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Frequency Based Radiance Cache for Rendering Animations

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Abstract

We propose a method to render animation sequences with direct distant lighting that only shades a fraction of the total pixels. We leverage frequency-based analyses of light transport to determine shading and image sampling rates across an animation using a samples cache. To do so, we derive frequency bandwidths that account for the complexity of distant lights, visibility, BRDF, and temporal coherence during animation. We finally apply a cross-bilateral filter when rendering our final images from sparse sets of shading points placed according to our frequency-based oracles (generally < 25\% of the pixels, per frame).

1. Introduction

In physically based rendering, pixel colors are computed by estimating a multi-dimensional integral. Pixels are usually computed independently, leading to potential redundancies. In the case of animations, where variation across frames may only change sparsely, the probability of performing redundant computation only increases. To exploit this redundancy, previous work has targeted more efficient computation of individual pixels by improving the underlying numerical integration routines (e.g., path-integral and density-estimation approaches [HPJ12, GJTS12] and importance sampling [VG95]), or by amortizing computation across image regions using adaptive methods based on predictive models of light transport (e.g., caching, interpolants and filters derived from frequency-space [DHS\textsuperscript{*}05] or first-order [RMB07] analyses).

We propose an adaptive approach to render animations with complex direct reflection and shadows from environment lighting. We perform adaptive sampling in image space, as well as during the numerical integration of each image sample. We further amortize shading cost by reusing previously computed image samples across the animation in a conservative manner, based on frequency analysis of light transport. To do so, we devise a caching scheme to store image samples using spatial, directional and temporal frequency bandwidths. Our intuition is that frequency bandwidths of moving objects or viewpoints can be translated into static spatio-angular bandwidths. For cases where this assumption fails (i.e., moving occluders), we derive tailored conservative estimates.

Our oracles builds on frequency-based light transport analysis [DHS\textsuperscript{*}05], and we additionally leverage them during final image reconstruction. We are able to render images using few image samples. Our frequency-analysis extends previous work to addi-
tionally treat temporally-varying occlusion changes, to support animation sequences using a lightweight caching scheme.

We perform rendering by applying three simple concepts:

1. we shade only a carefully chosen sparse subset of pixels,
2. when doing so, we adapt the spherical integration sampling rate according to variations in lighting, BRDF, and occlusion, and
3. for animations, we further reduce the image sampling rates by reusing samples from previous frames according to our frequency analysis, properly accounting for shading and occlusion variations over time.

Specifically, our work consists of the following contributions:

- a frequency analysis of sample placement and reuse, within and across frames, that amortizes shading computations over space, time, and integration domains,
- an adaptive integration scheme based on material, lighting, and visibility statistics devised from our new frequency-analysis,
- a practical lightweight caching scheme that reuses, discards, and recomputes shading and occlusion information over time, according to our frequency-analysis, in order to control error.

2. Previous Work

Frequency Analysis of Light Transport. Durand et al. [DHS05] proposed a frequency analysis of local lightfields for surface based shading. They applied it to a proof-of-concept adaptive image sampling and reconstruction application, where the numerical integration of the shading integral at each sample location had already been computed to (visual) convergence. This seminal work has motivated our frequency analysis, properly accounting for shading and occlusion variations over time.

Lightfield Reconstruction. Lehtinen et al. [LAC11, LALD12] reconstruct static images from a sparse set of lightfield samples, leveraging the structure of continuous lightfield space. These methods assume smoothness along reconstruction directions in the lightfield and rely on the user provided sampling rate to be adapted to the integrand’s bandlimit. On the contrary, we determine adequate sampling rates, and as such are better suited for higher frequency variations.

Deep Image Filtering. A related set of techniques [RKZ11, DSHL10, SD11, LWC12, BRM16] apply denoising filters directly to rendered images, using custom feature-space metrics, computed from unconverted path traced simulation. Instead of leveraging structure in the high-dimensional lightfield, these approaches formulate final rendering as a signal reconstruction problem ignoring, for the most part, higher-order structures, coherence or frequency-content of the shading. Still, they perform well in many complex scenarios. In certain cases, a limited form of temporal filtering is supported, but only between adjacent frames; our caching scheme adapts over an entire animation sequence.

Caching and Temporal Coherence. Irradiance and radiance caching [WH92, KGBP05] model local variations in indirect lighting to place sparse cache samples in a scene, and then compute smooth shading from the samples using first- or second-order [SJ11] interpolants. Again, here the shading integral is computed independently at each sample (without any adaptivity), and temporal coherence is not handled. Bala et al. [BDT99] devise radiance interpolants and cache re-use oracles based on error estimate bounds on (potentially reusable) shading samples, and the render cache system [WDP99, BWG03, VALBW06] reprojects previous shading samples for interactive preview. We also reproject cache samples for temporally coherent animations, but instead leverage frequency bandwidth estimates that take local geometry variation, spatial and temporal (spherical and camera) occlusion variations, reflectance and lighting changes into account. Meyer and Anderson [MA06] used a smooth basis to reproject stochastic samples both in space and time for smooth indirect illumination effects; our approach instead treats much sharper features not amenable to smooth, generalized basis-space techniques.
Figure 2: From an initial sparse image-space sampling ((a), showing part of our extended G-buffer) we estimate the necessary sampling density for artifact-free reconstruction (b). We then reproject & validate samples from our temporal cache ((c), in red) and compute the sampling density from this first set (d) and subtract it from (b). We distribute samples according to this difference in density (e) and reconstruct the final shading (f) using our frequency-space oracles. New samples are added to the cache for the next frame.

- coupling adaptive image/object-space sampling (Section 5), spherical sampling (Section 4) and final image reconstruction, using oracles devised from our frequency analysis (Section 6).
- accounting for shading variations due to camera and object motion in our bandwidth computation, e.g., adaptively sample all-frequency shadows and reflection effects, accounting for complex occlusion and appearance variations in animated sequences (Section 5), and by
- exploiting coherence in the spatio-angular outgoing radiance with a lightweight cache of frequency-space metrics (Section 5).

4. Adapting the Spherical Sampling Rate

Our approach adaptively places radiance points in a scene (and across animation frames) and, for each one of these point we use a spherical integration scheme that adapts the sampling rate when computing the outgoing radiance estimate.

