Uniqueness in the elastic bounce problem, II

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INTRODUCTION

In a recent paper (see [6]) uniqueness for the elastic bounce problem has been studied in a very general framework. More precisely given $T>0$ and $f \in C^3$ ($\mathbb{R}^n$; $\mathbb{R}$), a pair $(x, U) \in \mathrm{Lip}(0, T; \mathbb{R}^n) \times L^1(0, T; C^2(\mathbb{R}^n))$ is said to be a solution to the elastic bounce problem iff

$$
(\mathrm{P}) \quad \begin{cases}
\text{(i)} & f(x(t)) \geqslant \mathrm{o} \text{ in } [0, T] \\
\text{(ii)} & \text{there exists a bounded measure } \mu \geqslant 0 \text{ on } [0, T] \text{ such that } x(t) \\
& \text{is an extremal for the functional} \\
& F(y) = \displaystyle\int_0^T \{\tfrac{1}{2}|\dot{y}|^2 + U(t, y(t))\}\, dt + \int_0^T f(y(t))\, d\mu \\
& \text{and spt } \mu \subseteq \{t \in [0, T] : f(x(t)) = 0\}. \\
\text{(iii)} & \text{the function } \mathscr{E} : t \to |\dot{x}(t)|^2 \text{ is continuous on } [0, T].
\end{cases}
$$

In [6, 7] it was pointed out that the Cauchy problem for (P) admits a unique solution when certain inequalities (involving the Gaussian curvature of $\partial\Omega$ and the normal component with respect to $\partial\Omega$ of U is fulfilled (see [6, Thm. 2.2]).[1]

When these conditions are violated, uniqueness for (P) may fail as it is shown in [7, 8], where an example of $f \in C^\infty(\mathbb{R}^2)$ is constructed in such a way that $\Omega = \{x : f(x) \geqslant 0\}$ is convex and, as soon as one takes $U=0$, the solution of the Cauchy problem for (P) is not unique.

This example shows that—even in the absence of external forces and in two dimensions—boundaries having rapidly oscillating Gaussian curvature which vanishes at order infinity at some point may cause a loss of uniqueness.

[1] Here we set $\Omega = \{x : f(x) \geqslant 0\}$.

The aim of this paper is to show that if u is assumed to be real analytic and $U \equiv 0$ then these phenomena disappear and the solution to the elastic bounce problem is unique for every choice of the Cauchy data.

1. Statement of the Problem

Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be a real analytic function such that $df(x) \neq 0$ on the set $\{x \in \mathbb{R}^n : f(x) = 0\}$. We want to study the elastic bounce problem for a material point whose position at time t will be indicated by $x(t)$. This point moves in the region $\Omega = \{x : f(x) \geqslant 0\}$ and bounces against the boundary $\partial\Omega = \{x : f(x) = 0\}$. We shall assume that no external force is acting on the point and therefore given $T > 0$ we say that $x \in \mathrm{Lip}(0, T; \mathbb{R}^n)$ solves the elastic bounce problem (P) iff

(i) $f(x(t)) \geqslant 0$ for every $t \in [0, T]$

(ii) there exists a finite positive measure μ on $[0, T]$ such that $x(t)$ is an extremal for the functional

$$F(y) = \frac{1}{2} \int_0^T |\dot{y}|^2 \, dt + \int_0^T f(y(t)) \, d\mu$$

and $\operatorname{spt} \mu \subseteq \{t \in [0, T] : f(x(t)) = 0\}$

(iii) for every $t_1, t_2 \in [0, T]$ we have

$$|\dot{x}_\pm(t_1)|^2 = |\dot{x}_\pm(t_2)|^2,$$

where $\dot{x}_+$ and $\dot{x}_-$ respectively denote the right and left derivatives of x since $\dot{x}$ is a BV function (see [3, 4, 7]).

As we have seen in [3], a function $x \in \mathrm{Lip}(0, T; \mathbb{R}^n)$ satisfies (i), (ii), and (iii) if and only if it satisfies (i), (iii), and the following equality

$$\ddot{x} = \mu \nabla f(x(t)) \tag{1}$$

holds true in the sense of distributions and $\operatorname{spt} \mu \subseteq \{t \in [0, T] : f(x(t)) = 0\}$.

