The Helmholtz decomposition in weighted L^p spaces in cones

Serge Nicaise*

June 19, 2017

Abstract

In this paper we prove the Helmholtz decomposition in conical domains of $\mathbb{R}^3$ in weighted L^p_β spaces under a spectral condition on α, p . The basic bricks are the transformation of the original problem into a problem set in cylindrical domains and the combination of a priori bounds from [5] with the vector-valued multiplier theorem [29].

AMS (MOS) subject classification 35J05, 35J20.

Key Words Helmholtz decomposition, cones.

1 Introduction

The Helmholtz decomposition, namely the decomposition of vector fields into a solenoidal field and a gradient part, is an indispensable tool in the analysis of incompressible fluid flows. While in the L^2 -setting such a decomposition is easily obtained for any domains of $\mathbb{R}^d, d \geq 2$ using elementary Hilbert space properties [20, Lemma 2.5.1], the situation becomes more delicate in the L^p -setting. Nevertheless, there are numerous results for bounded domains [3, 10, 12, 20] or exterior domains (complement of a bounded domains) [17, 18, 28]; see also [6, 7, 23, 22] for such results in weighted spaces. A common feature of such domains is that they have a compact boundary. On the other hand there exist domains with non compact boundary (sector-like) for which the L^p -Helmholtz decomposition fails [1, 2, 16]. Nevertheless it is still valid for basic domains with a smooth boundary like half-spaces or perturbations of them [11, 13, 27], aperture domains [4, 17, 6], cylinders or layers [17, 21, 5, 19, 26, 25], or domains whose boundary is the graph of smooth functions [2, 11]. But to the best of our knowledge, the question of the validity of the L^p -Helmholtz decomposition for domains with a non compact and non-smooth boundary remains open. Nevertheless let us mention the paper [15] that obtains a Helmholtz decomposition in anisotropic L^p spaces for domains whose boundary is the graph of a globally Lipschitz functions (hence conical domains are admitted).

The goal of this paper is therefore to prove such a Helmholtz decomposition in conical domains of $\mathbb{R}^3$ in weighted L^p_β spaces (L^p -spaces being a particular one when $\beta = 0$) under a condition between α, p and the eigenvalues of the Laplace-Beltrami operator with Neumann boundary conditions on the intersection of the cone with the unit sphere. The basic tools are first to transform the problem into a problem set in cylindrical domains using the Euler change of variables, second by applying a formal Fourier transformation we found a problem that is a lower order perturbation of a problem

*Université de Valenciennes et du Hainaut Cambrésis, LAMAV, FR CNRS 2956, F-59313 - Valenciennes Cedex 9 France, Serge.Nicaise@univ-valenciennes.fr

considered in [5], hence combining a priori bounds from this paper with the vector-valued multiplier theorem [29], we conclude the result. Note that this result can be derived via a duality argument, we refer to [24] in this respect.

The paper is organized as follows. The second section deals with the weak Neumann problem in the Fourier space, namely we study the resolvent of the operator obtained after Euler change of variables and Fourier transform and show its $\mathcal{R}$ -boundedness. In section 3, we show the well-posedness of the weak Neumann problem in a cone and conclude with the Helmholtz decomposition. **Acknowledgement:** We are grateful to Professor P. Deuring (Univ. Littoral) to have pointed out the problem considered in this paper and for various discussions about it.

2 The weak Neumann problem in the Fourier space

Let us fix an open subset S of the unit sphere of $\mathbb{R}^3$ with a $C^{1,1}$ boundary. On such a domain, we want to consider a Neumann type problem with complex parameters ξ and on $L^p(S)$ spaces. Hence as in [5], in order to have uniform estimates in ξ and to use the vector-valued multiplier theorem [29], we enlarge the setting to weighted L^p spaces.

Before let us fix some notations. Let us denote by $(\mu_k)_{k \geq 0}$, the sequence of the eigenvalues (repeated according to their multiplicities) of the (non negative) Laplace-Beltrami operator L_{LB}^{Neu} on S with Neumann boundary conditions (obviously $\mu_0 = 0$ and $\mu_1 > 0$).

Let us further fix an atlas (U_ℓ, Φ_ℓ) , $\ell = 1, \dots, L$ of S in the sense that for each ℓ , U_ℓ is a bounded open set of $\mathbb{R}^2$ and

$$\Phi_\ell : U_\ell \rightarrow S$$

is an injective and C^1 mapping such that

$$\cup_{\ell=1}^L \Phi_\ell(U_\ell) = S.$$

Now a non-negative function $\omega \in L^1(S)$ is called an A_p -weight if and only if $\omega \circ \Phi_\ell$ is the restriction to U_ℓ of an A_p -weight of $\mathbb{R}^2$. Finally we introduce the space $L_\omega^p(S)$ defined by

$$L_\omega^p(S) := \{u \in L^1(S) : \|u\|_{p,\omega} = \left(\int_S |u|^p \omega \, d\vartheta \right)^{\frac{1}{p}} < \infty\}.$$

In the same manner, we define

$$\mathbf{L}_{T,\omega}^p(S) := \{u \in L_\omega^p(S) : u \cdot \vartheta = 0\},$$

as well as

$$H_{p,\omega}^1(S) := \{u \in L_\omega^p(S) : \nabla_T u \in \mathbf{L}_{T,\omega}^p(S)\},$$

equipped with their natural norm.