Algorithm 1: Our adaptive sampling & reconstruction method.

input : Empty deep image img; Current cache cache; Viewpoint v
output: Reconstructed image buffer img

// Uniformly sample screen space (Section 5)
pts = INITIALPIXELSAMPLING(v)

// Compute radiance with adaptive integration at these positions // and accumulate visibility statistics [µ, σ] (Section 4)
COMPUTEPOINTS(RADIANCE(pts, v)

// Save points for cache insertion after image generation
SAVEPOINTSINCACHE(cache, pts)

// Cache point verification and reprojection (Section 5)
for c in cache do
   if ISVALIDENTRY(c, v) do pts.add(c)
else cache.discard(c)
end

// For each pixel, reconstruct occlusion statistics and estimate // the required sampling density (Section 4)
N = COMPUTESAMPLINGDENSITY(pts, v)
npts = SAMPLEIMAGEFROMDISTRIBUTION(N, Nmax)

// Compute the radiance at sampled positions and accumulate // visibility statistics (Section 4)
COMPUTEPOINTS(RADIANCE(npts, v)

// Reconstruct final image (Section 6)
img = UPSAMPLE(npts, v)
Figure 3: We only consider the frequency content of the (distant) incident lighting and visibility within the view-evaluated BRDF footprint (in yellow), defined as a cone centered about the mean reflection direction of the BRDF slice $\mu_0$ with radius equal to three standard deviation $\sigma_0$ of the BRDF lobe. Our visibility statistics, comprised of its mean $\mu_v$ and three standard deviation $\sigma_v$, are also accumulated exclusively within this footprint.

Figure 4: We precompute the distant environment light’s bandwidth for multiple footprint sizes (left), at discrete footprint directions. During rendering, we query this structure to obtain the lighting’s bandwidth within the BRDF’s footprint (right).

Accounting for Visibility

It is impractical to directly incorporate the visibility’s bandwidth into the integrand’s bandwidth estimate $B$, since piece-wise constant (binary) functions in the primal (i.e., spherical) domain have infinite frequency bandwidth.

Instead, we motivate our solution by considering two spherical regions of integration, one that is largely occluded and another that is not: we note that the spherical sampling rate should not be modified in the unoccluded region since the sample rate already accounts for the frequency of the integrand, and these regions will contribute the most to the integral; we do, however, want to increase the sampling rate in occluded regions within the footprint, since any occlusions will likely increase the frequency bandwidth. As such, we opt to modulate the sampling rate determined by $B_{L_{\infty}}$ by a factor $f_v$ that accounts for the amount of occlusion in the solid angle subtended by the cone of integration, as follows:

$$N_\Omega = f_v + 4B_{L_{\infty}}^2.$$  

(3)

To determine the amount of occlusion, we accumulate a statistical representation of visibility during a first Monte Carlo estimation of Equation 1, which we compute using multiple importance sampling (MIS) [Ve97] and $N_e = 4B_{L_{\infty}}^2$ spherical samples. Our MIS implementation distributes samples evenly according to the pdfs of the environment light $p_f(\omega) \propto 1/\cos(\omega)$ and the view-evaluated BRDF $p_f(\omega) \propto \delta(p, \omega, \omega_0)$. During integration, we progressively accumulate the weighted arithmetic mean visibility vector $\mu_v$ and $3 \times 3$ visibility covariance matrix $\Sigma_v$ within the BRDF’s footprint,

$$\mu_v = \frac{1}{A_\Omega} \sum_{i=1}^{N_l} p_i \rho_i (1 - v_i),$$  

(4)

$$\Sigma_v = \frac{N}{(N_e - 1)A_\Omega} \sum_{i=1}^{N_l} (\rho_i (1 - v_i))^2 (p_i - \mu_v) (p_i - \mu_v)^T,$$

which we will use to adapt our spherical sampling rate. Here, the $p_i$ are the first hit-points visible from the shading position to the $i^{th}$ Monte Carlo (2D spherical) integration sample’s direction $\omega_i$ (expressed in the local coordinate frame about the BRDF cone’s central direction $\mu_0$) hitting a finite distance occluder, $v_i = V(\omega)$ is the binary visibility evaluated at $\omega$, and we weight the value by the view-evaluated BRDF’s pdf evaluated in the sampling direction.

Figure 5: We increase the number of spherical integration samples to properly account for shading variations due to occlusion. This reduces the error (right insets) with respect to the ground truth.
\( \rho_i = p_i(\mathbf{0}_i) \) to avoid considering visibility samples in regions outside the BRDF footprint (i.e., regions of little contribution to the final integral); finally, \( A_0 = \sum_{i=1}^{N} \rho_i (1 - v_i) \) is the normalization of BRDF weights for the occluded samples (\( v_i = 0 \)).

The visibility’s standard deviation \( \sigma_v \) corresponds roughly to an occlusion coverage measure within the BRDF’s footprint (see Figure 3) and is derived as

\[
\sigma_v = \max(\sqrt{\Sigma} \mu_p^T \Sigma \mu_v - \sqrt{\Sigma} \mu_p^T \Sigma \mu_p),
\]

where \( \mu_p \) and \( \mu_v \) are the unit tangent and binormal vector of the local coordinate frame around \( \mu_p \). We use the ratio of this coverage to the size of the BRDF footprint to estimate how many more samples are needed as \( f_v = f(\sigma_v/\sigma_p) \). We experimented with various easing functions and found that a Gaussian profile, \( f(x) = (N_{\text{max}} - N_{\Omega}) e^{-\frac{1}{2}(x-\Omega)^2} \) yields good results, where \( N_{\text{max}} \) is our maximum spherical sampling budget (Figure 5). We use the remaining integration samples to improve our Monte Carlo estimate of Equation 1 (with the same MIS estimator).

We use the frequency content of the BRDF and lighting, as well as occlusion statistics, to adapt the spherical integration cost for each radiance point. To further reduce render time, we will both reuse information from radiance points computed in previous frames (Section 5), as well as reconstructing the final image using only a sparse set of image-space radiance points (Section 6).