According to [3] we introduce the set $E = \{x \in \mathrm{Lip}(0, T; \mathbb{R}^n) : x \text{ solves (P)}\}$ and define the initial trace $\mathscr{T} : [0, T] \times E \to \mathbb{R}^{3n+2}$

$$\mathscr{T}(t, x) = (\tfrac{1}{2} |\dot{x}(t)|^2, x(t), \dot{x}_\tau(t), f(x(t))\, \dot{x}(t), 0),$$

where $\dot{x}(t) = |\nabla f(x(t))|^2 \dot{x}(t) - \langle \dot{x}, \nabla f(x(t)) \rangle \nabla f(x(t))$. Now, fixed $t_0 \in [0, T]$ and $b \in \mathscr{T}(\{t_0\} \times E) = \mathscr{B}$, we set

$$G(t_0, b) = \{x \in \mathrm{Lip}(0, T; \mathbb{R}^n) : x \in E, \mathscr{T}(t_0, x) = b\}.$$

As we have seen in [6] this set is non void and moreover we can state the following result (see [6]):

PROPOSITION I. *Let* $t_0 \in [0, T]$ *and* $b \in \mathscr{B}$ *such that* $f(b_2) = 0$ *and* $|b_3|^2 - 2b_1 |\nabla f(b_2)|^4 = 0$. *Suppose that for every* $x \in G(t_0, b)$ *there exists* δ *such that in* $(t_0, t_0 + \delta)$ *the inequality*

$$\left(\dot{x}_i \frac{\partial}{\partial x_i}\right)^2_{x(t)} f \geqslant 0$$

holds true. Then there exists $\sigma > 0$ *such that the set*

$$G(t_0, b) = \{x \in \mathrm{Lip}(t_0, t_0 + \sigma) : x \in E, \mathscr{T}(t_0, x) = b\}$$

is a singleton.

The aim of this paper is to prove the following result:

THEOREM II. *If* $f: \mathbb{R}^n \to \mathbb{R}$ *is real analytic, then for each* $t_0 \in [0, T]$ *and for each* $b \in \mathscr{B}$ *the set* $G(t_0, b)$ *is a singleton.*

In order to prove Theorem II we remark that f may be put into the form $f(x) = h(x_1, x_2, ..., x_{n-1}) - x_n$ is a suitable neighbourhood of $\bar{x} \in \partial\Omega$ with $\nabla h(\bar{x}_1, ..., \bar{x}_{n-1}) = 0$. Now, fixing $t_0 \in [0, T]$, we may consider—without loss of generality—only those $b \in \mathscr{B}$ such that $f(b_2) = 0$ and $|b_3|^2 - 2 |\nabla f(b_2)| b_1 = 0 = |b_3|^2 - 2b_1$; the last equality implies that $\langle \dot{x}_\pm(t_0), \nabla f(x(t_0)) \rangle = 0$ and therefore, from now on, we assume that $f(b_2) = 0$, $\nabla h(b_2) = 0$ $b_3 = (1, 0, ..., 0)$, $b_2 = (0, 0, ..., 0)$.

Moreover we put $\hat{x} = (x_2, ..., x_{n-1}) \in \mathbb{R}^{n-2}$, $\varphi(x) = \nabla_{x_1} f(x)$, $\tilde{x} = (x_1, \hat{x})$, $\nabla_{\hat{x}} f(x) = \Psi(\tilde{x})$ so that problem (P) can be rewritten as follows:

(P′)

(i) $h(x_1(t), \hat{x}(t)) \geqslant x_n(t)$ for every t $[0, T]$.

