilon_0 > 0$ such that

$$v_{\eta,\varepsilon}(z) + K_r \le \varphi^\varepsilon(z), \quad \forall z \in \partial B(\bar z, r),\ \forall \varepsilon \in (0, \varepsilon_0). \tag{2.54}$$

Indeed, since $\overline{v_\eta} - \varphi$ has a strict local maximum at $\bar z$ and since $\overline{v_\eta}(\bar z) = \varphi(\bar z)$, there exists a positive constant $\tilde K_r > 0$ such that $\overline{v_\eta}(z) + \tilde K_r \le \varphi(z)$ for any $z \in \partial B(\bar z, r)$. Since $\overline{v_\eta} = \limsup_\varepsilon{}^* v_{\eta,\varepsilon}$, there exists $\tilde\varepsilon_0 > 0$ such that

$$v_{\eta,\varepsilon}(z) + \frac{\tilde K_r}{2} \le \varphi(z) \quad \text{ for any } 0 < \varepsilon < \tilde\varepsilon_0 \text{ and } z \in \partial B(\bar z, r). \tag{2.55}$$

On the other hand, from (2.44) in Proposition 2.10,

$$\begin{aligned} &\psi(z_2 e_2 + \eta e_1) + W(\partial_{z_2}\psi(\bar z), z) \\ \ge &\psi(z_2 e_2 + \eta e_1) + \left(\overline{\Pi}^R(\partial_{z_2}\psi(\bar z))1_{z_1>\eta} + \widehat{\Pi}^M(\partial_{z_2}\psi(\bar z))1_{z_1<\eta}\right)(z_1 - \eta) = \varphi(z). \end{aligned} \tag{2.56}$$

Moreover, $z \mapsto \varphi^\varepsilon(z)$ converges locally uniformly to $z \mapsto \psi(\eta e_1 + z_2 e_2) + W(\partial_{z_2}\psi(\bar z), z)$ as ε tends to 0. By collecting the latter observation, (2.56) and (2.55), we get (2.54) for some constants $K_r > 0$ and $\varepsilon_0 > 0$.

Step 3 From the previous steps, we find by comparison that for r and ε small enough,

$$v_{\eta,\varepsilon}(z) + K_r \le \varphi^\varepsilon(z) \qquad \forall z \in B(\bar z, r).$$

Setting $z = \bar z$ and taking the lim sup as $\varepsilon \to 0$, we obtain

$$\overline{v_\eta}(\bar z) + K_r \le \psi(\bar z) = \varphi(\bar z) = \overline{v_\eta}(\bar z),$$

which cannot happen. The proof is completed. □

Remark 2.14. *For the proof of the supersolution inequality, the test-function φ should be chosen of the form*

$$\varphi(z + te_1) = \psi(z) + \left(\widehat{\Pi}^R\left(\partial_{z_2}\psi(\bar z)\right)1_{t>0} + \overline{\Pi}^M\left(\partial_{z_2}\psi(\bar z)\right)1_{t<0}\right)t, \quad \forall z \in \Gamma_\eta^{M,R}, t \in \mathbb{R},$$

where $\psi \in \mathcal{C}^1(\mathbb{R})$.

3 The second passage to the limit: η tends to 0

We now aim at passing to the limit in (2.8) as η tends to 0. Recall that Ω^L, Ω^R and Γ are defined in (1.19).

3.1 Main result

Theorem 3.1. *As $\eta \to 0$, v_η converges locally uniformly to v, the unique bounded viscosity solution of (1.20)- (1.22), (for short (1.24)).*
In the transmission condition (1.22), namely

$$\lambda v(z) + \max\left(E(\partial_{z_2}v(z)),\ H^{+,1,L}(Dv^L(z)),\ H^{-,1,R}(Dv^R(z)\right) = 0 \mathit{if}\ z \in \Gamma,$$

the effective flux limiter E is given for $p_2 \in \mathbb{R}$ by

$$E(p_2) = \max(E^{L,M}(p_2), E^{M,R}(p_2)), \tag{3.1}$$

where $E^{L,M}$ and $E^{M,R}$ are defined in §2.4, see (2.34).

Remark 3.2. *It is striking that, in (3.1), the effective flux limiter $E(p_2)$ can be deduced explicitly from the limiters $E^{L,M}(p_2)$ and $E^{M,R}(p_2)$ obtained in § 2.*

Let us consider the relaxed semi-limits

$$\overline{v}(z) = \limsup_{\eta}{}^* v_\eta(z) = \limsup_{z'\to z, \eta\to 0} v_\eta(z') \quad \text{ and } \quad \underline{v}(z) = \liminf_{\eta}{}_* v_\eta(z) = \liminf_{z'\to z, \eta\to 0} v_\eta(z'). \tag{3.2}$$

Note that $\overline{v}$ and $\underline{v}$ are well defined, since $(v_\eta)_\eta$ is uniformly bounded by M_ℓ/λ, see (1.6). It is classical to check that the functions $\overline{v}(z)$ and $\underline{v}(z)$ are respectively a bounded subsolution and a bounded supersolution in Ω^i of

$$\lambda u(z) + H^i(Du(z)) = 0. \tag{3.3}$$

To find the effective transmission on Γ, we shall proceed as in [4, 17, 3] and consider cell problems in larger and larger bounded domains.

3.2 Proof of Theorem 3.1

3.2.1 State-constrained problem in truncated domains

Let us fix $p_2 \in \mathbb{R}$. For $\rho > 1$, we consider the one dimensional *truncated cell problem*:

$$\begin{cases} H^L\left(\dfrac{du}{dy}(y) + p_2 e_2\right) \le \mu_\rho(p_2), & \text{if } y \in (-\rho, -1), \\ H^L\left(\dfrac{du}{dy}(y) + p_2 e_2\right) \ge \mu_\rho(p_2), & \text{if } y \in [-\rho, -1), \\ H^M\left(\dfrac{du}{dy}(y) + p_2 e_2\right) = \mu_\rho(p_2), & \text{if } y \in (-1, 1), \\ H^R\left(\dfrac{du}{dy}(y) + p_2 e_2\right) \le \mu_\rho(p_2), & \text{if } y \in (1, \rho), \\ H^R\left(\dfrac{du}{dy}(y) + p_2 e_2\right) \ge \mu_\rho(p_2), & \text{if } y \in (1, \rho], \\ \max\left(E^{L,M}(p_2), H^{L,M}\left(\dfrac{du^L}{dy}(-1^-) + p_2 e_2, \dfrac{du^M}{dy}(-1^+) + p_2 e_2\right)\right) = \mu_\rho(p_2), \\ \max\left(E^{M,R}(p_2), H^{M,R}\left(\dfrac{du^M}{dy}(1^-) + p_2 e_2, \dfrac{du^R}{dy}(1^+) + p_2 e_2\right)\right) = \mu_\rho(p_2). \end{cases} \tag{3.4}$$

Exactly as in [3], we can prove the following lemma:

Lemma 3.3. *There is a unique $\mu_\rho(p_2) \in \mathbb{R}$ such that (3.4) admits a bounded solution. For this choice of $\mu_\rho(p_2)$, there exists a solution $y \mapsto \psi_\rho(p_2, y)$ which is Lipschitz continuous with a Lipschitz constant L depending on p_2 only (independent of ρ).*

It is also possible to check that there exists a scalar constant K such that for all real numbers ρ_1 and ρ_2 such that $\rho_1 \le \rho_2$,

$$\mu_{\rho_1}(p_2) \le \mu_{\rho_2}(p_2) \le K.$$

From this property, it is possible to pass to the limit as $\rho \to +\infty$: the effective tangential Hamiltonian $E(p_2)$ is defined by

$$E(p_2) = \lim_{\rho\to\infty} \mu_\rho(p_2). \tag{3.5}$$

3.2.2 The global cell problem

Fixing $p_2 \in \mathbb{R}$, the *global cell-problem* reads

$$\begin{cases} H^L\left(\dfrac{du}{dy}(y)e_1 + p_2e_2\right) = E(p_2), & \text{if } y < -1, \\ H^M\left(\dfrac{du}{dy}(y)e_1 + p_2e_2\right) = E(p_2), & \text{if } y \in (-1,1), \\ H^R\left(\dfrac{du}{dy}(y)e_1 + p_2e_2\right) = E(p_2), & \text{if } y > 1, \\ \max\left(E^{L,M}(p_2), H^{L,M}\left(\dfrac{du}{dy}(-1^-)e_1 + p_2e_2, \dfrac{du}{dy}(-1^+)e_1 + p_2e_2\right)\right) = E(p_2), & \\ \max\left(E^{M,R}(p_2), H^{M,R}\left(\dfrac{du}{dy}(1^-)e_1 + p_2e_2, \dfrac{du^R}{dy}(1^+)e_1 + p_2e_2\right)\right) = E(p_2). & \end{cases} \tag{3.6}$$

Exactly as in [3], we obtain the existence of a solution of the global cell problem by passing to the limit in (3.4) as $\rho \to +\infty$:

Proposition 3.4 (Existence of a global corrector)**.** *For $p_2 \in \mathbb{R}$, there exists $\psi(p_2, \cdot)$ a Lipschitz continuous viscosity solution of (3.6) such that $\psi(p_2, 0) = 0$. For $\eta > 0$, setting $W_\eta(p_2, y) = \eta\psi(p_2, \frac{y}{\eta})$, there exists a sequence η_n such that $W_{\eta_n}(p_2, \cdot)$ converges locally uniformly to a Lipschitz function $y \mapsto W(p_2, y)$, with the same Lipschitz constant as ψ. The function W is a viscosity solution of*

$$H^i\left(\frac{du}{dy_1}(y_1)e_1 + p_2e_2\right) = E(p_2) \quad \text{if } y_1e_1 \in \Omega^i, \tag{3.7}$$

and satisfies $W(p_2, 0) = 0$. Moreover,

$$E(p_2) \geq \max\left\{E_0^L(p_2), E_0^R(p_2)\right\}.$$

3.2.3 Proof of (3.1)

In view of Proposition 3.4, the following numbers are well defined for all $p_2 \in \mathbb{R}$:

$$\overline{\pi}^L(p_2) = \min\left\{q \in \mathbb{R} : H^L(p_2e_2 + qe_1) = H^{-,1,L}(p_2e_2 + qe_1) = E(p_2)\right\}, \tag{3.8}$$

$$\widehat{\pi}^L(p_2) = \max\left\{q \in \mathbb{R} : H^L(p_2e_2 + qe_1) = H^{-,1,L}(p_2e_2 + qe_1) = E(p_2)\right\}, \tag{3.9}$$

$$\overline{\pi}^R(p_2) = \min\left\{q \in \mathbb{R} : H^R(p_2e_2 + qe_1) = H^{+,1,R}(p_2e_2 + qe_1) = E(p_2)\right\}, \tag{3.10}$$

$$\widehat{\pi}^R(p_2) = \max\left\{q \in \mathbb{R} : H^R(p_2e_2 + qe_1) = H^{+,1,R}(p_2e_2 + qe_1) = E(p_2)\right\}. \tag{3.11}$$

From the convexity of the Hamiltonians H^i, we deduce that for $i = L, R$, if $E_0^i(p_2) < E(p_2)$, then $\overline{\pi}^i(p_2) = \widehat{\pi}^i(p_2)$. In this case, we will use the notation

$$\pi^i(p_2) = \overline{\pi}^i(p_2) = \widehat{\pi}^i(p_2). \tag{3.12}$$

Lemma 3.5. *For any $p_2 \in \mathbb{R}$:*

- *if $E(p_2) > E_0^R(p_2)$, then $\psi(p_2, \cdot)$ is affine in the interval $(1, +\infty)$ and $\partial_y\psi(p_2, y) = \pi^R(p_2)$*
- *if $E(p_2) > E_0^L(p_2)$, then $\psi(p_2, \cdot)$ is affine in the interval $(-\infty, -1)$ and $\partial_y\psi(p_2, y) = \pi^L(p_2)$.*

Proof. If $E(p_2) > E_0^R(p_2)$, we prove, exactly as Proposition 2.9 that there exist $\rho^* = \rho^*(p_2) > 0$ and $M^* = M^*(p_2) \in \mathbb{R}$ such that, for all $y \in [\rho^*, +\infty)$, $h_1 \geq 0$,

$$\psi(p_2, y + h_1 e_1) - \psi(p_2, y) \geq \pi^R(p_2)h_1 - M^*. \tag{3.13}$$

From (3.13), classical arguments on viscosity solutions of one-dimensional equations with convex Hamiltonians yield the desired result for $y > 1$. The same kind of arguments are used for $y < -1$. □

Proposition 3.6. *The constant $E(p_2)$ defined in (3.5) satisfies (3.1).*

Proof. From the fourth and fifth equations in (3.6), we see that $E(p_2) \geq \max(E^{L,M}(p_2), E^{M,R}(p_2))$. Moreover, we know that $E^{M,R}(p_2) \geq E_0^{M,R}(p_2) = \max(E_0^M(p_2), E_0^R(p_2))$ from Proposition 2.7. Similarly $E^{L,M}(p_2) \geq E_0^{L,M}(p_2) = \max(E_0^L(p_2), E_0^M(p_2))$.
We make out two main cases:

1. If $E(p_2) = E_0^M(p_2)$, then using the observations above, we get that $E(p_2) = E^{L,M}(p_2) = E^{M,R}(p_2))$, which implies (3.1).
2. If $E(p_2) > E_0^M(p_2)$, then we can define two real numbers $\pi^{M,-} < \pi^{M,+}$ such that

$$\begin{aligned} H^M(\pi^{M,-}e_1 + p_2 e_2) = H^{-,1,M}(\pi^{M,-}e_1 + p_2 e_2) = E(p_2), \\ H^M(\pi^{M,+}e_1 + p_2 e_2) = H^{+,1,M}(\pi^{M,+}e_1 + p_2 e_2) = E(p_2), \end{aligned}$$

   and one and only one of the following three assertions is true:

   (a) the function $\psi(p_2, \cdot)$ defined in Proposition 3.4 is affine in $(-1, 1)$ with slope $\pi^{M,-}$: in this case, $H^{+,1,M}(\partial_y \psi(p_2, 1^-)e_1 + p_2 e_2) < E(p_2)$: using the fifth equation in (3.6), we deduce that

$$\max\left(E^{M,R}(p_2), H^{-,1,R}\left(\frac{d\psi}{dy}(1^+)e_1 + p_2 e_2\right)\right) = E(p_2);$$

   there are two subcases:

   i. if $E(p_2) = E_0^R(p_2)$, then using the fact that $E^{M,R}(p_2) \geq E_0^R(p_2)$, we get that $E^{M,R}(p_2) = E(p_2)$
   ii. if $E(p_2) > E_0^R(p_2)$, then as a consequence of Lemma 3.5, we see that $H^{-,1,R}(\partial_y \psi(p_2, 1^+)e_1 + p_2 e_2) < E(p_2)$, which again implies that $E^{M,R}(p_2) = E(p_2)$.

   Therefore $E^{M,R}(p_2) = E(p_2)$, and since $E^{L,M}(p_2) \leq E(p_2)$ from the fourth equation in (3.6), we obtain (3.1).

   (b) $\psi(p_2, \cdot)$ is affine in $(-1, 1)$ with slope $\pi^{M,+}$. The same arguments as in the previous case yield that $E^{L,M}(p_2) = E(p_2)$ then (3.1).

   (c) $\psi(p_2, \cdot)$ is piecewise affine in $(-1, 1)$, with the slope $\pi^{M,+}$ in $(-1, c)$ and the slope $\pi^{M,-}$ in $(c, 1)$, for some c with $|c| < 1$. Hence, $H^{-,1,M}(\partial_y \psi(p_2, -1^+)e_1 + p_2 e_2) < E(p_2)$ and $H^{+,1,M}(\partial_y \psi(p_2, 1^-)e_1 + p_2 e_2) < E(p_2)$: therefore,

$$\max\left(E^{L,M}(p_2), H^{+,1,L}(\partial_y \psi(p_2, -1^-)e_1 + p_2 e_2)\right) = E(p_2), \tag{3.14}$$

$$\max\left(E^{M,R}(p_2), H^{-,1,R}(\partial_y \psi(p_2, 1^+)e_1 + p_2 e_2)\right) = E(p_2). \tag{3.15}$$

   i. If $E(p_2) = E_0^R(p_2)$, then the very first observation in the proof imply that $E^{M,R}(p_2) = E(p_2)$.

ii. If $E(p_2) > E_0^R(p_2)$, then from Lemma 3.5, $H^{-,1,R}(\partial_y\psi(p_2,1^+)e_1+p_2e_2) < E(p_2)$, and (3.15) yields that $E^{M,R}(p_2) = E(p_2)$.

Similarly, using (3.14), we find that $E^{L,M}(p_2) = E(p_2)$, so $E^{L,M}(p_2) = E^{M,R}(p_2) = E(p_2)$, which yields (3.1).

□

Remark 3.7. *We have actually proved that $E(p_2)$ defined by (3.1) is the unique constant such that the global cell problem (3.6) has a Lipschitz continuous solution.*

3.2.4 End of the proof of Theorem 3.1

The end of the proof of Theorem 3.1 is completely similar to the proof of the main result in [3]. The general method was first proposed in [17] and uses Evans' method of perturbed test-functions with the particular test-functions proposed in [18], to which the correctors found in Proposition 3.4 are associated. For brevity, we do not repeat the proof here.

4 Simultaneous passage to the limit as $\eta = \varepsilon \to 0$

4.1 Main result

We now turn our attention to the case when $\eta = \varepsilon$. We are interested in the asymptotic behavior of the sequence $v_{\varepsilon,\varepsilon}$ as $\varepsilon \to 0$. The main result tells us that the limit is the same function v as the one defined in Theorem 3.1, i.e. obtained by two successive passages to the limit in $v_{\eta,\varepsilon}$, first by letting $\varepsilon \to 0$ then $\eta \to 0$.