5. A Sparse, Adaptive Radiance Cache

When rendering an image, we first uniformly distribute a small number of radiance points in image-space in order to gather the required information to compute the optimal sampling density: we estimate the outgoing radiance’s spatio-angular frequency bandwidth (Section 5.1). We then reuse radiance points from previous frames, leveraging a object-space cache (Section 5.2). Shading variations caused by camera and/or object motion can invalidate radiance points stored in the cache, and so we yet again leverage our frequency analysis to appropriately handle cache sample invalidation, resampling, and reuse (Section 5.3) in a manner that maintains image fidelity and temporal coherence.

5.1. Adaptive Image Space Sampling

The spatio-angular bandwidth of the outgoing radiance in image-space will be used to determine the (screen-space) density of radiance points required for artifact-free image reconstruction (discussed in Section 6). To compute this bandwidth we proceed similarly to previous work [DHS’05, BSS’13a] as illustrated in Figure 6: beginning with the bandwidth emitted within the solid angle of the (distant) light subtended by the BRDF’s footprint \( B_{L_{\infty}} \) (Figure 6(a)), we analyze the reflected bandwidth (Figure 6(b)) accounting for the local visibility and BRDF’s band-limit \( B_p \), to estimate the bandwidth of the shade point at the sensor location \( B_{S} \) (Figure 6(c)).

Frequency analysis methods typically operate either on the bandwidth \( B \) or the variance \( \sigma^2 \) of the spectrum. We interchange between these two measures in our discussion using 3 standard deviation (corresponding to the 99.9th percentile of a Gaussian) as the

Figure 6: To estimate the screen-space bandwidth \( B_{S} \), we start with the emitted (angular) bandwidth of the environment light \( B_{L_{\infty}} \) reduced to the BRDF’s footprint \( (a) \), we then apply the BRDF band-limit \( B_p \) \( (b) \) and modulated according to the curvature, foreshortening, and spatial transport \( (c) \). The angular bandwidth serves directly as the screen-space bandwidth estimate.

While accumulating the mean occlusion direction \( \mu_v \) and covariance \( \Sigma \) during radiance point integration (Section 4), we also compute the mean and variance of occluders’ distance in the BRDF cone, similarly to Equations 4. We project the occluders’ hit information on the mean reflection direction of the BRDF \( \mu_p \) instead of on its local tangents. We thus get a mean occlusion distance of \( \mu = ||\mu_p|| \) and a standard deviation of \( \sigma = \sqrt{\mu_p^T \Sigma \mu_p} \). We approximate the minimum distance to occluders as

\[
l_{\text{min}} = \mu - 3 \times \sigma.
\]

Moreover, assuming a pinhole camera, the bandwidth incident on the aperture is equal to the screen-space bandwidth we will use for sampling. And so, the screen-space spatio-angular \( 2 \times 2 \) covariance matrix is computed as

\[
\Sigma_{S,0} = T_{x \rightarrow V} (C_v \circ B_p \circ C_{L}) \left( \sigma_{L_{\infty}}^2 + T_{V \rightarrow x}(l_{\text{min}}) \right),
\]

where each of the five operators above are simply \( 2 \times 2 \) matrices [BSS’13a] (see Appendix A for details); \( C_v \) accounts for the local curvature and cosine factor between \( n \) and the mean BRDF direction \( \mu_p \); \( B_p \) bandlimits the bandwidth according to the view-evaluated BRDF; \( C_n \) accounts for the mirror reflection, curvature and cosine foreshortening between the shade point and the viewpoint; \( \sigma_{L_{\infty}}^2 \) is estimated using the bandwidth queried from our hierarchical structure; and \( T_{x \rightarrow V} \) transports the bandwidth from the shade point to the eye. When accounting for occlusion, an additional transport operator \( T_{V \rightarrow x} \) warps the spatio-angular occlusion bandwidth according to the minimum occluder distance between the shade point and the light as derived in Equation 6 [DHS’05, EDR’11].

From the covariance matrix \( \Sigma_{S,0} \) we retrieve the angular part \( \sigma_0^2 \) defining our final scalar image-space variance, and derive its band-
Figure 7: During reprojection, we test whether the new viewing direction remains inside the BRDF’s footprint and, if so, we project the sample onto the screen (plain green line); if not, we (optionally) discard the sample from the cache (dashed red line).

width as stated above as

$$B_S \approx 3 \sqrt{\frac{\sigma^2}{\theta^2}}$$  \hspace{1cm} (8)

**Sampling Density and Radiance Point Placement** We first uniformly sample pixels in the image plane with a low number of samples per pixel to compute per pixel visibility statistics. The number of directional samples used is proportional to the BRDF cone apex. We apply a simple depth- and normal-aware bilateral upsampling filter [SGNS07] with filter sizes proportional to the pixel’s unobscured angular variance (see Appendices A, A) to remove noise from the bandwidth and visibility statistic values, \(\{\mu, \Sigma\}\). We use these values to compute the screen-space bandwidth \(B_S\) using Equations 7 and 8, and determine the final pixel sampling densities by clamping the bandwidth between zero and one (see Section 7) which we use to sample the image space by rejection sampling: each pixel receiving one unique sample if a random value between zero and one falls lower or equal than the corresponding pixel’s sampling density, ensuring that a density of one always yields one and only one sample and one of zero will never be sampled.

As detailed below in Section 5.2, we can further reduce the rendering cost by only generating new radiance points in space where reprojected (valid) radiance point samples from our cache do not satisfy our pixel sampling density requirement. When introducing a new radiance point, we always perform spherical integration using the approach in Section 4.

**5.2. Caching and Reusing Radiance Points**

During animation, the number of new radiance points computed each frame can be significantly reduced if we carefully reuse radiance points from previous frames. As such, we propose an object-space cache to store, update, invalidate, and reuse (when suitable) radiance points across frames of an animation. Every scene object gets an associated cache for easier management of its associated samples. Each cache entry consists of a tuple comprising a 3D position, 2D viewing direction, surface normal, radiance value, time at which the sample was originally created, boolean flag identifying moving samples, spatial bandwidth \(B_x\), angular bandwidth \(B_\theta\), a temporal bandwidth \(B_t\) (see Section 5.3), and the occlusion ratio used to define \(f \_\varepsilon\). The spatial, angular and temporal bandwidths \((B_x, B_\theta, B_t)\) (see derivation in Appendix A) are used to define a circular surface patch, a cone of directions (see Figure 7, inset), and a time interval within which the sample’s radiance value is known to remain close to the correct value. In the case where the associated object is a specular mirror, the tuple receives an additional 3D position, normal and object ID describing the reflected object from the view direction it was sampled from. Three render-time parameters \(\epsilon \_\varepsilon\) are used (see below) to control the accuracy of our method.