(ii) there exists a bounded measure $\mu \geqslant 0$ on $[0, T]$ such that $\mathrm{spt}\, \mu \subseteq \{t \in [0, T] : f(x(t)) = 0\}$ and

$$\begin{cases} \ddot{x}_1 = \mu\varphi \\ \ddot{\hat{x}} = \mu\Psi \\ \ddot{x}_n = -\mu \end{cases}$$

in the sense of measures

(iii) for every $t_1, t_2 \in [0, T]$ we have

$$|\dot{x}_\pm(t_1)|^2 = |\dot{x}_\pm(t_2)|^2 = 1.$$

2. Proof of Theorem (II)

Let $t_0 \in [0, T]$, $b \in \mathscr{B}$, $x \in G(t_0, b)$ and assume that t_0 and b satisfy the hypothesis stated in the previous section. Moreover—by virtue of [6, Thm. 2.2]—it is no restrictive to assume that

$$\sum_{i=1}^{n} \frac{\partial^2 f}{\partial x_i \, \partial x_j} (x(t_0)) \, \dot{x}_i (t_0) \, \dot{x}_j (t_0) = 0$$

and as in [6] (see also [3]) it is not difficult to check that the following energy-relation holds:

$$\begin{aligned} \left| \frac{d}{dt} f(x(t)) \right|^2 &= 2 \int_{t_0}^{t} \left[\sum_{i=1}^{n} \frac{\partial^2 f}{\partial x_i \, \partial x_j} (x(s)) \, \dot{x}_i(s) \, \dot{x}_j(s) \right] \frac{d}{ds} f(x(s)) \, ds \\ &= 2 \left[\sum_{i=1}^{n} \frac{\partial^2 f}{\partial x_j \, \partial x_i} (x(t)) \, \dot{x}_i(t) \, \dot{x}_j(t) \right] f(x(t)) \\ &\quad - 2 \int_{t_0}^{t} (\langle \dot{x}(s), \nabla \rangle^3 f(x(s))(f(x(s))) \, ds. \end{aligned} \tag{2}$$

We put $f(x) = h(x_1, \hat{x}) - x_n$; hence when $n = 2$, (2) becomes

$$\begin{aligned} &\left| \frac{d}{dt} (h(x_1(t)) - x_2(t)) \right|^2 \\ &\quad = 2(h(x_1(t) - x_2(t)) \, h''(x_1(t)) \, \dot{x}_1^2(t) \\ &\quad\quad - 2 \int_{t_0}^{t} h'''(x_1(s)) \, \dot{x}_1^3(s)(h(x_1(s)) - x_2(s)) \, ds. \end{aligned} \tag{2'}$$

Since h is real analytic we may assume that there exists an index $p > 2$ such that $h^p(x_1(t_0)) \neq 0$ unless $h''(x_1)$ is identically zero; in the latter case from (2′) it follows $f(x(t)) \equiv 0$ in a neighbourhood of t_0. If $h^p(x_1(t_0)) > 0$ then $h''(x_1(t)) \geqslant 0$ so that $f(x(t))$ is convex in a suitable $[t_0, t_0 + \delta)$ and therefore or $z(t) \equiv 0$ in $[t_0, t_0 + \delta)$ or $z(t) > 0$ in $(t_0, t_0 + \delta)$. In both cases from [6, Lemma (2.1)] we argue the thesis.[2]

We claim that even in the case $h^{(p)}(x(t_0)) < 0$ the function $z(t) = h(x_1(t)) - x_2(t)$ is identically zero in some neighbourhood of t_0.

In fact we have (choosing $t_0 = 0$)

$$\begin{aligned} |\dot{z}(t)|^2 &= 2h''(x_1(t)) \, z(t) \, \dot{x}_1^2(t) - 2 \int_0^t h'''(x_1(t)) \, x_1(s) \, z(s) \, ds \\ &\leqslant C \left\{ t^{p-2} z(t) + t^{p-3} \int_0^t z(s) \, ds \right\} \end{aligned}$$

[2] Here $\langle \xi, \nabla \rangle^m (f)(x(t))$ denotes $\partial f / (\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}) \, \xi_1^{\alpha_1} \cdots \xi_n^{\alpha_n} \sum \alpha_i = m$.

and then, by using Hölder inequality,

$$\int_0^t |\dot z(s)|^2\,ds \leqslant C\left\{t^{p-2}\int_0^t z(s)\,ds+\int_0^t ds\int_0^s \tau^{p-3}z(\tau)\,d\tau\right\}$$

$$\leqslant 2Ct^{p-1/2}\left(\int_0^t |\dot z(s)|^2\,ds\right)^{1/2}$$

which yields

$$\left(\int_0^t |\dot z(s)|^2\,ds\right)^{1/2} \leqslant 2Ct^{(2p-1)/2} \tag{3}$$

and then $|z(t)| \leqslant 2Ct^p$.