Theorem 4.1. *As $\varepsilon \to 0$, $v_{\varepsilon,\varepsilon}$ converges locally uniformly to v, the unique bounded viscosity solution of (1.20)-(1.22), with $H^{L,R}$ given by (1.23) and the effective flux-limiter $E(p_2)$ given by (3.1).*

Remark 4.2. *Note that the same convergence result holds for the sequence $v_{\varepsilon,\varepsilon^q}$ where q is any positive number.*

4.2 Correctors

Let us consider the problem in the original geometry dilated by the factor $1/\varepsilon$. Defining $\Omega^L_{1,\varepsilon}, \Omega^R_{1,\varepsilon}, \Gamma_{1,\varepsilon}$ and $H_{\Gamma_{1,\varepsilon}}$ as in § 1.1 and § 1.2.2, and recalling that $Y^\rho = \{y \in \mathbb{R}^2 : |y_1| < \rho\}$, we consider the truncated cell problem

$$\begin{cases} H^L(Du(y)+p_2e_2) & \le E_{\varepsilon,\rho}(p_2) & \text{if } y \in \Omega^L_{1,\varepsilon} \cap Y^\rho, \\ H^L(Du(y)+p_2e_2) & \ge E_{\varepsilon,\rho}(p_2) & \text{if } y \in \Omega^L_{1,\varepsilon} \cap \overline{Y^\rho}, \\ H^R(Du(y)+p_2e_2) & \le E_{\varepsilon,\rho}(p_2) & \text{if } y \in \Omega^R_{1,\varepsilon} \cap Y^\rho, \\ H^R(Du(y)+p_2e_2) & \ge E_{\varepsilon,\rho}(p_2) & \text{if } y \in \Omega^R_{1,\varepsilon} \cap \overline{Y^\rho}, \\ H_{\Gamma_{1,\varepsilon}}(y, Du^L(y)+p_2e_2, Du^R(y)+p_2e_2) & = E_{\varepsilon,\rho}(p_2) & \text{if } y \in \Gamma_{1,\varepsilon}, \\ u \text{ is } \varepsilon \text{ periodic w.r.t. } y_2, & & \end{cases} \tag{4.1}$$

where ρ is large enough such that $\Gamma_{1,\varepsilon} \subset\subset Y^\rho$ and the inequations are understood in the sense of viscosity. The following lemma can be proved with the same ingredients as in § 2.3.2:

Lemma 4.3. *There is a unique $E_{\varepsilon,\rho}(p_2) \in \mathbb{R}$ such that (4.1) admits a viscosity solution. For this choice of $E_{\varepsilon,\rho}(p_2)$, there exists a solution $\xi_{\varepsilon,\rho}(p_2,\cdot)$ which is Lipschitz continuous with a Lipschitz constant L depending on p_2 only (independent of ε and ρ).*

As in [4, 3], using the optimal control interpretation of (4.1), it is easy to prove that for a positive K which may depend on p_2 but not on ρ and ε and for all $0 < \rho_1 \le \rho_2$,

$$E_{\varepsilon,\rho_1}(p_2) \le E_{\varepsilon,\rho_2}(p_2) \le K. \tag{4.2}$$

For $p_2 \in \mathbb{R}$, let $E_\varepsilon(p_2)$ be defined by

$$E_\varepsilon(p_2) = \lim_{\rho\to\infty} E_{\varepsilon,\rho}(p_2). \tag{4.3}$$

For a fixed $p_2 \in \mathbb{R}$, the *global cell-problem* reads

$$\begin{cases} H^L(Du(y) + p_2e_2) & = E_\varepsilon(p_2) \quad \text{if } y \in \Omega^L_{1,\varepsilon}, \\ H^R(Du(y) + p_2e_2) & = E_\varepsilon(p_2) \quad \text{if } y \in \Omega^R_{1,\varepsilon}, \\ H_{\Gamma_{1,\varepsilon}}(y, Du^L(y) + p_2e_2, Du^R(y) + p_2e_2) & = E_\varepsilon(p_2) \quad \text{if } y \in \Gamma_{1,\varepsilon}, \\ u \text{ is } \varepsilon \text{ periodic w.r.t. } y_2. \end{cases} \tag{4.4}$$

The following theorem can be obtained by using the same arguments as in § 2.4:

Theorem 4.4. *Let $\xi_{\varepsilon,\rho}(p_2,\cdot)$ be a sequence of uniformly Lipschitz continuous solutions of the truncated cell-problem (4.1) which converges to $\xi_\varepsilon(p_2,\cdot)$ locally uniformly on $\mathbb{R}^2$ as $\rho \to +\infty$. The function $\xi_\varepsilon(p_2,\cdot)$ is a Lipschitz continuous viscosity solution of the global cell-problem (4.4).*

Using the control interpretation of (4.1), we see that $E_\varepsilon(p_2)$ is bounded independently of ε. We may thus suppose that, possibly after the extraction of a subsequence, $\lim_{\varepsilon\to0} E_\varepsilon(p_2) = E(p_2)$. Moreover, if $E(p_2) > E_0^R(p_2)$, then for ε small enough, $E_\varepsilon(p_2) > E_0^R(p_2)$ and we can define $\pi^R(p_2)$ and $\pi^R_\varepsilon(p_2)$ as the unique real numbers such that

$$\begin{aligned} H^R(p_2e_2 + \pi^R(p_2)e_1) &= H^{+,1,R}(p_2e_2 + \pi^R(p_2)e_1) = E(p_2), \\ H^R(p_2e_2 + \pi^R_\varepsilon(p_2)e_1) &= H^{+,1,R}(p_2e_2 + \pi^R_\varepsilon(p_2)e_1) = E_\varepsilon(p_2). \end{aligned}$$

Note that $\pi^R(p_2) = \lim_{\varepsilon\to0} \pi^R_\varepsilon(p_2)$. Then we can prove exactly as Proposition 2.9 that if $E(p_2) > E_0^R(p_2)$, then, there exist $\rho^* = \rho^*(p_2) > 0$ and $M^* = M^*(p_2) \in \mathbb{R}$ such that, for all $\varepsilon > 0$ small enough, for all $(y_1, y_2) \in [\rho^*, +\infty) \times \mathbb{R}$, $h_1 \ge 0$ and $h_2 \in \mathbb{R}$,

$$\xi_\varepsilon(p_2, y + h_1e_1 + h_2e_2) - \xi_\varepsilon(p_2, y) \ge \pi^R_\varepsilon(p_2)h_1 - M^*. \tag{4.5}$$

Of course, the same observations can be made on the left side of the interface: if $E(p_2) > E_0^L(p_2)$, then for ε small enough, $E_\varepsilon(p_2) > E_0^L(p_2)$ and we can define $\pi^L(p_2)$ and $\pi^L_\varepsilon(p_2)$ as the unique real numbers such that

$$\begin{aligned} H^L(p_2e_2 + \pi^L(p_2)e_1) &= H^{-,1,L}(p_2e_2 + \pi^L(p_2)e_1) = E(p_2), \\ H^L(p_2e_2 + \pi^L_\varepsilon(p_2)e_1) &= H^{-,1,L}(p_2e_2 + \pi^L_\varepsilon(p_2)e_1) = E_\varepsilon(p_2). \end{aligned}$$

If $E(p_2) > E_0^L(p_2)$, then, there exist $\rho^* = \rho^*(p_2) > 0$ and $M^* = M^*(p_2) \in \mathbb{R}$ such that, for all $\varepsilon > 0$ small enough, for all $(y_1, y_2) \in (-\infty, -\rho^*] \times \mathbb{R}$, $h_1 \ge 0$ and $h_2 \in \mathbb{R}$,

$$\xi_\varepsilon(p_2, y + h_1e_1 + h_2e_2) - \xi_\varepsilon(p_2, y) \le \pi^L_\varepsilon(p_2)h_1 + M^*. \tag{4.6}$$

Using similar arguments to those in § 2.4 and Remark 3.7, we obtain the following results:

Theorem 4.5. *Let (ε_n) be a sequence of positive numbers tending to 0 such that the solution of (4.4) $(\xi_{\varepsilon_n}(p_2,\cdot), E_{\varepsilon_n}(p_2))$ satisfy: $E_{\varepsilon_n}(p_2) \to E(p_2)$ and $\xi_{\varepsilon_n}(p_2,\cdot) \to \xi(p_2,\cdot)$ locally uniformly. Then $\xi(p_2,\cdot)$ depends on y_1 only and is Lipchitz continuous, $(\xi(p_2,\cdot), E(p_2))$ is a solution of (3.6) and $E(p_2) = \max(E^{L,M}(p_2), E^{M,R}(p_2))$.*

Corollary 4.6. *As $\varepsilon \to 0$, the whole sequence $E_\varepsilon(p_2)$ tends to $\max(E^{L,M}(p_2), E^{M,R}(p_2))$.*

The construction of the function ξ has been useful to characterize $E(p_2)$ by (3.1). However, since ξ is the solution of (3.6), it is not directly connected to the oscillating interface $\Gamma_{\varepsilon,\varepsilon}$, and ξ will not be useful when applying Evans' method to prove Theorem 4.1. The function used in Evans' method will rather be ξ_ε and the following proposition will therefore be useful. We skip its proof, because it is very much similar to that of Proposition 2.10.

Proposition 4.7. *For any $p_2 > 0$, there exists a sequence (ε_n) of positive numbers tending to 0 such that $y \mapsto \varepsilon_n \xi_{\varepsilon_n}(p_2, \frac{y}{\varepsilon_n})$ converges locally uniformly to $y \mapsto W(p_2, y)$. The function $W(p_2, \cdot)$ does not depend on y_2 and is a Lipschitz continuous viscosity solution of (3.7). By adding a same constant to $W(p_2, \cdot)$ and $\xi_{\varepsilon_n}(p_2, \cdot)$, one can impose that $W(p_2, 0) = 0$. Moreover,*

$$-\widehat{\pi}^L(p_2)(y_1)^- + \overline{\pi}^R(p_2)(y_1)^+ \le W(p_2, y) \le -\overline{\pi}^L(p_2)(y_1)^- + \widehat{\pi}^R(p_2)(y_1)^+, \tag{4.7}$$

where for $i = L, R$, the values $\overline{\pi}^i(p_2)$ and $\widehat{\pi}^L(p_2)$ are defined in (3.8)-(3.11).