The main goal of this section is to study the next problem: given two real parameters η, δ , and functions $\mathbf{g} \in \mathbf{L}_{T,\omega}^p(S)$ and $g \in L_\omega^p(S)$, we look for a solution $\psi \in H_{p,\omega}^1(S)$ to

$$\begin{aligned} (2.1) \quad & \int_S \left(\nabla_T \psi \cdot \nabla_T \varphi - (i\xi - \eta)(i\xi + \delta)\psi \bar{\varphi} \right) d\vartheta \\ & = \int_S (\mathbf{g}(\vartheta) \cdot \nabla_T \bar{\varphi} - g(\vartheta)(i\xi + \delta)\bar{\varphi}) d\vartheta, \forall \varphi \in H_{p',\omega'}^1(S), \end{aligned}$$

where p' is the conjugate of p and $\omega' = \omega^{-\frac{1}{p-1}}$.

The case when $\delta = -\eta$ with $|\eta| < \mu_1$ is treated in [5, Theorem 3.6] (since Theorem 3.2 of [5] is valid by replacing Δ' by the operator

$$\sum_{i,j} a_{ij} \partial_i \partial_j,$$

with constant coefficients $a_{ij} = a_{ji}$ that is elliptic in the (usual) sense that the matrix $(a_{ij})_{1 \leq i,j \leq 2}$ is positive definite). We here extend this result under one condition on η, δ . Namely we have the next result.

Theorem 2.1 *Assume that*

$$(2.2) \quad \eta\delta \neq -\mu_k, \forall k \in \mathbb{N}.$$

Then for all $\mathbf{g} \in \mathbf{L}_{T,\omega}^p(S)$ and $g \in L_\omega^p(S)$, there exists a unique solution $\psi \in H_{p,\omega}^1(S)$ of (2.1) with the estimate

$$(2.3) \quad \|\nabla \psi\|_{p,\omega} + (1 + |\xi|)\|\psi\|_{p,\omega} \leq C(\|\mathbf{g}\|_{p,\omega} + \|g\|_{p,\omega}),$$

with a positive constant C that may depend on the A_p -constant $\mathcal{A}_p(\omega)$ but is independent of ξ .

Proof. We first prove the existence and uniqueness of the solution. For that purpose, we introduce the family of operators (compare with [5, p. 375])

$$T_{p,\omega}(\eta, \delta, \xi) : H_{p,\omega}^{1,0}(S) \rightarrow (H_{p',\omega'}^{1,0}(S))^* : \psi \rightarrow T_{p,\omega}(\eta, \delta, \xi)\psi,$$

where

$$H_{p,\omega}^{1,0}(S) := \{\psi \in H_{p,\omega}^1(S) : \int_S \psi \, d\vartheta = 0\},$$

and

$$\langle T_{p,\omega}(\eta, \delta, \xi)\psi, \varphi \rangle = \int_S \left(\nabla_T \psi \cdot \nabla_T \varphi - (i\xi - \eta)(i\xi + \delta)\psi\bar{\varphi} \right) d\vartheta, \forall \varphi \in H_{p',\omega'}^{1,0}(S).$$

The proof of Theorem 3.6 of [5] shows that $T_{p,\omega}(0, 0, \xi)$ is an isomorphism. Now as

$$(T_{p,\omega}(0, 0, \xi) - T_{p,\omega}(\eta, \delta, \xi))\psi = (\xi^2 + (i\xi - \eta)(i\xi + \delta))\psi,$$

and since by Proposition 2.5 of [5], $H_{p,\omega}^1(S)$ is compactly embedded into $L_\omega^p(S)$, we deduce that $T_{p,\omega}(0, 0, \xi) - T_{p,\omega}(\eta, \delta, \xi)$ is a compact operator. Therefore $T_{p,\omega}(\eta, \delta, \xi)$ is a Fredholm operator of index 0 and it will be an isomorphism if and only if it is injective. So let $\psi \in \ker T_{p,\omega}(\eta, \delta, \xi)$ or equivalently $\psi \in H_{p,\omega}^{1,0}(S)$ satisfies

$$(2.4) \quad \int_S \left(\nabla_T \psi \cdot \nabla_T \bar{\varphi} - (i\xi - \eta)(i\xi + \delta)\psi\bar{\varphi} \right) d\vartheta = 0, \forall \varphi \in H_{p',\omega'}^{1,0}(S).$$