Since $\dot x_1$ is decreasing and $\dot x_1(0)=1$ we have $|\dot x_1|\leqslant 1$ and then for t small enough

$$\int_0^t |h'''(x_1)|\,\dot x_1^3 z(s)\,ds \leqslant (1+\varepsilon_1(t))\,\frac{|h^p(0)|}{(p-3)!}\,x_1^{p-3}\int_0^t z(s)\,ds \tag{4}$$

$$|h''(x_1)|\,\dot x_1^2=\dot x_1^2\,\frac{|h^p(0)|}{(p-2)!}\,x_1^{p-2}(1+\varepsilon_2(t)), \tag{5}$$

where $\varepsilon_1(t)$ and $\varepsilon_2(t)$ goes to zero when $t\to 0$. From these two relations we obtain

$$|\dot z(t)|^2 \leqslant |h''(x_1)|\,\dot x_1^2\left\{-z(t)+(p-2)\,K(t)\,t^{-1}\int_0^t z(s)\,ds\right\},$$

where $K(t)=(1+\varepsilon_1(t))(1+\varepsilon_2(t))^{-1}t\dot x_1^{-2}x_1^{-1}$ goes to 1 as $t\to 0$ and therefore it is possible to choose δ in such a way that $K(t)<(p-1)(p-2)^{-1}$ for $0<t<\delta$.

Now, the latter inequality yields (for $t\in(0,\delta)$)

$$|\dot z(t)|^2 \leqslant |h''(x_1)|\,\dot x_1^2\left\{-z(t)+(p-1)\,t^{-1}\int_0^t z(s)\,ds\right\};$$

moreover we have $|z(t)|\leqslant ct^p$ and

$$\frac{d}{dt}\left(t^{1-p}\int_0^t z(s)\,ds\right)=t^{-p}\left(tz(t)-(p-1)\int_0^t z(s)\,ds\right)\leqslant 0$$

so that $t\to t^{1-p}\int_0^t z(s)\,ds$ is decreasing and then $\int_0^t z(s)\,ds\leqslant 0$ but $z\geqslant 0$ and therefore $z(t)\equiv 0$ in $(0,\delta)$; from [6, Lemma 2.1] we easily complete the proof of Theorem II in the case $n=2$. We deal now with the case $n>2$; to this aim it is useful to put $f(x)=h(x_1,0)+\langle \hat{x}, \hat{g}(x)\rangle - x_n$ so that

$$\nabla_{x_1} f(x)=\varphi(\tilde{x})=h_{x_1}(x_1,0)+\langle \hat{x}, \hat{g}_{x_1}(\tilde{x})\rangle \tag{6}$$

and

$$\nabla_{\hat{x}} f(x)=\Psi(\tilde{x})=\hat{g}(x_1,0)+A(\tilde{x})\hat{x}, \tag{6'}$$

where $A(x)$ is a suitable $(n-2)\times(n-2)$ matrix. Now we are able to state the following

LEMMA 1. *If $x\in G(t_0,b)$ then there exists $\sigma>0$ such that*

$$|\dot{\hat{x}}(t)|\leqslant K\,|\hat{g}(x_1(t),0)|\;|\dot{x}_n(t)| \tag{7}$$

for every $t\in[t_0,t_0+\sigma)$.

