The Theory of Braids and Energetic Lattices II - Constrained Optimization by Disseminated Energy

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1 Notation

E : space under study; x, y points of E; $\mathcal{P}(E)$: set of the subsets of E, also called classes;
S, A : classes of E; T_j the sibling classes of S
$\pi = \pi(E)$: partition of E;
$\pi(S)$: partial partition (in short p.p.) of support $S \in \mathcal{P}(E)$); the notation $\tau(S)$ is also used for p.p.
$\{S\}$ p.p. with unique class S (when there is no ambiguity, $\{S\}$ is just written S) ;
$\mathcal{D}(E)$ set of all p.p. for all supports $S \in \mathcal{P}(E)$;
$\leq, \wedge, \vee$: when applied to partitions, are relative to the refinement ordering;
$\sqcup$: concatenation of classes and p.p. i.e. $\pi(S) = S_1 \sqcup S_2 \Leftrightarrow S_1 \cup S_2 = S$ and $S_1 \cap S_2 = \varnothing$;
$H = \{\pi_i, i \in I\}$: hierarchy, i.e. family of increasing partitions;
cut π: partition of E into classes taken in H;
$\Pi(\leq, H)$: set of all cuts of H, viewed as a lattice for the refinement ordering
ω: energy, i.e. scalar function on $\mathcal{D}(E)$;
$\preceq_\omega, \curlywedge_\omega, \curlyvee_\omega$: ω-energetic ordering, infimum, and supremum, w.r.t. energy ω;
$\Pi(\omega, H)$: ω-energetic lattice on the cuts of H;
π^* minimal cut in an energetic lattice;
π^*_φ (resp. π^*_∂) minimal cut in the energetic lattice of energy ω_φ (resp. ω_∂).

2 Introduction

Two questions:
1/ How to introduce localization in constrained optimization (compare Everett with and without localization)
2/ Lagrange approach for hierarchies gives upperbounds only
Lines of thought (history of the ideas) and plan of the paper

3 Two reminders

3.1 Hierarchies, climbing energies, and energetic lattices

We denote a partition of space E by π and a partial partition (p.p.) of subset $S \subseteq E$ by $\pi(S)$ [9]. The family of all partial partitions of E is $\mathcal{D}(E)$. A hierarchy of partitions (HOP) is a finite chain of partitions $H = \{\pi_i,\ i \in [0, n]\}$, with $\pi_i \leq \pi_j, i < j$, where $\leq$ stands for the refinement ordering. The minimal element π_0 of H is called the leaves partition, while the maximal element is the one class partition $\{E\}$, called the root. A cut of hierarchy H is a partition of E whose elements are composed of classes in H. The set of all cuts of H is $\Pi(\leq, H)$.

An energy $\omega : \mathcal{D} \rightarrow \mathbb{R}^+$ is a non-negative function that is defined on the family of partial partitions. The energy of a partial partition $\pi(S)$ is often obtained by composition comp of the energies of its constituent classes T_j. If the composition is the addition, we have

$$\omega[\pi(S)] = \mathrm{comp}\{\omega(T_j),\quad T_j \sqsubseteq S\} = \sum \omega(T_j), \qquad T_j \sqsubseteq S$$

The additive composition law is the most investigated one, but some problems require other laws, as supremum [17], [1], or infimum (e.g. composition of distances functions, in Ch. 4 of [8]).

In [3], [10], and [6], where the composition law is the addition, one introduces the cut $\pi^{**} = \pi^{**}(E)$ of minimal energy. This cut is calculated by aggregating, over all classes $S \in H$, the local minima $\pi^{**}(S)$ which choose between the parent S and its children according to their energies, i.e.

$$\pi^{**}(S) = \begin{cases} \{S\}, & \text{if } \omega(S) \leq \mathrm{comp}\{\omega(T_j),\quad T_j \sqsubseteq S\} \\ \bigsqcup\{T_j, T_j \sqsubseteq S\} & \text{otherwise} \end{cases} \tag{1}$$

Cut $\pi^{**}(E)$ is obtained by scanning the classes of H following a lexicographic order, and taking the provisional minimum in equation (1) according to a dynamic program.

As it is presented, this dynamic program does not extend to non linear energies. Take for example the hierarchy of figure 1a. The indicated energies hold on classes, and one goes to the p.p. by supremum composition. Algorithm (1) yields the cut π^{**}, whose energy 5 is indeed the lowest one, but the three other cuts in dotted lines have the same energy.