After our initial sampling, but before computing the current pixel sampling density, we update (based on object motion) and reproject radiance points stored in our cache onto the image plane. We only reuse these points for final image reconstruction if:

1. they pass a z-buffer camera visibility test, and
2. their radiance value remains valid after reprojection.

The validity of reprojected radiance points is determined according to spatial, angular and temporal bandwidth tests in order to control the bias introduced in the final rendering (as detailed below and in Figure 7). Reprojected radiance points that do not pass these tests are not considered for reprojection nor reconstruction, and are flagged for discard from their respective cache (as illustrated in Figure 8(a)).

Namely, object motion affects cache point validity in four ways:

1. camera and object motion can lead to failed z-buffer (i.e., camera visibility) tests during reprojection,
2. object motion can influence the validity of cached radiance points that lie on the surface of the moving object,
3. temporal changes of distant lighting affects the product of the light and BRDF during integration, and
4. object motion can influence the accuracy of radiance points due to changes in the spherical visibility on all other objects.

We describe the latter case (visibility changes) in Section 5.3 and discuss here the remaining points.

**Validating Reprojected Cache Points** We use the spatial, and angular bandwidths to define a circular surface patch and directional cone with radii \(r_x = 2\pi \_x / B_x\) and \(r_\theta = 2\pi \_\theta / B_\theta\) where the directional...
cone is centered along \( \mathbf{0}_t \) (at the time of the radiance point’s insertion into the cache), \( \epsilon_{\Omega_{t+1}} = \arccos(1 - q_{t} \mathbf{0}_t \cdot \mathbf{0}_t) \) is based on quality thresholds \( q_{t} \mathbf{0}_t \cdot \mathbf{0}_t \) corresponding to the maximum relative error a cached point can cause during reprojection. Validating a reprojected radiance point corresponds to z-buffer testing the pixel onto which it projects (for camera occlusion) and ensuring that the view vector is inside its directional cone (Figure 7).

To treat temporal changes in the radiance value due to lighting and occlusion motion, we add a temporal validity period to each cache point based on temporal bandwidth \( \Delta t = 2\pi/\omega \). We apply Egan et al.’s [ETH09] bandwidth estimate to treat variations in lighting rotation, and we derive temporal bandwidths for changes in radiance due to occlusion motion in Section 5.3. The final temporal bandwidth \( B_t \) is the maximum of these two bandwidths.

**Image Resampling and Cache Updates** Once reprojected, cache samples contribute in decreasing the per-pixel density in their screen-space reprojection neighbourhood. Each reprojected sample is splatted on screen in a manner similar to the final reconstruction (Section 6) and their derived screen-space bandwidth is subtracted from \( B_t \). This difference density is used to distribute new samples with the same rejection sampling rules as the empty cache case (see Figure 8(b)). Given the increased start-up density due to cache re-projection, we are more likely to introduce samples in areas that are not covered by the reprojected sampling.

**Cache Point Discard and Merging** Each newly generated radiance point is added to its corresponding cache and uses the spherical sampling scheme of Section 4. If two cached radiance points reproject onto the same pixel they may be merged if their object IDs are identical and if their spatial and angular bandwidths agree with the angular and spatial distances between them (Figure 9). If a radiance point reprojects outside the view frustum, it is also flagged for discard. Once per frame, each cache will remove in a batch all the flagged-as-dismissed entries and set their slot as open again for further entries to be added on future frames. The cache memory size increases if no open slot is available for this object, which happens implicitly for the first frame, and less regularly during the rest of an animation. Cache size is grown by sample batches which happens implicitly for the first frame, and less regularly during the rest of an animation. Cache size is grown by sample batches which happens implicitly for the first frame, and less regularly during the rest of an animation. Cache size is grown by sample batches which happens implicitly for the first frame, and less regularly during the rest of an animation.

Each cache is grown at least once (for the first frame) and (re)allocations can thus occur a maximum of once per object per frame. Each cache is grown at least once (for the first frame) by the amount of slots the sampler gave each object for the initial frame.

**5.3. Handling Temporal Occlusion Changes**

Modeling the impact of object motion on the scene visibility is challenging. Starting from a time-dependent visibility formulation of the reflection equation (Equation 1), we will perform a frequency analysis of changes in the outgoing radiance due to changes in visibility. We will show that the temporal bandwidth \( B_t \) of the change in the final shading can be modeled using the projected angular velocity of occluders and the distant light’s bandwidth. We store this bandwidth \( B_t \) at each cached radiance point.

We consider the relative motion of occluders at a shade point in our time-dependent reflection equation (below; Equation 9), and assume that only the spherical visibility changes over time, but not the lighting nor reflectance:

\[
L(\mathbf{p}, \mathbf{\omega}_t, t) = \int_{V_t} L_{\infty}(\mathbf{\omega}_t) V(\mathbf{p}, \mathbf{\omega}_t, t) \mathbf{p}(\mathbf{p}, \mathbf{\omega}_t, \mathbf{\omega}_t) \cdot (\mathbf{\omega}_t \cdot \mathbf{n}) d\omega_t,
\]

Concurrently treating temporal variations in lighting and reflectance is a challenging problem that we leave to future work.

