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Pisano word, tesselation, plane-filling fractal

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Abstract

This paper presents a plane-filling curve with fractal properties, based on a signed version of the Fibonacci word. The development from word to curve involves a tesselation of Kepler triangles which has some remarkable properties of its own.

1 Introduction

Since he encountered the intriguing Fibonacci word fractal [1], now and again the author has wondered whether a plane-filling curve might not be designed in a similar vein. The present paper is the result. The author is convinced the statements in it are correct in essence, even where there is waving of hands or no proof at all. Hopefully, the observations will make up for the lack of mathematical rigor.

The Pisano word, a signed version of the Fibonacci word, is defined in Section 2, and a plane-filling curve based on this word is presented in Sections 3-4. The rationale behind the curve definition is given in Sections 5-7, using a tesselation in which Kepler triangles are attributed to letters in the Pisano word. Variant formulations of the curve are proposed (Section 8) and judged to be less satisfactory. Some remarkable properties of the tesselation and the curve are noted (Section 9). An analysis of frequency and position of letters and subwords in the Pisano word, an expression for the length of the Pisano curve and similar issues may be found in Sections 10-13.

2 Pisano word

The Fibonacci word $\tilde{w}_k$ can be defined via the substitution rule:

$$\tilde{\sigma}: \quad \begin{array}{l} 0 \to 1 \\ 1 \to 10 \end{array} \qquad \text{with} \qquad \tilde{w}_1 = 0\,.$$

The word w_k considered in this paper is a version of the Fibonacci word with associated signs:

$$\sigma: \quad \begin{array}{l} 0^- \to 1^- \\ 0^+ \to 1^+ \\ 1^- \to 1^+0^+ \\ 1^+ \to 1^-0^- \end{array} \qquad \text{with} \qquad w_1 = 0^+\,.$$

The first few instances are

$$\begin{aligned} w_1 &= 0^+\,, \\ w_2 &= 1^+\,, \\ w_3 &= 1^-0^-\,, \\ w_4 &= 1^+0^+1^-\,, \\ w_5 &= 1^-0^-1^+1^+0^+\,. \end{aligned}$$

This word will be referred to as the Pisano word, from Leonardo Pisano, one of the several names by which Fibonacci is known. The length of w_k equals the Fibonacci number F_k,

$$F_0 = 0\,, \quad F_1 = 1\,, \qquad F_k = F_{k-1} + F_{k-2}\,.$$

There are two infinite words. For even k we have

$$w_{\infty,\mathrm{even}} = 1^+0^+1^-1^-0^-1^-0^-1^+1^-0^-1^+1^+0^+1^-0^-1^+\ldots\,,$$

and $w_{\infty,\mathrm{odd}} = \bar{w}_{\infty,\mathrm{even}}$, where the overbar indicates sign reversion.

From $k = 3$ onwards, w_k is the concatenation of w_{k-1} and w_{k-2} with opposite signs. So an alternative definition of the Pisano word is the process

$$w_1 = 0^+\,, \quad w_2 = 1^+\,, \qquad w_k = \bar{w}_{k-1}\bar{w}_{k-2}\,.$$

Defining w_0 as the empty word, the concatenation property extends to $k = 2$.

In Pisano words of sufficient length, subwords occur as opposite pairs. This follows because w_k contains both the subwords and the opposite subwords of w_{k-2}:

$$w_k = \bar{w}_{k-1}\bar{w}_{k-2} = w_{k-2}w_{k-3}\bar{w}_{k-2}\,.$$

There are 10 different two-letter subwords. Because $\bar{w}_{k-1}$ and $\bar{w}_{k-2}$ each contain subwords from earlier levels (up to sign reversion), truly new two-letter subwords in w_k are formed only at the point where $\bar{w}_{k-1}$ and $\bar{w}_{k-2}$ are concatenated. This proceeds as follows:

k	$\bar{w}_{k-1}$	$\bar{w}_{k-2}$	new subword	opposite subword
3	1^-	0^-	$M_3 = 1^-0^-$	$\bar{M}_3 = 1^+0^+$
4	1^+0^+	1^-	$M_4 = 0^+1^-$	$\bar{M}_4 = 0^-1^+$
5	$1^-0^-1^+$	1^+0^+	$M_5 = 1^+1^+$	$\bar{M}_5 = 1^-1^-$
6	$\ldots 0^-$	$1^-\ldots$	$M_6 = 0^-1^-$	$\bar{M}_6 = 0^+1^+$
7	$\ldots 1^-$	$1^+\ldots$	$M_7 = 1^-1^+$	$\bar{M}_7 = 1^+1^-$
8	$\ldots 0^+$	$1^-\ldots$	-	

For easier reference in the following, the subwords are denoted by symbols reflecting the level at which they occur for the first time: M_k is formed at the concatenation border at level k, whereas $\bar{M}_k$ is first generated through sign reversion at level $k+1$.

New two-letter subwords occur at steps $k = 3, 4, 5, 6, 7$, but then repetition sets in. This is a result of

$$w_{k+4} = w_k w_{k-1} \bar{w}_k w_{k+1} w_{k+1} w_k\,.$$

After four steps, w_{k+4} starts and ends with w_k. As a result, the formation of two-letter subwords at the concatenation boundary is cyclic with period 4 (from $k = 4$ onwards). The final new subword 1^-1^+ at $k = 7$ becomes its opposite 1^+1^- at $k = 8$; at $k = 9$ all possible two-letter subwords are present in the Pisano word.

For three-letter subwords we need to consider two cases, one with two letters from $\bar{w}_{k-1}$ and the other with two letters from $\bar{w}_{k-2}$. Otherwise, the process is similar:

k	$\bar{w}_{k-1}$	$\bar{w}_{k-2}$	new subword	opposite subword
3	1^-	0^-	-	
4	1^+0^+	1^-	$N_4 = 1^+0^+1^-$	$\bar{N}_4 = 1^-0^-1^+$
5	$\ldots 0^-1^+$	$1^+\ldots$	$N_5 = 0^-1^+1^+$	$\bar{N}_5 = 0^+1^-1^-$
6	$\ldots 1^-0^-$	$1^-\ldots$	$N_6 = 1^-0^-1^-$	$\bar{N}_6 = 1^+0^+1^+$
7	$\ldots 0^+1^-$	$1^+\ldots$	$N_7 = 0^+1^-1^+$	$\bar{N}_7 = 0^-1^+1^-$
8	$\ldots 1^+0^+$	$1^-\ldots$	-	

k	$\bar{w}_{k-1}$	$\bar{w}_{k-2}$	new subword	opposite subword
3	1^-	0^-	-	
4	$\ldots 0^+$	1^-	-	
5	$\ldots 1^+$	1^+0^+	$N'_5 = 1^+1^+0^+$	$\bar{N}'_5 = 1^-1^-0^-$
6	$\ldots 0^-$	$1^-0^-\ldots$	$N'_6 = 0^-1^-0^-$	$\bar{N}'_6 = 0^+1^+0^+$
7	$\ldots 1^-$	$1^+0^+\ldots$	$N'_7 = 1^-1^+0^+$	$\bar{N}'_7 = 1^+1^-0^-$
8	$\ldots 0^+$	$1^-0^-\ldots$	$N'_8 = 0^+1^-0^-$	$\bar{N}'_8 = 0^-1^+0^+$
9	$\ldots 1^+$	$1^+0^+\ldots$	-	

This results in 16 different three-letter subwords.

In analogy to the subwords, symbols can be attributed to the letters: $L_2 = 1^+$, $\bar{L}_2 = 1^-$, $L_3 = 0^-$, $\bar{L}_3 = 0^+$. The subwords (apart from M_3 and $\bar{M}_3$) can be arranged in groups of four, e.g., N'_5, $\bar{N}'_5$, N'_7, $\bar{N}'_7$, which in the unsigned Fibonacci word would be identical. With increasing k, more different subwords become present in the Pisano word; notable are

- $k \geq 7$, at least one out of each pair of opposite two-letter subwords;
- $k \geq 8$, at least one out of each pair of opposite three-letter subwords;
- $k \geq 9$, all two-letter subwords;
- $k \geq 10$, all three-letter subwords.

3 Scale factor

When discussing anything Fibonacci, the golden ratio must be introduced at some point:

$$\Phi = \frac{1+\sqrt{5}}{2}\,.$$

In the following, negative half-integer powers of the golden ratio will be used repeatedly. To simplify notation, we define what will turn out to be a scale factor:

$$t = \Phi^{-\frac{1}{2}} = \sqrt{\frac{-1+\sqrt{5}}{2}}\,.$$

Identities involving the golden number can be rephrased in terms of t, the most relevant being:

$$t^2 + t^4 = 1\,, \qquad t^{-2} + t^4 = 2\,, \qquad t^{2k} = t^2 F_{-k} + F_{1-k}\,.$$

4 Plane-filling curve

Consider a piecewise linear curve consisting of vertices $x_{k,i}$ (to be associated with the letters $w_{k,i}$ in word w_k) connected by line segments. The length of segment i between vertices $x_{k,i}$ and $x_{k,i+1}$ is denoted by $r_{k,i}$; the vertex angle $\psi_{k,i}$ specifies the change of direction between segments i and $i+1$ at vertex $x_{k,i+1}$. The segment direction angles $\theta_{k,i}$ and the vertex coordinates of the curve follow as:

$$\begin{aligned}
\theta_{k,i} &= \theta_{k,1} + \sum_{j=1}^{i-1} \psi_{k,j}\,, && i = 2, \ldots, F_{k-1}\,, \\
x_{k,i} &= x_{k,1} + \sum_{j=1}^{i-1} \begin{bmatrix} r_{k,j}\cos\theta_{k,j} \\ r_{k,j}\sin\theta_{k,j} \end{bmatrix}, && i = 2, \ldots, F_k\,.
\end{aligned}$$

We will now define a curve of this type by relating $r_{k,i}$ and $\psi_{k,i}$ to the two- and three-letter subwords of the Pisano word. In the following, we intend to show that for $k \to \infty$ this is a plane-filling curve which does not intersect itself. The segment length is determined from the two-letter subwords in the Pisano word:

$$r_{k,i} = \begin{cases}
t^{k+2}/\sqrt{2} & \text{if } w_{k,i}w_{k,i+1} \text{ equals } M_3\,, \bar{M}_3\,, \\
t^{k-1}/\sqrt{2} & \text{if } w_{k,i}w_{k,i+1} \text{ equals } M_4\,, \bar{M}_4\,, \\
t^{k}\ /\sqrt{2} & \text{if } w_{k,i}w_{k,i+1} \text{ equals } M_5\,, \bar{M}_5\,, M_7\,, \bar{M}_7\,, \\
t^{k+1}/\sqrt{2} & \text{if } w_{k,i}w_{k,i+1} \text{ equals } M_6\,, \bar{M}_6\,.
\end{cases} \tag{1}$$

