

REAL SPECTRUM VERSUS ℓ -SPECTRUM VIA BRUMFIEL SPECTRUM

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ABSTRACT. It is well known that the real spectrum of any commutative unital ring, and the ℓ -spectrum of any Abelian lattice-ordered group with order-unit, are all completely normal spectral spaces. We prove the following results:

- (1) Every real spectrum can be embedded, as a spectral subspace, into some ℓ -spectrum.
- (2) Not every real spectrum is an ℓ -spectrum.
- (3) A spectral subspace of a real spectrum may not be a real spectrum.
- (4) Not every ℓ -spectrum can be embedded, as a spectral subspace, into a real spectrum.
- (5) There exists a completely normal spectral space which cannot be embedded, as a spectral subspace, into any ℓ -spectrum.

The commutative unital rings and Abelian lattice-ordered groups in (2), (3), (4) all have cardinality \aleph_1 , while the spectral space of (5) has a basis of cardinality \aleph_2 . Moreover, (3) solves a problem by Mellor and Tressl.

1. INTRODUCTION

Denote by \mathbf{SX} the class of all spectral subspaces of members of a class \mathbf{X} of spectral spaces. Most of the paper is devoted to proving the containments and non-containments, between classes of spectral spaces, represented in Figure 1.1. The classes in question are the following:

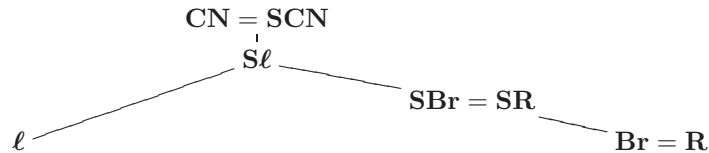


FIGURE 1.1. Classes of completely normal spectral spaces

- \mathbf{CN} , the class of all completely normal spectral spaces;
- ℓ , the class of ℓ -spectra of all Abelian ℓ -groups with order-unit;
- \mathbf{Br} , the class of Brumfiel spectra of all commutative unital f -rings;
- \mathbf{R} , the class of real spectra of all commutative unital rings.

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The context of our work is the following. The classical construction of the Zariski spectrum of a commutative unital ring (cf. Subsection 4.1) extends to many contexts, including distributive lattices, *lattice-ordered groups* (ℓ -groups for short), *partially ordered rings*, yielding *Stone duality*, the ℓ -spectrum, and the *real spectrum*, respectively. All the topological spaces thus obtained are *spectral spaces*, that is, sober spaces in which the compact open subsets are a basis of the topology, closed under finite intersection. Conversely, every spectral space is the spectrum of some bounded distributive lattice (Stone [30]) and also of some commutative unital ring (Hochster [16]).

The paper will focus on the ℓ -spectrum of an Abelian ℓ -group (cf. Subsection 4.2) and the *real spectrum* of a commutative unital ring (cf. Subsection 4.4). Those two frameworks are connected by the *Brumfiel spectrum* of a commutative f -ring (cf. Subsection 4.3). All the spectral spaces thus obtained are *completely normal*, that is, for all elements x and y in the closure of a singleton $\{z\}$, either x belongs to the closure of $\{y\}$ or y belongs to the closure of $\{x\}$.

Prior to the present paper, part of the picture (Figure 1.1) was already known:

- Delzell and Madden [8] proved that $\ell \subsetneq \mathbf{CN}$ and $\mathbf{R} \subsetneq \mathbf{CN}$.
- Delzell and Madden's result got amplified in Mellor and Tressl [22], who established that *any class of spectral spaces containing \mathbf{R} , whose Stone dual lattices are definable by a class of $\mathcal{L}_{\infty, \lambda}$ -formulas for some infinite cardinal λ , has a member outside \mathbf{SR}* . In particular, the class of all Stone duals of the spaces from \mathbf{R} (resp., \mathbf{SR}) are not $\mathcal{L}_{\infty, \lambda}$ -definable. Further, $\mathbf{SR} \subsetneq \mathbf{CN}$.
- Delzell and Madden [9, Proposition 3.3] observed that $\mathbf{R} \subseteq \mathbf{Br}$.
- It follows easily from Madden and Schwartz [29] and Schwartz [28] that $\mathbf{Br} \subseteq \mathbf{R}$. Consequently, $\mathbf{Br} = \mathbf{R}$ (cf. Corollary 4.17).
- The author proved in [33] that *every second countable completely normal spectral space is in ℓ* , and that moreover, *the class of all Stone duals of spaces from ℓ is not $\mathcal{L}_{\infty, \omega}$ -definable*.
- The author provided an example in [33, § 5] showing that $\ell \subsetneq \mathbf{S}\ell$.

The missing pieces provided in the present paper are the following:

- Every Brumfiel spectrum, thus also every real spectrum, can be embedded, as a spectral subspace, into some ℓ -spectrum. This is stated in Corollary 5.7. Hence, $\mathbf{SBr} \subseteq \mathbf{S}\ell$.
- Not every real spectrum is an ℓ -spectrum. This is established in Theorem 5.4, *via* the construction of a *condensate*. Hence, $\mathbf{R} \not\subseteq \ell$.
- A spectral subspace of a real spectrum may not be a Brumfiel spectrum (thus also not a real spectrum). This is stated in Corollary 5.10, *via* the construction of a condensate. It follows that $\mathbf{SR} \not\subseteq \mathbf{Br}$. This solves a problem of Mellor and Tressl [22].
- Not every ℓ -spectrum can be embedded, as a spectral subspace, into a Brumfiel spectrum (thus also not into a real spectrum). Hence, $\ell \not\subseteq \mathbf{SBr}$. This is stated in Corollary 6.8.
- There exists a completely normal spectral space which cannot be embedded, as a spectral subspace, into any ℓ -spectrum. This is stated in Corollary 7.6. Hence, $\mathbf{S}\ell \subsetneq \mathbf{CN}$. The spectral space constructed there has \aleph_2 compact open members. The proof begins by coining a class of infinitary statements satisfied by all homomorphic images of ℓ -representable lattices

(Lemma 7.1). Then the proof technique loosely follows the one introduced by the author in [31] for solving representation problems of congruence lattices of lattices and nonstable K-theory of von Neumann regular rings.

We point out that although not every ℓ -spectrum is a real spectrum (cf. Corollary 6.8), there is a formally related problem, on the interaction between Abelian ℓ -groups and commutative rings, with a well known positive solution. Let D be an integral domain with group of units U and field of fractions K . Denote by K^\times the multiplicative group of all nonzero elements of K . The *group of divisibility of D* (cf. Močkoř [24]) is the quotient group K^\times/U , endowed with the unique translation-invariant partial ordering with positive cone D/U . Every Abelian ℓ -group is the group of divisibility of some integral domain, which, in addition, can be taken a *Bezout domain* (cf. Anderson [1, page 4], where the result is credited to Krull, Jaffard, Kaplansky, and Ohm). Thus, Corollary 6.8 illustrates the gap between the group of divisibility and the real spectrum.

Since Stone duality is more conveniently stated with *bounded* distributive lattices, our results on spectral spaces are mostly formulated for commutative, *unital* rings and Abelian ℓ -groups *with order-unit*. On the other hand, most of our lattice-theoretical results are valid for lattices without top element, and thus formulated in that more general context.

2. BASIC CONCEPTS

For ℓ -groups and f -rings, we refer the reader to Bigard, Keimel, and Wolfenstein [6] or Anderson and Feil [2]. For any partially ordered Abelian group G , we set $G^+ \stackrel{\text{def}}{=} \{x \in G \mid x \geq 0\}$ (the *positive cone* of G) and $G^{++} \stackrel{\text{def}}{=} \{x \in G \mid x > 0\}$. For $a, b \in G^+$, let $a \ll b$ hold if $ka \leq b$ for every positive integer k .

For partially ordered Abelian groups G and H , the *lexicographical product* of G by H , denoted $G \times_{\text{lex}} H$, is the product group $G \times H$, endowed with the positive cone consisting of all pairs (x, y) with either $x > 0$ or $(x = 0 \text{ and } y \geq 0)$.

For any chain Λ , we denote by $\mathbb{Z}\langle\Lambda\rangle$ the lexicographical power, of the chain \mathbb{Z} of all integers, by Λ . Hence the elements of $\mathbb{Z}\langle\Lambda\rangle$ have the form $x = \sum_{i=1}^n k_i c_{\xi_i}$, where each $k_i \in \mathbb{Z} \setminus \{0\}$ and $\xi_1 < \dots < \xi_n$ in Λ , and x belongs to the positive cone of $\mathbb{Z}\langle\Lambda\rangle$ iff either $n = 0$ (i.e., $x = 0$) or $k_n > 0$. This endows $\mathbb{Z}\langle\Lambda\rangle$ with a structure of a totally ordered Abelian group.

A *lattice-ordered group*, or ℓ -group for short, is a group endowed with a translation-invariant lattice ordering. All our ℓ -groups will be Abelian and will thus be denoted additively. Elements x and y , in an ℓ -group, are *orthogonal* if $x \wedge y = 0$.

A subset I in an Abelian ℓ -group G is an ℓ -ideal if it simultaneously a subgroup of G and an order-convex sublattice of G .

For any elements a and b in an Abelian ℓ -group G , we will set $a^+ \stackrel{\text{def}}{=} a \vee 0$, $a^- \stackrel{\text{def}}{=} (-a) \vee 0$, $|a| \stackrel{\text{def}}{=} a \vee (-a)$, and $a \setminus b \stackrel{\text{def}}{=} (a - b)^+ = a - (a \wedge b)$.

A *lattice-ordered ring* is a ring endowed with a lattice ordering invariant under additive translations and preserved by multiplicative translations by positive elements. A lattice-ordered ring A is an *f -ring* if $x \wedge y = 0$ implies that $x \wedge yz = x \wedge zy = 0$ whenever $x, y, z \in A^+$ and $x \wedge y = 0$. Equivalently, A is a subdirect product of totally ordered (not necessarily unital) rings (cf. Bigard, Keimel, and Wolfenstein [6, Théorème 9.1.2]).

Lemma 2.1 (folklore). *Let G be an Abelian ℓ -group and let $a, b, c \in G$. Then*

- $a \setminus c \leq (a \setminus b) + (b \setminus c)$.
- $(a \setminus b) \wedge (b \setminus a) = 0$.
- If, in addition, G is the underlying additive ℓ -group of an f -ring A and $c \in A^+$, then $ca \setminus cb = c(a \setminus b)$ and $ac \setminus bc = (a \setminus b)c$.

A subset I in an f -ring A is an ℓ -ideal if it is, simultaneously, an ideal of the underlying ring of A and an order-convex sublattice of A .

Totally ordered rings are particular cases of f -rings. About those, we will need the following lemma.