But as ψ is of mean zero, we have

$$\int_S \left(\nabla_T \psi \cdot \nabla_T 1 - (i\xi - \eta)(i\xi + \delta)\psi 1 \right) d\vartheta = -(i\xi - \eta)(i\xi + \delta) \int_S \psi \, d\vartheta = 0,$$

and therefore (2.4) extends to the whole $H_{p',\omega'}^1(S)$, namely

$$(2.5) \quad \int_S \left(\nabla_T \psi \cdot \nabla_T \bar{\varphi} - (i\xi - \eta)(i\xi + \delta)\psi\bar{\varphi} \right) d\vartheta = 0, \forall \varphi \in H_{p',\omega'}^1(S).$$

But Lemma 2.2 of [8] guarantees that $H_{p,\omega}^1(S)$ is continuously embedded into $W^{1,s}(S)$ for some $s > 1$ and by the Sobolev embedding theorem, $W^{1,s}(S)$ being continuously embedded into $L^2(S)$, we deduce that $\psi \in L^2(S)$.

Now for any $h \in L^2(S)$, as by our assumption $(-i\xi - \eta)(-i\xi + \delta)$ is never an eigenvalue of L_{LB}^{Neu} , there exists a unique solution $\varphi \in H^2(S)$ solution of

$$\left(L_{LB}^{\text{Neu}} - (-i\xi - \eta)(-i\xi + \delta)\right)\varphi = h.$$

Since $H^2(S)$ is continuously embedded into $W^{1,q}(S)$ for all $q > 1$, by Lemma 2.4 of [5], $H^2(S)$ is continuously embedded into $H_{p',\omega'}^1(S)$. Consequently the above identity implies that

$$\int_S \left(\nabla_T \psi \cdot \nabla_T \bar{\varphi} - (i\xi - \eta)(i\xi + \delta)\psi \bar{\varphi} \right) d\vartheta = \int_S \psi \bar{h} d\vartheta.$$

Comparing with (2.5), we see that

$$\int_S \psi \bar{h} d\vartheta = 0,$$

for any $h \in L^2(S)$ and we conclude that $\psi = 0$.

At this stage it remains to show the estimate (2.3). Let us then fix $\mathbf{g} \in \mathbf{L}_{T,\omega}^p(S)$ and $g \in L_\omega^p(S)$ and the unique solution $\psi \in H_{p,\omega}^1(S)$ of (2.1). Then ψ can be viewed as the solution of

$$\begin{aligned} \int_S \left(\nabla_T \psi \cdot \nabla_T \bar{\varphi} + (\xi + i\alpha)^2 \psi \bar{\varphi} d\vartheta \right) &= \int_S \left((\mathbf{g}(\vartheta) \cdot \nabla_T \bar{\varphi} - g(\vartheta)(i\xi + \delta)\bar{\varphi}) \right. \\ &\quad \left. + (i\xi(\delta - \eta + \alpha) - \eta\kappa - \alpha^2)\psi \bar{\varphi} \right) d\vartheta, \forall \varphi \in H_{p',\omega'}^1(S), \end{aligned}$$

with a fixed $\alpha \in (0, \sqrt{\mu_1})$, or equivalently

$$\int_S \left(\nabla_T \psi \cdot \nabla_T \bar{\varphi} + (\xi + i\alpha)^2 \psi \bar{\varphi} d\vartheta \right) = \int_S \left(\mathbf{g}(\vartheta) \cdot \nabla_T \bar{\varphi} - i(\xi + i\alpha)h(\vartheta)\bar{\varphi} \right) d\vartheta, \forall \varphi \in H_{p',\omega'}^1(S),$$

where h is given by

$$h(\vartheta) = -\frac{i\xi(\delta - \eta + \alpha) - \eta\kappa - \alpha^2}{i\xi - \alpha}\psi(\vartheta) + \frac{i\xi + \delta}{i\xi - \alpha}g(\vartheta).$$

Hence using Theorem 3.6 of [5] to this last problem, we find that

$$\|\nabla \psi\|_{p,\omega} + |\xi|\|\psi\|_{p,\omega} \leq C(\|\mathbf{g}\|_{p,\omega} + \|h\|_{p,\omega}),$$

with a positive constant C that may depend on the A_p -constant $\mathcal{A}_p(\omega)$ but is independent of ξ . From the form of h , we find that

$$\|\nabla \psi\|_{p,\omega} + |\xi|\|\psi\|_{p,\omega} \leq C(\|\mathbf{g}\|_{p,\omega} + \|g\|_{p,\omega} + \|\psi\|_{p,\omega}),$$

since the ratios $\frac{i\xi(\delta - \eta + \alpha) - \eta\kappa - \alpha^2}{i\xi - \alpha}$ and $\frac{i\xi + \delta}{i\xi - \alpha}$ are bounded for any $\xi \in \mathbb{R}$. This proves (2.3) for $|\xi|$ large. For $|\xi|$ small, the previous estimate (compare with the estimate (3.11) of [5]) allows to use the arguments of the proof of Theorem 3.6 of [5] to deduce that (2.3) is also valid (using our assumption (2.2)). ■