Opposite subwords specify the same segment length. The vertex angle is determined from the three-letter subwords:

$$\psi_{k,i} = \begin{cases}
-\frac{1}{2}\pi & \text{if } w_{k,i}w_{k,i+1}w_{k,i+2} \text{ equals } N_4\,, N_5\,, N_6\,, N_7\,, N_6'\,, N_8'\,, \\
0 & \text{if } w_{k,i}w_{k,i+1}w_{k,i+2} \text{ equals } N_5'\,, N_7'\,, \bar{N}_5'\,, \bar{N}_7'\,, \\
+\frac{1}{2}\pi & \text{if } w_{k,i}w_{k,i+1}w_{k,i+2} \text{ equals } \bar{N}_4\,, \bar{N}_5\,, \bar{N}_6\,, \bar{N}_7\,, \bar{N}_6'\,, \bar{N}_8'\,.
\end{cases} \tag{2}$$

Opposite subwords specify opposite vertex angles, and in fact all overbarred subwords correspond to non-negative vertex angles. The curve definition is completed by the direction angle of the first segment,

$$\theta_{k,1} = \begin{cases}
-\arctan t^3 & \text{if } k \text{ is odd}, \\
\arctan t + \arctan t^3 & \text{if } k \text{ is even},
\end{cases}$$

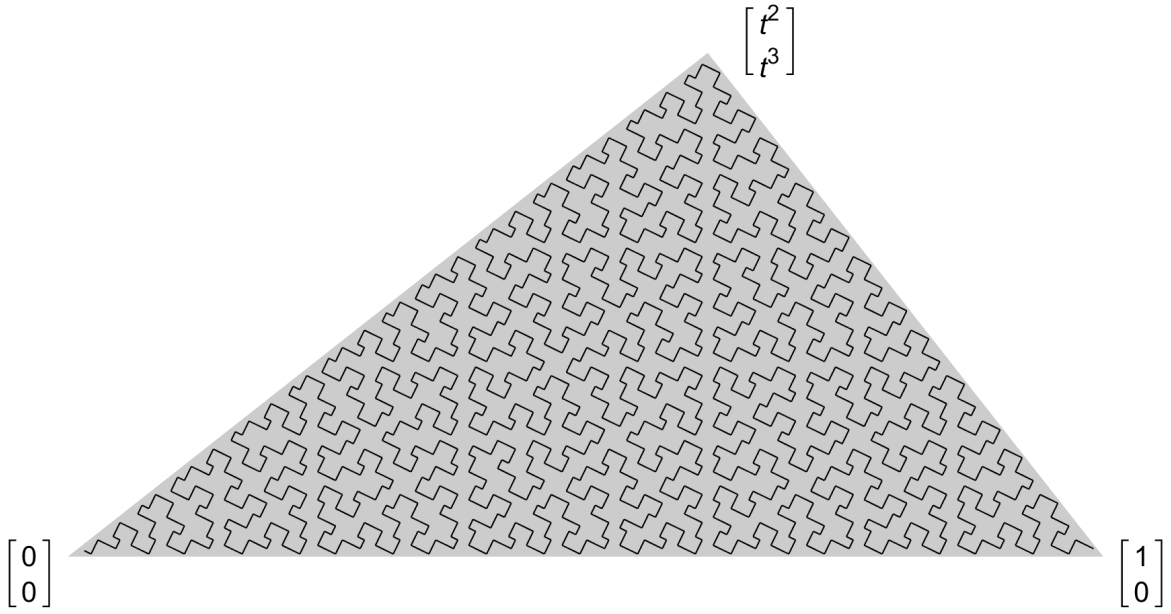


Figure 1: Pisano curve ($k = 17$).

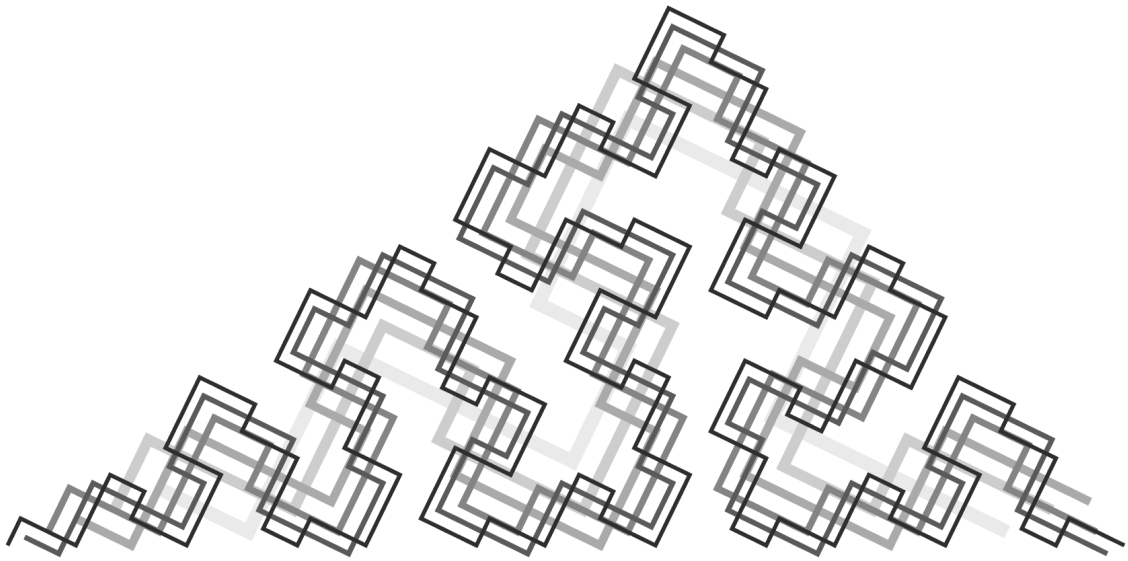


Figure 2: Superimposed Pisano curves ($k = 7$ to $k = 12$). For clarity, the curves are shown with a finite width.

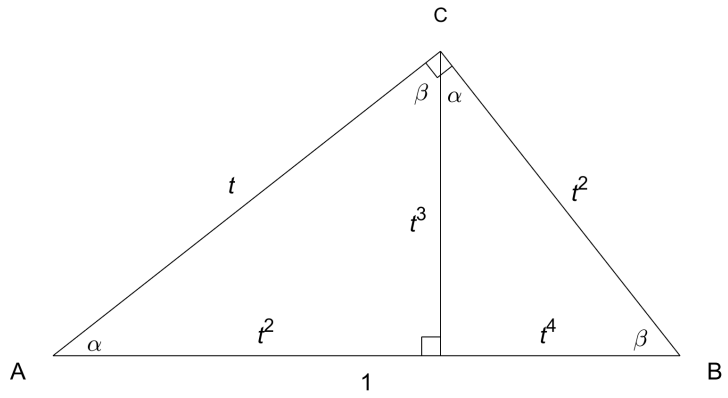


Figure 3: Kepler triangle divided into two mirrored Kepler triangles.

and the initial vertex,

$$x_{k,1} = \begin{cases} \dfrac{t^{k-2}}{4} \begin{bmatrix} 3t \\ -1+3t^2 \end{bmatrix} & \text{if } k \text{ is odd,} \\ \dfrac{t^{k-2}}{4} \begin{bmatrix} 3-t^2 \\ t \end{bmatrix} & \text{if } k \text{ is even.} \end{cases}$$

Figure 1 gives an example of the curve thus defined, and Figure 2 gives an impression of the development of the curve for increasing k.

5 From word to tesselation

As an intermediate step, a geometric interpretation of the Pisano word is proposed. This involves Kepler triangles, which are triangles with sides in the ratio $t^2 : t : 1$, and therefore right-angled. A Kepler triangle with sides $(t^2, t, 1)$ can be divided into Kepler triangles with sides (t^3, t^2, t) and (t^4, t^3, t^2), both of which are mirrored relative to the original. This is illustrated in Figure 3, together with the naming convention for the corners and for the angles $\alpha = \arctan t$, $\beta = \arctan t^{-1}$ and $\gamma = \pi/2$. In the present context, a Kepler triangle is characterised by two aspects: the length of its hypotenuse, and whether or not it is mirrored relative to the reference triangle of Figure 3. In the following, 'triangle' means Kepler triangle and 'magnitude' means hypotenuse length.

Triangles are associated with the letters of the Pisano word as follows. Letters 1^- and 1^+ in word w_k correspond to 'large' triangles of magnitude t^{k-2}, whereas 0^- and 0^+ correspond to 'small' triangles of magnitude t^{k-1}. The triangles for 1^- and 0^- are mirrored relative to the reference triangle; those for 1^+ and 0^+ are not. This association has the following consequences:

- 1-triangles are $t^{-2} = \Phi$ times larger in area than 0-triangles of the same level k;
- 1-triangles at level k are Φ times larger in area than 1-triangles at level $k+1$;

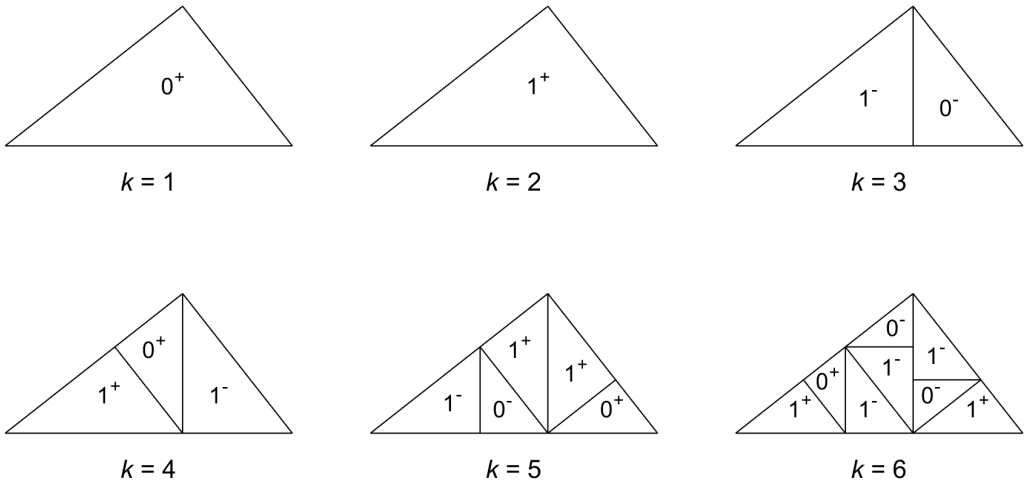


Figure 4: First steps of the triangle division process.