Lemma 2.2. *Let A be a totally ordered (not necessarily unital) commutative domain and let I be a proper order-convex ideal of A . Then for every $x \in I$ and every $a \in A$, the relation $|xa| \ll |a|$ holds.*

Proof. We will use repeatedly the fact that for every $c \in A^{++}$, the assignment $t \mapsto tc$ defines an order-embedding of A into itself. Since A is totally ordered, we may assume that $a \geq 0$ and $x \geq 0$. Let $n < \omega$ and suppose that $nxa > a$ (so $a > 0$). Then for every $b \in A^+$, $nxab \geq ab \geq 0$, thus (as $a > 0$) $nxb \geq b \geq 0$. Since $x \in I$ and I is an ideal of A , we get $nxb \in I$. Since I is order-convex, it follows that $b \in I$. This holds for every $b \in A^+$, whence $I = A$, a contradiction. Since A is totally ordered, it follows that $nxa \leq a$. \square

For lattice theory we refer the reader to Grätzer [15], Johnstone [18]. For any elements a and b in a distributive lattice D with zero, a *splitting* of (a, b) is a pair (x, y) of elements of D such that $a \vee b = a \vee y = x \vee b$ and $x \wedge y = 0$. Observe that in that case, $x \leq a$ and $y \leq b$. We say that D is *completely normal* if every pair of elements in D has a splitting.

We denote by P^{op} the *opposite poset* of a poset P . For any functions f and g with common domain X , we set

$$\llbracket f \neq g \rrbracket \stackrel{\text{def}}{=} \{x \in X \mid f(x) \neq g(x)\}.$$

We denote by $\mathfrak{P}(X)$ the powerset of any set X , ordered under set inclusion. We denote by ω_α , or \aleph_α according to the context (“ordinal versus cardinal”), the α th infinite cardinal, and we set $\omega \stackrel{\text{def}}{=} \omega_0 = \{0, 1, 2, \dots\}$.

Throughout the paper, “countable” means “at most countable”.

3. STONE DUALITY BETWEEN DISTRIBUTIVE LATTICES WITH ZERO AND GENERALIZED SPECTRAL SPACES

In this section we recall a few well known facts on Stone duality for bounded distributive lattices. For references and more details, see Johnstone [18, § II.3], Grätzer [15, § II.5].

Definition 3.1. For a topological space X , we denote by $\overset{\circ}{\mathcal{K}}(X)$ the set of all compact¹ open subsets of X , ordered under set inclusion. We say that X is

- *sober*, if every join-irreducible member, of the lattice of all closed subsets of X , is the closure of a unique singleton²;

¹Throughout the paper, “compact” means what some other references call “quasicompact”; in particular, it does not imply Hausdorff.

²Due to the uniqueness, every sober space is T_0 (not all references assume this).

- *generalized spectral*, if it is sober, $\mathring{\mathcal{K}}(X)$ is a basis of the topology of X , and $U \cap V$ is compact whenever U and V are compact open subsets of X ;
- *spectral*, if it is simultaneously compact and generalized spectral.

The *specialization preorder* on X is defined by

$$x \leq y \text{ if } y \in \text{cl}_X(\{x\}), \quad \text{for all } x, y \in X.$$

The *spectrum* $\text{Spec } D$, of a distributive lattice D with zero, is defined as the set of all (proper) prime ideals of D , endowed with the closed sets $\{P \in \text{Spec } D \mid I \subseteq P\}$, for subsets (equivalently, ideals) I of D . The specialization order on $\text{Spec } D$ is just set-theoretical inclusion. The correspondence between distributive lattices with zero and generalized spectral spaces is spelled out in the following result, originating in Stone [30].

Theorem 3.2 (Stone).

- For every distributive lattice D with zero, the space $\text{Spec } D$ is generalized spectral and the assignment $a \mapsto \{P \in \text{Spec } D \mid a \notin P\}$ defines an isomorphism $\alpha_D: D \rightarrow \mathring{\mathcal{K}}(\text{Spec } D)$.
- For every generalized spectral space X , $\mathring{\mathcal{K}}(X)$ is a distributive lattice with zero and the assignment $x \mapsto \left\{ U \in \mathring{\mathcal{K}}(X) \mid x \notin U \right\}$ defines a homeomorphism $\xi_X: X \rightarrow \text{Spec } \mathring{\mathcal{K}}(X)$.

For distributive lattices D and E , a 0-lattice homomorphism $f: D \rightarrow E$, and $Q \in \text{Spec } E$, the inverse image $f^{-1}[Q]$ may be the whole of D , in which case it does not belong to $\text{Spec } D$ (prime ideals are assumed to be proper). This does not happen if we assume the map f to be *cofinal*, that is, every element of E is bounded above by some element of the range of f .

Say that a map $\varphi: X \rightarrow Y$, between topological spaces, is *spectral* if the inverse image under φ , of any compact open subset of Y , is a compact open subset of X . If X and Y are both generalized spectral, then the map $\mathring{\mathcal{K}}(\varphi): \mathring{\mathcal{K}}(Y) \rightarrow \mathring{\mathcal{K}}(X)$, $V \mapsto \varphi^{-1}[V]$ is a cofinal 0-lattice homomorphism. Hence we obtain the following statement of Stone's duality (spelled out in Rump and Yang [26, page 63]), which extends the classical Stone duality between bounded distributive lattices and spectral spaces.

Theorem 3.3 (Stone). *The category of all distributive lattices with zero, with cofinal 0-lattice homomorphisms, and the category of all generalized spectral spaces, with spectral maps, are dual, with respect to the natural transformations α and ξ given in Theorem 3.2 and the functors given as follows:*

- The dual of a distributive lattice D with zero is its spectrum $\text{Spec } D$. The dual of a 0-lattice homomorphism $f: D \rightarrow E$ is the map $\text{Spec } f: \text{Spec } E \rightarrow \text{Spec } D$, $Q \mapsto f^{-1}[Q]$.
- The dual of a generalized spectral space X is the lattice $\mathring{\mathcal{K}}(X)$. The dual of a spectral map $\varphi: X \rightarrow Y$ is the map $\mathring{\mathcal{K}}(\varphi): \mathring{\mathcal{K}}(Y) \rightarrow \mathring{\mathcal{K}}(X)$, $V \mapsto \varphi^{-1}[V]$.

Remark 3.4. The case where φ is the inclusion map from a generalized spectral space X into a generalized spectral space Y is interesting. We say that X is a

spectral subspace of Y if the topology of X is the topology induced by the topology of Y and the inclusion map from X into Y is spectral. In that case, the dual map $\mathring{\mathcal{K}}(Y) \rightarrow \mathring{\mathcal{K}}(X)$, $V \mapsto X \cap V$ is a surjective lattice homomorphism. Conversely, for every surjective lattice homomorphism $f: D \rightarrow E$, the spectral map $\text{Spec } f: \text{Spec } E \rightarrow \text{Spec } D$ is a *spectral embedding*, that is, it embeds $\text{Spec } E$ into $\text{Spec } D$ as a spectral subspace. Hence, *spectral subspaces correspond, via Stone duality, to surjective lattice homomorphisms*.

The generalized spectral spaces X that we will consider in this paper will mostly be *completely normal*. By Monteiro [23, Théorème V.3.1], this is equivalent to saying that the dual lattice $\mathring{\mathcal{K}}(X)$ is completely normal (cf. Section 2 for the definition of completely normal lattices).

4. ZARISKI, ℓ , BRUMFIEL, REAL: SPECTRA AND LATTICES

In this section we recall some well known facts on the various sorts of spectra and distributive lattices that will intervene in the paper. We also include a few new results, such as Lemma 4.11. For more details and references, we refer the reader to Delzell and Madden [9], Johnstone [18, Chapter 5], Keimel [19], Coste and Roy [7], Dickmann [10, Chapter 6].

4.1. Zariski spectrum. The (*Zariski*) *spectrum* of a commutative unital ring A is defined as the set $\text{Spec } A$ of all prime ideals of A , endowed with the topology whose closed sets are exactly the sets $\text{Spec}(A, I) \stackrel{\text{def}}{=} \{P \in \text{Spec } A \mid I \subseteq P\}$, for subsets (equivalently, radical ideals) I of A .

Denote by $\langle a_1, \dots, a_m \rangle^r$ the radical ideal of A generated by elements a_1, \dots, a_m of A , and denote³ by $\text{Id}_c^r A$ the set of all ideals of A of the form $\langle a_1, \dots, a_m \rangle^r$ (*finitely generated radical ideals*), ordered by set inclusion. Due to the formulas

$$\langle a_1, \dots, a_m \rangle^r \vee \langle b_1, \dots, b_n \rangle^r = \langle a_1, \dots, a_m, b_1, \dots, b_n \rangle^r, \quad (4.1)$$

$$\langle a_1, \dots, a_m \rangle^r \cap \langle b_1, \dots, b_n \rangle^r = \langle a_i b_j \mid 1 \leq i \leq m \text{ and } 1 \leq j \leq n \rangle^r, \quad (4.2)$$

(where \vee stands for the join in the lattice of all radical ideals of A), $\text{Id}_c^r A$ is a 0-sublattice of the distributive lattice of all radical ideals of A .

Since every radical ideal of A is the intersection of all prime ideals containing it, $\text{Id}_c^r A$ is the Stone dual of $\text{Spec } A$ (cf. Delzell and Madden [9, page 115]):

Proposition 4.1. *The Zariski spectrum $\text{Spec } A$, of a commutative unital ring A , is a spectral space, and the assignment $I \mapsto \{P \in \text{Spec } A \mid I \not\subseteq P\}$ defines an isomorphism from $\text{Id}_c^r A$ onto the Stone dual $\mathring{\mathcal{K}}(\text{Spec } A)$ of $\text{Spec } A$.*

Due to the following deep result by Hochster [16], there is no need to give a name to the class of all lattices of the form $\text{Id}_c^r A$.

Theorem 4.2 (Hochster). *Every spectral space is homeomorphic to the Zariski spectrum of some commutative unital ring. Hence, every bounded distributive lattice is isomorphic to $\text{Id}_c^r A$ for some commutative unital ring A .*

³The subscript “c” stands for “compact”, which is the lattice-theoretical formalization of “finitely generated”. The superscript “r” stands for “radical”.

4.2. ℓ -spectrum and ℓ -representable lattices. The ℓ -spectrum of an Abelian ℓ -group G is defined as the set $\text{Spec}_\ell G$ of all prime ℓ -ideals of G , endowed with the topology whose closed sets are exactly the $\{P \in \text{Spec}_\ell G \mid I \subseteq P\}$, for subsets (equivalently, ℓ -ideals) I of G .

Denote by $\langle a_1, \dots, a_m \rangle^\ell$, or $\langle a_1, \dots, a_m \rangle_G^\ell$ if G needs to be specified, the ℓ -ideal of G generated by elements a_1, \dots, a_m of G , and denote by $\text{Id}_c^\ell G$ the set of all ℓ -ideals of G of the form $\langle a_1, \dots, a_m \rangle^\ell$ (finitely generated ℓ -ideals), ordered by set inclusion. Since $\langle a_1, \dots, a_m \rangle^\ell = \langle a \rangle^\ell$ where $a \stackrel{\text{def}}{=} \sum_{i=1}^m |a_i|$, we get $\text{Id}_c^\ell G = \left\{ \langle a \rangle^\ell \mid a \in G^+ \right\}$. Due to the formulas

$$\langle a \rangle^\ell \vee \langle b \rangle^\ell = \langle a + b \rangle^\ell \text{ and } \langle a \rangle^\ell \cap \langle b \rangle^\ell = \langle a \wedge b \rangle^\ell, \quad \text{for all } a, b \in G^+ \quad (4.3)$$

(where \vee stands for the join in the lattice of all ℓ -ideals of A), $\text{Id}_c^\ell G$ is a 0-sublattice of the distributive lattice of all ℓ -ideals of G . It has a top element iff G has an order-unit.