Proof. From (6) and (6′) we deduce

$$\ddot{\hat{x}}=-(\hat{g}(x_1(t),0)+A(x(t))\,\hat{x}(t))\ddot{x}_n \tag{8}$$

and then

$$\dot{\hat{x}}=-(\hat{g}(x_1(t),0)+A(x(t))\,\hat{x}(t))\,\dot{x}_n+\int_{t_0}^{t}\lambda(s)\,ds, \tag{9}$$

where $\lambda(s)=[x_1(s)\,\hat{g}(x_1(s),0)+(A(x(s))\,\hat{x}(s))']\,\dot{x}_n(s)$.

By using equality (9) we obtain

$$|\dot{\hat{x}}|\leqslant|\hat{g}(x_1,0)|\;|\dot{x}_n|+c_1\int_{t_0}^{t}|\hat{x}(s)|\,ds$$

$$+\int_{t_0}^{t}\{|\hat{g}(x_1,0)|\;|\dot{x}_n|+c_2\,|\hat{x}|+c_1\,|\dot{\hat{x}}|\}\,ds,$$

where we have used that $|A(x(t)|\leqslant c_1$ and $|\dot{A}(x(t)|\leqslant c_2$. From the previous inequality we argue

$$|\dot{\hat{x}}|^2\leqslant c_3\,|\hat{g}(x_1,0)|^2\dot{x}_n^2+c_4\int_{t_0}^{t}|\dot{\hat{x}}(s)|^2\,ds$$

for every $t\in[t_0,T]$.

Since $t\to[\hat{g}(x_1(t),0)|^2\dot{x}_n^2(t)$ is increasing in a suitable interval

$[t_0, t_0+\sigma)$, we can apply Gronwall's Lemma to the latter inequality and we obtain

$$|\dot{\hat{x}}(t)|^2 \leqslant e^{c_4} c_3 \, |\hat{g}(x_1(t), 0)|^2 \dot{x}_n^2$$

which is precisely (7).

The method consists now in proving that the sign of $\langle \dot{x}, \nabla \rangle^m (f)(x(t))$ depends only on the sign of $\partial_{x_1}^m h(x_1, 0)$ in a neighbourhood of t_0. To this aim we consider first the case in which $\hat{g}(x_1, 0)$ goes to zero faster than $h_{x_1}(x_1, 0)$ as $t \to t_0$, which leads to

$$|\Psi| \text{ goes to zero faster than } \varphi \text{ as } t \to t_0.$$

In this case, by using Lemma 1, a direct computation shows that for every $k \geqslant 2$ and $t - t_0$ small enough

$$\langle \dot{x}, \nabla \rangle^k (f)(x(t)) = \dot{x}_1^k \, \partial_{x_1}^k h(x_1, 0)[1 + \sigma(t)]$$

holds true with $\sigma(t) \leqslant c \, |t - t_0|$.

By the previous equality, we may reduce ourselves to the two-dimensional case and, by using the same techniques, the proof of Theorem II can be easily achieved in this case.

In order to complete the proof, from now on we suppose that

$$|h_{x_1}(x_1, 0)| \leqslant C \, |\hat{g}(x_1, 0)|.$$

Again from Lemma 1 if $\hat{g}(x_1, 0) \equiv 0$ in some neighbourhood of $(x_1(t_0), 0)$, then $\hat{x}(t) \equiv 0$ in some neighbourhood of t_0 and we fall in the two-dimensional case; therefore $\hat{g}(x_1, 0)$ is assumed to be different from zero in some neighbourhood of $(x_1(t_0), 0)$.

LEMMA 2. *If $x \in G(t_0, b)$ then there exists $\delta > 0$ such that*

$$\langle \Psi(x(t)), \dot{\hat{x}}(t) \rangle \geqslant 0 \tag{10}$$

$$\left\langle \frac{d}{dt} (\Psi(x(t)), \dot{\hat{x}}(t) \right\rangle \geqslant 0 \tag{11}$$

for every $t \in [t_0, t_0 + \delta]$.