- a 0-triangle at level k has the same area as a 1-triangle at level $k+1$ (i.e., $\frac{1}{2}t^{2k+1}$);
- a 1-triangle at level k has the same area as a 1-triangle plus a 0-triangle at level $k+1$ (i.e., $\frac{1}{2}t^{2k-1} = \frac{1}{2}t^{2k+1} + \frac{1}{2}t^{2k+3}$).

The latter two properties form the key to a geometrical interpretation of the Pisano word. A 0-triangle at one level can be interpreted as a 1-triangle at the next level, hence the substitution rules $0^- \to 1^-$ and $0^+ \to 1^+$ translate to the geometrical rule

$$k \to k+1: \quad \text{keep triangles of magnitude } t^{k-1} \text{ unchanged}.$$

When a 1-triangle is split into two triangles, these can be interpreted as a mirrored 1-triangle and a mirrored 0-triangle of the next level. Hence the geometrical analogon of $1^- \to 1^+0^+$ and $1^+ \to 1^-0^-$ is

$$k \to k+1: \quad \begin{array}{c}\text{split triangles of magnitude } t^{k-2} \text{ into} \\ \text{mirrored triangles of magnitude } t^{k-1} \text{ and } t^k\ .\end{array}$$

The effect of the geometrical rules is that with each step, 'small' triangles are retained, thereby becoming 'large' triangles, whereas 'large' triangles are split to become a mirrored 'large' triangle plus a mirrored 'small' triangle of the next level. The procedure transforms the Pisano word into a Kepler triangle tesselation. The process is shown in Figure 4 for the first few steps. Figure 5 illustrates the relation between word and tesselation.

An implicit, important property is that two-letter subwords translate to ordered triangle pairs *in a specific configuration*. For example, the two 1^+-triangles for $M_5 = 1^+1^+$ are always found hypotenuse against hypotenuse. Such fixed configurations will be called subtesselations. This property can be proven in a straightforward manner, evaluating how the two-triangle patterns arise at the concatenation boundary. It holds also for sequences of three and more triangles. These can be thought of as sequences of overlapping two-triangle subtesselations,

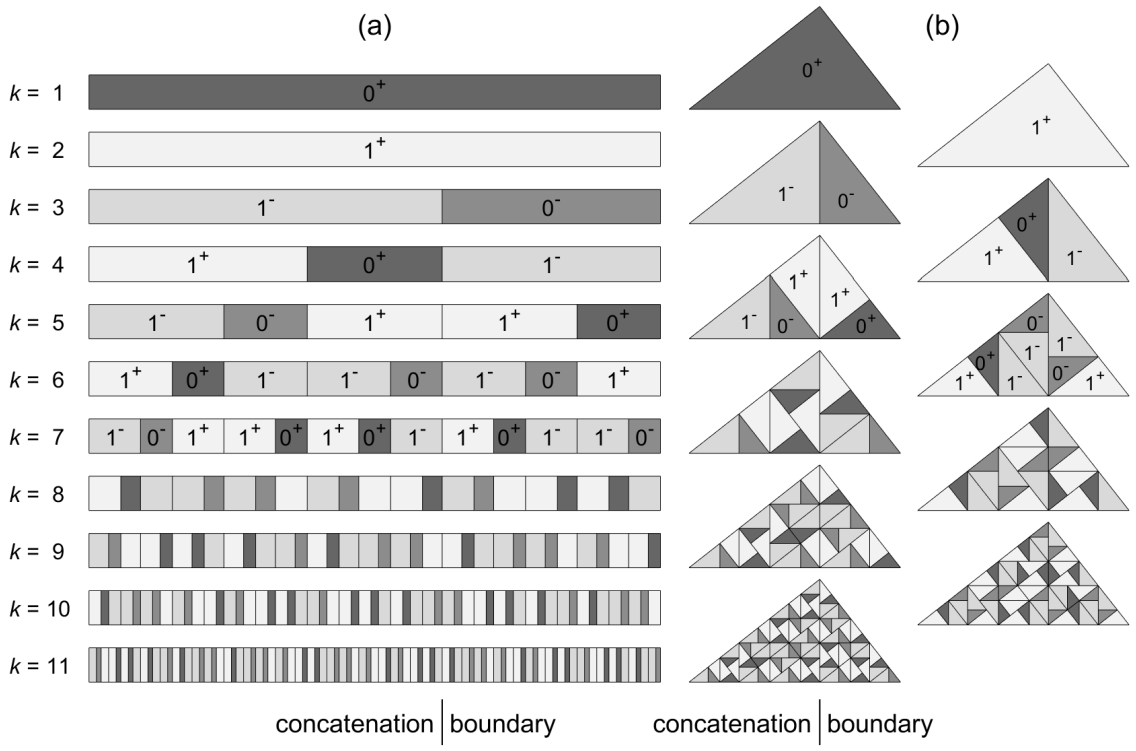


Figure 5: (a) Representation of the Pisano word as the division of a line segment. At each level, the length ratio of 1-segments and 0-segments equals Φ. (b) The Pisano word as a tesselation. At each level, the area ratio of 1-triangles and 0-triangles equals Φ.

from which they inherit the configuration. The two-triangle subtesselations are shown in Figure 6.

6 From tesselation to plane-filling curve

One may take a representative point within each triangle of the tesselation, and connect these *in the order of the Pisano word*. This results in a curve consisting of F_k vertices connected by straight line segments, running from the triangle at A to the triangle at B. It is evident from the construction that such a curve is bounded by the initial triangle. Also, any point within the bounding triangle can not be farther from the curve than the hypotenuse length of the largest triangle in the tesselation, or t^{k-2}. This maximum distance can be made arbitrarily small by increasing k.

There is line contact between all consecutive triangles, as triangle pairs are in contact by either a full side (subtesselations M_3, M_4 and M_5, see Figure 5) or by a substantial part of their sides (subtesselations M_6 and M_7). Hence the individual triangles, which by construction do not overlap, form a continuous strip with non-zero minimum width. It is topologically impossible for such a triangle strip to intersect itself.

If each segment connecting the curve vertices of two consecutive triangles lies completely within these two triangles, then the resulting curve can not self-intersect either. This is not

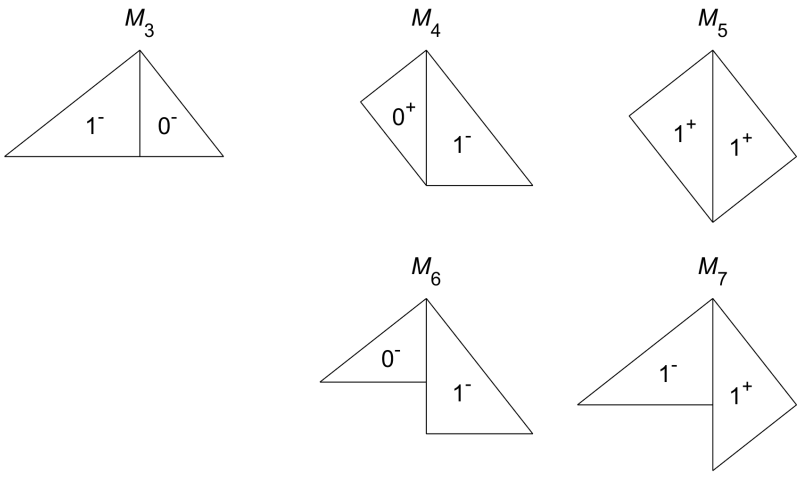


Figure 6: The five different subtesselations (up to mirroring) corresponding to two-letter subwords in the Pisano word. Triangles are oriented such that the word order is from left to right.

a very restrictive condition in view of the large contact regions between the triangles. It is fulfilled, for example, if the curve vertices are taken to be the centroids of the triangles (Figure 7).

Alternatively, one may prove the absence of self-intersection via induction, ensuring that at each step $k \to k+1$ the splitting of existing triangles does not give rise to self-intersection of the involved segments. Then the curve, which for $k = 3$ is a single straight line segment, will not self-intersect at higher levels. This applies to a wider range of cases, but is much more laborious in its details.

7 Selection of suitable vertices

Barycentric coordinates are used to describe the position of the curve vertex $x_{k,i}$ within its triangle. We use one set of barycentric coordinates for 0-triangles and another for 1-triangles:

$$x_{k,i} = \begin{cases} a_{k,i}\kappa_{a0} + b_{k,i}\kappa_{b0} + c_{k,i}\kappa_{c0} & \text{if } w_{k,i} = 0^-, 0^+ \,, \\ a_{k,i}\kappa_{a1} + b_{k,i}\kappa_{b1} + c_{k,i}\kappa_{c1} & \text{if } w_{k,i} = 1^-, 1^+ \,. \end{cases}$$

where $a_{k,i}$, $b_{k,i}$ and $c_{k,i}$ are the positions of corners A, B and C of triangle i. The barycentric coordinates are constrained by

$$\kappa_{a0} + \kappa_{b0} + \kappa_{c0} = 1 \,,$$
$$\kappa_{a1} + \kappa_{b1} + \kappa_{c1} = 1 \,.$$

This leaves 4 independent κ-parameters. An arbitrary choice of these results in a curve with segments of 5 different lengths (from 10 two-triangle subtesselations, mirror pairs yielding identical lengths) and 8 pairs of opposite direction angles at a vertex (from 16 three-triangle subtesselations, mirror pairs yielding opposite angles).