Since every ℓ -ideal of G is the intersection of all prime ℓ -ideals containing it, $\text{Id}_c^\ell G$ is the Stone dual of $\text{Spec}_\ell G$:

Proposition 4.3. *The ℓ -spectrum $\text{Spec}_\ell G$, of any Abelian ℓ -group G , is a generalized spectral space, and the assignment $I \mapsto \{P \in \text{Spec}_\ell G \mid I \not\subseteq P\}$ defines an isomorphism from $\text{Id}_c^\ell G$ onto the Stone dual $\mathcal{K}(\text{Spec}_\ell G)$ of $\text{Spec}_\ell G$.*

Following terminology from Iberkleid, Martínez, and McGovern [17] and Wehrung [33], we recall the following definition.

Definition 4.4. For distributive lattices A and B , a map $f: A \rightarrow B$ is *closed* if for all $a_0, a_1 \in A$ and all $b \in B$, if $f(a_0) \leq f(a_1) \vee b$, then there exists $x \in A$ such that $a_0 \leq a_1 \vee x$ and $f(x) \leq b$.

The following lemma is established in Wehrung [33, § 3].

Lemma 4.5. *Let A and B be Abelian ℓ -groups and let $f: A \rightarrow B$ be an ℓ -homomorphism. Then the map $\text{Id}_c^\ell f: \text{Id}_c^\ell A \rightarrow \text{Id}_c^\ell B$, $\langle x \rangle^\ell \mapsto \langle f(x) \rangle^\ell$ is a closed 0-lattice homomorphism.*

In particular, the assignments $G \mapsto \text{Id}_c^\ell G$, $f \mapsto \text{Id}_c^\ell f$ define a functor, from the category of all Abelian ℓ -groups with ℓ -homomorphisms, to the category of all distributive lattices with zero with closed 0-lattice homomorphisms. It is well known that this functor preserves nonempty finite direct products and directed colimits.

Say that a lattice D is *ℓ -representable* if it is isomorphic to $\text{Id}_c^\ell G$ for some Abelian ℓ -group G . Equivalently, the spectrum of D is homeomorphic to the ℓ -spectrum of some Abelian ℓ -group. This terminology is extended to diagrams \vec{D} of distributive lattices with zero and 0-lattice homomorphisms, by saying that $\vec{D} \cong \text{Id}_c^\ell \vec{G}$ for some diagram \vec{G} of Abelian ℓ -groups and ℓ -homomorphisms.

It is well known that every ℓ -representable lattice is completely normal. The author established the following result in [33].

Theorem 4.6. *Every countable completely normal distributive lattice with zero is ℓ -representable. On the other hand, the class of all ℓ -representable lattices cannot be defined by a class of $\mathcal{L}_{\infty, \omega}$ -formulas of lattice theory.*

The following easy lemma is established in Wehrung [33, § 3].

Lemma 4.7. *Let G be an Abelian ℓ -group, let S be a distributive lattice with zero, and let $\varphi: \text{Id}_c^\ell G \rightarrow S$ be a closed surjective 0-lattice homomorphism. Then $I \stackrel{\text{def}}{=}} \{x \in G \mid \varphi(\langle x \rangle^\ell) = 0\}$ is an ℓ -ideal of G , and there is a unique isomorphism $\psi: \text{Id}_c^\ell(G/I) \rightarrow S$ such that $\psi(\langle x + I \rangle^\ell) = \varphi(\langle x \rangle^\ell)$ for every $x \in G^+$.*

4.3. Brumfiel spectrum and Brumfiel-representable lattices. For any commutative f -ring A , we say that a (proper) ℓ -ideal P is *prime* if it is both an ℓ -ideal and prime as a ring ideal. Then P is also prime as an ℓ -ideal of the underlying additive ℓ -group of A , that is, $x \wedge y \in P$ implies that either $x \in P$ or $y \in P$, whenever $x, y \in A$.

The *Brumfiel spectrum* of a commutative f -ring A is defined as the set $\text{Spec}_B A$ of all prime ℓ -ideals of A , endowed with the topology whose closed sets are exactly the $\{P \in \text{Spec}_B A \mid I \subseteq P\}$, for subsets (equivalently, radical ℓ -ideals) I of A .

Denote by $\langle a_1, \dots, a_m \rangle^r$, or $\langle a_1, \dots, a_m \rangle_A^r$ if A needs to be specified, the radical ℓ -ideal of A generated by elements a_1, \dots, a_m of A , and denote by $\text{Id}_c^r A$ the set of all ideals of A of the form $\langle a_1, \dots, a_m \rangle^r$ (*finitely generated radical ideals*),⁴ ordered by set inclusion. Since $\langle a_1, \dots, a_m \rangle^r = \langle a \rangle^\ell$ where $a \stackrel{\text{def}}{=} \sum_{i=1}^m |a_i|$, we get $\text{Id}_c^r A = \{\langle a \rangle^r \mid a \in A^+\}$. Due to the formulas

$$\langle a \rangle^r \vee \langle b \rangle^r = \langle |a| + |b| \rangle^r \text{ and } \langle a \rangle^r \cap \langle b \rangle^r = \langle |a| \wedge |b| \rangle^r = \langle ab \rangle^r, \quad \text{for all } a, b \in A \quad (4.4)$$

(where \vee stands for the join in the lattice of all radical ℓ -ideals of A), $\text{Id}_c^r A$ is a 0-sublattice of the distributive lattice of all radical ℓ -ideals of A . If A is unital, then $\text{Id}_c^r A$ has a top element.

Since every radical ℓ -ideal of A is the intersection of all prime ℓ -ideals containing it, $\text{Id}_c^r A$ is the Stone dual of $\text{Spec}_B A$ (cf. Delzell and Madden [9, Proposition 4.2]):

Proposition 4.8. *The Brumfiel spectrum $\text{Spec}_B A$, of a commutative f -ring A , is a generalized spectral space, and the assignment $I \mapsto \{P \in \text{Spec}_B A \mid I \not\subseteq P\}$ defines an isomorphism from $\text{Id}_c^r A$ onto the Stone dual $\overset{\circ}{\mathcal{K}}(\text{Spec}_B A)$ of $\text{Spec}_B A$.*

Say that a lattice D is *Brumfiel-representable* if it is isomorphic to $\text{Id}_c^r A$ for some commutative f -ring A . Equivalently, the spectrum of D is homeomorphic to the Brumfiel spectrum of some commutative f -ring. As in Subsection 4.2, this terminology is extended to diagrams of lattices, in a standard fashion.

It is well known that every Brumfiel-representable lattice is completely normal. We will see, with Corollary 6.8 in the present paper, that not every ℓ -representable lattice (thus, *a fortiori*, not every completely normal distributive lattice) is Brumfiel-representable.

The following result is an analogue of Lemma 4.7 for f -rings. Its proof is similar and we omit it.

Lemma 4.9. *Let A be a commutative f -ring, let S be a distributive lattice with zero, and let $\varphi: \text{Id}_c^r A \rightarrow S$ be a closed surjective 0-lattice homomorphism. Then $I \stackrel{\text{def}}{=} \{x \in A \mid \varphi(\langle x \rangle^r) = 0\}$ is a radical ℓ -ideal of A , and there is a unique isomorphism $\psi: \text{Id}_c^r(A/I) \rightarrow S$ such that $\psi(\langle x + I \rangle^r) = \varphi(\langle x \rangle^r)$ for every $x \in A^+$.*

⁴Although we are using, for radical ℓ -ideals, the same notation as the one in Subsection 4.1 for radical ideals, the context will always make it clear which concept is used.

Definition 4.10. Let A and B be distributive lattices with zero. A 0-lattice homomorphism $f: A \rightarrow B$ is *convex* if for all $P \in \text{Spec } A$, all $Q_0 \in \text{Spec } B$, and every proper ideal J of B , if $Q_0 \subseteq J$ and $f^{-1}[Q_0] \subseteq P \subseteq f^{-1}[J]$, then there exists $Q \in \text{Spec } B$ such that $Q_0 \subseteq Q \subseteq J$ and $P = f^{-1}[Q]$.

The following result extends to the Brumfiel spectrum functor a result originally established for the real spectrum functor in Korollar 4, pages 133–134 of Knebusch and Scheiderer [20]. Our proof is a straightforward modification of its analogue for real spectra, stated in the forthcoming monograph Dickmann, Schwartz, and Tressl [12, Theorem 12.3.12].

Lemma 4.11. *Let A and B be commutative f -rings and let $f: A \rightarrow B$ be a homomorphism of f -rings. Then the map $\text{Id}_c^f: \text{Id}_c^f A \rightarrow \text{Id}_c^f B$ is convex.*

Proof. We must prove that for every $P \in \text{Spec}_B A$, all $Q_0 \in \text{Spec}_B B$, every proper radical ℓ -ideal J of B , if $Q_0 \subseteq J$ and $f^{-1}[Q_0] \subseteq P \subseteq f^{-1}[J]$, then there exists $Q \in \text{Spec}_B B$ such that $Q_0 \subseteq Q \subseteq J$ and $P = f^{-1}[Q]$. We may replace A by $A/f^{-1}[Q_0]$, B by B/Q_0 , J by J/Q_0 , and f by the canonical embedding $A/f^{-1}[Q_0] \rightarrow B/Q_0$. Hence, we may assume that A is an ordered subring of a totally ordered (not necessarily unital) commutative domain B , $Q_0 = \{0\}$, f is the inclusion map from A into B , $P \in \text{Spec}_B A$, J is a proper radical ℓ -ideal of B , and $P \subseteq J$. We must find $Q \in \text{Spec}_B B$ such that $P = Q \cap A$ and $Q \subseteq J$. We set

$$Q \stackrel{\text{def}}{=} \{y \in J \mid (\exists n < \omega)(\exists x \in P)(|y|^n \leq x)\}.$$

We claim that Q is a prime ℓ -ideal of B . It is obvious that Q is an order-convex ℓ -subgroup of B . Now let $y \in Q$ and $b \in B$. We must prove that $yb \in Q$. Since B is totally ordered, we may assume that $y, b \in B^+$. Since J is an ideal of B , $yb \in J$. By assumption, there are $n < \omega$ and $x \in P$ such that $y^n \leq x$. It follows that $y^n \in J$, thus, since J is a radical ideal of B , $y \in J$, and thus $yb^{n+1} \in J$. Since J is a proper ℓ -ideal of B , it follows, using Lemma 2.2, that $(yb)^{n+1} = y^n(yb^{n+1}) \leq y^n \leq x$, whence $yb \in Q$. This completes the proof that Q is an ℓ -ideal of B .

Let $x, y \in B$ such that $xy \in Q$, we must prove that $x \in Q$ or $y \in Q$. Since B is totally ordered, we may assume that $0 \leq x \leq y$. There are $n < \omega$ and $p \in P$ such that $(xy)^n \leq p$. It follows that $x^{2n} \leq (xy)^n \leq p$, whence $x \in Q$, thus completing the proof that Q is prime.