At this stage for a fixed pair of real numbers (δ, η) fulfilling (2.2), for any $\xi \in \mathbb{R}$ and any $(\mathbf{g}, g) \in X_{p,\omega} := \mathbf{L}_{T,\omega}^p(S) \times L_\omega^p(S)$, we set

$$(2.6) \quad \mathcal{M}_{\delta,\eta}(\xi)(\mathbf{g}, g) = (\nabla_T \psi, i\xi \psi),$$

where $\psi \in H_{p,\omega}^1(S)$ is the unique solution of (2.1). According to Theorem 2.1, the operator $\mathcal{M}_{\delta,\eta}(\xi)$ is bounded from $X_{p,\omega}$ into itself. For $T \in \mathcal{L}(X_{p,\omega})$ denote by $|||T|||$ its operator norm. Then the previous Theorem mainly allows to obtain the next result.

Corollary 2.2 *Given $p \in (1, \infty)$, fix a pair of real numbers (δ, η) fulfilling (2.2). Then $\mathcal{M}_{\delta,\eta}(\xi)$ is Fréchet differentiable with respect to $\xi \in \mathbb{R}$ and there exists an A_p -consistent constant c (depending on (δ, η)) such that*

$$(2.7) \quad |||\mathcal{M}_{\delta,\eta}(\xi)||| + |\xi| |||\mathcal{M}'_{\delta,\eta}(\xi)||| \leq c, \forall \xi \in \mathbb{R}.$$

Proof. The estimate of $|||\mathcal{M}_{\delta,\eta}(\xi)|||$ is nothing else than (2.3). But it is easy to check that the Fréchet derivative $\frac{\partial}{\partial \xi} \mathcal{M}_{\delta,\eta}(\xi)(\mathbf{g}, g) = (\nabla_T \psi', i\xi \psi') + (0, i\psi)$, where $\psi \in H_{p,\omega}^1(S)$ is solution of (2.1) and $\psi' \in H_{p,\omega}^1(S)$ is solution of (compare with (2.1))

$$\begin{aligned} \int_S \left(\nabla_T \psi' \cdot \nabla_T \varphi - (i\xi - \eta)(i\xi + \delta)\psi' \bar{\varphi} \right) d\vartheta \\ = -i \int_S (g(\vartheta) + (2i\xi + \delta - \eta)\psi \bar{\varphi}) d\vartheta, \forall \varphi \in H_{p',\omega'}^1(S). \end{aligned}$$

Hence applying Theorem 2.1 to this problem, we find that

$$\|\nabla \psi'\|_{p,\omega} + |\xi| \|\psi'\|_{p,\omega} \leq C(\|\psi\|_{p,\omega} + \frac{1}{1+|\xi|} \|g\|_{p,\omega}),$$

with a positive constant C that may depend on the A_p -constant $\mathcal{A}_p(\omega)$ but is independent of ξ . By the estimate (2.3) satisfied by ψ , we find that

$$\|\nabla \psi'\|_{p,\omega} + |\xi| \|\psi'\|_{p,\omega} \leq \frac{C_1}{1+|\xi|} (\|\mathbf{g}\|_{p,\omega} + \|g\|_{p,\omega}),$$

with a positive constant C_1 that may depend on the A_p -constant $\mathcal{A}_p(\omega)$ but is independent of ξ , which yields the requested on $|||\mathcal{M}'_{\delta,\eta}(\xi)|||$. ■

Remark 2.3 Notice that the statement of Corollary 2.2 remains valid for the operator $\widetilde{\mathcal{M}}_{\delta,\eta}(\xi)$ defined by

$$\widetilde{\mathcal{M}}_{\delta,\eta}(\xi)(\mathbf{g}, g) = (\nabla_T \psi, (1 + i\xi)\psi),$$

for any $\xi \in \mathbb{R}$ and any $(\mathbf{g}, g) \in X_{p,\omega}$.

To end up our preliminary results, we need an extrapolation property on $X_{p,\omega}$ and its consequence concerning $\mathcal{R}$ -boundedness, see Theorem 4.3 of [9] in $L_\omega^p(\Omega)$ or Theorem 4.3 of [5] in $L_\omega^p(\Omega)^n$ for an open set of $\mathbb{R}^n$, $n \in \mathbb{N}^*$.