Proof. We prove only (11) since (10) can be proven in an analogous way. We have

$$\left\langle \frac{d}{dt} (\Psi(x(t))), \dot{\hat{x}}(t) \right\rangle = \left\langle \hat{g}_{x_1}(x_1(t), 0), \int_{t_0}^{t} \hat{g}(x_1(s), 0) \, d\mu \right\rangle + \theta(t)$$

with $|\theta(t)| \leqslant \varepsilon(t) \, |\hat{g}(x_1(t), 0)| \int_{t_0}^{t} |\hat{g}(x_1(s), 0)| \, d\mu$ and $\varepsilon(t) \to 0$ as $t \to t_0$.

By the analyticity of $\hat{g}$ we may suppose

$$\hat{g}_i(x_1, 0) = \alpha_i x_1^{m_i} + O(x_1^{m_i}) \tag{12}$$

and it is easy to check that, when $t \to t_0$, $\theta(t)$ goes to zero faster than

$$\left\langle \hat{g}_{x_1}(x_1(t), 0), \int_{t_0}^{t} \hat{g}(x_1(s), 0)\, d\mu \right\rangle.$$

This fact easily implies (11).

The crucial point of the proof is the following

LEMMA 3. *If* $x \in G(t_0, b)$ *then there exists* $M > 0$ *such that*

$$\dot{x}_n^2(t) \leqslant M \frac{\varphi^2(x(t))}{|\Psi(x(t))|^2} + 2 \int_{t_0}^{t} \left\langle \frac{d}{dt}\left(\frac{\varphi\Psi}{|\psi|^2}\right), \hat{x}(s) \right\rangle ds \tag{13}$$

for every $t \in [0, T]$.

Proof. From (P′) we obtain

$$\varphi(x(t))\hat{x} = x_1 \Psi(x(t)) \tag{14}$$

and then

$$\varphi(x(t))\langle \ddot{\hat{x}}\ \Psi \rangle = \ddot{x}_1 |\Psi|^2 \tag{15}$$

which implies

$$\dot{x}_1(t) = 1 + \left\langle \dot{\hat{x}}(t), \frac{\Psi\varphi}{|\Psi|^2} \right\rangle - \int_{t_0}^{t} \left\langle \frac{d}{dt}\left(\frac{\varphi\Psi}{|\Psi|^2}\right), \dot{\hat{x}}(s) \right\rangle ds.$$

Since $\dot{x}_1^2(t) + |\dot{\hat{x}}(t)|^2 + \dot{x}_n^2(t) = 1$, setting $I(t) = \int_{t_0}^{t} \langle (d/dt)(\varphi\Psi/|\psi|^2), \dot{\hat{x}}(s) \rangle\, ds$, we obtain

$$\left\langle \dot{\hat{x}} \frac{\varphi\Psi}{|\Psi|^2} \right\rangle + (1 - I(t))^2 + 2(1 - I(t)) \left\langle \dot{\hat{x}} \frac{\Psi\varphi}{|\Psi|^2} \right\rangle + |\dot{\hat{x}}|^2 + \dot{x}_n^2 = 1. \tag{16}$$

By a standard argument from (16) we obtain

$$\frac{\varphi^2}{|\Psi|^2} \frac{(1 - I(t))^2}{(1 + \varphi^2/|\Psi|^2)} - (1 - I(t))^2 - \dot{x}_n^2 + 1 \geqslant 0$$

which easily yields (13).

We recall now that, since $b_2 = (0, ..., 0)$ and $b_1 = (1, 0, ..., 0)$, to say $(\langle \dot{x}(t_0), \nabla \rangle^2 f)(x(t_0)) = 0$ is equivalent to saying that $h_{x_1 x_1}(0, 0) = 0$ and therefore there exists an index $p>2$ such that $(\partial^p h/\partial x_1^p)(0,0)\neq 0$ unless $h_{x_1}x_1(x_1,0)\equiv 0$ in some neighbourhood of $(0,0)$; in the latter case we agree to put $p=\infty$.

First we assume $p<\infty$, $(\partial^p f/\partial x_1^p)(0,0)<0$ and as in (13) we set

$$\hat{g}_i(x_1,0)=\alpha_i x_1^{m_i}+O(x_1^{m_i})$$

and $m=\min\{m_i\colon i=2,\dots,n-1\}$; we now prove the following.