Now it is obvious that $Q \subseteq J$ and $P = Q \cap A$. □

4.4. Real spectrum and real-representable lattices. Let A be a commutative unital ring. A subset C of A is a *cone* if it is both an additive and a multiplicative submonoid of A , containing all squares in A . A cone P of A is *prime* if $A = P \cup (-P)$ and the *support* $P \cap (-P)$ is a prime ideal of A . For a prime cone P , $-1 \notin P$ (otherwise $1 \in P \cap (-P)$, thus $P \cap (-P) = A$, a contradiction). We denote by $\text{Spec}_r A$ the set of all prime cones of A , endowed with the topology generated by all subsets of the form $\{P \in \text{Spec}_r A \mid a \notin P\}$, for $a \in A$, and we call $\text{Spec}_r A$ the *real spectrum of A* .

It is not so straightforward to describe directly the Stone dual of $\text{Spec}_r A$. However, it is possible to reduce it to the Brumfiel spectrum, as follows. The *universal f -ring* $F(A)$ of A is a commutative unital f -ring. The first statement in the following result is established in Delzell and Madden [9, Proposition 3.3]. The second statement follows by using Proposition 4.8.

Theorem 4.12 (Delzell and Madden). *Let A be a commutative unital ring. The canonical homomorphism $A \rightarrow F(A)$ induces a homeomorphism between the real spectrum of A and the Brumfiel spectrum of $F(A)$. Hence, the Stone dual of $\text{Spec}_r A$ is $\text{Id}_c^r F(A)$.*

Say that a lattice is *real-representable* if it is isomorphic to the Stone dual of the real spectrum of some commutative unital ring. As in Subsection 4.2, this terminology is extended to diagrams of lattices, in a standard fashion. It follows from Theorem 4.12 that every real-representable lattice is isomorphic to $\text{Id}_c^r A$ for some commutative unital f -ring A (thus it is Brumfiel-representable). We will see shortly that the converse holds (cf. Corollary 4.17).

Every real-representable lattice is completely normal. By Delzell and Madden [8], not every completely normal bounded distributive lattice can be represented in this way. In fact, Mellor and Tressl established in [22] the following result.

Theorem 4.13 (Mellor and Tressl). *For every infinite cardinal λ , the class of all real-representable lattices cannot be defined by any class of $\mathcal{L}_{\infty, \lambda}$ -formulas of lattice theory.*

The real spectrum can also be reduced to the Zariski spectrum, as follows. A commutative unital ring A is *real-closed* (cf. Schwartz [27, 28], Prestel and Schwartz [25]) if it has no nonzero nilpotent elements, the squares in A form the positive cone of a structure of f -ring on A , $0 \leq a \leq b$ implies that $a^2 \in Ab$, and for every prime ideal P of A , the quotient field $A(P)$ of A/P is real closed, and A/P is integrally closed in $A(P)$. Every commutative unital ring has a “real closure” $C_r(A)$, which is a real-closed ring together with a unital ring homomorphism $A \rightarrow C_r(A)$. The following result is contained in Theorem I.3.10, Propositions I.3.19 and I.3.23, and the top of page 27, in Schwartz [28].

Theorem 4.14 (Schwartz). *For any commutative, unital ring A , the canonical homomorphism $A \rightarrow C_r(A)$ induces a homeomorphism $\text{Spec}_r C_r(A) \rightarrow \text{Spec}_r A$. Moreover, if A is real-closed, then the support map $P \mapsto P \cap (-P)$ induces a homeomorphism $\text{Spec}_r A \rightarrow \text{Spec } A$.*

Corollary 4.15. *For any commutative, unital ring A , the Stone dual of the real spectrum of A is isomorphic to the lattice $\text{Id}_c^r C_r(A)$ of all finitely generated radical ideals of $C_r(A)$.*

Although the two following corollaries are probably well known, we could not find them explicitly stated anywhere, so we include proofs for convenience.

Corollary 4.16. *Every closed subspace of a real spectrum is a real spectrum.*

Proof. By Theorem 4.14, every real spectrum has the form $\text{Spec } A$ for some real-closed ring A . By definition, any closed subspace of $\text{Spec } A$ has the form $\text{Spec}(A, I) \stackrel{\text{def}}{=} \{P \in \text{Spec } A \mid I \subseteq P\}$, for a subset I of A , which we may assume to be a radical ideal of A . It follows that the assignment $P \mapsto P/I$ defines a homeomorphism from $\text{Spec}(A, I)$ onto $\text{Spec}(A/I)$. Now it follows from Schwartz [28, Theorem I.4.5] that the ring A/I is real-closed. By the second part of Theorem 4.14, it follows that $\text{Spec}(A/I)$ is homeomorphic to $\text{Spec}_r(A/I)$. \square

Corollary 4.17. *The class of real spectra of all commutative unital rings and the class of Brumfiel spectra of all commutative unital f -rings are identical.*

Proof. It follows from Theorem 4.12 that every real spectrum is the Brumfiel spectrum of some commutative unital f -ring. Conversely, for every commutative unital f -ring A , the assignment $P \mapsto A^+ + P$ defines a homeomorphism from $\text{Spec}_B A$ onto $\text{Spec}_r(A, A^+) \stackrel{\text{def}}{=} \{Q \in \text{Spec}_r A \mid A^+ \subseteq Q\}$, with inverse the support map $Q \mapsto Q \cap (-Q)$ (cf. Madden and Schwartz [29, page 49]). Since $\text{Spec}_r(A, A^+)$ is, by definition, a closed subspace of $\text{Spec}_r A$, it follows from Corollary 4.16 that $\text{Spec}_B A$ is the real spectrum of some commutative unital ring. \square

While real-representability makes sense only for bounded lattices, Brumfiel-representability can also be defined for unbounded lattices. According to the following corollary, the two concepts agree on bounded lattices.

Corollary 4.18. *A bounded distributive lattice is real-representable iff it is Brumfiel-representable.*

Proof. By Stone duality and Corollary 4.17, it suffices to prove that if a bounded distributive lattice is Brumfiel-representable, then it can be represented by a commutative unital f -ring. Let A be a commutative f -ring such that $\text{Id}_c^r A$ has a top element. Thus, $A = \langle u \rangle_A^r$ for some $u \in A^+$. The localization $A[u^{-1}]$ of A with respect to the multiplicative subset $S \stackrel{\text{def}}{=} \{u^n \mid 0 < n < \omega\}$, endowed with the positive cone $A[u^{-1}]^+ \stackrel{\text{def}}{=} \{x/u^n \mid x \in A^+, 0 < n < \omega\}$, is a commutative unital⁵ f -ring, for which the canonical homomorphism $\kappa: A \rightarrow A[u^{-1}]$, $x \mapsto xu/u$ is an f -ring homomorphism. Obviously, the induced map $\kappa: \text{Id}_c^r A \rightarrow \text{Id}_c^r A[u^{-1}]$ is surjective. We claim that κ is one-to-one. Let $x, y \in A^+$ such that $\kappa(\langle x \rangle_A^r) \subseteq \kappa(\langle y \rangle_A^r)$. This means that $\langle \kappa(x) \rangle_{A[u^{-1}]}^r \subseteq \langle \kappa(y) \rangle_{A[u^{-1}]}^r$, that is, there are $z \in A^+$ and positive integers m, n such that $\kappa(x)^m \leq (z/u^n)\kappa(y)$. It follows that there are positive integers k, l such that $x^m u^k \leq zy u^l$ within A . Since $\langle u^k \rangle_A^r = \langle u \rangle_A^r = A$, we obtain, using (4.4),

$$\langle x \rangle_A^r = \langle x^m \rangle_A^r = \langle x^m \rangle_A^r \cap \langle u^k \rangle_A^r = \langle x^m u^k \rangle_A^r \subseteq \langle zy u^l \rangle_A^r \subseteq \langle y \rangle_A^r,$$

as required. Therefore, $\text{Id}_c^r A \cong \text{Id}_c^r A[u^{-1}]$, with $A[u^{-1}]$ a commutative unital f -ring. \square

5. COUNTEREXAMPLES CONSTRUCTED FROM CONDENSATES

In the present section we shall apply the construction of a *condensate*, put to use in the author's paper [32] for one arrow and studied in depth in Gillibert and Wehrung [14] for more complicated diagrams. This construction enables us to construct non-representable *objects* from non-representable *arrows* (cf. Theorems 5.4 and 5.9), and it runs as follows.

Definition 5.1. Let A and B be universal algebras (in a given similarity type) and let I be a set. The *I -condensate* of a homomorphism $\varphi: A \rightarrow B$ is the following subalgebra of $A \times B^I$:

$$\text{Cond}(\varphi, I) \stackrel{\text{def}}{=} \{(x, y) \in A \times B^I \mid y_i = \varphi(x) \text{ for all but finitely many } i\}.$$

Lemma 5.2. *The condensate $\text{Cond}(\varphi, I)$ is a directed union of copies of algebras of the form $A \times B^J$ for finite $J \subseteq I$.*

⁵This also applies to the degenerate case where u is nilpotent, in which case $A[u^{-1}]$ collapses to the zero element.

Proof. For each finite $J \subseteq I$, denote by C_J the subalgebra of $\text{Cond}(\varphi, I)$ consisting of all pairs $(x, y) \in A \times B^J$ such that y is constant on $I \setminus J$, with value $\varphi(x)$. Then $\text{Cond}(\varphi, I)$ is the directed union of all C_J . Clearly, $C_J \cong A \times B^J$. \square

In particular, it follows from Lemma 5.2 that if A and B are members of a class \mathcal{C} of algebras, closed under nonempty finite direct products and directed colimits, then so is the condensate $\text{Cond}(\varphi, I)$. For example:

- Whenever A and B are distributive lattices with zero and φ is a 0-lattice homomorphism, then $\text{Cond}(\varphi, I)$ is a distributive lattice with zero.
- Whenever A and B are Abelian ℓ -groups and φ is an ℓ -homomorphism, then $\text{Cond}(\varphi, I)$ is an Abelian ℓ -group.
- Whenever A and B are real-closed rings and φ is a unital ring homomorphism, then $\text{Cond}(\varphi, I)$ is a real-closed ring.

The proof of the following lemma is an extension of the one of Wehrung [32, Theorem 9.3]. It is also an instance of a much more general result, called the *Condensate Lifting Lemma (CLL)*, established in Gillibert and Wehrung [14], that enables to infer representability of lattice-indexed diagrams from representability of certain larger objects, also called there condensates. In particular, as we will see shortly, Lemma 5.3 can be extended to (many) other functors than Id_c^ℓ . The use of CLL requires quite an amount of technical steps and although it may be unavoidable for complicated diagrams, the case of one arrow can be handled directly; thus we include here a self-contained proof for convenience.