Theorem 2.4 *Let $1 < p, s < \infty$, $\omega \in A_p$. Furthermore, let $\mathcal{T}$ be a family of $\mathcal{L}(X_{p,\omega})$ that satisfies*

$$(2.8) \quad \|TF\|_{s,\nu} \leq C\|F\|_{s,\nu}, \forall T \in \mathcal{T}, F \in X_{p,\omega} \cap X_{s,\nu},$$

for every $\nu \in A_s$ with a constant C that depends only on the A_s -constant $\mathcal{A}_s(\nu)$. Then $\mathcal{T}$ is $\mathcal{R}$ -bounded on $\mathcal{L}(X_{p,\omega})$, in the sense that there exists $C > 0$ such that for any $N \in \mathbb{N}$, $T_j \in \mathcal{T}$ and $F_j \in X_{p,\omega}$, it holds

$$\left\| \left(\sum_{j=1}^N |T_j F_j|^2 \right)^{\frac{1}{2}} \right\|_{p,\omega} \leq C \left\| \left(\sum_{j=1}^N |F_j|^2 \right)^{\frac{1}{2}} \right\|_{p,\omega}.$$

Proof. First by using an atlas of S , we notice that the result is valid in $L_\omega^p(S)^4$, consequently it suffices to use an extension argument from $X_{p,\omega}$ to $L_\omega^p(S)^4$. For that purpose, for $T \in \mathcal{T}$, we introduce the operator $\tilde{T}$ as follows: for any $(\mathbf{g}, g) \in L_\omega^p(S)^4$, we denote by

$$\tilde{T}(\mathbf{g}, g) := T(\mathbf{g} - (\mathbf{g} \cdot \theta)\theta, g).$$

Note that $\tilde{T}$ is well-defined since $\mathbf{g} - (\mathbf{g} \cdot \theta)\theta$ belongs to $\mathbf{L}_{T,\omega}^p(S)$. Hence we find a family $\tilde{\mathcal{T}} = \{\tilde{T} : T \in \mathcal{T}\}$ of elements of $\mathcal{L}(L_\omega^p(S)^4)$. Furthermore, by our assumption (2.8), one has

$$\|\tilde{T}(\mathbf{g}, g)\|_{s,\nu} \leq C \|(\mathbf{g} - (\mathbf{g} \cdot \theta)\theta, g)\|_{s,\nu},$$

with $C > 0$ from (2.8). As there exists a positive constant C_1 depending only on s and S (and that we may suppose to be ≥ 1) such that

$$\|\mathbf{g} - (\mathbf{g} \cdot \theta)\theta\|_{s,\nu} \leq C_1 \|\mathbf{g}\|_{s,\nu},$$

we deduce that

$$\|\tilde{T}(\mathbf{g}, g)\|_{s,\nu} \leq CC_1 \|(\mathbf{g}, g)\|_{s,\nu}.$$

Hence $\tilde{T}$ satisfies the statement of the Theorem in $L_\omega^p(S)^4$, hence it is $\mathcal{R}$ -bounded.

Let us now show that this property implies a similar property for T . Indeed, taking any $N \in \mathbb{N}$, $T_j \in \mathcal{T}$ and $F_j \in X_{p,\omega}$, we notice that

$$T_j F_j = \tilde{T}_j F_j,$$

and therefore

$$\begin{aligned} \left\| \left(\sum_{j=1}^N |T_j F_j|^2 \right)^{\frac{1}{2}} \right\|_{p,\omega} &= \left\| \left(\sum_{j=1}^N |\tilde{T}_j F_j|^2 \right)^{\frac{1}{2}} \right\|_{p,\omega} \\ &\leq C \left\| \left(\sum_{j=1}^N |F_j|^2 \right)^{\frac{1}{2}} \right\|_{p,\omega}, \end{aligned}$$

for some $C > 0$ due to the $\mathcal{R}$ -boundedness of the family $\tilde{\mathcal{T}}$. ■

This Theorem and Corollary 2.2 directly implies the next result.

Theorem 2.5 *Given $p \in (1, \infty)$, fix a pair of real numbers (δ, η) fulfilling (2.2). Then the sets $\{\mathcal{M}_{\delta,\eta}(\xi) : \xi \in \mathbb{R}\}$ and $\{\xi \mathcal{M}'_{\delta,\eta}(\xi) : \xi \in \mathbb{R}\}$ are $\mathcal{R}$ -bounded on $\mathcal{L}(X_{p,\omega})$.*

3 The weak Neumann problem in a cone

In the whole section let Γ be a three-dimensional cone with a section S with a $C^{1,1}$ boundary, in the sense that

$$\Gamma = \{x \in \mathbb{R}^3 : \vartheta = \frac{x}{|x|} \in S\}.$$

On such domain, we recall that the weighted Sobolev space of Kondrat'ev's type $V_\alpha^{k,p}(\Omega)$ with $k \in \mathbb{N}$, $p \in (1, \infty)$ and $\beta \in \mathbb{R}$ is defined by

$$V_\beta^{k,p}(\Gamma) = \{u \in L_{\text{loc}}^2(\Gamma) : \|u\|_{V_\beta^{k,p}(\Gamma)} < \infty\},$$

where

$$\|u\|_{V_\beta^{k,p}(\Gamma)}^p = \sum_{|\alpha| \leq k} \|r^{\beta+|\alpha-k} D^\alpha u\|_{L^p(\Gamma)}^p,$$

$r(x) = |x|$ being the distance from x to the origin, that is the corner point of Γ . In particular, we write $L_\beta^p(\Gamma) = V_\beta^{0,p}(\Gamma)$. Let us also recall that the singular functions of the Laplace operator with Neumann boundary conditions are in the form

$$r^{\lambda_{k\pm}} \varphi_k,$$

where φ_k is the eigenvector of L_{LB}^{Neu} of eigenvalue μ_k and

$$(3.1) \quad \lambda_{k\pm} = \frac{-1 \pm \sqrt{1 + 4\mu_k^2}}{2}.$$

In this section, we prove the next result.