Next, we model the change in outgoing radiance \( \Delta L \) due to the motion \( V(t) \) of a small moving occluder patch\( \ddagger \); we will later generalize this model to large occluders. The outgoing radiance “removed” due to the (potentially moving) occluder at time \( t \) is

\[
\Delta L(t) = \int_{V_t} L_{\infty}(\mathbf{\omega}_t) V(\mathbf{p}, \mathbf{\omega}_t, t) \mathbf{p}(\mathbf{p}, \mathbf{\omega}_t, \mathbf{\omega}_t) \cdot (\mathbf{\omega}_t \cdot \mathbf{n}) d\omega_t,
\]

where we denote the set of all occluded directions at time \( t \) as \( V_t = \{ \mathbf{\omega}_t | V(\mathbf{p}, \mathbf{\omega}_t, t) = 0 \} \). We can now rewrite the outgoing radiance at any moment of time as the sum of a temporally constant (unoccluded) component, and the change in outgoing radiance that does vary with time:

\[
L(\mathbf{p}, \mathbf{\omega}_t, t) = \left[ L(\mathbf{p}, \mathbf{\omega}_t, 0) - \Delta L(0) \right] + \Delta L(t). \tag{11}
\]

Here, \( \Delta L(t) \) can be thought of as time-dependent antiradiance [DSDD07]. We perform a frequency analysis of this formulation in order to reason about the temporal bandwidth of changes to the outgoing radiance at a radiance point and, given this segmentation, the Fourier transform of the outgoing radiance is equal to the Fourier transform of the change in the outgoing radiance, \( \hat{L}(\Omega_t) = \Delta \hat{L}(\Omega_t) \), as:

\[
\hat{L}(\Omega_t) = \hat{F}[L(\mathbf{p}, \mathbf{\omega}_t, t)] = \hat{F}\left[ L(\mathbf{p}, \mathbf{\omega}_t, 0) - \Delta L(0) \right] + \Delta L(t) \tag{12} = A \delta(\Omega_t) + \hat{F}[\Delta L(t)],
\]

where \( A \) is a constant offset we can ignore since it has a bandwidth equal to zero; as such, we only need consider the last term in our bandwidth derivation:

\[
\hat{F}[\Delta L(t)] = \Delta \hat{L}(\Omega_t) = \hat{F}\left[ \int_{V_t} L_{\infty}(\mathbf{\omega}_t) V(\mathbf{p}, \mathbf{\omega}_t, t) \mathbf{p}(\mathbf{p}, \mathbf{\omega}_t, \mathbf{\omega}_t) \cdot (\mathbf{\omega}_t \cdot \mathbf{n}) d\omega_t \right].
\]

\( \ddagger \) We occasionally omit location \( \mathbf{p} \) and view \( \mathbf{\omega}_t \) parameters for brevity.
Figure 10: We illustrate the need for our moving occluders bandwidth to discard samples during animations in the KILLEROO scene after 10 frames. Without this bandwidth, the cache incorrectly reprojected shadows (bottom inset) onto the scene which result in a dark region in at the feet of the Killeroo.

We will directly relate the Fourier transform of the (change in) outgoing radiance to the Fourier transform of the distant lighting, allowing us to rewrite the Fourier transform in spherical angular motion of the occluder as a motion relative to the occluder's amplitude, where the amplitude is reasonable given our small occluder and small motion assumptions, assuming that the projected (spherical) area of the moving occluder is constant over the projected occluder patch; this is reasonable given our small occluder and small motion assumptions, and similar to reflectance-constancy assumptions used in previous work [ETH09, EDR11, RAMN12]. We additionally model the spherical angular motion of the occluder as a motion relative to the distant illumination, allowing us to rewrite the Fourier transform in Equation 12 as an integration of the occluding patch in its original configuration at $V(0)$, but now lit under the distant lighting rotated according to the appropriate inverse (angular) rotation of the occluder motion:

$$\hat{\Delta L}(\Omega_t) \approx \mathcal{F} \left[ \hat{\rho} \int_{V(0)} L_{\infty}(\omega + \nu \cdot v) d\omega \right],$$

where $\nu$ is the angular velocity of the occluder patch, we introduce a notational abuse of the $\hat{\nu}$ operator to denote angular rotation of the lighting direction $\omega$, for simplicity, and

$$\hat{\rho} = \frac{1}{|V(0)|} \int_{V(0)} \rho(p, \omega, \omega_o) |\omega_o \cdot n| d\omega,$$

is the average cosine-weighted BRDF value over the occluder.

We can interchange the order of the integral and Fourier transform (due to linearity) in Equation 13, allowing us to express the Fourier transform of the (change in) outgoing radiance as the product of the Fourier transform of the lighting, the ratio of the average cosine-weighted BRDF and the angular occluder motion magnitude, and a phase term:

$$\hat{\Delta L}(\Omega_t) \approx \hat{\rho} \int_{V(0)} e^{i\hat{\nu}(\omega \cdot n + \omega_o \cdot \phi)} \left| \hat{\rho} \right| \hat{L}_{\infty}(\nu \cdot \hat{\omega}) d\omega,$$

$$= \frac{\hat{\rho}^2}{|\nu|} \hat{L}_{\infty}(\nu \cdot \hat{\omega}) \left[ \int_{V(0)} e^{i\hat{\nu}(\omega \cdot n + \omega_o \cdot \phi)} d\omega \right],$$

where the bracketed term is the phase component of the Fourier transform which we conservatively bound below.

We can bound the bandwidth of $\hat{\Delta L}(\Omega_t)$ by analyzing the spectrum’s amplitude, where the amplitude is $|\cdot| = \sqrt{\text{Re}(\cdot)^2 + \text{Im}(\cdot)^2}$; we first bound the amplitude of the phase term as the projected solid angle of the occluder $V_{\Omega_o}$.

$$\left| \int_{V(0)} e^{i\hat{\nu}(\omega \cdot n + \omega_o \cdot \phi)} d\omega \right| \leq \int_{V(0)} d\omega.$$

since $|e^{i\phi}| \leq 1$, and we can then bound the amplitude of $\hat{\Delta L}(\Omega_t)$ in Equation 15 as

$$|\hat{\Delta L}(\Omega_t)| \leq \frac{V_{\Omega_o} \hat{\rho}^2}{|\nu|} \hat{L}_{\infty}(\nu \cdot \hat{\omega}) \right].$$