Lemma 5.3. *Let A and B be distributive lattices with zero, with A countable, and let $\varphi: A \rightarrow B$ be a non- ℓ -representable 0-lattice homomorphism. Then for any set I , if the lattice homomorphism φ is ℓ -representable, then the distributive lattice $\text{Cond}(\varphi, I)$ is ℓ -representable. If I is uncountable, then the converse holds.*

Proof. Suppose first that φ is ℓ -representable. This means that there are Abelian ℓ -groups G and H , together with an ℓ -homomorphism $f: G \rightarrow H$, such that $\text{Id}_c^\ell f \cong \varphi$ (as arrows). Since the functor Id_c^ℓ commutes with nonempty finite direct products and with directed colimits, it sends the representation

$$\text{Cond}(f, I) = \varinjlim_{J \subset I \text{ finite}} (G \times H^J),$$

given by Lemma 5.2, to a representation

$$\text{Id}_c^\ell \text{Cond}(f, I) \cong \varinjlim_{J \subset I \text{ finite}} (A \times B^J) \cong \text{Cond}(\varphi, I).$$

Hence, $\text{Cond}(\varphi, I)$ is ℓ -representable.

Suppose, conversely, that I is uncountable and the distributive lattice $E \stackrel{\text{def}}{=} \text{Cond}(\varphi, I)$ is ℓ -representable. This means that there are an Abelian ℓ -group G and an isomorphism $\varepsilon: \text{Id}_c^\ell G \rightarrow E$. Denote by $p: E \rightarrow A$ and $q_i: E \rightarrow B$, for $i \in I$, the canonical projections from E . The set $U \stackrel{\text{def}}{=} \left\{ x \in G \mid (p \circ \varepsilon)(\langle x \rangle_G^\ell) = 0 \right\}$ is an ℓ -ideal of G ; denote by $u: G \rightarrow G/U$ the canonical projection. Since p is easily seen to be a closed map (cf. Definition 4.4), it follows from Lemma 4.7 that there is a unique isomorphism $\alpha: \text{Id}_c^\ell(G/U) \rightarrow A$ such that

$$p \circ \varepsilon = \alpha \circ (\text{Id}_c^\ell u). \tag{5.1}$$

Likewise, for every $i \in I$, the set $V_i \stackrel{\text{def}}{=} \{x \in G \mid (q_i \circ \varepsilon)(\langle x \rangle_G^\ell) = 0\}$ is an ℓ -ideal of G ; denote by $v_i: G \rightarrow G/V_i$ the canonical projection. As for U and α , there is a unique isomorphism $\beta_i: \text{Id}_c^\ell(G/V_i) \rightarrow B$ such that

$$q_i \circ \varepsilon = \beta_i \circ (\text{Id}_c^\ell v_i). \quad (5.2)$$

Since A is countable, a standard Löwenheim-Skolem type argument shows that there exists a countable ℓ -subgroup H of G such that α induces an isomorphism $\alpha': \text{Id}_c^\ell(H/U) \rightarrow A$ (where we set $H/U \stackrel{\text{def}}{=} \{x + U \mid x \in H\}$). We are going to argue that for every index i outside a certain countable subset of I , the ℓ -homomorphism $f: H/U \rightarrow G/V_i$, $x + U \mapsto x + V_i$ is well-defined and represents the lattice homomorphism $\varphi: A \rightarrow B$. Our argument is partly illustrated in Figure 5.1. The diagram of Figure 5.1 is not commutative, because $\varphi \circ p \neq q_i$ as a rule.

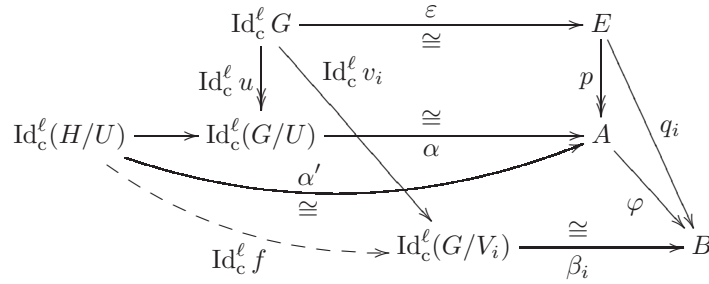


FIGURE 5.1. Illustrating the proof of Lemma 5.3

For each $x \in H$, the element $\varepsilon(\langle x \rangle_G^\ell)$ belongs to E , that is, $(\varphi \circ p)(\varepsilon(\langle x \rangle_G^\ell)) = q_i(\varepsilon(\langle x \rangle_G^\ell))$ for all but finitely many i . Since H is countable, there exists a countable subset J of I such that

$$(\varphi \circ p)(\varepsilon(\langle x \rangle_G^\ell)) = q_i(\varepsilon(\langle x \rangle_G^\ell)) \text{ for all } x \in H \text{ and all } i \in I \setminus J. \quad (5.3)$$

Pick $i \in I \setminus J$. We claim that $H \cap U \subseteq V_i$. Every $x \in H \cap U$ satisfies the equations

$$(\text{Id}_c^\ell u)(\langle x \rangle_G^\ell) = \langle x + U \rangle_{G/U}^\ell = 0,$$

thus, using (5.1), $p(\varepsilon(\langle x \rangle_G^\ell)) = (\alpha \circ \text{Id}_c^\ell u)(\langle x \rangle_G^\ell) = 0$. By (5.3), it follows that $q_i(\varepsilon(\langle x \rangle_G^\ell)) = 0$. By (5.2), this means that $\beta_i(\langle x + V_i \rangle_{G/V_i}^\ell) = 0$, that is, since β_i is an isomorphism, $x \in V_i$, thus completing the proof of our claim.

It follows that there exists a unique ℓ -homomorphism $f: H/U \rightarrow G/V_i$ such that $f(x+U) = x+V_i$ for every $x \in H$. For every $x \in H$,

$$\begin{aligned}
(\varphi \circ \alpha')(\langle x+U \rangle_{H/U}^\ell) &= (\varphi \circ \alpha)(\langle x+U \rangle_{G/U}^\ell) \\
&= (\varphi \circ \alpha \circ \text{Id}_c^\ell u)(\langle x \rangle_G^\ell) \\
&= (\varphi \circ p \circ \varepsilon)(\langle x \rangle_G^\ell) && \text{(use (5.1))} \\
&= (q_i \circ \varepsilon)(\langle x \rangle_G^\ell) && \text{(use (5.3))} \\
&= (\beta_i \circ \text{Id}_c^\ell v_i)(\langle x \rangle_G^\ell) && \text{(use (5.2))} \\
&= \beta_i(\langle x+V_i \rangle_{G/V_i}^\ell) \\
&= (\beta_i \circ \text{Id}_c^\ell f)(\langle x+U \rangle_{H/U}^\ell),
\end{aligned}$$

so $\varphi \circ \alpha' = \beta_i \circ \text{Id}_c^\ell f$. Therefore, f represents φ . \square

Theorem 5.4. *There exists a real-closed ring, of cardinality \aleph_1 , whose real spectrum is not homeomorphic to the ℓ -spectrum of any Abelian ℓ -group.*

Proof. Let K be any countable, non-Archimedean real-closed field. The subset

$$A \stackrel{\text{def}}{=} \{x \in K \mid (\exists n < \omega)(-n \cdot 1_K \leq x \leq n \cdot 1_K)\}$$

is an order-convex unital subring of K , thus it is a real-closed ring. Hence, denoting by $\varepsilon: A \hookrightarrow K$ the inclusion map, it follows from Lemma 5.2 that the condensate $R = \text{Cond}(\varepsilon, \omega_1)$ is a real-closed ring. Observe that the cardinality of R is \aleph_1 .

Suppose that the real spectrum of R is homeomorphic to the ℓ -spectrum of an Abelian ℓ -group G . By Proposition 3.2, it follows that $\text{Id}_c^r R \cong \text{Id}_c^\ell G$. Since the functor Id_c^r commutes with nonempty finite direct products and directed colimits, it follows from Lemma 5.2 that $\text{Id}_c^r R \cong \text{Cond}(\text{Id}_c^r \varepsilon, \omega_1)$. In particular, the distributive lattice $\text{Cond}(\text{Id}_c^r \varepsilon, \omega_1)$ is ℓ -representable. By Lemma 5.3, it follows that the lattice homomorphism $\text{Id}_c^r \varepsilon$ is ℓ -representable. By Lemma 4.5, it follows that the map $\varepsilon = \text{Id}_c^r \varepsilon$ is closed.

However, $\text{Id}_c^r A$ is a chain with more than two elements, $\text{Id}_c^r K$ is the two-element chain, and ε is the unique zero-separating map $\text{Id}_c^r A \rightarrow \text{Id}_c^r K$. In particular, if $0 < u < 1$ in $\text{Id}_c^r A$, then $\varepsilon(1) = \varepsilon(u) \vee 0$ but there is no $x \in \text{Id}_c^r A$ such that $1 \leq u \vee x$ and $\varepsilon(x) \leq 0$. Hence, ε is not closed. \square

Remark 5.5. By (the proof of) Dickmann, Gluschankof, and Lucas [11, Proposition 1.1], the field K and the ring A , of the proof of Theorem 5.4, can be constructed in such a way that $\text{Id}_c^r A$ is the three-element chain. By a simple Löwenheim-Skolem type argument, K may be taken countable. Then the lattice $\text{Cond}(\text{Id}_c^r \varepsilon, \omega_1)$ is isomorphic to the lattice \mathbf{D}_{ω_1} introduced in Wehrung [33, § 5]. As observed in the final example of Wehrung [33, § 5], \mathbf{D}_{ω_1} is a homomorphic image of an ℓ -representable distributive lattice, without being ℓ -representable itself. By Theorem 3.3, this means that the spectrum of \mathbf{D}_{ω_1} can be embedded, as a spectral subspace, into the ℓ -spectrum of an Abelian ℓ -group, without being an ℓ -spectrum itself.

The negative property satisfied by the counterexample R of Theorem 5.4 cannot be strengthened further by replacing “ ℓ -spectrum” by “spectral subspace of an ℓ -spectrum”. The reason for this is the following easy observation, which ought to be well known but for which we could not locate a reference.

Proposition 5.6. *For every commutative f -ring A , there are an Abelian ℓ -group G , which can be taken with order-unit if A is unital, together with a surjective lattice homomorphism $\mu: \text{Id}_c^\ell G \rightarrow \text{Id}_c^r A$. Hence, every Brumfiel-representable lattice is a homomorphic image of some ℓ -representable lattice.*

Proof. Denote by G the underlying additive ℓ -group of A . It is easy to verify that the map $\text{Id}_c^\ell G \rightarrow \text{Id}_c^r A$, $\langle x \rangle^\ell \mapsto \langle x \rangle^r$ is a well defined lattice homomorphism (use (4.3) and (4.4)). It is, of course, surjective.

Now suppose that A is unital. Since the multiplicative unit 1 of A may not be an order-unit of A , the construction above of G must be modified. To this end, define G as the underlying ℓ -group of $\{x \in A \mid (\exists n < \omega)(-n \cdot 1 \leq x \leq n \cdot 1)\}$, and define again $\mu: \langle x \rangle^\ell \mapsto \langle x \rangle^r$. We need to prove that μ is surjective. For every $x \in A^+$, the relations $x = (x \vee 1) \cdot (x \wedge 1)$ and $0 \leq x \wedge 1 \leq x$ imply that x and $x \wedge 1$ generate the same ℓ -ideal, and thus, *a fortiori*, $\langle x \rangle^r = \langle x \wedge 1 \rangle^r = \mu(\langle x \wedge 1 \rangle^\ell)$. \square

By applying Theorem 3.3 and Remark 3.4 to Proposition 5.6, we thus get

Corollary 5.7. *The Brumfiel spectrum of any commutative unital f -ring (thus also every real spectrum) is a spectral subspace of the ℓ -spectrum of some Abelian ℓ -group with order-unit.*

Moving to f -rings, a *mutatis mutandis* modification of the proof of Lemma 5.3, using Lemma 4.9 instead of Lemma 4.7, leads to the following result.