Theorem 3.1 *Let $p \in (1, \infty)$ and $\beta \in \mathbb{R}$ be such that*

$$(3.2) \quad 1 - \frac{3}{p} - \beta \neq \lambda_{k\pm}, \forall k \in \mathbb{N}.$$

Then for all $\mathbf{f} \in L_\beta^p(\Gamma)^3$ there exists a unique solution $u \in V_\beta^{1,p}(\Gamma)$ of

$$(3.3) \quad \int_\Gamma \nabla u \cdot \nabla \bar{v} \, dx = \int_\Gamma \mathbf{f} \cdot \nabla \bar{v} \, dx, \forall v \in V_{-\beta}^{1,q}(\Gamma),$$

where $q > 1$ is the conjugate of p , i.e., $\frac{1}{p} + \frac{1}{q} = 1$. Furthermore there exists $C > 0$ (independent of u and $\mathbf{f}$) such that

$$(3.4) \quad \|u\|_{V_\beta^{1,p}(\Gamma)} \leq C \|\mathbf{f}\|_{L_\beta^p(\Gamma)^3},$$

Proof. Assume that such a solution u exists. We perform the Euler change of variables $r = e^t$ that transform Γ into the strip $B = \mathbb{R} \times S$ and for any $v \in V_{-\beta}^{1,q}(\Gamma)$, set

$$U(t, \vartheta) = e^{\eta t} u(e^t, \vartheta), V(t, \vartheta) = e^{\delta t} v(e^t, \vartheta),$$

with $\eta = -(1 - \frac{3}{p} - \beta)$ and $\delta = 1 - \eta$. By the arguments of [14, p. 193] (easily extended to the L^p -setting), we deduce equivalently that U belongs to $W^{1,p}(B)$ and satisfies

$$(3.5) \quad \begin{aligned} & \int_B \left(\nabla_T U \cdot \nabla_T V + (\partial_t U - \eta U)(\partial_t \bar{V} - \delta \bar{V}) \right) d\vartheta \\ &= \int_B \left(\mathbf{h}_T(t, \vartheta) \cdot \nabla_T \bar{V} + h_r(t, \vartheta)(\partial_t \bar{V} - \delta \bar{V}) \right) d\vartheta, \forall V \in W^{1,q}(B), \end{aligned}$$

where we have set

$$h_r(t, \vartheta) = e^{(2-\delta)t} \mathbf{f}(e^t, \vartheta) \cdot \vartheta, \quad \mathbf{h}_T(t, \vartheta) = e^{(2-\delta)t} \mathbf{f}(e^t, \vartheta) - h_r(t, \vartheta) \vartheta,$$

that respectively belong to $L^p(B)$ and $\mathbf{L}_T^p(B) := \{\mathbf{h} \in L^p(B)^3 : \mathbf{h} \cdot \vartheta = 0\}$ with

$$(3.6) \quad \|\mathbf{h}_T\|_{\mathbf{L}_T^p(B)} + \|h_r\|_{L^p(B)} \leq C \|\mathbf{f}\|_{L_\beta^p(\Gamma)^3},$$

for some positive constant C . Hence it suffices to show that (3.5) has a unique solution $U \in W^{1,p}(B)$ for $(\mathbf{h}_T, h_r) \in \mathbf{L}_T^p(B) \times L^p(B)$.

Performing a formal Fourier transform $\mathcal{F}$ in t , we see that $\hat{U}(\xi, \vartheta) = (\mathcal{F}_{t \rightarrow \xi} U(\cdot, \vartheta))(\xi)$ satisfies

$$(3.7) \quad \int_S \left(\nabla_T \hat{U} \cdot \nabla_T \varphi + (i\xi - \eta)(-i\xi - \delta) \hat{U} \bar{\varphi} \right) d\vartheta = \int_S \left(\hat{\mathbf{h}}_T(\xi, \vartheta) \cdot \nabla_T \bar{\varphi} - \hat{h}_r(\xi, \vartheta)(i\xi + \delta) \bar{\varphi} \right) d\vartheta,$$