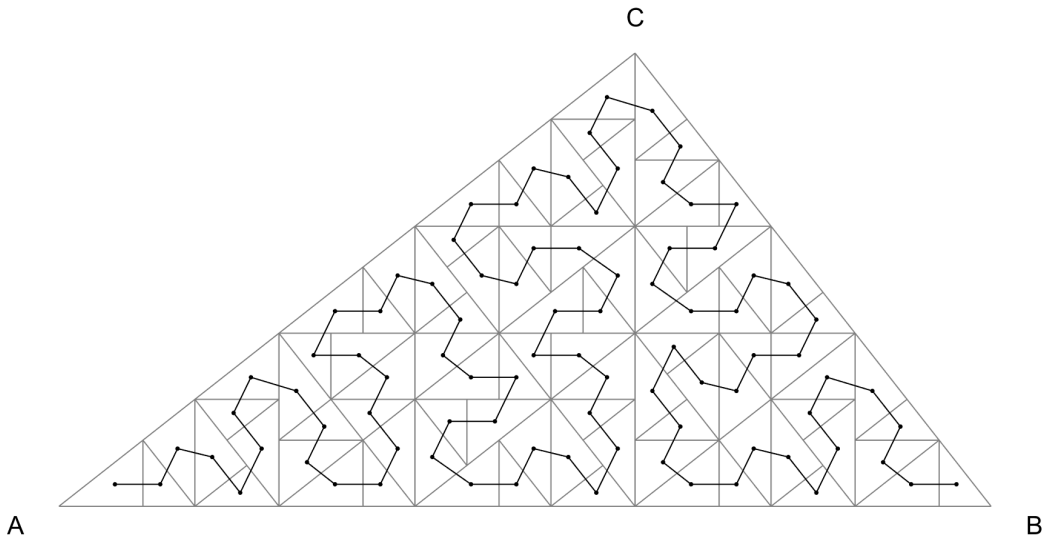


Figure 7: Pisano-like curve with underlying tesselation. The curve vertex for each triangle is located at the centroid. Segments between consecutive vertices lie completely within the two associated triangles.

The curve will be more pleasing to the eye, and simpler to describe, if the number of different segment lengths and vertex angles is reduced. To achieve this, one must make use of the inherent symmetries of the tesselation. The two-triangle subtesselations of Figure 6 are shown again in Figure 8a in a manner that reflects these symmetries. The pair of 1-triangles occurs mirrored around the bisectrix of their common angle β, and the pair of 0-triangles has the same symmetry. The two bisectrix directions are perpendicular, and in fact will turn out to form a rectangular grid with interesting properties (see Section 9). The selection of suitable representative points for the triangles is a semi-intuitive process, which is not easily presented in a concise manner. Here only the outcome is given (the next section discusses some alternatives):

$$(\kappa_{a0}, \kappa_{b0}, \kappa_{c0}) = \tfrac{1}{4}\left(t^2, t^4, 3\right) ,$$
$$(\kappa_{a1}, \kappa_{b1}, \kappa_{c1}) = \tfrac{1}{4}\left(1, 1+t^4, 1+t^2\right) .$$

The curve vertices are shown in Figure 8b in relation to their respective triangles, and in Figure 9 together with a part of the ensuing curve.

To arrive at the form of the Pisano curve as presented in Section 4, one must determine the values for all $r_{k,i}$ and $\psi_{k,i}$ from the curve, associate these with the two- and three-letter subwords, and establish expressions for the initial point and initial direction of the curve. This is straightforward but requires repeated application of identities in higher powers of t.

The curve has 4 different segment lengths (out of a possible 5) and 3 different vertex angles (0 and $\pm\frac{1}{2}\pi$, out of a possible 16). Due to its construction, it may be considered to consist of many parts that are copies or mirrored copies of each other, and in this sense it has a fractal nature for $k \to \infty$.

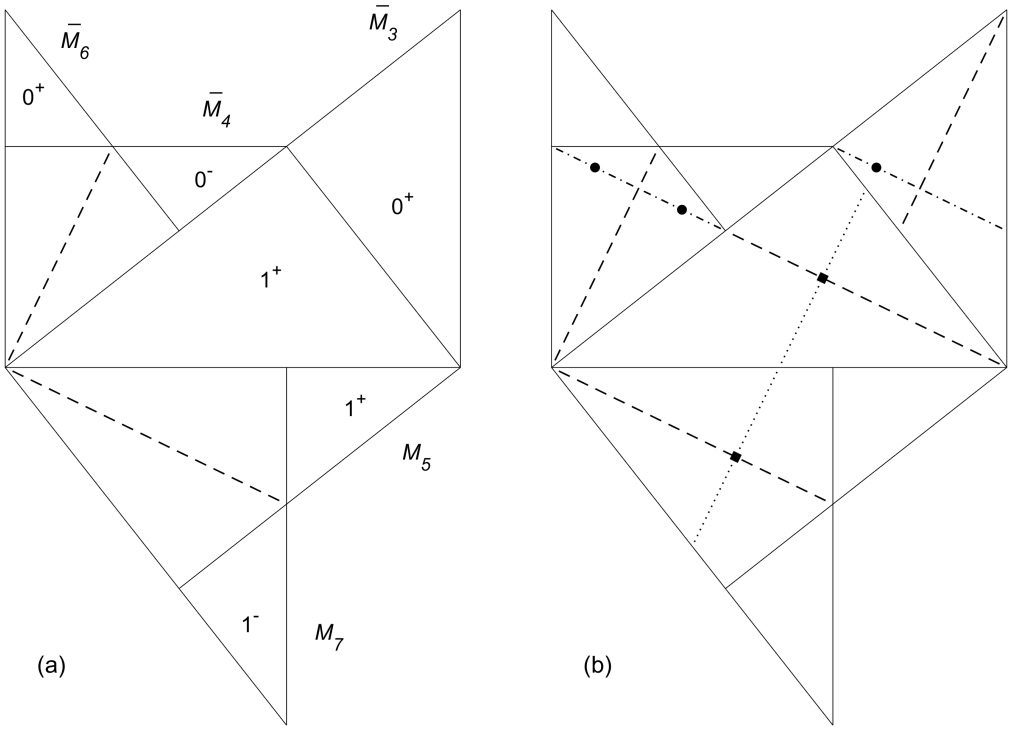


Figure 8: (a) Superimposed subtesselations for $\bar{M}_3 = 1^+0^+$, $\bar{M}_4 = 0^-1^+$, $M_5 = 1^+1^+$, $\bar{M}_6 = 0^+1^+$ and $M_7 = 1^-1^+$, grouped around a common 1^+-triangle. The 0-triangles from $\bar{M}_4$ and $\bar{M}_6$ form a mirror pair with respect to the bisectrix of the angle β. The same holds for the 1-triangles from M_5 and M_7. Both bisectrices (broken lines) are perpendicular. (b) Location of the curve vertices in the triangles (squares: vertices in 1-triangles, circles: vertices in 0-triangles). Dash-dot lines indicate the continuation of the 1-triangle bisectrix in the 0-triangles. The dotted line is a line through the midpoint of the hypotenuse of the 1-triangles, perpendicular to the bisectrix. Lines from the periferal vertices to the vertex in the central 1^+-triangle are mutually perpendicular and agree with the bisectrix directions.

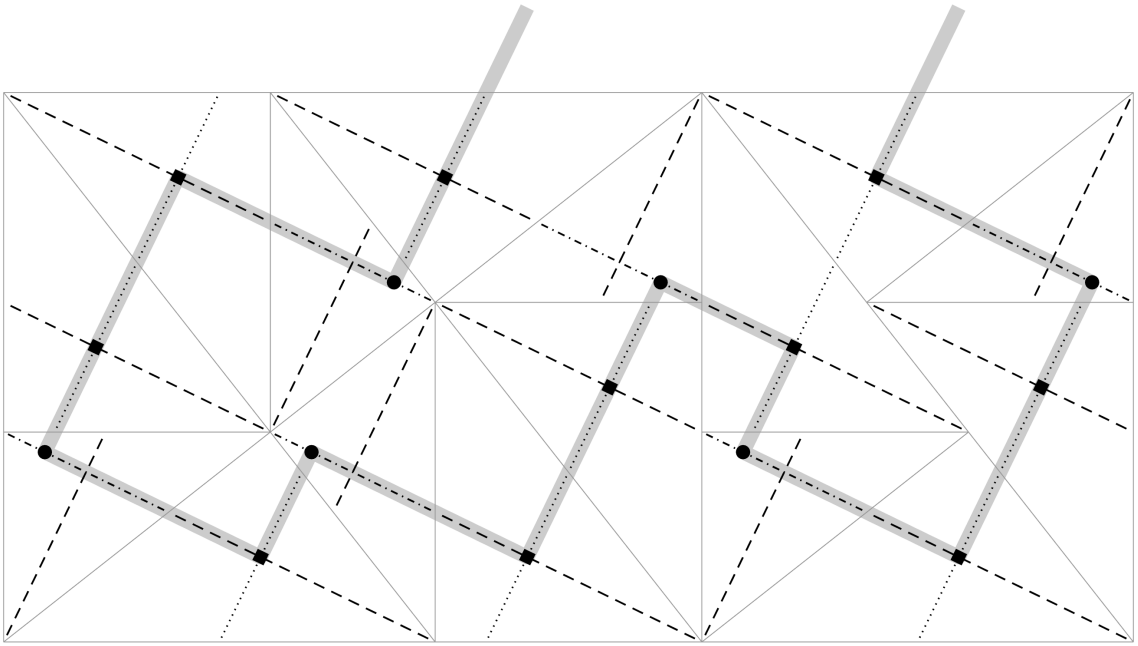


Figure 9: The Pisano curve (gray) together with the directions indicated in Figure 8b.

8 Variant curves

Other choices for the curve vertices are possible, some having appealing qualities of their own. Desirable properties are:

- m_r, the number of different segment lengths in the curve, is small;
- m_ψ, the number of different direction changes between the segments, is small;
- m_θ, the number of different orientation angles of the segments (modulo π), is small.