LEMMA 4. *Let $x\in G(t_0,b)$; then there exists $\tau>0$ such that*

$$\dot{x}_n^2(t)\leqslant \tilde{K}x_1(t)^{2(p-1-m)} \tag{17}$$

for every $t\in[t_0,t_0+\tau]$ and for a suitable constant $\tilde{K}>0$.

Proof. Let $p\leqslant 2m+1$; from Lemma 3 we have

$$\dot{x}_n^2(t)\leqslant M\frac{\varphi^2}{|\Psi|^2}+2\int_{t_0}^{t}\frac{1}{|\psi|^4}\{\langle\Psi,\ddot{x}\rangle(\dot{\varphi}\,|\Psi|^2$$
$$-2\varphi\langle\Psi,\dot{\Psi}\rangle)+\langle\dot{\Psi},\dot{x}\rangle\,|\Psi|^2\}\,ds$$

and from (6), (6′), and Lemma 1 we argue

$$\dot{x}_n^2(t)\leqslant L(x_1(t)^{2(p-1-m)}+x_1(t)^{(p-1)}\omega(t)),$$

where $\omega(t)$ goes to zero as $t\to t_0$ and having taken $p\leqslant 2m+1$ the inequality (17) holds true in a suitable interval $[t_0,t_0+\tau]$. If $p>2m+1$, setting $|\hat{g}(x_1,0)|^2=\alpha^2x_1^{2m}+O(x_1^{2m})$ and $h(x_1,0)=\beta x_1^p+O(x_1^p)$ we obtain

$$\begin{aligned}[\dot{\varphi}\,|\Psi|^2&-2\varphi\langle\Psi,\dot{\Psi}\rangle]\langle\Psi,\dot{x}\rangle+\varphi\,|\Psi|^2\,\langle\dot{\psi},\dot{x}\rangle\\ &=\dot{x}_1^2\{\alpha^2\beta p(p-1-2(m-1))\,x_1^{2m-2+p}+O(x_1^{2m-2+p}))\}\langle\Psi,\dot{x}\rangle\\ &\quad+\alpha^2\beta p(x_1^{2m+p-2}+O(x_1^{p+2m-2}))\langle\dot{\Psi},\dot{x}\rangle.\end{aligned}$$

Since we have assumed $\beta<0$ from Lemma 2 we argue the sum on the second member is negative and this fact yields

$$\dot{x}_n^2(t)\leqslant M\frac{\varphi^2}{|\Psi|^2}$$

for all t sufficiently close to t_0 and so (17) is completely proved.

From the previous lemmas we easily obtain the following.

PROPOSITION 5. *For every $k \geqslant 2$ we have*

$$(\langle \dot{x}(t), \nabla \rangle^k)(f)(x(t)) = \frac{\partial^k h}{\partial x_1^k}(x_1(t), 0)(1+\varepsilon(t))$$

for all t sufficiently close to t_0 and with $\varepsilon(t) \to 0$ as $t \to t_0$.

Now by using the same techniques of the case $n=2$ we easily achieve the proof of the following.

PROPOSITION 6. *Let $x \in G(t_0, b)$ and asume that $(\partial^p h/\partial x_1^p)(0,0) < 0$; then the function $z(t) = h(x_1(t), \hat{x}(t)) - x_n(t)$ is identically zero in a suitable interval $[t_0, t_0 + \gamma]$.*

Combining Proposition 6 with [6, Lemma 2.1] we prove local uniqueness in this first case.