for all suitable test-functions φ . Comparing with (2.1), we see that formally $(\nabla_T \hat{U}, i\xi \hat{U}) = \{\mathcal{M}_{\delta, \eta}(\xi)(\hat{\mathbf{h}}_T, \hat{h}_r)$. To be more correct, reminding the definition (2.6), as our assumption (3.2) guarantees that the pair (δ, η) , defined above, satisfies (2.2), by Theorem 2.5, the sets $\{\mathcal{M}_{\delta, \eta}(\xi) : \xi \in \mathbb{R}\}$ and $\{\xi \mathcal{M}'_{\delta, \eta}(\xi) : \xi \in \mathbb{R}\}$ are $\mathcal{R}$ -bounded on $L^p(S)$. Consequently by the vector-valued multiplier Theorem (see [29, Theorem 3.4]), we deduce that $\mathcal{F}^{-1} \mathcal{M}_{\delta, \eta}(\cdot) \mathcal{F}$ defines a bounded linear operator on $L^p(\mathbb{R}; X_{p,1})$ into itself. Consequently the vector-valued function $(\mathbf{u}, v) = (\mathcal{F}^{-1} \mathcal{M}_{\delta, \eta} \mathcal{F})(\mathbf{h}_T, h_r)$ belongs to $L^p(\mathbb{R}; L_T^p(S) \times L^p(S))$. By taking a sequence of functions $\varphi_n \in \mathcal{D}(\mathbb{R}, X_{p,1})$ that converges to $(\mathbf{h}_T, h_r)$, using Corollary 2.2, Remark 2.3 and passing to the limit the next results can be obtained. There exists $U \in W^{1,p}(B)$ such that

$$\mathbf{u} = \nabla_T U, v = i\partial_t U,$$

with

$$\|U\|_{W^{1,p}(B)} \leq C(\|\mathbf{h}_T\|_{\mathbf{L}_T^p(B)} + \|h_r\|_{L^p(B)}),$$

for some $C > 0$. Furthermore it will satisfy (3.5) since $\mathcal{D}(\mathbb{R}; W^{1,q}(S))$ is dense in $W^{1,q}(B)$. Finally as

$$\|u\|_{V_\beta^{1,p}(\Gamma)} \leq C\|U\|_{W^{1,p}(B)},$$

for some $C > 0$, with (3.6) we conclude that (3.4) holds. ■

Corollary 3.2 *Let $p \in (1, \infty)$ and $\beta \in \mathbb{R}$ satisfy (3.2). Then any $\mathbf{f} \in L_\beta^p(\Gamma)^3$ admits the Helmholtz decomposition*

$$\mathbf{f} = \mathbf{f}_0 + \nabla u,$$

with $\mathbf{f}_0 \in L_\beta^p(\Gamma)^3$ divergence free and $u \in V_\beta^{1,p}(\Gamma)$ such that

$$(3.8) \quad \|u\|_{V_\beta^{1,p}(\Gamma)} + \|\mathbf{f}_0\|_{L_\beta^p(\Gamma)^3} \leq C\|\mathbf{f}\|_{L_\beta^p(\Gamma)^3},$$

for some positive constant C independent of $\mathbf{f}$.

Proof. Given $\mathbf{f} \in L_\beta^p(\Gamma)^3$, the previous Theorem furnishes $u \in V_\beta^{1,p}(\Gamma)$ solution of (3.3) and we conclude by setting

$$\mathbf{f}_0 = \mathbf{f} - \nabla u,$$

that is obviously divergence free by taking test-functions $v \in \mathcal{D}(\Gamma)$ in (3.3) and satisfies (3.8) owing to (3.4). ■