The Pisano curve has segments that are either parallel or perpendicular to the bisectrix of the angle β of the bounding triangle (these directions together are denoted as the bisectrix directions). Here we note the a number of alternative possibilities, illustrated in Figure 10.
1) Four variants with similar expressions for the barycentric coordinates:

$$\begin{aligned}
&\text{a)} && (\kappa_{a0},\kappa_{b0},\kappa_{c0}) = \tfrac{1}{2}\left(1,t^2,t^4\right)\ , && (\kappa_{a1},\kappa_{b1},\kappa_{c1}) = \tfrac{1}{2}\left(t^4,1,t^2\right)\ ,\\
&\text{b)} && (\kappa_{a0},\kappa_{b0},\kappa_{c0}) = \tfrac{1}{2}\left(t^2,t^4,1\right)\ , && (\kappa_{a1},\kappa_{b1},\kappa_{c1}) = \tfrac{1}{2}\left(1,t^2,t^4\right)\ ,\\
&\text{c)} && (\kappa_{a0},\kappa_{b0},\kappa_{c0}) = \tfrac{1}{2}\left(t^4,1,t^2\right)\ , && (\kappa_{a1},\kappa_{b1},\kappa_{c1}) = \tfrac{1}{2}\left(t^2,t^4,1\right)\ ,\\
&\text{d)} && (\kappa_{a0},\kappa_{b0},\kappa_{c0}) = \tfrac{1}{2}\left(t^2,t^4,1\right)\ , && (\kappa_{a1},\kappa_{b1},\kappa_{c1}) = \tfrac{1}{2}\left(t^4,1,t^2\right)\ .
\end{aligned}$$

These variants all have segments that are aligned to sides of the bounding triangle, the line perpendicular to the hypotenuse of the bounding triangle, or the bisectrix directions.
2) Two variants in which all vertex angles $\psi_{k,i}$ are integer multiples of $\pi/3$:

$$(\kappa_{a0},\kappa_{b0},\kappa_{c0}) = \left(\lambda_0 t^2,\lambda_0 t^4,1-\lambda_0\right)\ , \qquad (\kappa_{a1},\kappa_{b1},\kappa_{c1}) = \left(\lambda_1 t^2,\lambda_1 t^4,1-\lambda_1\right)\ ,$$

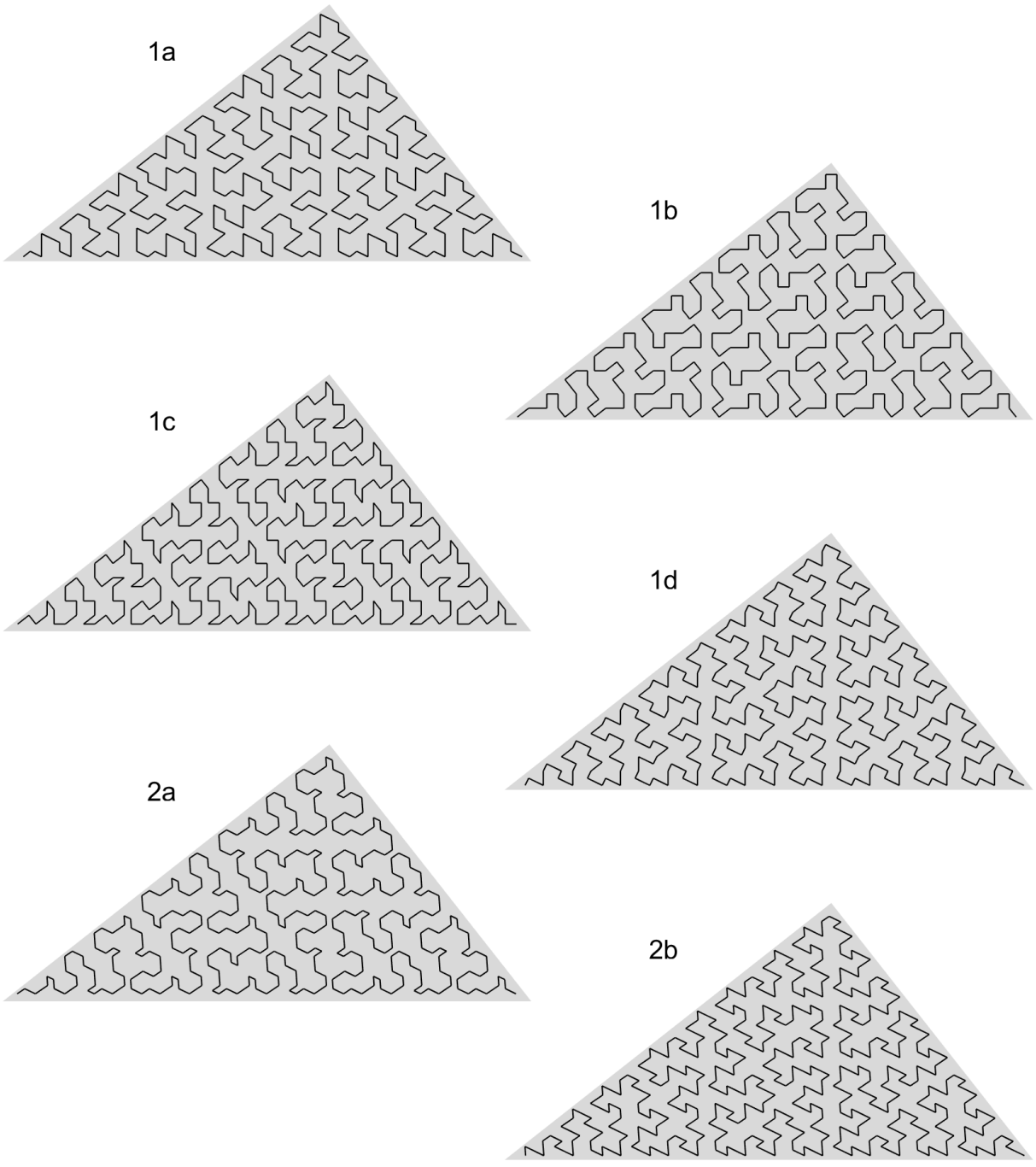


Figure 10: Variants of the plane-filling curve, as described in the text ($k = 14$).

with

$$\text{a)} \qquad \lambda_0 = \frac{1}{4}\left(1 + \frac{t^3}{\sqrt{3}}\right), \qquad \lambda_1 = \frac{1}{4t^4}\left(1 + \frac{t^3}{\sqrt{3}}\right),$$

$$\text{b)} \qquad \lambda_0 = \frac{1}{4}\left(1 - \frac{t^3}{\sqrt{3}}\right), \qquad \lambda_1 = \frac{1}{4t^4}\left(1 - \frac{t^3}{\sqrt{3}}\right).$$

The segments run in three directions, one of which is a bisectrix direction. These curves are remarkable for their quasi-hexagonal structure, even when the triangles of the underlying tesselation do not contain related angles.

The numbers m_r, m_ψ and m_θ for each variant are as as follows:

variant	m_r	m_ψ	m_θ
0	4	3	2
1a	4	5	3
1b	3	7	4
1c	2	8	4
1d	3	6	4
2a, 2b	4	4	3
maximum	5	16	10

The null-variant—the Pisano curve—does appear to have the most appealing properties, in particular the alignment of all segments to the bisectrix directions.

9 Bisectrix grid and curve grid

The bisectrices of the angle β of the individual triangles in the tesselation form an orthogonal grid, which is interesting in itself and because its properties are transferred in part to the Pisano curve.

In Figure 11, the bounding triangle is shown rotated over ζ, with

$$\zeta = \tfrac{1}{2}\beta = \tfrac{1}{2}\arctan t^{-1} = \arctan t^3\,,$$

which causes the bisectrix of the angle β to be parallel to the horizontal axis. The horizontal and vertical dimensions in this figure demonstrate a general principle: distances in the bisectrix grid are naturally expressed in terms of $t^j/\sqrt{2}$ with j integer (compare the segment lengths of Eq. (1)).

We define 'the 1-bisectrices' as the bisectrices of the angle β of all 1-triangles in the tesselation, and 'the 0-bisectrices' analogously. The bisectrix grid is the combination of both. The essence of the bisectrix grid development is observed at the step from $k = 3$ to $k = 4$ (Figure 12). The 0-bisectrix becomes a 1-bisectrix (via $0^\pm \to 1^\pm$), whereas the 1-bisectrix becomes a 0-bisectrix and in addition 'splits off' a perpendicular 1-bisectrix (via $1^\pm \to 1^\mp 0^\mp$). This exemplifies the general pattern: at each step 0-bisectrices become 1-bisectrices, 1-bisectrices become 0-bisectrices, and an additional set of 1-bisectrices is generated. All 1-bisectrices are parallel to each other, and perpendicular to the 0-bisectrices. The 1-bisectrices are horizontal

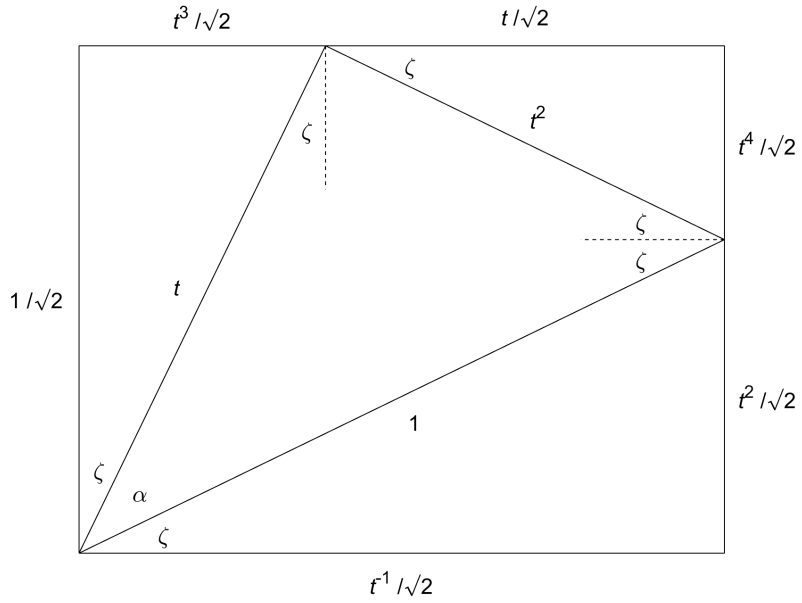


Figure 11: The bounding triangle rotated over an angle $\zeta = \beta/2$.

for even k and vertical for odd k. For k odd $\rightarrow$ even, new horizontal bisectrices are generated, for k even $\rightarrow$ odd new vertical ones.