Lemma 5.8. *Let A and B be distributive lattices with zero, with A countable, and let $\varphi: A \rightarrow B$ be a non-Brumfiel-representable 0 -lattice homomorphism. Then for any set I , if the lattice homomorphism φ is Brumfiel-representable, then the distributive lattice $\text{Cond}(\varphi, I)$ is Brumfiel-representable. If I is uncountable, then the converse holds.*

Theorem 5.9. *There exists a real-representable distributive lattice E , of cardinality \aleph_1 , with a non-Brumfiel-representable (thus non-real-representable) homomorphic image.*

Proof. It follows from Dickmann, Gluschankof, and Lucas [11, Proposition 1.1] that there exists a real-closed domain A with exactly three prime ideals $P_1 \subset P_2 \subset P_3$. Hence, $\text{Id}_c^r A$ is isomorphic to the four-element chain $\mathbf{4} =_{\text{def}} \{0, 1, 2, 3\}$. By a simple Löwenheim-Skolem type argument, A may be taken countable. Denote by R the ring of all almost constant families $(x_\xi \mid \xi < \omega_1)$ of elements of A . Then the lattice E of all almost constant ω_1 -sequences of elements in $\mathbf{4}$ is isomorphic to $\text{Id}_c^r R$, thus it is real-representable.

Consider the chains $\mathbf{3} =_{\text{def}} \{0, 1, 2\}$ and $\mathbf{4} =_{\text{def}} \{0, 1, 2, 3\}$. The (surjective) dual homomorphism $\varphi: \mathbf{4} \rightarrow \mathbf{3}$, of the map $\{1, 2\} \rightarrow \{1, 2, 3\}$ sending 1 to 1 and 2 to 3 , is not convex, thus, by Lemma 4.11, not Brumfiel-representable. By Lemma 5.8, it follows that the lattice $\text{Cond}(\varphi, \omega_1)$ is not Brumfiel-representable.

On the other hand, the assignment $(x_\xi \mid \xi < \omega_1) \mapsto (x_\infty, (\varphi(x_\xi) \mid \xi < \omega_1))$ defines a surjective lattice homomorphism from E onto $\text{Cond}(\varphi, \omega_1)$. \square

By Stone duality (cf. Theorem 3.3 and Remark 3.4), it follows that the spectrum of the bounded distributive lattice $\text{Cond}(\varphi, \omega_1)$ witnesses the following corollary.

Corollary 5.10. *There exists a real spectrum with a spectral subspace which is not a Brumfiel spectrum (thus also not a real spectrum).*

6. AN ℓ -REPRESENTABLE, NON-BRUMFIEL REPRESENTABLE LATTICE

Although the proof of the present section's main negative result, Theorem 6.4, arises from a lattice-theoretical investigation of the argument of Delzell and Madden [8, Lemma 2], the construction of its counterexample, which is the ℓ -group that we will denote by $G_{\omega_1^{\text{op}}}$, is somehow simpler. Moreover, the constructions of Delzell and Madden [8], Mellor and Tressl [22] yield lattices with 2^{\aleph_1} elements *a priori*, while our construction yields the smaller size \aleph_1 .

Notation 6.1. We denote by F the free Abelian ℓ -group defined by generators a and b , subjected to the relations $a \geq 0$ and $b \geq 0$. Moreover, we set $G_\Lambda \stackrel{\text{def}}{=} \mathbb{Z}\langle\Lambda\rangle \times_{\text{lex}} F$, for any chain Λ .

Since $\mathbb{Z}\langle\Lambda\rangle$ is a totally ordered group and F is an Abelian ℓ -group, G_Λ is also an Abelian ℓ -group. It has an order-unit iff Λ has a largest element. We shall occasionally identify F with the ℓ -ideal $\{0\} \times F$ of G_Λ .

Lemma 6.2. *The ℓ -ideal $\langle a + b \rangle^\ell$ is directly indecomposable in the lattice $\text{Id}_c^\ell G_\Lambda$. That is, there are no nonzero $\mathbf{x}, \mathbf{y} \in \text{Id}_c^\ell G_\Lambda$ such that $\langle a + b \rangle^\ell = \mathbf{x} \vee \mathbf{y}$ and $\mathbf{x} \cap \mathbf{y} = \{0\}$.*

Proof. Since $a, b \in F$ and F is an ideal of G_Λ , it suffices to prove that $\langle a + b \rangle^\ell$ is directly indecomposable in the lattice $\text{Id}_c^\ell F$. The right closed upper quadrant

$$\Omega \stackrel{\text{def}}{=} \{(x, y) \in \mathbb{R}^2 \mid x \geq 0 \text{ and } y \geq 0\}$$

is a convex subset of \mathbb{R}^2 . Further, by the Baker-Beynon duality (cf. Baker [3], Beynon [4, 5]), F is isomorphic to the ℓ -subgroup of \mathbb{R}^Ω generated by the canonical projections $a: (x, y) \mapsto x$ and $b: (x, y) \mapsto y$, and there exists a unique lattice embedding ι from $\text{Id}_c^\ell F$ to the lattice of all relative open subsets of Ω such that $\iota(\langle x \rangle^\ell) = \llbracket x \neq 0 \rrbracket$ whenever $x \in F$. Hence, in order to prove that $\langle a + b \rangle^\ell$ is indecomposable in $\text{Id}_c^\ell F$, it suffices to prove that $\llbracket a + b \neq 0 \rrbracket$ is a connected subset of \mathbb{R}^2 . This, in turn, follows from the relation $\llbracket a + b \neq 0 \rrbracket = \{(x, y) \in \Omega \mid x + y > 0\}$, which implies that $\llbracket a + b \neq 0 \rrbracket$ is a convex (thus connected) subset of Ω , thus of \mathbb{R}^2 . \square

Lemma 6.3. *Let Λ be a chain. Then every pairwise orthogonal subset of G_Λ^{++} is countable.*

Proof. Since $\mathbb{Z}\langle\Lambda\rangle$ is a chain, every pairwise orthogonal subset X of G_Λ^{++} , with more than one element, is a subset of F . The latter being countable, X is countable. \square

Theorem 6.4. *There are no commutative f -ring A and no surjective lattice homomorphism $\mu: \text{Id}_c^r A \rightarrow \text{Id}_c^\ell G_{\omega_1^{\text{op}}}$.*

Until the end of the proof of Theorem 6.4, we shall assume, by way of contradiction, that there are a commutative f -ring A and a surjective lattice homomorphism $\mu: \text{Id}_c^r A \rightarrow \text{Id}_c^\ell G_{\omega_1^{\text{op}}}$.

Pick $x, y \in A^+$ such that $\langle a \rangle^\ell = \mu\langle x \rangle^r$ and $\langle b \rangle^\ell = \mu\langle y \rangle^r$. Moreover, for each $\xi < \omega_1$, pick $z_\xi \in A^+$ such that

$$\langle c_\xi \rangle^\ell = \mu\langle z_\xi \rangle^r. \quad (6.1)$$

In particular,

$$\langle a + b \rangle^\ell \subseteq \mu\langle z_\xi \rangle^r, \quad \text{whenever } \xi < \omega_1. \quad (6.2)$$

Let $\xi < \eta < \omega_1$. Since $z_\xi \leq z_\eta + (z_\xi \setminus z_\eta)$, with z_ξ , z_η , and $z_\xi \setminus z_\eta$ all in A^+ , we get

$$\langle z_\xi \rangle^r \subseteq \langle z_\eta \rangle^r \vee \langle z_\xi \setminus z_\eta \rangle^r,$$

whence, applying the lattice homomorphism μ and by (6.1),

$$c_\xi \in \langle c_\eta \rangle^\ell \vee \mu \langle z_\xi \setminus z_\eta \rangle^r.$$

Since $c_\eta \ll c_\xi$ (within $G_{\omega_1^{\text{op}}}$), it follows that

$$c_\xi \in \mu \langle z_\xi \setminus z_\eta \rangle^r.$$

Hence we obtain, *a fortiori*, that

$$\langle a + b \rangle^\ell \subseteq \mu \langle z_\xi \setminus z_\eta \rangle^r, \quad \text{whenever } \xi < \eta < \omega_1. \quad (6.3)$$

Lemma 6.5. *For all $\xi \leq \eta < \omega_1$, the relation $\langle a + b \rangle^\ell \cap \mu \langle z_\eta \setminus z_\xi \rangle^r = \{0\}$ holds.*

Proof. The conclusion is trivial if $\xi = \eta$. Now suppose that $\xi < \eta$. We compute:

$$\begin{aligned} \langle a + b \rangle^\ell \cap \mu \langle z_\eta \setminus z_\xi \rangle^r &\subseteq \mu \langle z_\xi \setminus z_\eta \rangle^r \cap \mu \langle z_\eta \setminus z_\xi \rangle^r \quad (\text{use (6.3)}) \\ &= \mu(\langle z_\xi \setminus z_\eta \rangle^r \cap \langle z_\eta \setminus z_\xi \rangle^r) \\ &= \mu(\{0\}) \quad (\text{use (4.4) and Lemma 2.1}) \\ &= \{0\}. \quad \square \end{aligned}$$

For each $\xi < \omega_1$, we set

$$\begin{aligned} x_\xi &\stackrel{\text{def}}{=} (x + y) \wedge (xz_0 \setminus (x + y)z_{\xi+1}), \\ y_\xi &\stackrel{\text{def}}{=} (x + y) \wedge ((x + y)z_\xi \setminus xz_0), \\ \mathbf{x}_\xi &\stackrel{\text{def}}{=} \mu \langle x_\xi \rangle^r, \\ \mathbf{y}_\xi &\stackrel{\text{def}}{=} \mu \langle y_\xi \rangle^r. \end{aligned}$$

Lemma 6.6. *The following relations hold, whenever $\xi < \eta < \omega_1$:*

- (1) $\mathbf{x}_\xi \subseteq \langle a \rangle^\ell$;
- (2) $\mathbf{y}_\xi \subseteq \langle b \rangle^\ell$;
- (3) $\langle a + b \rangle^\ell = \mathbf{x}_\xi \vee \mathbf{y}_\xi$;
- (4) $\mathbf{x}_\xi \cap \mathbf{y}_\eta = \{0\}$.

Proof. *Ad* (1). From $x + y \geq 0$ and $z_\xi \geq 0$ it follows that $0 \leq x_\xi \leq xz_0$, whence $\langle x_\xi \rangle^r \subseteq \langle xz_0 \rangle^r \subseteq \langle x \rangle^r$. Apply the homomorphism μ .