References

- [1] K. Abe, Y. Giga, K. Schade, and T. Suzuki. On the Stokes semigroup in some non-Helmholtz domains. *Arch. Math. (Basel)*, 104(2):177–187, 2015.
- [2] M. E. Bogovskii. Decomposition of $L_p(\Omega; \mathbf{R}^n)$ into a direct sum of subspaces of solenoidal and potential vector fields. *Dokl. Akad. Nauk SSSR*, 286(4):781–786, 1986.
- [3] E. Fabes, O. Mendez, and M. Mitrea. Boundary layers on Sobolev-Besov spaces and Poisson’s equation for the Laplacian in Lipschitz domains. *J. Funct. Anal.*, 159(2):323–368, 1998.
- [4] R. Farwig. The weak Neumann problem and the Helmholtz decomposition in general aperture domains. In *Progress in partial differential equations: the Metz surveys, 2 (1992)*, volume 296 of *Pitman Res. Notes Math. Ser.*, pages 86–96. 1993.
- [5] R. Farwig. Weighted L^q -Helmholtz decompositions in infinite cylinders and in infinite layers. *Adv. Differential Equations*, 8(3):357–384, 2003.
- [6] R. Farwig and H. Sohr. Weighted L^q -theory for the Stokes resolvent in exterior domains. *J. Math. Soc. Japan*, 49(2):251–288, 1997.
- [7] A. Fröhlich. The Helmholtz decomposition of weighted L^q -spaces for Muckenhoupt weights. *Ann. Univ. Ferrara Sez. VII (N.S.)*, 46:11–19, 2000. Navier-Stokes equations and related nonlinear problems (Ferrara, 1999).
- [8] A. Fröhlich. The Stokes operator in weighted L^q -spaces. I. Weighted estimates for the Stokes resolvent problem in a half space. *J. Math. Fluid Mech.*, 5(2):166–199, 2003.
- [9] A. Fröhlich. The Stokes operator in weighted L^q -spaces. II. Weighted resolvent estimates and maximal L^p -regularity. *Math. Ann.*, 339(2):287–316, 2007.
- [10] D. Fujiwara and H. Morimoto. An L_r -theorem of the Helmholtz decomposition of vector fields. *J. Fac. Sci. Univ. Tokyo Sect. IA Math.*, 24(3):685–700, 1977.
- [11] G. P. Galdi. *An introduction to the mathematical theory of the Navier-Stokes equations*. Springer Monographs in Mathematics. Springer, New York, second edition, 2011. Steady-state problems.
- [12] J. Geng and Z. Shen. The Neumann problem and Helmholtz decomposition in convex domains. *J. Funct. Anal.*, 259(8):2147–2164, 2010.
- [13] T. Kobayashi and T. Kubo. Weighted $L^p - L^q$ estimates of the Stokes semigroup in some unbounded domains. *Tsukuba J. Math.*, 37(2):179–205, 2013.
- [14] V. A. Kozlov, V. G. Maz’ya, and J. Rossmann. *Elliptic boundary value problems in domains with point singularities*, volume 52 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1997.
- [15] Y. Maekawa and H. Miura. Remark on the Helmholtz decomposition in domains with non-compact boundary. *Math. Ann.*, 359(3-4):1077–1095, 2014.
- [16] V. N. Maslennikova and M. E. Bogovskii. Elliptic boundary value problems in unbounded domains with noncompact and nonsmooth boundaries. *Rend. Sem. Mat. Fis. Milano*, 56:125–138 (1988), 1986.

- [17] T. Miyakawa. The Helmholtz decomposition of vector fields in some unbounded domains. *Math. J. Toyama Univ.*, 17:115–149, 1994.
- [18] C. G. Simader and H. Sohr. A new approach to the Helmholtz decomposition and the Neumann problem in L^q -spaces for bounded and exterior domains. In *Mathematical problems relating to the Navier-Stokes equation*, volume 11 of *Ser. Adv. Math. Appl. Sci.*, pages 1–35. World Sci. Publ., River Edge, NJ, 1992.
- [19] C. G. Simader and T. Ziegler. The weak Dirichlet and Neumann problems in L^q for the Laplacian and the Helmholtz decomposition in infinite cylinders and layers. In *Recent developments in partial differential equations*, volume 2 of *Quad. Mat.*, pages 37–161. Dept. Math., Seconda Univ. Napoli, Caserta, 1998.
- [20] H. Sohr. *The Navier-Stokes equations*. Birkhäuser Advanced Texts: Basler Lehrbücher. [Birkhäuser Advanced Texts: Basel Textbooks]. Birkhäuser Verlag, Basel, 2001. An elementary functional analytic approach.
- [21] H. Sohr and G. Thäter. Imaginary powers of second order differential operators and L^q -Helmholtz decomposition in the infinite cylinder. *Math. Ann.*, 311(3):577–602, 1998.
- [22] M. Specovius-Neugebauer. The Helmholtz decomposition of weighted L^r -spaces. *Comm. Partial Differential Equations*, 15(3):273–288, 1990.
- [23] M. Specovius-Neugebauer. The weak Neumann problem and the Helmholtz decomposition of two-dimensional vector fields in weighted L^r -spaces. In *Navier-Stokes equations and related nonlinear problems (Funchal, 1994)*, pages 105–116. Plenum, New York, 1995.
- [24] M. Specovius-Neugebauer. Helmholtz decomposition in a cone. Private communication, 2017.
- [25] G. Thäter. The Neumann problem, cylindrical outlets and Sobolev spaces. *Math. Methods Appl. Sci.*, 25(10):875–894, 2002.
- [26] G. Thäter. Neumann problem in domains with outlets of bounded diameter. *Acta Appl. Math.*, 73(3):251–274, 2002.
- [27] M. T. Tun and T. Miyakawa. A proof of the Helmholtz decomposition of vector fields over the half-space. *Adv. Math. Sci. Appl.*, 18(1):199–217, 2008.
- [28] W. von Wahl. Abschätzungen für das Neumann-Problem und die Helmholtz-Zerlegung von L^p . *Nachr. Akad. Wiss. Göttingen Math.-Phys. Kl. II*, (2):29, 1990.
- [29] L. Weis. Operator-valued Fourier multiplier theorems and maximal L_p -regularity. *Math. Ann.*, 319(4):735–758, 2001.