Keeping track of the bisectrices as they appear (Figure 13), the distances between them develop according to a specific pattern. The basis of this pattern is the substitution rule

$$\begin{aligned} c &\rightarrow b \\ b &\rightarrow a \\ a &\rightarrow bcb\,, \end{aligned}$$

where the letters a, b and c can be interpreted as distances in the ratio $1 : t^2 : t^4$ or $\Phi^2 : \Phi : 1$. The distance scale is reduced by a factor t at each step, or by Φ^{-1} every other step. To describe in detail what is going on at the borders, separate rules are required for the left, right, bottom and top parts of the rotated triangle:

position	left	right	bottom	top
substitution rule	$c \rightarrow b$ $b \rightarrow a$ $a \rightarrow bcb$ $L \rightarrow Lb$	$c \rightarrow b$ $b \rightarrow a$ $a \rightarrow bcb$ $R \rightarrow aR$	$c \rightarrow b$ $b \rightarrow a$ $a \rightarrow bcb$ $B \rightarrow Bb$	$c \rightarrow b$ $b \rightarrow a$ $a \rightarrow bcb$ $T \rightarrow bT$
start	L at $k = 3$	R at $k = 3$	B at $k = 2$	T at $k = 4$
update	k even $\rightarrow$ odd		k odd $\rightarrow$ even	

Letters L, B and T correspond in length to $a = b + c$, whereas R corresponds in length to $b + a = b + c + b$ combined. Ignoring the details at the borders, a generalized pattern can be

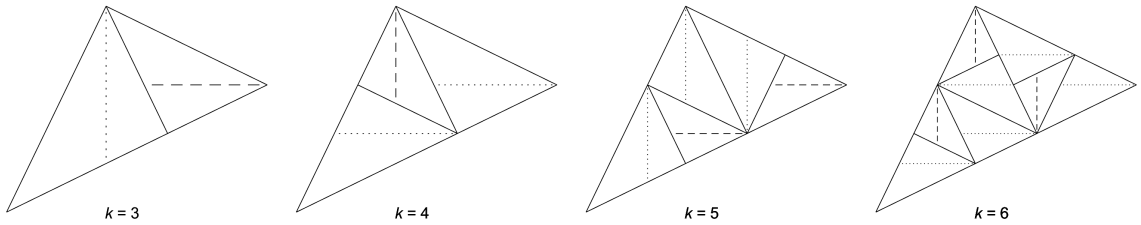


Figure 12: Bisectrices of angle β of all 0-triangles (broken lines) and 1-triangles (dotted lines)

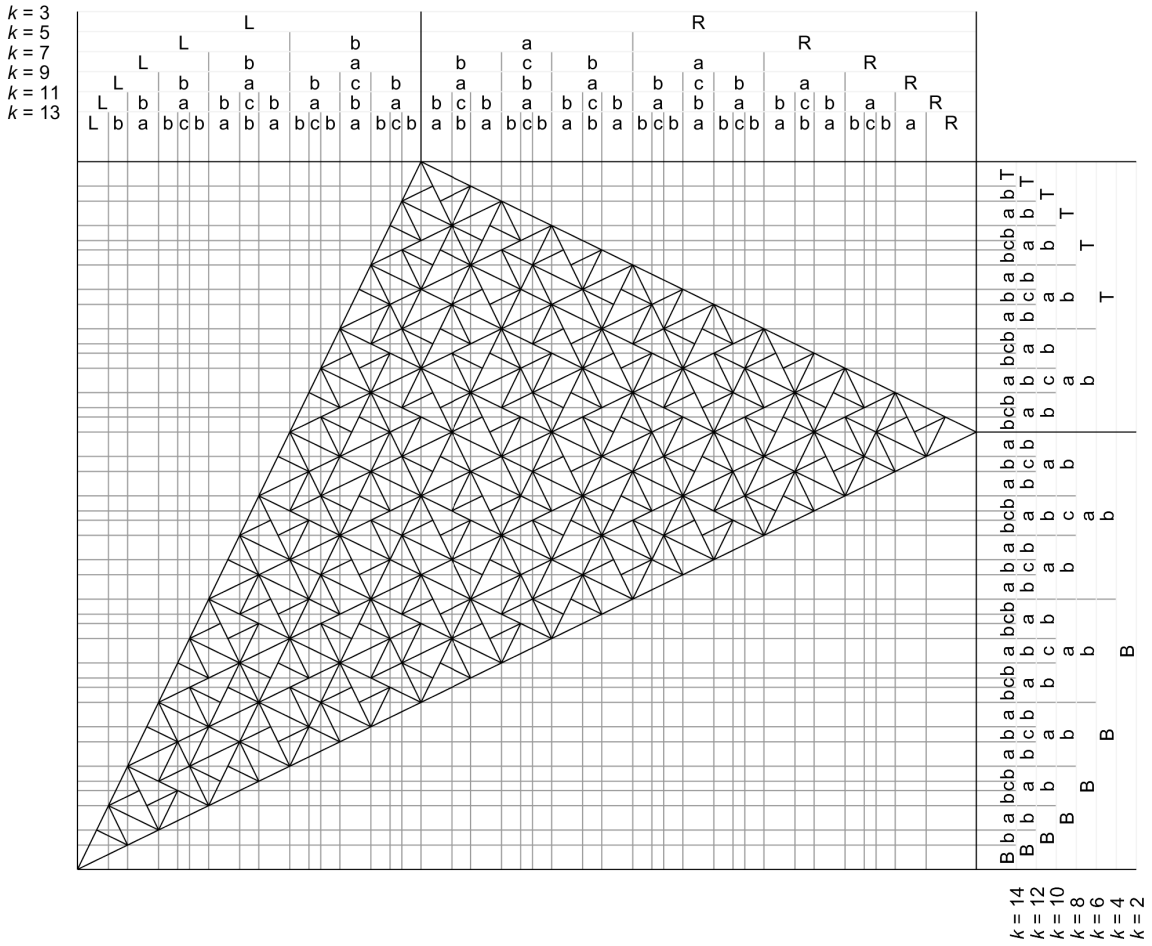


Figure 13: Bisectrices of angle β (grey lines) of all triangles of the tesselation (black), as they are generated with increasing k. The progression of the substitution rules generating the pattern of distances is indicated between the bisectrices.

defined, starting with a at $k = -1$ (for left and right combined) or at $k = 0$ (for bottom and top combined). This produces all actual bisectrix locations, plus a few virtual ones.

What does this mean for the Pisano curve? Figure 14 shows the curve in the rotated coordinate system, for even k. Also here a natural grid consisting of two sets of perpendicular lines is found, denoted by 1-lines (through the curve vertices, parallel to the 1-bisectrices) and 0-lines (through the curve vertices, parallel to the 0-bisectrices). All curve vertices lie on one of the 1-bisectrices (see Figures 8b and 9). As a result, 1-lines of the curve grid coincide with 1-bisectrices. The curve vertices do not lie on 0-bisectrices, however.

The distances between the 0-lines assume two distinct values which correspond to a and b of the bisectrix grid. The pattern of these distances *matches the pattern of the letters in the Fibonacci word*. Given the strongly meandering nature of the Pisano curve, it is not immediately obvious how this comes about. With increasing k, ever more 'letters' of the infinite Fibonacci word are added to the right. In this sense each level of the Pisano curve has an associated Fibonacci subword, which for Pisano curve k has length $F_{\lfloor k/2 \rfloor + 2} - 2$ and is palindromic. (For odd k the situation is essentially the same, only in that case the vertical curve grid lines coincide with the bisectrix grid.)

It can be shown that each curve vertex has the same distance to the nearest 0-bisectrix; this distance is one half of c from the bisectrix grid. The relation between 0-lines and 0-bisectrices can be expressed loosely as: if one takes the 0-bisectrices, and shifts those at distinct locations alternatingly to the left and to the right by $c/2$, then they are at the locations of the 0-lines of the curve grid. (This alternating pattern is a direct consequence of the tesselation process and a sensible choice of the curve vertices.) This implies that if we take the distances between the 0-bisectrices, and alternatingly add and subtract c, we should end up with the distances between the 0-lines. The table illustrates the first instances:

0-b.	b	a	b	c	b	a	b	a	b	c	b	a	b	c	b	$\ldots$
Δ	$+c$	$-c$	$+c$	$-c$	$+c$	$-c$	$+c$	$-c$	$+c$	$-c$	$+c$	$-c$	$+c$	$-c$	$+c$	$\ldots$
0-l.	a	b	a	0	a	b	a	b	a	0	a	b	a	0	a	$\ldots$

The last line features the Fibonacci word in at least three different guises. First, ignoring the zeros, the pattern $abaababaabaa\ldots$ is indeed the Fibonacci word pattern of distances between the 0-lines (distances of zero not being noticeable). The even positions in the last line, $b0bb0b0\ldots$, form the beginning of a second Fibonacci word pattern. And although the sequence shown here is too short to make this plausible, the distances between the positions of the zeros assume the values $6, 4, 6, 6, 4, 6, 4, 6, 6, 4, 6, 6, 4, \ldots$, a third Fibonacci word pattern.

All above observations stand in need of a decent proof by someone else than the author.

10 Letter and subword frequencies

The Pisano word without the signs reduces to the Fibonacci word. As a result, w_k contains F_k letters, F_{k-1} of which are 1 and F_{k-2} are 0. But what can be said about the number of occurrences of 0^+, 0^-, 1^+ and 1^- individually, and of subwords consisting of these letters? A simple scheme, consisting of two entangled Fibonacci number relations, will be useful to evaluate this. The scheme involves two variables: P_k, which is to represent the frequency of a certain subword in w_k, and $\bar{P}_k$, the frequency of the opposite subword. This subword may be

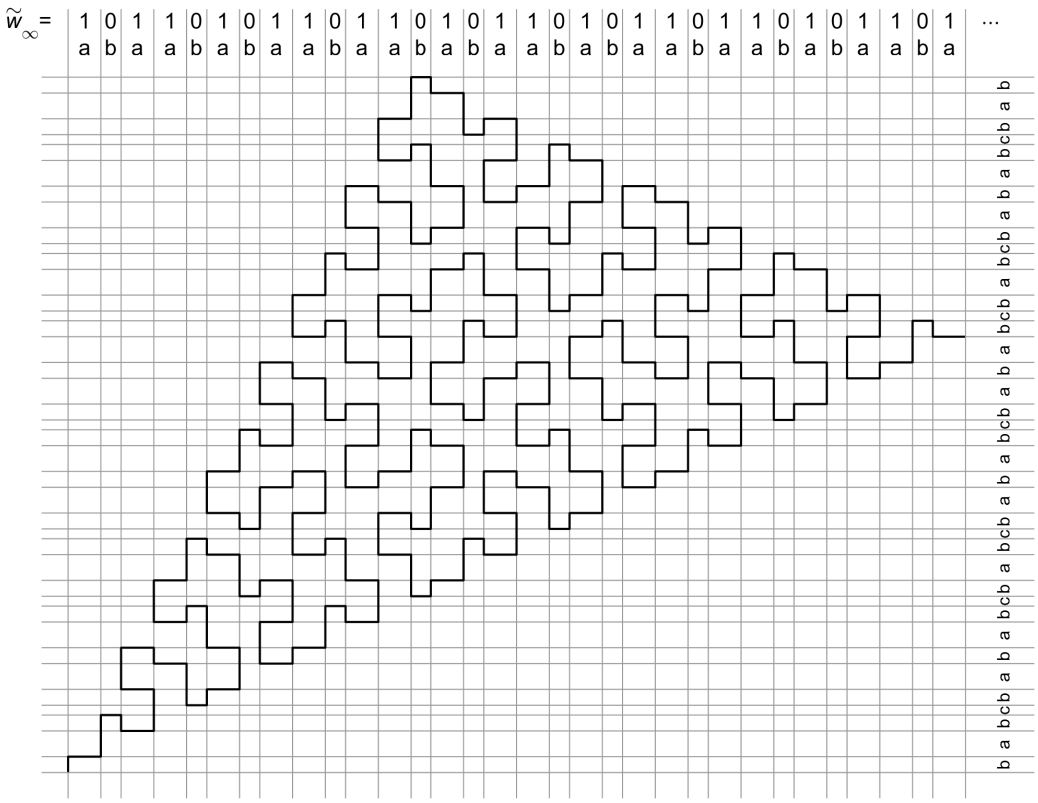


Figure 14: Grid through the vertices of the Pisano curve ($k = 14$). The horizontal distances correspond to a palindromic initial section of the infinite Fibonacci word.

a single letter. We will call these variables here for short Pisano numbers (not to be confused with the Pisano period).