Given this bound on the amplitude we can conservatively bound the bandwidth $B_{single}$ of the (change in) outgoing radiance due to a small occluding patch: since the temporal rotation (i.e., shift) in the primal domain of $L_{\infty}$ in Equation 13 results in a scaling of the spectrum $L_{\infty}$ by a factor of $|\nu|$ in Equation 15, the lighting’s original bandwidth $B_{L_{\infty}}$ is similarly scaled to $|\nu|B_{L_{\infty}}$ under the occluder’s relative motion. Thus, the bandwidth $B_{single}$ of $\hat{\Delta L}$ is bounded as

$$B_{single} \leq |\nu|B_{L_{\infty}}.$$

Our analysis above considers the motion of only a single small occluding patch, and we conceptually estimate the bandwidth due to the motion of all the occluders by taking the maximum bandwidth across all the small occluding patches: during spherical radiance point integration (Section 4), we compute (and cache; Section 5.1) the mean angular velocity of occluding samples times the light bandwidth (estimated using our hierarchical structure; Section 4) $\mu(|\nu|B_{L_{\infty}})$ and its variance $\sigma(|\nu|B_{L_{\infty}})$, in a manner similar to the accumulation of the visibility statistics in Equation 4. We then approximate the final temporal bandwidth, and thus the maximum bandwidth across all occluding patches, as the mean plus 3 times the standard deviation,

$$B_t = \mu(|\nu|B_{L_{\infty}}) + 3\sigma(|\nu|B_{L_{\infty}}).$$

This conservative bandwidth estimate models the change in outgoing radiance that a radiance point will undergo in time due to occluder motion; while it uses a conservative bound on the amplitude of the integrated phase term, we do not need to explicitly account for the phase term during its calculation. This bandwidth can be interpreted as associating a lifespan to radiance points in the cache, and we simply extend our cache validity checks to accept reprojected cache samples only if they are “young” enough; if not, we discard them. We also discard samples which occlusion ratio is different from more than 0.125 with the occlusion ratio computed at the pixel the reproject to (occlusion ratio are ranging from 0 to 1). This further incorporates a change of viewpoint for the cache sample.

While our bandwidth estimate is far from accurate, it improves the quality of all-frequency shadowed regions and behaves consistently: without occluder motion, cached samples will have infinite lifespan and only be discarded if their spatial or angular bandwidth tests are not satisfied resulting in blurred shadows (see Figure 10).
6. Image Reconstruction

We generate the final image(s) with a reconstruction using all our (sparse) screen-space samples. We apply texture maps after shading reconstruction to avoid having to account for their potentially high-frequency content in our frequency analysis.

The reconstructed shading $L_i$ at a pixel $i$ is a weighted sum of the radiance points’ values $L_{p'}$ (where $p'$ denotes a projected pixel position) in its local neighborhood $N_i$, where we only consider radiance points that reside on the same object as $\text{the radiance points' values}$.

$$L_i = T_i \sum_{p' \in N_i} W_{p' \rightarrow i} L_{p'}, \quad (20)$$

where $T_i$ is the texture value for pixel $i$. We use a standard cross-bilateral filter, as in previous shading reconstruction works [DHS’05], tailored to our predicted bandwidth estimates:

$$W_{p' \rightarrow i} = \exp \left[ -\frac{1}{2\sigma_t^2} || f_i - f_{p'} ||^2 \right], \quad (21)$$

where $f_{p'} = (\mathbf{p}, \mathbf{n}, \rho, \mathbf{x})$ is our feature vector, so that the reconstruction will not combine values that differ much along any of the axes in this vector, and $\sigma_t$ is a user defined standard deviation. In the case of the pixel feature vector, $f_i$, we use the center of the pixel. Since our feature vector contains the frequency estimate, it is capable of reconstructing both hard and soft transitions in image-space caused by shadows and reflections.

7. Results and Implementation

Our implementation is built directly atop Intel’s Embree raytracing engine and operates completely on the CPU, although it is readily parallelizable on the GPU. Our implementation runs in parallel using OpenMP and a screen buffer separated into tiles of 16 by 16 pixels. We used the a screen-space curvature estimator depending on object-space normals and positions when computing an object’s curvature at eye-ray hitpoints. Our MIS estimator of Equation 1 (Section 4) uses the power heuristic [VG95] and evenly distributes samples between the light and BRDF pdf’s.

To construct the environment map’s bandwidth hierarchy (Section 4), we use a 2D fast-Fourier transform [CT65] and, for footprints smaller than $16 \times 16$ pixels, we return the maximum possible frequency (one sample per texel). The bandwidth is computed by taking the 95th percentile of the 2D spectrum. If the footprint spans several pixels at the target resolution, we conservatively take the maximum value over the resulting pixels.

To prepare the screen-space bandwidth of Equation 8 for sampling by rejection, we first modulate it according to the screen’s aspect ratio, and then bound it between zero and one. So that the sampling density conforms to the aspect ratio, we multiply it by the maximum of the horizontal and vertical modifiers, so that $B_S = B_S \max \left[ \frac{f_y}{\pi}, \frac{f_x}{\pi} \right]$, in pixels$^{-1}$, where $f_x$ and $f_y$ are the horizontal and vertical fields of view, for a $W \times H$ sized image. We expose the maximum filter radius $f_{\max}$ for reconstruction as a user parameter, from which we can derive the minimum bandwidth we allow according to Shannon’s theorem as $B_{\min} = \frac{\text{width}}{2f_{\max}}$, $B_{\min}$ is fixed at 1 so that we never sample a pixel more than once. This higher bound could be extended to add support for sub-pixel supersampling.

If an object’s associated cache needs to accommodate more samples than its current capacity during an animation, we resize it to be 1.5 time its current capacity so that we avoid the constant reallocations that could theoretically occur.

We add anti-aliasing as a post-process operation for the results of all the compared techniques and which processing time is not included in the resulting times since it isn’t relevant to the measure. For that reason, we use sub-pixel information for it about the normals, depths and object IDs for a more precise detection of edges.

**Results.** We have tested our method on the following scenes: Helmet (Figure 1) shows a rotating rotor blade in a static scene with simple Whitted-style indirect specular effects and texture mapping.
CAR (Figure 12) is an example of glossy, diffuse and mirror objects under camera rotation. KILLEROO (Figure 11) combines diffuse and glossy BRDFs with meshes that exhibit high curvature, and so are particularly challenging for resolving reflections. All reflections and glossy BRDFs with meshes that exhibit high curvature, and so are particularly challenging for resolving reflections. All results are generated on an Intel i7 CPU 930 2.80 GHz at a resolution of 1280 × 720, with the following parameter settings $f_{\text{max}} = 8$, $q_{\epsilon} = q_{\eta} = 1 \times 10^{-4}$ and $q_{\theta} = 1 \times 10^{-7}$.