4.3 Proof of Theorem 4.1

The proof of Theorem 4.1 is similar to that of Theorem 2.1. Let us consider the relaxed semi-limits

$$\overline{v}(z) = \limsup_{\varepsilon}{}^* v_{\varepsilon,\varepsilon}(z) = \limsup_{z' \to z, \varepsilon \to 0} v_{\varepsilon,\varepsilon}(z') \quad \text{and} \quad \underline{v}(z) = \liminf_{\varepsilon}{}_* v_{\varepsilon,\varepsilon}(z) = \liminf_{z' \to z, \varepsilon \to 0} v_{\varepsilon,\varepsilon}(z'). \tag{4.8}$$

Note that $\overline{v}$ and $\underline{v}$ are well defined, since $(v_{\varepsilon,\varepsilon})_\varepsilon$ is uniformly bounded. It is classical to check that the functions $\overline{v}(z)$ and $\underline{v}(z)$ are respectively a bounded subsolution and a bounded supersolution in Ω^i, $i = L, R$, of

$$\lambda u(z) + H^i(Du(z)) = 0. \tag{4.9}$$

We will prove that $\overline{v}$ and $\underline{v}$ are respectively a subsolution and a supersolution of (1.24). From the comparison theorem proved in [9, 18, 25], this will imply that $\overline{v} = \underline{v} = v = \lim_{\varepsilon \to 0} v_{\varepsilon,\varepsilon}$. We just have to check the transmission condition (1.22).
Take $\bar{z} = (0, \bar{z}_2) \in \Gamma$. It is possible to use the counterpart of Theorem 2.12 because $\overline{v}$ is Lipschitz continuous, see Remark 2.13.
Take a test-function of the form

$$\varphi(z + te_1) = \psi(z) + \left(\overline{\pi}^R\left(\partial_{z_2}\psi(\bar{z})\right) 1_{t>0} + \widehat{\pi}^L\left(\partial_{z_2}\psi(\bar{z})\right) 1_{t<0}\right) t, \quad \forall z \in \Gamma, t \in \mathbb{R}, \tag{4.10}$$

for a $\mathcal{C}^1$ function $\psi : \Gamma \to \mathbb{R}$, such that $\overline{v} - \varphi$ has a strict local maximum at $\bar{z}$ and that $\overline{v}(\bar{z}) = \varphi(\bar{z})$. Let us argue by contradiction and assume that

$$\lambda\varphi(\bar{z}) + \max\left(E(\partial_{z_2}\varphi(\bar{z})), H^{L,R}(D\varphi^L(\bar{z}), D\varphi^R(\bar{z}))\right) = \theta > 0. \tag{4.11}$$

From (4.10), we see that $H^{L,R}(D\varphi^L(\bar{z}), D\varphi^R(\bar{z})) \le E(\partial_{z_2}\varphi(\bar{z}))$ and (4.11) is equivalent to

$$\lambda\psi(\bar{z}) + E(\partial_{z_2}\psi(\bar{z})) = \theta > 0. \tag{4.12}$$

Step 1 Consider a sequence $(\xi_{\varepsilon_n})_n$ as in Proposition 4.7, that we note (ξ_ε) for short. We claim that for ε and r small enough, the function φ^ε:

$$\varphi^\varepsilon(z) = \psi(z_2 e_2) + \varepsilon \xi_\varepsilon(\partial_{z_2}\psi(\bar z), \frac{z}{\varepsilon})$$

is a viscosity supersolution of

$$\left\{ \begin{array}{rcll} \lambda\varphi^\varepsilon(z) + H^i(D\varphi^\varepsilon(z)) & \geq & \frac{\theta}{2} & \text{if } z \in \Omega^i_{\varepsilon,\varepsilon} \cap B(\bar z, r),\ i = L, R, \\ \lambda\varphi^\varepsilon(z) + H_{\Gamma_{\varepsilon,\varepsilon}}(z, D\left(\varphi^\varepsilon\right)^L(z), D\left(\varphi^\varepsilon\right)^R(z)) & \geq & \frac{\theta}{2} & \text{if } z \in \Gamma_{\varepsilon,\varepsilon} \cap B(\bar z, r). \end{array} \right. \tag{4.13}$$

Indeed, if ν is a test-function in $\mathcal{R}_{\varepsilon,\varepsilon}$ such that $\varphi^\varepsilon - \nu$ has a local minimum at $z^\star \in B(\bar z, r)$, then, from the definition of φ^ε, $y \mapsto \xi_\varepsilon(\partial_{z_2}\psi(\bar z), y) - \frac{1}{\varepsilon}(\nu(\varepsilon y) - \psi(\varepsilon y_2 e_2))$ has a local minimum at $\frac{z^\star}{\varepsilon}$. If $\frac{z^\star}{\varepsilon} \in \Omega^i_{1,\varepsilon}$, for $i = L$ or R, then , from (4.4), $H^i(D\nu(z^\star) - \partial_{z_2}\psi(z_2^\star e_2)e_2 + \partial_{z_2}\psi(\bar z)e_2) \geq E_\varepsilon(\partial_{z_2}\psi(\bar z))$. From the regularity properties of H^i,

$$H^i(D\nu(z^\star) - \partial_{z_2}\psi(z_2^\star e_2)e_2 + \partial_{z_2}\psi(\bar z)e_2) = H^i(D\nu(z^\star)) + o_{r\to 0}(1),$$

thus, using also the convergence of E_ε to E,

$$\lambda\varphi^\varepsilon(z^\star) + H^i(D\nu(z^\star)) \geq E(\partial_{z_2}\psi(\bar z)) + \lambda\left(\psi(z_2^\star e_2) + \varepsilon\xi_\varepsilon(\partial_{z_2}\psi(\bar z), \frac{z^\star}{\varepsilon})\right) + o_{r\to 0}(1) + o_{\varepsilon\to 0}(1).$$

From (4.12), this implies that

$$\lambda\varphi^\varepsilon(z^\star) + H^i(D\nu(z^\star)) \geq \theta + \lambda\varepsilon\xi_\varepsilon(\partial_{z_2}\psi(\bar z), \frac{z^\star}{\varepsilon}) + o_{r\to 0}(1) + o_{\varepsilon\to 0}(1).$$

From the Lipschitz continuity of $\xi_\varepsilon(\partial_{z_2}\psi(\bar z), \cdot)$ with a constant independent of ε, we get that $\varepsilon\xi_\varepsilon(\partial_{z_2}\psi(\bar z), \frac{z^\star}{\varepsilon}) = \varepsilon\xi_\varepsilon(\partial_{z_2}\psi(\bar z), \frac{\bar z}{\varepsilon}) + o_{r\to 0}(1)$. Moreover it is easy to check that $\varepsilon\xi_\varepsilon(\partial_{z_2}\psi(\bar z), \frac{\bar z}{\varepsilon}) = o_{\varepsilon\to 0}(1)$. Therefore, for r and ε small enough, $\lambda\varphi^\varepsilon(z^\star) + H^i(D\nu(z^\star)) \geq \theta/2$.
If $\frac{z^\star}{\varepsilon} \in \Gamma_{1,\varepsilon}$, then we have

$$H^{+,L}_{\Gamma_{1,\varepsilon}}(\frac{z^\star}{\varepsilon}, D\nu^L(z^\star) - \partial_{z_2}\psi(z_2^\star e_2)e_2 + \partial_{z_2}\psi(\bar z)e_2,) \geq E_\varepsilon(\partial_{z_2}\psi(\bar z))$$

or

$$H^{+,R}_{\Gamma_{1,\varepsilon}}(\frac{z^\star}{\varepsilon}, D\nu^R(z^\star) - \partial_{z_2}\psi(z_2^\star e_2)e_2 + \partial_{z_2}\psi(\bar z)e_2,) \geq E_\varepsilon(\partial_{z_2}\psi(\bar z))$$

Since the Hamiltonians $H^{\pm,i}_{\Gamma_{1,\varepsilon}}$ enjoys the same regularity properties as $H^{\pm i}$, it is possible to use the same arguments as in the case when $\frac{z^\star}{\varepsilon} \in \Omega^i_{1,\varepsilon}$. For r and ε small enough,

$$\lambda\varphi^\varepsilon(z^\star) + H_{\Gamma_{\varepsilon,\varepsilon}}(z^\star, D\left(\varphi^\varepsilon\right)^L(z^\star), D\left(\varphi^\varepsilon\right)^R(z^\star)) \geq \frac{\theta}{2}.$$

The claim that φ^ε is a supersolution of (4.13) is proved.