Assume now that $(\partial^p h/\partial x_1^p)(0,0) > 0$ and $2m+1 < p < \infty$. From the first part of the proof of Lemma 4 we argue

$$\dot{x}_n^2(t) \leqslant \tilde{L} x_1(t)^{p-1} \tag{18}$$

when $t - t_0$ is small enough; on the other hand it is easy to verify (as in Lemma 2) that

$$\langle \nabla_{\hat{x}} \varphi, \hat{x}(t) \rangle \geqslant 0 \tag{19}$$

and then

$$\begin{aligned}(\langle \dot{x}(t), \nabla \rangle^2)(f)(x(t)) \\ = h_{x_1x_1}(x_1, 0)\, \dot{x}_1^2 + \langle \hat{x}, \hat{g}_{x_1x_1} \rangle\, \dot{x}_1^2 + 2\langle \nabla_{\hat{x}} \varphi, \dot{\hat{x}} \rangle\, \dot{x}_1 + h_{\hat{x}_i\hat{x}_j} \dot{x}_i \dot{x}_j.\end{aligned}$$

Again we can prove that $\langle \hat{x}, \hat{g}_{x_1x_1}(x_1, 0) \rangle \geqslant 0$ for $t - t_0$ small enough and by using (18), (19) from the latter equality we obtain

$$(\langle \dot{x}(t), \nabla \rangle^2)(f)(x(t)) \geqslant 0$$

for all t such that $t - t_0$ is sufficiently small.

When $p \leqslant 2m+1$ we may proceed as in the first part of Lemma 4 (which does not depend on the sign of $(\partial^p h/\partial x_1^p)(0,0)$) and we obtain

$$\dot{x}_n^2(t) \leqslant K x_1(t)^{2(p-1-m)} \tag{20}$$

as $t \to t_0$. By using (20) we obtain for $t - t_0$ small enough

$$(\langle \dot{x}(t), \nabla \rangle^2)(f)(x(t)) = h_{x_1x_1}(x_1, 0)\, \dot{x}_1^2 (1 + \nu(t)),$$

where $\nu(t) \to 0$ as $t \to t_0$.

We have only to consider the case $p=\infty$, i.e., $h(x_1,0)\equiv 0$; from (6) and (6′) we obtain $\varphi(x)=\langle \hat{x}, \hat{g}_{x_1}(x)\rangle$ and so Lemma 1 and Lemma 3 yield

$$\begin{aligned} \dot{x}_n^2(t) &\leqslant M\frac{\varphi^2}{|\Psi|^2}+C\int_{t_0}^{t}|\hat{x}(s)|^2\,ds \\ &\leqslant M\frac{\varphi^2}{|\Psi|^2}+C\int_{t_0}^{t}|\hat{g}(x_1(s),0)|^2\,\dot{x}_n^2(s)\,ds \end{aligned}$$

when $t-t_0$ is small enough. But for the same t we have

$$\begin{aligned} |\varphi(x(t))| &\leqslant c_1\,|\hat{x}|^2\,|\hat{g}_{x_1}(x_1,0)|^2 \\ &\leqslant \sigma(t)\,|\hat{g}(x_1,0)|^2\,|\hat{g}_{x_1}(x_1,0)|^2\,\dot{x}_n^2(t), \end{aligned}$$

where $\sigma(t)\to 0$ as $t\to t_0$. Since $|\Psi(x_1(t),0)|^2\geqslant m\,|\hat{g}(x_1(t),0)|^2$ we obtain

$$\dot{x}_n^2(t)\leqslant k\int_{t_0}^{t}|\hat{g}(x_1(s),0)|^2\,\dot{x}_n^2(s)\,ds$$

and so $\dot{x}_n(t)\equiv 0$ in a suitable interval $[t_0, t_0+\eta]$.

Therefore, by virtue of Lemma 1, we obtain $\ddot{x}(t)\equiv 0$ in the same interval and the equality

$$(\langle \dot{x}(t),\nabla\rangle^2)(f)(x(t))=0$$

holds true in $[t_0, t_0+\eta]$.

We have proved that when $(\partial^p h/\partial x_1^p)(0,0)>0$ or $f(x_1,0)\equiv 0$ then

$$(\langle \dot{x}(t),\nabla\rangle^2)(f)(x(t))\geqslant 0$$

holds true in a suitable interval $[t_0, t_0+\bar{\eta}]$; this fact, by applying Proposition I completes the proof of local uniqueness and so Theorem II is completely proven. ∎

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