Our method consistently reconstructs image sequences using only a fraction of the pixels on a screen, and our results are temporally coherent (see video): we typically reuse between 10 and 25\% of the pixels to render any given frame of an animation. Our cache size ranges on tested scenes from 45K to 500K radiance points, but could potentially go further than that (1M) for long, complex animations over convoluted scenes; corresponding to a maximum size of 6, 68 and 136 Mb respectively.

We perform an equal-quality comparison of an animation sequence generated with our method to that of Durand et al.’s [DHS05] prototype (Figure 14); we also exhibit sublinear scaling of the rendering time with respect to the number of rendered frames.

Finally, we perform an equal-time comparison of a still frame to the SURE-based optimization for Adaptive Sampling and Reconstruction from Li et al. [LWC12] (Figure 13). The results were rendered with the parameter set put forth by the authors, that is $\sigma_{f} = 0.4$, $0.125$ and $0.3$ for normal, texture color and depth respectively. We adapted the number of filter bank iterations for the final pass as $\sigma_{\epsilon} = \{0.1, 2, 4, 8\}$ for performance purposes to reduce the execution time of their technique to something closer to ours. The SURE-based adaptive sampling method performs a lot better than raw MIS perceptually, especially in low-frequency regions where their algorithm makes good use of larger filters. While it performs quantitatively better than our method here, Li et al. technique appear noisier, especially when handling all-frequency shadows, as present in the HELICO scene.

8. Conclusion and Future Work

We presented an adaptive sampling, signal-tailored integration and reconstruction technique for all-frequency direct illumination, rendering images and animations using a fraction of the cost of standard techniques. We develop new frequency bandwidth estimates to appropriately sample occlusion, reflectance, and lighting variations. In the case of animation sequences, we further amortize rendering cost with a lightweight caching scheme that also exploits our frequency analysis.

Discussion

Our temporal bandwidth derivation for moving occluders assumes that the projected size of a small occluding patch does not change over time; this only holds for purely rotational occluder motion (from the shading point’s perspective), but still provides a good approximation in the case of small motions. Alleviating this constraint could further improve our visibility bandwidth, and we would like to investigate coupling the effects of temporal changes in the reflectance and lighting together with the occlusion. Similarly, we only consider linear object motion, as is common in many rendering approaches, so modeling shading variations due to rotational motion could be an interesting avenue of future work.

We do not model the temporally varying depth complexity of occluders, which could lead to shading variations when occluders subtend the same spherical region at certain moments in time during there motion; in the case where one moving occluder completely blocks another, this may lead temporal bandwidth underestimation in our current approach, however we have not noticed a case where this results in any artifacts during our investigation.

Another more subtle issue arises when trying to simply multiply the reconstructed shading by a texture value after the fact (Equation 20): this only holds when the texture value at a pixel does not vary over time; this is clearly not the case for objects under relative motion w.r.t. the camera. The correct solution would be to project the texture, masked by the pixel’s spatial footprint, across time and
While our caching scheme and bandwidth estimates can be readily integrated into other frequency-based shading approaches to account for distribution effects such as defocus [SSD*09, BSS*13b] or motion blur [ETH*09, BSS*13b], our algorithm and analysis are currently restricted to single-bounce direct illumination (and Whitted-style recursive effects, e.g. mirror reflections). Extensions to global illumination are not trivial, but one direction would be to formulate a progressively-accumulated bandwidth estimate to deal with multiple bounces in e.g. a path-tracing estimator.

Lastly, while one of the benefits of our cache system is that it is very lightweight, we do not currently share or reuse integration samples across radiance points; previous approaches [EDR11, EHCR11] have treated similar problems using a heavyweight ray-space cache. We find our proposed solution to be a reasonable middle-ground solution, using higher-order statistics to compare and reason about nearby radiance points, but we never share this information across radiance points to influence the integration process. Doing so is left to future work and could further improve the efficiency of a technique like ours.

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Appendix A: Image-space spatio-angular variance

We derive the 2D spatial and angular variances of the incident light field in object and image space to direct our sampling and reconstruction algorithms. We do this by analysing how the light field function changes along the light path from the light source to the eye, and work with $2 \times 2$ matrix operators to facilitate derivation. Our analysis is based on previous works [DHS 07, EDR11, BSS 2007] and our analysis is based on the Phong BRDF, which band-limits the signal by an amount inversely-proportional to the BRDF's shininess. We only consider time- and space-invariant isotropic BRDFs and our analysis is based on the Phong BRDF.

Unoccluded Case We only consider light from distant illumination modelled by environment maps, which has no spatial variation and only non-zero angular variance, as

$$\Sigma = \begin{bmatrix} 0 & 0 \\ 0 & \sigma_{\infty}^2 \end{bmatrix}. \quad (22)$$

Light travels from the source to the surface point $x$ in a straight line (since we do not tackle participating media). This results in an angular shear with a magnitude equal to the distance travelled $d_i$. Since we use infinitely distant light with only angular variation, this operation has no effect on the signal and isn’t represented in Equation 7. The operator is defined as

$$T_{d_i} = \begin{bmatrix} 1 & -d_i \\ 0 & 1 \end{bmatrix}. \quad (23)$$

Reflection of the incident light field at the surface point $x$ in the direction of the eye is composed of a series of transformations [DHS 05]:

1. The re-parametrization of the light field in the surface’s local frame, where the light field is first scaled by $c_i = \cos \theta_i$ to account for the shortening of the incident ray, and then sheared by the effect of the curvature $k$ of the surface. The scale due to the incident angle is spatial and has no effect on our purely angular light signal here. In matrix form, $\Sigma$ is modified by

$$C_L = \begin{bmatrix} c_i & 0 \\ k & 1 \end{bmatrix}. \quad (24)$$

2. The angular convolution of the re-parametrized light field by the surface’s BRDF, which band-limits the signal by an amount inversely-proportional to the BRDF’s shininess. We only consider time- and space-invariant isotropic BRDFs and our analysis is based on the Phong BRDF.
for which we obtain the covariance \( \text{cov}_{\theta,\psi}(\rho) = s/4\pi^2 \) for a Phong exponent (or shininess) of \( s \) [BSS*13b], which can be rewritten in matrix form, with \( b = 1/\text{cov}_{\theta,\psi}(\rho) \), as

\[
B_k = \begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix}.
\]  