We assume for now that the subword and its opposite are absent at $k = 0$, and that at $k = 1$ the subword, but not its opposite, is generated for the first and only time:

$$\begin{aligned} P_0 &= 0\,, \qquad P_1 = 1\,, \\ \bar{P}_0 &= 0\,, \qquad \bar{P}_1 = 0\,. \end{aligned}$$

Under these simplifying assumptions, the frequencies of subword and opposite are determined exclusively by the word concatenation rule,

$$\begin{aligned} w_k &= \bar{w}_{k-1}\bar{w}_{k-2}\,, \\ \bar{w}_k &= w_{k-1}w_{k-2}\,, \end{aligned}$$

which translates directly to number relations,

$$\begin{aligned} P_k &= \bar{P}_{k-1} + \bar{P}_{k-2}\,, \\ \bar{P}_k &= P_{k-1} + P_{k-2}\,. \end{aligned}$$

These relations can be written in matrix-vector form,

$$\begin{bmatrix} P_k \\ \bar{P}_k \\ P_{k-1} \\ \bar{P}_{k-1} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} P_{k-1} \\ \bar{P}_{k-1} \\ P_{k-2} \\ \bar{P}_{k-2} \end{bmatrix}, \qquad \begin{bmatrix} P_1 \\ \bar{P}_1 \\ P_0 \\ \bar{P}_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

An eigenvalue decomposition of the matrix in this expression results in an eigenvalue matrix Λ and a matrix of right eigenvectors V:

$$\Lambda = \begin{bmatrix} \Phi & 0 & 0 & 0 \\ 0 & -\Phi^{-1} & 0 & 0 \\ 0 & 0 & e^{\frac{2\pi}{3}i} & 0 \\ 0 & 0 & 0 & e^{-\frac{2\pi}{3}i} \end{bmatrix}, \qquad V = \begin{bmatrix} \Phi & -\Phi^{-1} & -e^{\frac{2\pi}{3}i} & -e^{-\frac{2\pi}{3}i} \\ \Phi & -\Phi^{-1} & e^{\frac{2\pi}{3}i} & e^{-\frac{2\pi}{3}i} \\ 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

These form the basis for expressions for P_k and $\bar{P}_k$. Such expressions take the form of weighted sums of the eigenvalues to the power k. In first instance, the complex pair of eigenvalues gives rise to a complex-valued expression, which can be rephrased in terms of sinusoids, giving

$$\begin{aligned} P_k &= \frac{1}{2}\left(\frac{1}{\sqrt{5}}\Phi^k - \frac{1}{\sqrt{5}}(-\Phi^{-1})^k + \frac{2}{\sqrt{3}}\sin\left(\frac{2\pi}{3}k\right)\right), \\ \bar{P}_k &= \frac{1}{2}\left(\frac{1}{\sqrt{5}}\Phi^k - \frac{1}{\sqrt{5}}(-\Phi^{-1})^k - \frac{2}{\sqrt{3}}\sin\left(\frac{2\pi}{3}k\right)\right). \end{aligned}$$

The terms in Φ are Binet's formula for F_k. For integer k, the sine term assumes three integer values:

$$\frac{2}{\sqrt{3}}\sin\left(\frac{2\pi}{3}k\right) = \begin{cases} 0 & \text{if } k \bmod 3 = 0\,, \\ 1 & \text{if } k \bmod 3 = 1\,, \\ \text{-1} & \text{if } k \bmod 3 = 2\,. \end{cases}$$

This allows for simple expressions,

$$P_k = \tfrac{1}{2}\,(F_k + \delta_k)\ ,$$
$$\bar{P}_k = \tfrac{1}{2}\,(F_k - \delta_k)\ , \qquad \text{with} \qquad \delta_k = (k+1) \bmod 3 - 1\,.$$

The sum and difference are obvious yet elegant,

$$P_k + \bar{P}_k = F_k\,, \qquad P_k - \bar{P}_k = \delta_k\,.$$

The numbers extend naturally to negative k:

k	...	-8	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8	...
F_k	...	-21	13	-8	5	-3	2	-1	1	0	1	1	2	3	5	8	13	21	...
δ_k	...	1	-1	0	1	-1	0	1	-1	0	1	-1	0	1	-1	0	1	-1	...
P_k	...	-10	6	-4	3	-2	1	0	0	0	1	0	1	2	2	4	7	10	...
$\bar{P}_k$	...	-11	7	-4	2	-1	1	-1	1	0	0	1	1	1	3	4	6	11	...

In the current context only the values for non-negative indices are required.

Both F_k and δ_k are coefficients in the expansion of simple rational functions [3],

$$\frac{x}{1-x-x^2} = \sum_{j=0}^{\infty} F_j\, x^j\,, \qquad \frac{x}{1+x+x^2} = \sum_{j=0}^{\infty} \delta_j\, x^j\,,$$

from which we find for the Pisano numbers

$$\frac{x}{(1-x-x^2)(1+x+x^2)} = \sum_{j=0}^{\infty} P_j\, x^j\,, \qquad \frac{x^2+x^3}{(1-x-x^2)(1+x+x^2)} = \sum_{j=0}^{\infty} \bar{P}_j\, x^j\,.$$

The poles of these functions correspond to the eigenvalues in Λ.

Pisano numbers are directly applicable to determine the frequency of individual letters and non-cyclic subwords. For example, consider $M_3 = 1^-0^-$ with its opposite $\bar{M}_3 = 1^+0^+$. These conform to the above scheme, except that M_3 occurs for the first time at $k = 3$ rather than at $k = 1$; so in Pisano word w_k there are P_{k-2} occurrences of M_3 and $\bar{P}_{k-2}$ occurrences of $\bar{M}_3$. The scheme also applies to the letters 1^+ and 1^-. For the letters $0^\pm$ there is a minor anomaly: the first regular occurrence of 0^- is at $k = 3$, whereas the initial word $w_1 = 0^+$ falls outside the pattern.

To determine the frequency of the other two- and three-letter words, which are formed cyclically at the concatenation boundary with period 4, the several generations must be added. This results in cumulative Pisano numbers:

$$Q_k = \sum_{j=0}^{\lfloor k/4 \rfloor} P_{k-4j}\,, \qquad \bar{Q}_k = \sum_{j=0}^{\lfloor k/4 \rfloor} \bar{P}_{k-4j}\,.$$

The upper value for j is such that the index of P does not become negative. The sum of the cumulative Pisano numbers can be expressed as a cumulative Fibonacci number:

$$G_k = Q_k + \bar{Q}_k = \sum_{j=0}^{\lfloor k/4 \rfloor} F_{k-4j} = \sum_{j=0}^{\lfloor k/4 \rfloor} F_{k \bmod 4 + 4j}\,.$$

These expressions are applicable to all two-letter subwords except M_3 and $\bar{M}_3$, and to all three-letter subwords, if one makes allowance for the level k at which the subword first occurs.

In overview, within Pisano word w_k the letters L, the two-letter subwords M and the three-letter subwords N or N' have a frequency specified by Pisano numbers $P_k, \bar{P}_k$ or cumulative Pisano numbers $Q_k, \bar{Q}_k$:

L	M	N	N'	frequency
$L_2 = 1^+$				P_{k-1}
$\bar{L}_2 = 1^-$				$\bar{P}_{k-1}$
$L_3 = 0^-$	$M_3 = 1^-0^-$			P_{k-2}
$\bar{L}_3 = 0^+$	$\bar{M}_3 = 1^+0^+$			$\bar{P}_{k-2}$
	$M_4 = 0^+1^-$	$N_4 = 1^+0^+1^-$		Q_{k-3}
	$\bar{M}_4 = 0^-1^+$	$\bar{N}_4 = 1^-0^-1^+$		$\bar{Q}_{k-3}$
	$M_5 = 1^+1^+$	$N_5 = 0^-1^+1^+$	$N'_5 = 1^+1^+0^+$	Q_{k-4}
	$\bar{M}_5 = 1^-1^-$	$\bar{N}_5 = 0^+1^-1^-$	$\bar{N}'_5 = 1^-1^-0^-$	$\bar{Q}_{k-4}$
	$M_6 = 0^-1^-$	$N_6 = 1^-0^-1^-$	$N'_6 = 0^-1^-0^-$	Q_{k-5}
	$\bar{M}_6 = 0^+1^+$	$\bar{N}_6 = 1^+0^+1^+$	$\bar{N}'_6 = 0^+1^+0^+$	$\bar{Q}_{k-5}$
	$M_7 = 1^-1^+$	$N_7 = 0^+1^-1^+$	$N'_7 = 1^-1^+0^+$	Q_{k-6}
	$\bar{M}_7 = 1^+1^-$	$\bar{N}_7 = 0^-1^+1^-$	$\bar{N}'_7 = 1^+1^-0^-$	$\bar{Q}_{k-6}$
			$N'_8 = 0^+1^-1^+$	Q_{k-7}
			$\bar{N}'_8 = 0^-1^+1^-$	$\bar{Q}_{k-7}$

As all subwords with two or three letters are listed in this table, one concludes that

$$\begin{aligned} F_{k-2} + G_{k-3} + G_{k-4} + G_{k-5} + G_{k-6} &= F_k - 1\,, \\ G_{k-3} + 2G_{k-4} + 2G_{k-5} + 2G_{k-6} + G_{k-7} &= F_k - 2\,, \end{aligned}$$

which implies

$$G_{k-2} + G_{k-3} + G_{k-4} + G_{k-5} = F_k - 1\,.$$

From the definition of G_k it is seen that the sum on the left is in fact the sum of $F_0, F_1, \ldots, F_{k-2}$, which by a well-known identity [2] equals $F_k - 1$.