Step 2 Let us prove that there exist some positive constants $K_r > 0$ and $\varepsilon_0 > 0$ such that

$$v_{\varepsilon,\varepsilon}(z) + K_r \leq \varphi^\varepsilon(z), \quad \forall z \in \partial B(\bar z, r),\ \forall \varepsilon \in (0, \varepsilon_0). \tag{4.14}$$

Indeed, since $\overline{v} - \varphi$ has a strict local maximum at $\bar z$ and since $\overline{v}(\bar z) = \varphi(\bar z)$, there exists a positive constant $\tilde K_r > 0$ such that $\overline{v}(z) + \tilde K_r \leq \varphi(z)$ for any $z \in \partial B(\bar z, r)$. Since $\overline{v} = \limsup_\varepsilon{}^* v_{\varepsilon,\varepsilon}$, there exists $\tilde\varepsilon_0 > 0$ such that

$$v_{\varepsilon,\varepsilon}(z) + \frac{\tilde K_r}{2} \leq \varphi(z) \quad \text{for any } 0 < \varepsilon < \tilde\varepsilon_0 \text{ and } z \in \partial B(\bar z, r). \tag{4.15}$$

On the other hand, from Proposition 4.7,

$$\psi(z_2e_2)+W(\partial_{z_2}\psi(\bar z),z)\geq \psi(z_2e_2)+\left(\overline{\pi}^R(\partial_{z_2}\psi(\bar z))1_{z_1>0}+\widehat{\pi}^L(\partial_{z_2}\psi(\bar z))1_{z_1<0}\right)z_1=\varphi(z). \quad (4.16)$$

Moreover, $z\mapsto\varphi^\varepsilon(z)$ converges locally uniformly to $z\mapsto\psi(z_2e_2)+W(\partial_{z_2}\psi(\bar z),z)$ as ε tends to 0. By collecting the latter observation, (4.16) and (4.15), we get (4.14) for some constants $K_r>0$ and $\varepsilon_0>0$.

Step 3 From the previous steps, we find by comparison that for r and ε small enough,

$$v_{\varepsilon,\varepsilon}(z)+K_r\leq\varphi^\varepsilon(z)\qquad\forall z\in B(\bar z,r).$$

Taking the lim sup as $z=\bar z$ and $\varepsilon\to0$, we obtain

$$\overline{v}(\bar z)+K_r\leq\psi(\bar z)=\varphi(\bar z)=\overline{v}(\bar z),$$

which cannot happen. The proof is completed. □

A Proofs of Propositions 2.9 and 2.10

Lemma A.1 (Control of slopes on the truncated domain)**.** *With $E^{M,R}$ and E_0^R respectively defined in (2.27) and (2.20), let $p_2\in\mathbb{R}$ be such that $E^{M,R}(p_2)>E_0^R(p_2)$. There exists $\rho^*=\rho^*(p_2)>0$, $\delta^*=\delta^*(p_2)>0$, $m(p_2,\cdot):[\rho^*,+\infty)\times[0,\delta^*]\to\mathbb{R}_+$ satisfying $\lim_{\delta\to0^+}\lim_{\rho\to+\infty}m(p_2,\rho,\delta)=0$ and $M^*=M^*(p_2)$, such that for all $\delta\in(0,\delta^*]$, $\rho\geq\rho^*$, $(y_1,y_2)\in[\rho^*,\rho]\times\mathbb{R}$, $h_1\in[0,\rho-y_1]$ and $h_2\in\mathbb{R}$,*

$$\chi_\rho(p_2,y+h_1e_1+h_2e_2)-\chi_\rho(p_2,y)\geq(\Pi^R(p_2)-m(p_2,\rho,\delta))h_1-M^*, \quad (A.1)$$

where $\Pi^R(p_2)$ is given by (2.39) *and $\chi_\rho(p_2,\cdot)$ is a solution of* (2.26) *given by Lemma 2.4.*

Similarly, let $p_2\in\mathbb{R}$ be such that $E^{M,R}(p_2)>E_0^M(p_2)$. There exists $\rho^*>0$, $\delta^*>0$, $m(p_2,\cdot)$ and M^* as above, such that for all $\delta\in(0,\delta^*]$, $\rho\geq\rho^*$, $(y_1,y_2)\in[-\rho,-\rho^*]\times\mathbb{R}$, $h_1\in[0,\rho+y_1]$ and $h_2\in\mathbb{R}$,

$$\chi_\rho(p_2,y-h_1e_1+h_2e_2)-\chi_\rho(p_2,y)\geq-(\Pi^M(p_2)+m(p_2,\rho,\delta))h_1-M^*. \quad (A.2)$$

Proof. Let us focus on (A.2), since the proof of (A.1) is similar and even simpler. Recall that $\rho\mapsto\lambda_\rho(p_2)$ is nondecreasing and tends to $E^{M,R}(p_2)$ as $\rho\to+\infty$. Choose $\rho^*=\rho^*(p_2)>0$ s.t. $E^{M,R}(p_2)>\lambda_\rho(p_2)>E_0^M(p_2)$ for any $\rho\geq\rho^*$. Then, choose $\delta^*=\delta^*(p_2)>0$ s.t. $\lambda_\rho(p_2)-\delta>E_0^M(p_2)$ for any $\delta\in(0,\delta^*]$ and $\rho\geq\rho^*$.

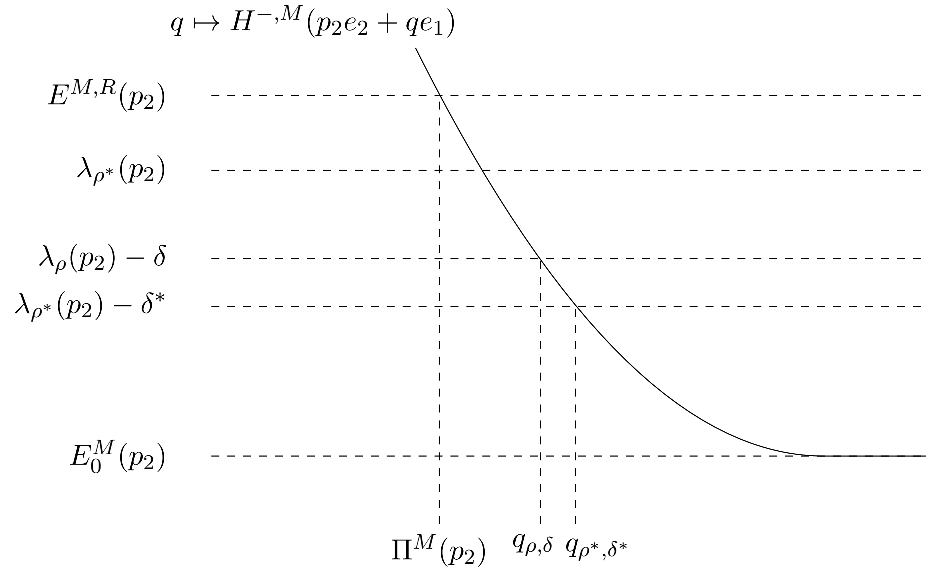

Figure 4: Construction of ρ^*, δ^* and q_δ: here $\lambda_\rho(p_2) - \delta < \lambda_{\rho^*}(p_2)$ but the opposite situation is possible.

Let us fix $\rho > \rho^*$, $\delta \in (0, \delta^*]$ and $\bar{y} = (\bar{y}_1, \bar{y}_2) \in [-\rho, -\rho^*] \times \mathbb{R}$. Consider $y \mapsto \chi_\rho(p_2, y)$ a solution of (2.26) as in Lemma 2.4. The function $\chi_\rho(p_2, \cdot)$ is η-periodic with respect to y_2 and Lipschitz continuous with constant $L = L(p_2)$. Thus, for any $(y_1, y_2) \in \{\bar{y}_1\} \times \mathbb{R}$,

$$\chi_\rho(p_2, y) - \chi_\rho(p_2, \bar{y}) \geq -L\eta.$$

Let us define

$$\tilde{v}(y) = \chi_\rho(p_2, y) - \chi_\rho(p_2, \bar{y}). \tag{A.3}$$

It is a supersolution of

$$\begin{cases} H^L(D\tilde{v}(y) + p_2e_2) & \geq \lambda_\rho(p_2) \quad \text{if } y \in \widetilde{\Omega}_\eta^L \text{ and } -\rho \leq y_1 < \bar{y}_1, \\ H^R(D\tilde{v}(y) + p_2e_2) & \geq \lambda_\rho(p_2) \quad \text{if } y \in \widetilde{\Omega}_\eta^R \text{ and } -\rho \leq y_1 < \bar{y}_1, \\ H_{\widetilde{\Gamma}_\eta}(D\tilde{v}^L(y) + p_2e_2, D\tilde{v}^R(y) + p_2e_2, y) & \geq \lambda_\rho(p_2) \quad \text{if } y \in \widetilde{\Gamma}_\eta \text{ and } -\rho \leq y_1 < \bar{y}_1, \\ \tilde{v}(y) \geq -L\eta, & \quad\quad\quad\quad\quad\; \text{if } y_1 = \bar{y}_1, \\ \tilde{v} \text{ is 1-periodic w.r.t. } y_2/\eta. & \end{cases} \tag{A.4}$$

On the other hand, since $\rho \geq \rho^*$ and $\delta \in (0, \delta^*]$, there exists a unique $q_{\rho,\delta} \in \mathbb{R}$, see Figure 4, such that

$$\lambda_\rho(p_2) - \delta = H^M(p_2e_2 + q_{\rho,\delta}e_1) = H^{-,1,M}(p_2e_2 + q_{\rho,\delta}e_1). \tag{A.5}$$

Observe that $q_{\rho^*,\delta^*} \geq q_{\rho,\delta^*} \geq q_{\rho,\delta} \geq \Pi^M(p_2)$ and that $\lim_{\delta \to 0+} \lim_{\rho \to +\infty} q_{\rho,\delta} = \Pi^M(p_2)$. Choose $m(p_2, \rho, \delta) = q_{\rho,\delta} - \Pi^M(p_2) \geq 0$ and consider the function w:

$$w(y) = q_{\rho,\delta}y_1 + \zeta(q_{\rho,\delta}e_1 + p_2e_2, y_2). \tag{A.6}$$