(25)

3. The re-parametrization in the outgoing direction to the eye, similar to the first step, is first a mirror reflection in the spatial domain, followed by the inverse curvature shear by \( -k \) and the scale by \( c_s = 1/\cos \theta_r \), as

\[
C_s = \begin{bmatrix} c_s & 0 \\ c_s k & -1 \end{bmatrix}.
\]  

(26)

Finally, light travels from \( \mathbf{x} \) to the eye by a distance \( d \), and the signal is modified one last time by the operator

\[
T_{x \rightarrow y} = \begin{bmatrix} 1 & -d \\ 0 & 1 \end{bmatrix}.
\]  

(27)

We put everything together to get the spatial and angular variances in object and image space. Let \( \Sigma \) be the covariance of the incident light field, we get the object space covariance \( \Sigma_{o,\theta} \) by the succession of operations involving the matrix operators defined above such as

\[
\Sigma_{o,\theta} = C_f^T \Sigma C_l \quad \Sigma_\theta = \Sigma_o - \frac{\Sigma_f^T B_{sp} \Sigma}{1 + T_r^T \Sigma_f B_{sp}} \quad \Sigma_{\theta,\theta} = C_f^T \Sigma C_r,
\]  

(28)

and the image space covariance \( \Sigma_{o,\theta} \) by applying the travel operator

\[
\Sigma_{o,\theta} = T_{x \rightarrow y}^T \Sigma_{o,\theta} T_{x \rightarrow y}.
\]  

(29)

**Occluded Case** When accounting for occlusion, an additional transport operator \( T_{V \rightarrow x} \) warps the spatio-angular occlusion bandwidth according to the minimum occluder distance \( t_{\text{min}} \) between the shade point and the light [DHS*05, EDR11]

\[
T_{V \rightarrow x} = \begin{bmatrix} 1 & -t_{\text{min}} \\ 0 & 1 \end{bmatrix}.
\]  

(30)

Since we employ distant environmental illumination, this operation wouldn’t affect the covariance as it only has non-zero angular variance \( \sigma_{3,\infty}^2 \) at emission. We add an undefined spatial frequency content \( a \) in the matrix at emission and apply the transport shear before the interaction with the surface, the BRDF and the re-projection to the eye, so that our covariance matrix at emission is defined as

\[
\Sigma = \begin{bmatrix} a & 0 \\ 0 & \sigma_{3,\infty}^2 \end{bmatrix}.
\]  

(31)

The rest of the derivation of the object and image space covariances is done in the same manner as in the unoccluded case above, except that

\[
\Sigma_{o,\theta} = C_f^T \Sigma C_l, \quad \text{with} \quad \Sigma = T_{V \rightarrow x}^T \Sigma T_{V \rightarrow x}.
\]  

(32)

To get defined results, we compute the covariance in the limit where the spatial content \( a \) of our emitted light tends to infinity.

**Scalar variance & Bandwidth** The matrix operations above are used to determine the object space spatial and angular bandwidths \( (B_s, B_\theta) \) to control the validity of re-projected cache samples in Section 5.2; and the image space angular variance \( \sigma_\theta^2 \) in Equation 8 of Section 5.1.

In practice, the occluded derivation is always used to get \( \sigma_\theta^2, B_s \) and \( B_\theta \). The only case where the unoccluded derivation is used is when determining filter sizes of the bilateral filtering pass used to smooth the visibility statistics \( \{\rho_\psi, \Sigma_\psi\} \) in Section 5.1.

When computing the occluded variables, in the frequent case where no actual occlusion occur between the surface and the light, \( t_{\text{min}} \) is undefined and we should optimally switch to the unoccluded variables. However, we found that by using a large enough distance as \( t_{\text{min}} \) when there is no occlusion gives us smoother transitions between occluded and unoccluded areas and avoids the use of a hard switch. This maximum distance is scene dependent and is defined proportionally to the scene’s bounds. By considering every direction as occluded, we first bias our sampling scheme and reconstruction filter sizes to distribute more samples and be smaller, respectively, than with the hard switch; and second make our cache re-projection policy stricter. We allow those biases since they can only improve quality, and because the maximum distance used is large enough as to make any of those effect imperceptible.

Below are the scalar variables used in practice, evaluated from the derivations above.

**Unoccluded Case**

\[
(\sigma_s^2)^2 = \Sigma_{3,\theta}[1,1] = (4c_s^2 k^2 \sigma_{3,\infty}^2)/(1 + b \sigma_{3,\infty}^2),
\]

\[
(\sigma_\theta^2)^2 = \Sigma_{3,\theta}[2,2] = (\sigma_{3,\infty}^2)/(1 + b \sigma_{3,\infty}^2), \quad \text{and}
\]

\[
\sigma_\theta^2 = \Sigma_{3,\theta}[2,2] = (1 - 2c_s k^2)^2 \sigma_{3,\infty}^2/(1 + b \sigma_{3,\infty}^2).\]

**Occluded Case**

\[
(\sigma_s^2)^2 = \lim_{a \rightarrow \infty} \Sigma_{3,\theta}[1,1] = (c_s + 2 t_{\text{min}} k)^2 + b c_s^2 \sigma_{3,\infty}^2)/b t_{\text{min}}^2,
\]

\[
(\sigma_\theta^2)^2 = \lim_{a \rightarrow \infty} \Sigma_{3,\theta}[2,2] = b^{-1}, \quad \text{and}
\]

\[
\sigma_\theta^2 = \lim_{a \rightarrow \infty} \Sigma_{3,\theta}[2,2] = (t_{\text{min}} - c_s d (c_s + 2 t_{\text{min}} k)^2 + b c_s^2 c_s^2 d^2 \sigma_{3,\infty}^2)/b t_{\text{min}}^2.
\]

We derive bandwidths \( B_s \) and \( B_\theta \) in the same manner (see Section 5.1) as

\[
B_s = 3 \sqrt{\sigma_s^2} \quad \text{and} \quad B_\theta = 3 \sqrt{\sigma_\theta^2}.
\]