The above table can be summarized as

subword			first occurrence	frequency in word w_k
L_{k_0} ,	M_{k_0}		$k_0 \le 3$	P_{k+1-k_0}
$\bar{L}_{k_0}$,	$\bar{M}_{k_0}$		$k_0 \le 3$	$\bar{P}_{k+1-k_0}$
L_{k_0} or $\bar{L}_{k_0}$,	M_{k_0} or $\bar{M}_{k_0}$		$k_0 \le 3$	F_{k+1-k_0}
M_{k_0} ,	N_{k_0} ,	N'_{k_0}	$k_0 \ge 4$	Q_{k+1-k_0}
$\bar{M}_{k_0}$,	$\bar{N}_{k_0}$,	$\bar{N}'_{k_0}$	$k_0 \ge 4$	$\bar{Q}_{k+1-k_0}$
M_{k_0} or $\bar{M}_{k_0}$,	N_{k_0} or $\bar{N}_{k_0}$,	N'_{k_0} or $\bar{N}'_{k_0}$	$k_0 \ge 4$	G_{k+1-k_0}

(3)

These expressions are valid on condition that $k + 1 - k_0 \ge 0$.

11 Indices of letters and subwords

More direct expressions for the location of the subwords within w_k can be obtained. First we consider the position of all occurrences in w_k of a given subword together with its opposite. This suffices to locate the segments with a given length in the Pisano curve, because segment lengths associated with opposite two-letter words are equal (Eq. (1)). The essential function in this context is

$$I(k_0, l) = F_{k_0-1} + \left\lfloor \frac{l}{\Phi^2} \right\rfloor F_{k_0} + \left\lfloor \frac{l}{\Phi} \right\rfloor F_{k_0+1} \,,$$

where k_0 represents the level at which the subword first appears, and l counts the occurrences of subword-or-opposite.

The occurrences of pairs of opposite letters or non-cyclic subwords can be expressed in terms of this function as:

$$\begin{aligned}
1^+ \text{ or } 1^- &\quad \text{is found at} \quad i = I(2,l) &&\quad \text{for } l = 1, \ldots, F_{k+1-2} \,, \\
0^- \text{ or } 0^+ &\quad \text{is found at} \quad i = I(3,l) + 1 &&\quad \text{for } l = 1, \ldots, F_{k+1-3} \,, \\
1^-0^- \text{ or } 1^+0^+ &\quad \text{starts at} \quad i = I(3,l) &&\quad \text{for } l = 1, \ldots, F_{k+1-3} \,.
\end{aligned}$$

For the cyclic subwords ($k_0 \geq 4$) we have:

$$\begin{aligned}
M_{k_0} \text{ or } \bar{M}_{k_0} &\quad \text{starts at} \quad i = I(k_0 + 4j, l) &&\quad , \\
N_{k_0} \text{ or } \bar{N}_{k_0} &\quad \text{starts at} \quad i = I(k_0 + 4j, l) - 1 &&\quad , \\
N'_{k_0} \text{ or } \bar{N}'_{k_0} &\quad \text{starts at} \quad i = I(k_0 + 4j, l) &&\quad , \\
&\quad \text{for } j = 0, \ldots, \lfloor (k - k_0)/4 \rfloor, \quad l = 1, \ldots, F_{k+1-k_0-4j} \,.
\end{aligned}$$

This can be established along the following lines. One assumes that a given subword is generated once at step k_0. Following the steps of the concatenatenation process, one observes that the distances between the occurrences (of the subword or its opposite) are always either F_{k_0} or F_{k_0+1}. These distances occur in a pattern that follows the letters of the Fibonacci word $\tilde{w}_k$, so that they can be written as $(1 - \tilde{w}_{k,l})F_{k_0} + \tilde{w}_{k,l}F_{k_0+1}$. The cumulative sum of these distances then follows as $\left\lfloor \frac{l}{\Phi^2} \right\rfloor F_{k_0} + \left\lfloor \frac{l}{\Phi} \right\rfloor F_{k_0+1}$. The first occurrence has approximate index F_{k_0-1}, although, dependent on the precise position of the initial subword relative to the concatenation boundary, a constant shift ± 1 may need to be applied. For subwords that occur cyclically at the concatenation boundary, the repetition is incorporated.

Can we find similar expressions for the subword and its opposite individually? This would be required to distinguish the positions of 'right turns' and 'left turns' of the Pisano curve (Eq. (2)). To this end we consider the process:

$$z_1 = 0 \,, \quad z_2 = 1 \,, \qquad z_k = z_{k-1}\bar{z}_{k-2} \,.$$

This is a variant of the process introduced in [4] and investigated in [5], and the ensuing word is related to sequences OEIS A095076 and A095111 [6]. Word z_k has length F_k. The infinite word reads:

$$z_\infty = 01110100100011000101110001011011 10 \ldots$$

If one looks at the subwords-or-opposites at the index positions as specified above (non-cyclic), they follow the pattern of the 0's and 1's in z_∞. To be precise, a 0 in z_{k+1-k_0} corresponds to the subword if $k - k_0$ is even, and to the opposite if $k - k_0$ is odd. However, as yet no expression has been found for the indices of the letters 0 and 1 in z_∞.

12 Asymptotic frequencies

The Pisano numbers have asymptotic approximations

$$k \to \infty : \qquad P_k, \bar{P}_k \to \frac{\Phi^k}{2\sqrt{5}} ,$$
$$k \to \infty : \qquad Q_k, \bar{Q}_k \to \sum_{j=0}^{\infty} \frac{\Phi^{k-4j}}{2\sqrt{5}} = \frac{\Phi^k}{2\sqrt{5}} \frac{\Phi^4}{\Phi^4 - 1} = \frac{\Phi^{k+2}}{10} .$$

These lead to asymptotic frequencies for letters and subwords:

letter	$1^+, 1^-$	$0^-, 0^+$			
asymptotic frequency	$\frac{\Phi^{k-1}}{2\sqrt{5}}$	$\frac{\Phi^{k-2}}{2\sqrt{5}}$			
asymptotic relative frequency	$\frac{1}{2\Phi}$	$\frac{1}{2\Phi^2}$			
two-letter subword	$M_3, \bar{M}_3$	$M_4, \bar{M}_4$	$M_5, \bar{M}_5$	$M_6, \bar{M}_6$	$M_7, \bar{M}_7$
asymptotic frequency	$\frac{\Phi^{k-2}}{2\sqrt{5}}$	$\frac{\Phi^{k-1}}{10}$	$\frac{\Phi^{k-2}}{10}$	$\frac{\Phi^{k-3}}{10}$	$\frac{\Phi^{k-4}}{10}$
asymptotic relative frequency	$\frac{1}{2\,\Phi^2}$	$\frac{1}{2\sqrt{5}\,\Phi}$	$\frac{1}{2\sqrt{5}\,\Phi^2}$	$\frac{1}{2\sqrt{5}\,\Phi^3}$	$\frac{1}{2\sqrt{5}\,\Phi^4}$
three-letter subword	$N_4, \bar{N}_4$	$N_5, \bar{N}_5$ $N'_5, \bar{N}'_5$	$N_6, \bar{N}_6$ $N'_6, \bar{N}'_6$	$N_7, \bar{N}_7$ $N'_7, \bar{N}'_7$	$N'_8, \bar{N}'_8$
asymptotic frequency	$\frac{\Phi^{k-1}}{10}$	$\frac{\Phi^{k-2}}{10}$	$\frac{\Phi^{k-3}}{10}$	$\frac{\Phi^{k-4}}{10}$	$\frac{\Phi^{k-5}}{10}$
asymptotic relative frequency	$\frac{1}{2\sqrt{5}\,\Phi}$	$\frac{1}{2\sqrt{5}\,\Phi^2}$	$\frac{1}{2\sqrt{5}\,\Phi^3}$	$\frac{1}{2\sqrt{5}\,\Phi^4}$	$\frac{1}{2\sqrt{5}\,\Phi^5}$

13 Length of the Pisano curve

The length R_k of the Pisano curve is found by multiplication of the segment frequencies (Eq. (3)) with the segment lengths (Eq. (1)),

$$R_k = \left(F_{k-2}\, t^2 + G_{k-5}\, t + G_{k-4} + G_{k-6} + G_{k-3}\, t^{-1}\right) \frac{t^k}{\sqrt{2}} .$$

The approximation of this length for large k is

$$k \to \infty : \quad R_k \to \left(\frac{\Phi^{k-2}}{\sqrt{5}} t^2 + \frac{\Phi^{k-3}}{5} t + \frac{\Phi^{k-2} + \Phi^{k-4}}{5} + \frac{\Phi^{k-1}}{5} t^{-1}\right) \frac{t^k}{\sqrt{2}} = \frac{t^3\sqrt{2} + t^6\sqrt{10}}{5}\, \Phi^{k/2} .$$

Asymptotically, with each step the curve length increases by a factor $\sqrt{\Phi}$.

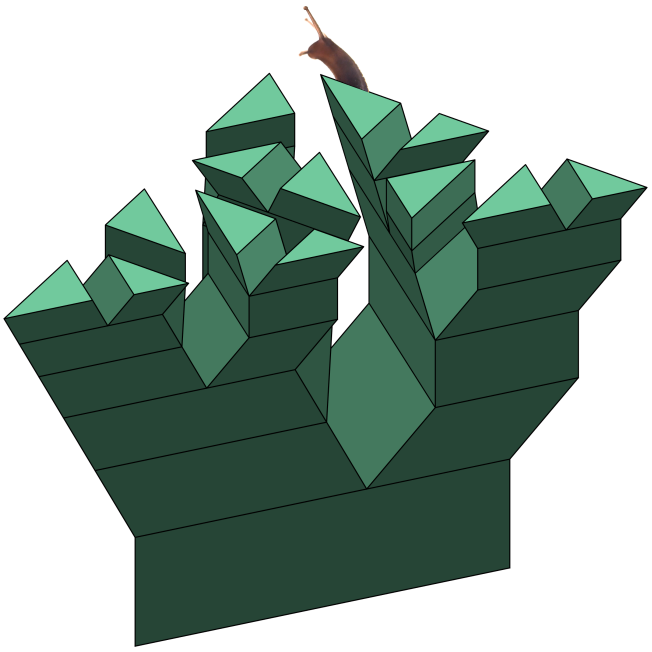


Figure 15: The triangle division process represented as the branching of a tree structure, from k=1 (bottom) to k=7 (top). Triangles have been reduced in size for clarity and effect.

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