It satisfies

$$\begin{cases} H^L(Dw(y) + p_2e_2) = H^M(q_{\rho,\delta}e_1 + p_2e_2) = \lambda_\rho(p_2) - \delta, & \text{if } y \in \widetilde{\Omega}_\eta^L \text{ and } -\rho < y_1 < \bar{y}_1, \\ H^R(Dw(y) + p_2e_2) = H^M(q_{\rho,\delta}e_1 + p_2e_2) = \lambda_\rho(p_2) - \delta, & \text{if } y \in \widetilde{\Omega}_\eta^R \text{ and } -\rho < y_1 < \bar{y}_1, \\ H_{\widetilde{\Gamma}_\eta}(Dw^L(y) + p_2e_2, Dw^R(y) + p_2e_2, y) = H^M(q_{\rho,\delta}e_1 + p_2e_2) & = \lambda_\rho(p_2) - \delta \\ & \text{if } y \in \widetilde{\Gamma}_\eta \text{ and } -\rho < y_1 < \bar{y}_1. \end{cases} \tag{A.7}$$

Remark A.2. *Note that w also satisfies*

$$\begin{cases} H^{-,1,L}(Dw(y)+p_2e_2) \le H^M(q_{\rho,\delta}e_1+p_2e_2)=\lambda_\rho(p_2)-\delta, & \text{if } y\in \widetilde{\Omega}^L_\eta \text{ and } y_1=-\rho,\\ H^{-,1,R}(Dw(y)+p_2e_2) \le H^M(q_{\rho,\delta}e_1+p_2e_2)=\lambda_\rho(p_2)-\delta, & \text{if } y\in \widetilde{\Omega}^R_\eta \text{ and } y_1=-\rho,\\ H^{-,1}_{\widetilde{\Gamma}_\eta}(Dw^L(y)+p_2e_2, Dw^R(y)+p_2e_2,y) \le H^M(q_{\rho,\delta}e_1+p_2e_2) & =\lambda_\rho(p_2)-\delta\\ & \text{if } y\in\widetilde{\Gamma}_\eta \text{ and } y_1=-\rho, \end{cases} \tag{A.8}$$

in the sense of viscosity, where $H^{-,1}_{\widetilde{\Gamma}_\eta}(p,q,y)$ is defined for $y_1<0$ and $p,q\in\mathbb{R}^2$ such that $p_1=q_1$ as the nonincreasing part of $p_1\mapsto H_{\widetilde{\Gamma}_\eta}(p_1e_1+p_2e_2,p_1e_1+q_2e_2,y)$.

Moreover, for any $(y_1,y_2)\in\{\bar{y}_1\}\times\mathbb{R}$,

$$w(y)-q_{\rho,\delta}\bar{y}_1\le C=C(p_2)$$

so the function u defined on $[-\rho,\bar{y}_1]\times\mathbb{R}$ by

$$u(y)=w(y)-q_{\rho,\delta}\bar{y}_1-C-L\eta \tag{A.9}$$

is a subsolution of (A.7), (A.8) and is such that $u(\bar{y}_1,\cdot)\le -L\eta$.
By a comparison result whose proof is sketched below, for all $y\in[-\rho,\bar{y}_1]\times\mathbb{R}$,

$$v(y)\ge u(y)\ge(\Pi^M(p_2)+m(\rho,\delta))(y_1-\bar{y}_1)-M^*, \tag{A.10}$$

where M^* is a constant depending only of x and p_2. This is the desired result. There remains to prove the comparison result.

Proof of (A.10) Call $m=\max_{-\rho\le y_1\le\bar{y}_1}(u(y)-v(y))$ and assume by contradiction that $m>0$. Then, since $u(\bar{y}_1,\cdot)<v(\bar{y}_1,\cdot)$, the maximum m is achieved at some point z such that $z_1<\bar{y}_1$. We make out three cases:

1. If $z_1>-\rho$, then we can reproduce the arguments of Imbert and Monneau contained in [18, Appendix 2] and find a contradiction, (note that in the region $-\rho<y_1\le\bar{y}_1$, the interface $\widetilde{\Gamma}_\eta$ is made of straight lines, so the arguments in [18] can be applied in a straightforward manner). Alternatively, it is possible to use the different methods proposed in either [9] or in [10]. It is also possible to use the arguments contained in the very recent work of Lions and Souganidis [24]. We therefore skip this part of the proof.

2. if $z_1=-\rho$ and $z\notin\widetilde{\Gamma}_\eta$, then after a suitable localization, we can apply the now classical arguments of Soner for state constrained boundary conditions and reach a contradiction, see [26, 27, 12, 7]. We also skip the details for brevity.

3. We will thus focus on the case when the maximum is reached at $z\in\widetilde{\Gamma}_\eta$ such that $z_1=-\rho$, because it contains additional difficulties.

We will make the following steps:

1. Localize around z: in the domain of interest, the interface will be made of only one straight line. Moreover, it will be convenient to modify the Hamiltonians for large values of q, which is always possible since u and v are Lipschitz continuous.

2. Recall the definition of the vertex test-function of Imbert-Monneau, see [18], which will be named $G^{\gamma,z}(x,y)$ below. This function will play the role of the penalty term $|x-y|^2$ in the classical arguments consisting of doubling the variables when there is no interface.

3. Adapt Soner's arguments for state constrained boundary condition to the present case. In the arguments consisting of doubling the variables, we will use Soner's ideas to ensure that the viscosity inequalities for u can be written, i.e. that the maximum point $\hat{x}$ be such that $\hat{x}_1 > -\rho$.

Step 1: localization and modification of the Hamiltonians We are going to localize the problem around z: near z, $\widetilde{\Gamma}_\eta$ coincides with a straight line that we name Δ. Changing the coordinates if necessary, we can assume that $\Delta = \{y : y_2 = 0\}$, so $z = (-\rho, 0)$. It is not restrictive to assume that for $r > 0$ small enough, $\Omega_\eta^R \cap \overline{B}(z,r) = \{y : y_2 > 0\} \cap \overline{B}(z,r)$ and that $\Omega_\eta^L \cap \overline{B}(z,r) = \{y : y_2 < 0\} \cap \overline{B}(z,r)$. Therefore, the Hamiltonian is H^R in $\{y : y_2 > 0\} \cap \overline{B}(z,r)$ and H^L in $\{y : y_2 < 0\} \cap \overline{B}(z,r)$, so it does not depend on y_1. In what follows, we will always suppose that $r > 0$ is small enough so that $B(z,r) \subset \{y_1 < \bar{y}_1\}$ and that we are in the situation described above.
Moreover, noting that u and v are both Lipschitz continuous with a constant L which may depend on p_2 but not on ρ and η, we can modify the Hamiltonians H^L and H^R in such a way:

- $q \mapsto H^L(p_2 e_2 + q)$ and $q \mapsto H^R(p_2 e_2 + q)$ are kept unchanged in the ball $|q| \le 2L$
- $q \mapsto H^L(p_2 e_2 + q)$ and $q \mapsto H^R(p_2 e_2 + q)$ become second order polynomials in q in a neighborhood of $|q| = +\infty$

For that, it is enough to replace H^i by $q \mapsto \max(H^i(p_2 e_2 + q), a|q|^2 - b)$ for well chosen positive constants a and b.
Let us name $\mathbb{H}^i$, $i = L, R$ the modified Hamiltonians. With the usual notations, we see that v is a supersolution of

$$\begin{cases} \mathbb{H}^L(Dv(y)) & \ge \lambda_\rho(p_2) \quad \text{if } y_1 \ge -\rho,\ y_2 < 0,\ \text{and } y \in \overline{B}(z,r), \\ \mathbb{H}^R(Dv(y)) & \ge \lambda_\rho(p_2) \quad \text{if } y_1 \ge -\rho,\ y_2 > 0,\ \text{and } y \in \overline{B}(z,r), \\ \max\{\mathbb{H}^{+,2,L}(Dv^L(y)), \mathbb{H}^{-,2,R}(Dv^R(y))\} & \ge \lambda_\rho(p_2) \quad \text{if } y_1 \ge -\rho,\ y_2 = 0,\ \text{and } y \in \overline{B}(z,r), \end{cases} \tag{A.11}$$

and that u is a subsolution of

$$\begin{cases} \mathbb{H}^L(Du(y)) & \le \lambda_\rho(p_2) - \delta \quad \text{if } y_1 > -\rho,\ y_2 < 0,\ \text{and } y \in \overline{B}(z,r), \\ \mathbb{H}^R(Du(y)) & \le \lambda_\rho(p_2) - \delta \quad \text{if } y_1 > -\rho,\ y_2 > 0,\ \text{and } y \in \overline{B}(z,r), \\ \max\{\mathbb{H}^{+,2,L}(Du^L(y)), \mathbb{H}^{-,2,R}(Du^R(y))\} & \le \lambda_\rho(p_2) - \delta \quad \text{if } y_1 > -\rho,\ y_2 = 0,\ \text{and } y \in \overline{B}(z,r). \end{cases} \tag{A.12}$$