Ad (2). From (6.3) it follows that $\langle a + b \rangle \subseteq \mu \langle z_0 \setminus z_\xi \rangle^r$. Hence,

$$\begin{aligned} \langle a + b \rangle \cap \mu \langle xz_\xi \setminus xz_0 \rangle^r &\subseteq \langle a + b \rangle \cap \mu \langle z_\xi \setminus z_0 \rangle^r \quad (6.4) \\ &= \{0\} \quad \text{by Lemma 6.5.} \quad (6.5) \end{aligned}$$

Using Lemma 2.1, we get $(x + y)z_\xi \setminus xz_0 \leq yz_\xi + (xz_\xi \setminus xz_0)$, thus

$$y_\xi \leq (x + y) \wedge (yz_\xi + (xz_\xi \setminus xz_0)),$$

and thus, applying the homomorphism μ together with (6.4)–(6.5),

$$\mathbf{y}_\xi = \mu \langle y_\xi \rangle^r \subseteq \mu \langle (x + y) \wedge yz_\xi \rangle^r \vee \left(\langle a + b \rangle^\ell \cap \mu \langle xz_\xi \setminus xz_0 \rangle^r \right) \subseteq \mu \langle y \rangle^r = \langle b \rangle^\ell.$$

Ad (3). We compute:

$$\begin{aligned}
\langle a + b \rangle^\ell &= \mu(\langle x + y \rangle^r \cap \langle z_\xi \setminus z_{\xi+1} \rangle^r) && \text{(apply (6.3))} \\
&= \mu(\langle x + y \rangle(z_\xi \setminus z_{\xi+1}))^r && \text{(use (4.4))} \\
&= \mu(\langle x + y \rangle z_\xi \setminus \langle x + y \rangle z_{\xi+1})^r && \text{(use Lemma 2.1)} \\
&\subseteq \mu(\langle x + y \rangle z_\xi \setminus xz_0)^r \vee \mu(xz_0 \setminus \langle x + y \rangle z_{\xi+1})^r && \text{(use Lemma 2.1)}.
\end{aligned}$$

Since $\langle a + b \rangle^\ell = \mu\langle x + y \rangle^r$, the desired conclusion follows from the distributivity of the lattice $\text{Id}_c^\ell G_{\omega_1^{\text{op}}}$.

Ad (4). It follows from Lemma 6.5 that $\langle a + b \rangle \cap \mu\langle z_\eta \setminus z_{\xi+1} \rangle^r = \{0\}$ and hence, *a fortiori*, that

$$\langle a + b \rangle \cap \mu\langle (x + y)z_\eta \setminus (x + y)z_{\xi+1} \rangle^r = \{0\}. \quad (6.6)$$

Hence, using Lemma 2.1,

$$\begin{aligned}
\mathbf{y}_\eta &= \langle a + b \rangle \cap \mu\langle (x + y)z_\eta \setminus xz_0 \rangle^r \\
&\subseteq \langle a + b \rangle \cap \left(\mu\langle (x + y)z_\eta \setminus (x + y)z_{\xi+1} \rangle^r \vee \mu\langle (x + y)z_{\xi+1} \setminus xz_0 \rangle^r \right) \\
&= \langle a + b \rangle \cap \mu\langle (x + y)z_{\xi+1} \setminus xz_0 \rangle^r \quad \text{(use (6.6))}.
\end{aligned}$$

It follows that

$$\begin{aligned}
\mathbf{x}_\xi \cap \mathbf{y}_\eta &\subseteq \mu\langle xz_0 \setminus (x + y)z_{\xi+1} \rangle^r \cap \mu\langle (x + y)z_{\xi+1} \setminus xz_0 \rangle^r \\
&= \mu(\{0\}) \quad \text{(use (4.4) and Lemma 2.1)} \\
&= \{0\}. \quad \square
\end{aligned}$$

Set $u_\xi \stackrel{\text{def}}{=} x_\xi \wedge y_\xi$ and $\mathbf{u}_\xi \stackrel{\text{def}}{=} \mu\langle u_\xi \rangle^r$, for all $\xi < \omega_1$.

Lemma 6.7. *The principal ℓ -ideals \mathbf{u}_ξ of $G_{\omega_1^{\text{op}}}$, for $\xi < \omega_1$, are all nonzero and pairwise orthogonal.*

Proof. The statement of pairwise orthogonality follows from Lemma 6.6(4). If $\mathbf{u}_\xi = \{0\}$, then $\mathbf{x}_\xi \cap \mathbf{y}_\xi = \{0\}$, thus, by Lemmas 6.2 and 6.6(3), either $\mathbf{x}_\xi = \{0\}$ or $\mathbf{y}_\xi = \{0\}$, thus either $\mathbf{x}_\xi = \langle a + b \rangle^\ell$ or $\mathbf{y}_\xi = \langle a + b \rangle^\ell$, and thus, by items (1) and (2) of Lemma 6.6, either $a + b \in \langle a \rangle^\ell$ or $a + b \in \langle b \rangle^\ell$, a contradiction. \square

End of the proof of Theorem 6.4. By Lemma 6.7, $\text{Id}_c^\ell G_{\omega_1^{\text{op}}}$ has an uncountable, pairwise orthogonal set of nonzero ideals. Picking positive generators of the \mathbf{u}_ξ , we get an uncountable, pairwise orthogonal set of nonzero elements in $G_{\omega_1^{\text{op}}}$, in contradiction with Lemma 6.3. \square

As mentioned before, the real spectrum of any commutative unital ring A is homeomorphic to the Brumfiel spectrum of the universal f -ring of A (cf. Theorem 4.12). By applying Theorem 3.3, we get the following corollary.

Corollary 6.8. *The ℓ -spectrum of the unital Abelian ℓ -group $G_{\omega_1^{\text{op}}}$ cannot be embedded, as a spectral subspace, into the Brumfiel spectrum of any commutative f -ring. Hence, it also cannot be embedded, as a spectral subspace, into the real spectrum of any commutative unital ring.*

7. OMITTING HOMOMORPHIC IMAGES OF ℓ -REPRESENTABLE LATTICES

Although it is well known, since Delzell and Madden [8], that there are non- ℓ -representable completely normal bounded distributive lattices, the corresponding result for *homomorphic images* of ℓ -representable lattices was not known. In this section, we shall fill that gap by constructing a completely normal bounded distributive lattice, of cardinality \aleph_2 , which is not a homomorphic image of any ℓ -representable lattice. The method used, in particular the part involving Kuratowski's Free Set Theorem, originates in the author's paper [31]. A crucial step consists of coining a property satisfied by all homomorphic images of ℓ -representable lattices.

Lemma 7.1. *For every set I , every homomorphic image of an ℓ -representable lattice satisfies the following infinitary statement:*

For every family $(\mathbf{a}_i \mid i \in I)$, there exists a family $(\mathbf{c}_{i,j} \mid (i,j) \in I \times I)$ such that each $\mathbf{a}_i = (\mathbf{a}_i \wedge \mathbf{a}_j) \vee \mathbf{c}_{i,j}$, each $\mathbf{c}_{i,j} \wedge \mathbf{c}_{j,i} = 0$, and each $\mathbf{c}_{i,k} \leq \mathbf{c}_{i,j} \vee \mathbf{c}_{j,k}$.

$$(7.1)$$

Proof. Since (7.1) is obviously preserved under homomorphic images, it suffices to prove that $\text{Id}_c^\ell G$ satisfies (7.1), whenever G is an Abelian ℓ -group. Write $\mathbf{a}_i = \langle a_i \rangle^\ell$, where $a_i \in G^+$, for all $i \in I$. The principal ℓ -ideals $\mathbf{c}_{i,j} = \langle a_i \searrow a_j \rangle_{\text{def}}^\ell$, for $i, j \in I$, satisfy the required conditions. \square

As usual, we set $\mathbf{2} =_{\text{def}} \{0, 1\}$ and $\mathbf{3} =_{\text{def}} \{0, 1, 2\}$. We set $\bar{0} = 0$, $\bar{1} = \bar{2} = 2$, we denote by $e: \mathbf{2} \hookrightarrow \mathbf{3}$ the map sending 0 to 0 and 1 to 2, and we denote by $r: \mathbf{3} \rightarrow \mathbf{2}$ the map sending 0 to 0 and any nonzero element to 1. Let $f, g: \mathbf{3} \hookrightarrow \mathbf{3}^2$ and $a, b, c: \mathbf{3}^2 \hookrightarrow \mathbf{3}^3 \times \mathbf{2}$ be the maps defined by

$$\begin{aligned} f(x) &= (\bar{x}, x), \\ g(x) &= (x, \bar{x}), \end{aligned}$$

for all $x \in \mathbf{3}$, and

$$\begin{aligned} a(x, y) &= (\bar{x}, x, y, r(y)), \\ b(x, y) &= (x, \bar{x}, y, r(x)), \\ c(x, y) &= (x, y, \bar{y}, r(y)), \end{aligned}$$

for all $(x, y) \in \mathbf{3}^2$. The mappings e, f, g, a, b, c form a commutative diagram of finite distributive lattices with 0, 1-lattice embeddings, represented in Figure 7.1 as indexed by the cube $(\mathfrak{P}(\{1, 2, 3\}), \subseteq)$. Observe that none of the maps f, g, a, b, c is closed.

We leave to the reader the straightforward verification of the following lemma.

Lemma 7.2. *Every square as on Figure 7.2, between three consecutive levels in the cube of Figure 7.1, is a strong amalgam, that is, setting $\mathbf{h} =_{\text{def}} g_1 \circ f_1 = g_2 \circ f_2$, the relation $g_1[E_1] \cap g_2[E_2] = \mathbf{h}[E_0]$ holds.*

Notation 7.3. We denote by \mathcal{L} the similarity type $(\vee, \wedge, \searrow, 0, 1)$, where \vee, \wedge , and \searrow are binary operation symbols and 0, 1 are constant symbols. Moreover, we denote by \mathcal{V}_0 the variety of \mathcal{L} -structures obtained by stating the identities defining

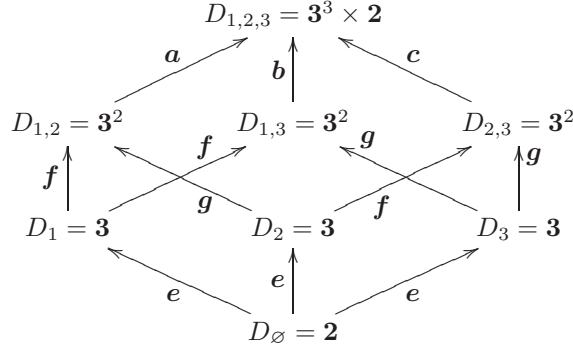


FIGURE 7.1. A cube of 0,1-lattice embeddings

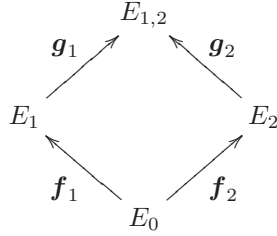


FIGURE 7.2. A commutative square for Lemma 7.2

bounded distributive lattices on $(\vee, \wedge, 0, 1)$, together with the identities

$$(x \wedge y) \vee (x \searrow y) = x, \quad (7.2)$$

$$(x \searrow y) \wedge (y \searrow x) = 0. \quad (7.3)$$

Evidently, (the lattice reduct of) any member of \mathcal{V}_0 is a completely normal bounded distributive lattice.