For brevity, we make an abuse of notation and rewrite the three inequalities in (A.11) and (A.12) as follows:

$$\begin{aligned} &\mathbb{H}(y_2, Dv) \ge \lambda_\rho(p_2) \quad \text{for } -\rho \le y_1 \text{ and } y \in \overline{B}(z,r), \\ &\mathbb{H}(y_2, Du) \le \lambda_\rho(p_2) - \delta \quad \text{for } -\rho < y_1 \text{ and } y \in \overline{B}(z,r), \end{aligned} \tag{A.13}$$

where for any $y \in \mathbb{R}^2$,

$$\mathbb{H}(y_2, p) = \mathbb{H}^R(p) \quad \text{if } y_2 > 0, \tag{A.14}$$

$$\mathbb{H}(y_2, p) = \mathbb{H}^L(p) \quad \text{if } y_2 < 0, \tag{A.15}$$

$$\mathbb{H}(y_2, (p^L, p^R)) = \max\{\mathbb{H}^{+,2,L}(p^L), \mathbb{H}^{-,2,R}(p^R)\} \text{ if } y_2 = 0. \tag{A.16}$$

Step 2: the test-function of Imbert-Monneau Following [18, Theorem 3.1], we are going to use the so-called vertex test-function at z: for γ, $0 < \gamma < 1$, there exists a function $G^{\gamma,z}$: $\mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ with the following properties:

1. (Regularity)
$$G^{\gamma,z} \in \mathcal{C}(\mathbb{R}^2 \times \mathbb{R}^2) \text{ and } \begin{cases} G^{\gamma,z}(X,\cdot) \in \mathcal{R} & \text{for all} \quad X \in \mathbb{R}^2, \\ G^{\gamma,z}(\cdot,Y) \in \mathcal{R} & \text{for all} \quad Y \in \mathbb{R}^2 \end{cases}$$
where $\mathcal{R}$ is the set of continuous functions on $\mathbb{R}^2$ whose restrictions to $\mathbb{R} \times [0, \pm\infty)$ are $\mathcal{C}^1$. If $f \in \mathcal{R}$ and $x \in \Delta$, $Df(x)$ denotes the pair $(Df^L(x), Df^R(x) \in \mathbb{R}^2 \times \mathbb{R}^2$.

2. (Bound from below) $G^{\gamma,z} \geq 0 = G^{\gamma,z}(z,z)$

3. (Compatibility condition on the diagonal) For all $X \in \mathbb{R}^2$,
$$0 \leq G^{\gamma,z}(X,X) = G^{\gamma,z}(X,X) - G^{\gamma,z}(z,z) \leq \gamma \tag{A.17}$$

4. (Compatibility condition on the gradients) For all $X, Y \in \mathbb{R}^2$ and $K > 0$ with $|X-Y| \leq K$,
$$\mathbb{H}(Y_2, -D_Y G^{\gamma,z}(X,Y)) - \mathbb{H}(X_2, D_X G^{\gamma,z}(X,Y)) \leq \omega_{C_K}(\gamma C_K) \tag{A.18}$$
with C_K given in (A.20) below and ω_{C_K} is a modulus of continuity defined on $[0, C_K]$. Here we have used the notations given in (A.14)-(A.16), so if $Y_2 = 0$, $D_Y G^{\gamma,z}(X,Y)$ is a pair of vectors in $\mathbb{R}^2$ with the same first component.

5. (Superlinearity) There exists $g : [0, +\infty) \to \mathbb{R}$ nondecreasing and such that for all $X, Y \in \mathbb{R}^2$,
$$g(|X-Y|) \leq G^{\gamma,z}(X,Y) \tag{A.19}$$
and $a \mapsto g(a)$ can be chosen to be quadratic ($g(a) = c_1 a^2 + c_2$) for a large enough.

6. (Gradient bounds) For all $K > 0$, there exists $C_K > 0$ independent of γ, such that for all $X, Y \in \mathbb{R}^2$
$$|X-Y| \leq K \quad \Rightarrow \quad |G^{\gamma,z}_X(X,Y)| + |G^{\gamma,z}_Y(X,Y)| \leq C_K. \tag{A.20}$$

Remark A.3. *The fact that $g(a)$ can be chosen quadratic for a large enough comes from the fact that $\mathbb{H}^i(q)$ are second order polynomials in $|q|$ for $|q|$ large enough, see the proof of Proposition 3.3, step 4, in [18, §3.4].*

Step 3: doubling the variables Let us introduce
$$m_{\tau,\gamma} = \max_{\substack{x, y \in \overline{B(z,r)}, \\ -\rho \leq x_1, y_1}} \left(u(x) - v(y) - G^{\gamma,z}_\tau(x, y + \kappa(\tau)e_1) - \phi(x)\right), \tag{A.21}$$
where
$$G^{\gamma,z}_\tau(x,y) = \tau G^{\gamma,z}\left(z + \frac{x-z}{\tau}, z + \frac{y-z}{\tau}\right) \tag{A.22}$$
and
$$\phi(x) = \frac{1}{2}|x-z|^2. \tag{A.23}$$

Finally $\kappa(\tau)$ is a power of τ with a positive exponent that will be chosen later.
For τ small enough, taking $x = z + \kappa(\tau)e_1$ and $y = z$, we see that

$$\begin{aligned} m_{\tau,\gamma} &\geq u(z+\kappa(\tau)e_1) - v(z) - G_\tau^{\gamma,z}(z+\kappa(\tau)e_1, z+\kappa(\tau)e_1) - \phi(z+\kappa(\tau)e_1) \\ &\geq u(z) - v(z) - L\kappa(\tau) - \tau\gamma - \frac{1}{2}\kappa^2(\tau) \\ &= m - L\kappa(\tau) - \tau\gamma - \frac{1}{2}\kappa^2(\tau) \end{aligned}$$

Hence, there exists $0 < \bar{\gamma} < 1$ and $0 < \bar{\tau}$ such that for all $0 < \gamma < \bar{\gamma}$ and $0 < \tau < \bar{\tau}$, $m_{\tau,\gamma} \geq \frac{m}{2} > 0$.
On the other hand,

$$\begin{aligned} u(x) - v(y) - G_\tau^{\gamma,z}(x, y+\kappa(\tau)e_1) - \phi(x) &\leq u(y) - v(y) + L|x-y| - G_\tau^{\gamma,z}(x, y+\kappa(\tau)e_1) - \phi(x) \\ &\leq m + L|x-y| - G_\tau^{\gamma,z}(x, y+\kappa(\tau)e_1) - \phi(x). \end{aligned}$$

Therefore, if $\hat{x}$ and $\hat{y}$ achieve the maximum in (A.21), then

$$G_\tau^{\gamma,z}(\hat{x}, \hat{y}+\kappa(\tau)e_1) + \phi(\hat{x}) \leq \tau\gamma + L\kappa(\tau) + \frac{1}{2}\kappa^2(\tau) + L|\hat{x}-\hat{y}|.$$

From (A.19), this implies that

$$\begin{aligned} \tau g\left(\frac{|\hat{x}-\hat{y}-\kappa(\tau)e_1|}{\tau}\right) + \phi(\hat{x}) &\leq \tau\gamma + L\kappa(\tau) + \frac{1}{2}\kappa^2(\tau) + L|\hat{x}-\hat{y}| \\ &\leq \tau\gamma + 2L\kappa(\tau) + \frac{1}{2}\kappa^2(\tau) + L\tau\frac{|\hat{x}-\hat{y}-\kappa(\tau)e_1|}{\tau}. \end{aligned} \tag{A.24}$$

Using the superlinear behavior of g at infinity, we see that there exists a constant $C > 0$ independent of γ and τ such that

$$\tau g\left(\frac{|\hat{x}-\hat{y}-\kappa(\tau)e_1|}{\tau}\right) \leq C.$$

If for a subsequence still called τ, $|\hat{x}-\hat{y}-\kappa(\tau)e_1| > 0$, then, setting $d_{\gamma,\tau} = |\hat{x}-\hat{y}-\kappa(\tau)e_1|$, the latter inequality can be written

$$\frac{\tau}{d_{\gamma,\tau}} g\left(\frac{d_{\gamma,\tau}}{\tau}\right) \leq \frac{C}{d_{\gamma,\tau}}.$$

From the quadratic behavior of g away from the origin, we know that there exist two positive constants D and c such that $g(d) \geq cd^2$ for $d > D$. If $d_{\gamma,\tau}/\tau > D$, then $\frac{C}{d_{\gamma,\tau}} \geq \frac{\tau}{d_{\gamma,\tau}} g\left(\frac{d}{\tau}\right) \geq c\frac{d_{\gamma,\tau}}{\tau}$.
We can choose $D = 1/\sqrt{\tau}$ for τ small enough, which yields that $d_{\gamma,\tau}$ is bounded by a quantity of the order of $\sqrt{\tau}$.
We have proved that

1. $m_{\tau,\gamma} > m/2 > 0$ for all $0 < \gamma < \bar{\gamma}$ and $0 < \tau < \bar{\tau}$
2. $|\hat{x}-\hat{y}-\kappa(\tau)e_1| \leq C\sqrt{\tau}$, for a positive constant C independent of γ, $0 < \gamma < \bar{\gamma}$
3. $\lim_{\tau\to 0} |\hat{x}-\hat{y}| = 0$, uniformly in $0 < \gamma < \bar{\gamma}$
4. From (A.24), we see that $\lim_{(\gamma,\tau)\to(0,0)} \hat{x} = z$. Hence, for τ and γ small enough, $\hat{x} \in B(z,r)$ and $\hat{y} \in B(z,r)$.

Moreover, choosing $\kappa(\tau)=\tau^{1/3}$ for example, we find that $\hat{x}_1>-\rho$ for τ small enough. This allows us to write the following viscosity inequalities:

$$\begin{aligned}\mathbb{H}(\hat{x}_2, p_X^{\tau,\gamma}+\hat{x}-z) &\le \lambda_\rho(p_2)-\delta, &(A.25)\\ \mathbb{H}(\hat{y}_2, p_Y^{\tau,\gamma}) &\ge \lambda_\rho(p_2), &(A.26)\end{aligned}$$

(with the notations introduced in (A.14)-(A.16)), where

$$\begin{aligned}p_X^{\tau,\gamma} &= G_X^{\gamma,z}\left(z+\frac{\hat{x}-z}{\tau}, z+\frac{\hat{y}+\kappa(\tau)e_1-z}{\tau}\right), &(A.27)\\ p_Y^{\tau,\gamma} &= -G_Y^{\gamma,z}\left(z+\frac{\hat{x}-z}{\tau}, z+\frac{\hat{y}+\kappa(\tau)e_1-z}{\tau}\right). &(A.28)\end{aligned}$$

1. If $\hat{x}\notin\Delta$, then the coercivity of the Hamiltonians $\mathbb{H}^L$ and $\mathbb{H}^R$ implies that, for a constant C independent of τ and γ,
$$|p_X^{\tau,\gamma}|\le C. \tag{A.29}$$
Subtracting (A.26) and (A.25) yields that
$$\mathbb{H}(\hat{y}_2, p_Y^{\tau,\gamma})-\mathbb{H}(\hat{x}_2, p_X^{\tau,\gamma}+\hat{x}-z)\ge\delta, \tag{A.30}$$
which is equivalent to
$$\mathbb{H}(\frac{\hat{y}_2}{\tau}, p_Y^{\tau,\gamma})-\mathbb{H}(\frac{\hat{x}_2}{\tau}, p_X^{\tau,\gamma}+\hat{x}-z)\ge\delta.$$
Note that $\frac{\hat{y}_2}{\tau}$ is also the second component of $z+\frac{\hat{y}-z+\kappa(\tau)e_1}{\tau}$ and that $\frac{\hat{x}_2}{\tau}$ is the second component of $z+\frac{\hat{x}-z}{\tau}$. Then, using (A.18) and the fact that $|\hat{x}-\hat{y}-\kappa(\tau)e_1|\le C\sqrt{\tau}$, we see that
$$\mathbb{H}(\frac{\hat{y}_2}{\tau}, p_Y^{\tau,\gamma})-\mathbb{H}(\frac{\hat{x}_2}{\tau}, p_X^{\tau,\gamma})\le\omega_{\frac{C}{\sqrt{\tau}}}(\frac{C}{\sqrt{\tau}}\gamma)$$
or equivalently,
$$\mathbb{H}(\hat{y}_2, p_Y^{\tau,\gamma})-\mathbb{H}(\hat{x}_2, p_X^{\tau,\gamma})\le\omega_{\frac{C}{\sqrt{\tau}}}(\frac{C}{\sqrt{\tau}}\gamma). \tag{A.31}$$
Adding and subtracting $\mathbb{H}(\hat{x}_2, p_X^{\tau,\gamma})$ in (A.30) and using (A.31) yields
$$\mathbb{H}(\hat{x}_2, p_X^{\tau,\gamma})-\mathbb{H}(\hat{x}_2, p_X^{\tau,\gamma}+\hat{x}-z)+\omega_{\frac{C}{\sqrt{\tau}}}(\frac{C}{\sqrt{\tau}}\gamma)\ge\delta.$$
Using the properties of the Hamiltonians and (A.29), we get that, for some constant $\tilde{C}$ independent of τ and γ,
$$\tilde{C}|\hat{x}-z|+\omega_{\frac{C}{\sqrt{\tau}}}(\frac{C}{\sqrt{\tau}}\gamma)\ge\delta.$$
This yields a contradiction by having $\gamma\to0$ then $\tau\to0$.

2. If $\hat{x}\in\Delta$ or equivalently $\hat{x}_2=0$, we see that $\hat{x}-z$ is colinear to e_1 and that $\max\{\mathbb{H}^{+,2,L}((p_X^{\tau,\gamma})^L+\hat{x}-z), \mathbb{H}^{-,2,R}((p_X^{\tau,\gamma})^R+\hat{x}-z)\}\le\lambda_\rho(p_2)-\delta$. This implies that for a constant $C>0$ independent of τ and γ,
$$|p_{X,1}^{\tau,\gamma}|+\max\left(0,-(p_{X,2}^{\tau,\gamma})^R\right)+\max\left(0,(p_{X,2}^{\tau,\gamma})^L\right)\le C. \tag{A.32}$$

where $p_{X,1}^{\tau,\gamma}$ stands for the first coordinate of both $(p_X^{\tau,\gamma})^L$ and $(p_X^{\tau,\gamma})^R$. Then using the arguments of Imbert and Monneau in [19, §5.5], we can find a constant K independent of τ and γ such that

$$\mathbb{H}(0,\bar{p}_X^{\tau,\gamma}) = \mathbb{H}(0,p_X^{\tau,\gamma}) \quad \text{and} \quad \mathbb{H}(0,\bar{p}_X^{\tau,\gamma}+\hat{x}-z) = \mathbb{H}(0,p_X^{\tau,\gamma}+\hat{x}-z),$$

where $\bar{p}_{X,1}^{\tau,\gamma} = p_{X,1}^{\tau,\gamma}$, $(\bar{p}_{X,2}^{\tau,\gamma})^R = \min\left(K,(p_{X,2}^{\tau,\gamma})^R\right)$ and $(\bar{p}_{X,2}^{\tau,\gamma})^L = \max\left(-K,(p_{X,2}^{\tau,\gamma})^L\right)$. Note that we have used the fact that $\hat{x}-z$ is colinear to e_1 and bounded independently of τ and γ. Since $|\bar{p}_X^{\tau,\gamma}| \le C$ for a constant C independent of τ and γ, there exists a constant $\tilde{C}$ such that

$$|\mathbb{H}(0,p_X^{\tau,\gamma}) - \mathbb{H}(0,p_X^{\tau,\gamma}+\hat{x}-z)| = |\mathbb{H}(0,\bar{p}_X^{\tau,\gamma}) - \mathbb{H}(0,\bar{p}_X^{\tau,\gamma}+\hat{x}-z)| \le \tilde{C}|\hat{x}-z|. \quad \text{(A.33)}$$

On the other hand, we have, exactly as above, that

$$|\mathbb{H}(\hat{x}_2,p_X^{\tau,\gamma}) - \mathbb{H}(\hat{y}_2,p_Y^{\tau,\gamma})| = |\mathbb{H}(\frac{\hat{x}_2}{\tau},p_X^{\tau,\gamma}) - \mathbb{H}(\frac{\hat{y}_2}{\tau},p_Y^{\tau,\gamma})| \le \omega_{\frac{C}{\sqrt{\tau}}}(\frac{C}{\sqrt{\tau}}\gamma). \quad \text{(A.34)}$$

Subtracting (A.26) and (A.25), then using (A.33) and (A.34), and letting γ tend to 0 then τ tend to 0 yields the desired contradiction.

□

Proof of Proposition 2.9 The proof follows easily from Lemma A.1 and the local uniform convergence of the sequence $\chi_\rho(p_2,\cdot)$ toward $\chi(p_2,\cdot)$, by letting ρ tend $+\infty$ and δ tend to 0. □

Proof of Proposition 2.10 From Lemma 2.6, we see that $y \mapsto W(p_2,y)$ is Lipschitz continuous w.r.t. y_1 and independent of y_2, and satisfies

$$H^R(\partial_{y_1}W(p_2,y)e_1+p_2e_2) = E^{M,R}(p_2) \quad \text{for a.a. } y_1>\eta. \quad \text{(A.35)}$$

Consider first the case when $E^{M,R}(p_2) > E_0^R(p_2)$; from the convexity and coercivity of H^R, the observations above yield that almost everywhere in y, $\partial_{y_1}W(p_2,y)$ can be either $\Pi^R(p_2)$ (the unique real number such that $H^{+,R}(qe_1+p_2e_2)=E^{M,R}(p_2)$), or the unique real number q (depending on (p_2)) such that $H^{-,R}(qe_1+p_2e_2)=E^{M,R}(p_2)$. Note that $q<\Pi^R(p_2)$. But from Proposition 2.9 and the local uniform convergence of $W_\eta(p_2,\cdot)$ toward $W(p_2,y)$, we see that that for any $y_1>\eta$ and $h_1\ge 0$,

$$W(p_2,y+h_1e_1)-W(p_2,y) \ge \Pi^R(p_2)h_1,$$

which implies that almost everywhere, $\partial_{y_1}W(p_2,y) \ge \Pi^R(p_2) > q$. Therefore, $\partial_{y_1}W(p_2,\cdot) = \Pi^R(p_2)$ for almost all $y_1>\eta$.
In the case when $E^{M,R}(p_2) = E_0^R(p_2)$, we deduce from (A.35) that for almost all $y_1>\eta$, $\overline{\Pi}^R(p_2) \le \partial_{y_1}W(p_2,y) \le \widehat{\Pi}^R(p_2)$.
We have proved (2.42). The proof of (2.43) is identical. Finally, (2.44) comes from (2.42), (2.43) and from the fact that $W(p_2,\eta e_1)=0$. □

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