Lemma 7.4. *The commutative diagram, of bounded lattice embeddings, represented in Figure 7.1, can be expanded to a commutative diagram of embeddings in \mathcal{V}_0 .*

Proof. Represent the diagram by lattices D_p and arrows $\mathbf{f}_p^q: D_p \rightarrow D_q$, where $p \subseteq q \subseteq \{1, 2, 3\}$ (cf. Figure 7.1). We define inductively the “difference operation” \searrow_{D_p} on D_p , assuming that it has already been defined on all D_q , for $q \subsetneq p$. Let $x_1, x_2 \in D_p$. If x_1 and x_2 belong to the range of \mathbf{f}_q^p for some $q \subsetneq p$, then due to Lemma 7.2, there is a smallest such q ; then let $y_1, y_2 \in D_q$ such that each $x_i = \mathbf{f}_q^p(y_i)$, and define $x_1 \searrow_{D_p} x_2 \stackrel{\text{def}}{=} \mathbf{f}_q^p(y_1 \searrow_{D_q} y_2)$. If x_1 and x_2 do not belong to the range of any \mathbf{f}_q^p where $q \subsetneq p$, then pick any splitting (u, v) of (x_1, x_2) in D_p , then define $x_1 \searrow_{D_p} x_2 \stackrel{\text{def}}{=} u$ and $x_2 \searrow_{D_p} x_1 \stackrel{\text{def}}{=} v$. By construction, D_p is thus expanded to a member of \mathcal{V}_0 , and each \mathbf{f}_p^q is an \mathcal{L} -embedding. \square

For the remainder of Section 7, we shall denote by $\vec{D} = (D_p, \mathbf{f}_p^q \mid p \subseteq q \subseteq \{1, 2, 3\})$ the cube of \mathcal{V}_0 obtained from Lemma 7.4, and by $D \stackrel{\text{def}}{=} D_{1,2,3} = (\mathbf{3}^3 \times \mathbf{2}, \vee, \wedge, \searrow, 0, 1)$

its top member. We also pick any variety \mathcal{V} of \mathcal{L} -algebras such that $D \in \mathcal{V}$ and $\mathcal{V} \subseteq \mathcal{V}_0$, and we denote by $F_{\mathcal{V}}(X)$ the free \mathcal{V} -algebra on X , for any set X . We will identify $F_{\mathcal{V}}(X)$ with its canonical copy in $F_{\mathcal{V}}(Y)$, whenever $X \subseteq Y$.

Theorem 7.5. *For every set I , the underlying lattice of $F_{\mathcal{V}}(I)$ is a completely normal bounded distributive lattice. Moreover, if $\text{card } I \geq \aleph_2$, then $F_{\mathcal{V}}(I)$ is not a homomorphic image of any ℓ -representable lattice.*

Proof. The first statement, that $F_{\mathcal{V}}(I)$ is completely normal, is obvious. Suppose, from now on, that $\text{card } I \geq \aleph_2$, and denote by a_i , for $i \in I$, the canonical generators of $F_{\mathcal{V}}(I)$. By Lemma 7.1, it suffices to prove that $F_{\mathcal{V}}(I)$ does not satisfy the infinitary statement (7.1). Suppose otherwise. Then there exists a family $(c_{i,j} \mid (i,j) \in I \times I)$, of elements in $F_{\mathcal{V}}(I)$, satisfying the conditions stated in (7.1).

For all $i, j \in I$, there exists a finite subset $\Phi(\{i, j\})$ of I such that $\{c_{i,j}, c_{j,i}\} \subseteq F_{\mathcal{V}}(\Phi(\{i, j\}))$. Since $\text{card } I \geq \aleph_2$ and by Kuratowski's Free Set Theorem (cf. Kuratowski [21], Erdős *et al.* [13, Theorem 46.1]), there are distinct elements in I , that we may denote by 1, 2, 3, such that

$$1 \notin \Phi(\{2, 3\}), \quad 2 \notin \Phi(\{1, 3\}), \quad \text{and} \quad 3 \notin \Phi(\{1, 2\}). \quad (7.4)$$

Then $J = \{1, 2, 3\}$ is a subset of I . We shall now define maps $\rho_X: F_{\mathcal{V}}(X) \rightarrow D_X$, for $X \subseteq J$, as follows:

- Denote by $\rho_{\emptyset}: F_{\mathcal{V}}(\emptyset) \rightarrow D_{\emptyset} = \mathbf{2}$ the unique isomorphism.
- For $i \in J$, denote by $\rho_i: F_{\mathcal{V}}(\{i\}) \rightarrow D_i = \mathbf{3}$ the unique \mathcal{L} -homomorphism sending a_i to 1.
- For $1 \leq i < j \leq 3$, denote by $\rho_{i,j} = \rho_{j,i}: F_{\mathcal{V}}(\{i, j\}) \rightarrow D_{i,j} = \mathbf{3}^2$ the unique \mathcal{L} -homomorphism sending a_i to $(2, 1)$ and a_j to $(1, 2)$.
- Finally, denote by $\rho_{1,2,3}: F_{\mathcal{V}}(J) \rightarrow D_{1,2,3} = \mathbf{3}^3 \times \mathbf{2}$ the unique \mathcal{L} -homomorphism sending a_1 to $(2, 2, 1, 1)$, a_2 to $(2, 1, 2, 1)$, and a_3 to $(1, 2, 2, 1)$.

It is straightforward to verify that $\vec{\rho} = (\rho_X \mid X \subseteq J)$ is a natural transformation from the diagram \vec{F} of all $F_{\mathcal{V}}(X)$, for $X \subseteq J$, to \vec{D} (it is sufficient to check the required equations on the generators a_i). The diagrams \vec{F} and \vec{D} , together with the natural transformation $\vec{\rho}$, are represented in Figure 7.3.

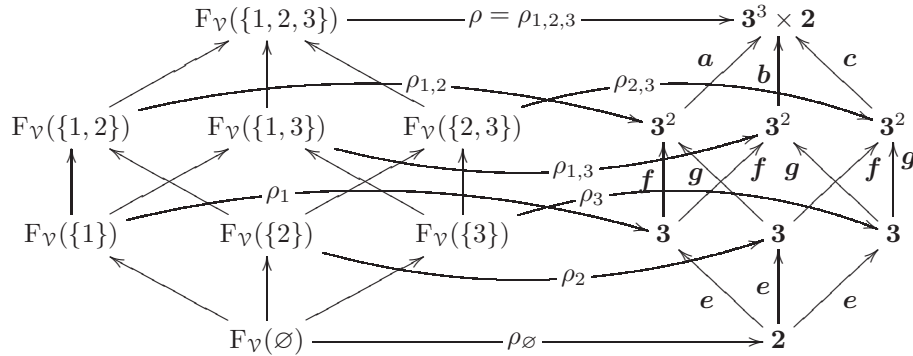


FIGURE 7.3. The natural transformation $\vec{\rho}$

Setting $\rho \stackrel{\text{def}}{=} \rho_{1,2,3}$, we obtain, in particular, the equations

$$\mathbf{a} \circ \rho_{1,2} = \rho \upharpoonright_{F_V(\{1,2\})}, \quad \mathbf{b} \circ \rho_{1,3} = \rho \upharpoonright_{F_V(\{1,3\})}, \quad \mathbf{c} \circ \rho_{2,3} = \rho \upharpoonright_{F_V(\{2,3\})}. \quad (7.5)$$

Denote by $\pi: F_V(I) \rightarrow F_V(J)$ the unique \mathcal{L} -homomorphism such that

$$\pi(a_i) \stackrel{\text{def}}{=} \begin{cases} a_i, & \text{if } i \in J, \\ 0, & \text{otherwise,} \end{cases} \quad \text{for every } i \in I. \quad (7.6)$$

The element $d_{i,j} \stackrel{\text{def}}{=} \pi(c_{i,j})$, for $i, j \in J$, belongs to $F_V(J)$. Moreover, it follows from (7.6), together with the assumptions on the $c_{i,j}$, that the $d_{i,j}$ satisfy the following relations:

$$a_i = (a_i \wedge a_j) \vee d_{i,j}, \quad \text{whenever } i, j \in J; \quad (7.7)$$

$$d_{i,j} \wedge d_{j,i} = 0, \quad \text{whenever } i, j \in J; \quad (7.8)$$

$$d_{i,k} \leq d_{i,j} \vee d_{j,k}, \quad \text{whenever } i, j, k \in J. \quad (7.9)$$

Moreover, for distinct $i, j \in J$, the element $c_{i,j}$ belongs to $F_V(\Phi(\{i, j\}))$, that is, $c_{i,j} = \mathbf{t}(a_k \mid k \in \Phi(\{i, j\}))$ for an \mathcal{L} -term \mathbf{t} . Since π is an \mathcal{L} -homomorphism, $d_{i,j} = \mathbf{t}(\pi(a_k) \mid k \in \Phi(\{i, j\}))$. By (7.4) and (7.6), each $\pi(a_k)$ is either 0 (if $k \notin J$) or belongs to $\{a_i, a_j\}$ (if $k \in J$). It follows that $d_{i,j} \in F_V(\{i, j\})$.

Let $1 \leq i < j \leq 3$. Since $\rho_{i,j}(a_i) = (2, 1)$ and $\rho_{i,j}(a_j) = (1, 2)$, it follows from (7.7) and (7.8) that

$$\begin{aligned} (2, 1) &= (1, 1) \vee \rho_{i,j}(d_{i,j}), \\ (1, 2) &= (1, 1) \vee \rho_{i,j}(d_{j,i}), \\ (0, 0) &= \rho_{i,j}(d_{i,j}) \wedge \rho_{i,j}(d_{j,i}), \end{aligned}$$

which leaves the only possibility

$$\rho_{i,j}(d_{i,j}) = (2, 0) \text{ and } \rho_{i,j}(d_{j,i}) = (0, 2). \quad (7.10)$$

By applying the homomorphisms \mathbf{a} , \mathbf{b} , \mathbf{c} , respectively, to the instances $(i, j) = (1, 2)$, $(i, j) = (1, 3)$, $(i, j) = (2, 3)$, respectively, of (7.10), we obtain, using (7.5),

$$\rho(d_{1,2}) = (2, 2, 0, 0), \quad \rho(d_{1,3}) = (2, 2, 0, 1), \quad \rho(d_{2,3}) = (2, 0, 0, 0),$$

whence (projecting on the last coordinate) $\rho(d_{1,3}) \not\leq \rho(d_{1,2}) \vee \rho(d_{2,3})$. On the other hand, by applying the homomorphism ρ to the inequality (7.9), we obtain $\rho(d_{1,3}) \leq \rho(d_{1,2}) \vee \rho(d_{2,3})$; a contradiction. \square

By applying Stone duality, we thus obtain the following result.

Corollary 7.6. *For every set I , the spectrum Ω_I of the underlying distributive lattice of $F_V(I)$ is a completely normal spectral space. Moreover, if $\text{card } I \geq \aleph_2$, then Ω_I cannot be embedded, as a spectral subspace, into any ℓ -spectrum.*

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