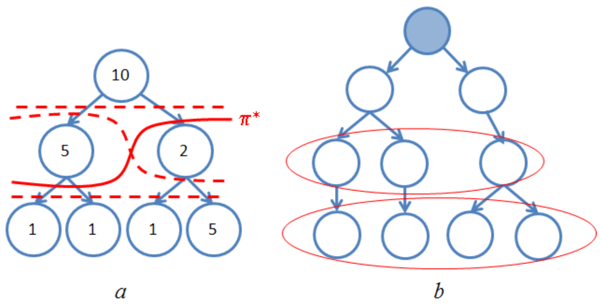
If we want to extend the dynamic program to non linear energies, it has to be presented differently, from the two notions of singularity and h-increasingness [11] [12]:

Definition 1. A energy ω is said to be singular when for all p.p. $\pi(S)$ we have

$$\text{either } \omega(\{S\}) < \omega(\pi(S)), \text{ or } \omega(\{S\}) > \omega(\pi(S)) \qquad S \subseteq E.$$

It is said to be h-increasing when

$$\omega(\pi(S)) \leq \omega(\pi'(S)) \Rightarrow \quad \omega(\pi(S) \sqcup \pi_0) \leq \omega(\pi'(S) \sqcup \pi_0), \quad \forall S \subseteq E. \tag{2}$$



Number of classes	energy
1	1
2	2
> 2	0

partial partition energy of b

Figure 1: a)In case of $\vee$-composition, there is a unique minimal cut $\pi*$ and its energy is minimal, but other cuts can have the same energy. b) and c) depict a non h -increasing energy $\omega(\pi)$ which depends on the number of classes of the p.p. π. The two cuts surrounded by ellipses are minimal (hence no energetic lattice here), but none of them is obtained by dynamic program, which gives the cut in grey.

where the symbol $\sqcup$ indicates the concatenation of any p.p π_0 with support that is disjoint with S). A singular and h-increasing energy is said to be *climbing*. It is strictly climbing when the $\leq$ relations become $<$ in implication (2).

The energies composed by addition, by supremum, by infimum, and many other laws are are climbing [12] [14].

In algebra, a classical way to ensure the existence and the unicity of a minimum element in a set consists in providing the set with a lattice structure. For the set of cuts $\Pi(\leq, H)$ [12] of hierarchy H we can proceed as follows. Given H and an energy ω, the cut $\pi_1 \in \Pi(\leq, H)$ is said to be less energetic than $\pi_2 \in \Pi(\leq, H)$, and one writes $\pi_1 \preceq_\omega \pi_2$ when in each class S of $\pi_1 \vee \pi_2$ the energy of the partial partition of π_1 is smaller or equal to that of $\pi_{2:}$:

$$\pi_1 \preceq_\omega \pi_2 \quad \Leftrightarrow \quad \{S \in \pi_1 \vee \pi_2 \quad \Rightarrow \quad \omega(\pi_1 \sqcap \{S\}) \leq \ \omega(\pi_2 \sqcap \{S\})\} \tag{3}$$

When the energy ω is climbing, the relation $\preceq_\omega$ turns out to be an ordering, and we can state [12] [14]:

Proposition 2. *The set of all cuts of H forms a complete lattice $\Pi(\omega, H)$ for the energetic ordering $\preceq_\omega$ if and only if the energy ω is climbing.*

When the energy ω is not climbing the property disappears as shown by the counter example of figure 1b and c. The minimal element π^* of lattice $\Pi(\omega, H)$ is unique, and coincides with *the* cut of minimal energy π^{**} only when ω is strictly h-increasing.

Proposition 2 provides a theoretical background to dynamic programming since [3], [10], [6] speak of cut of minimal energy, but their algorithm calculates the minimal cut π^* in the sense of the energetic lattice. Figure 2 depicts energetic order and lattice on a toy example.

The domain of validity of this approach encompasses the hierarchies and applies to the more general class of the braids, i.e. to families of partitions of E which is monitored by H, in the following sense [14]:

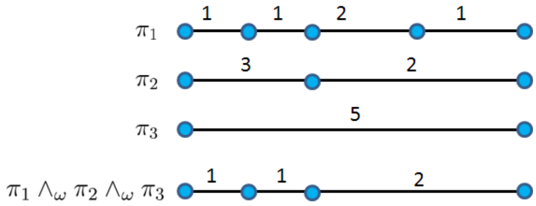


Figure 2: Three cuts of a hierarchy. The energies ω are indicated above the classes are composed by sum. We have $\pi_3 \preceq_\omega \pi_2$ and $\pi_3 \preceq_\omega \pi_1$, but π_1 and π_2 are not comparable.

Definition 3. (Braid of Partitions) A *braid* B of monitor H is a family in $\Pi(E)$ where the refinement supremum of any pair π_1, π_2 in B is a cut of H, other than $\{E\}$:

$$\forall \pi_1, \pi_2 \in B \quad \Rightarrow \quad \pi_1 \vee \pi_2 \in \Pi(E, H) \setminus \{E\} \tag{4}$$

3.2 inf-modularity

Inf-modularity is the concern of a monotonicity condition on energies [8]:

Definition 4. An energy $\omega_\partial : \mathcal{D}(E) \to \mathbb{R}^+$on partial partitions is said inf-modular when for each p.p. $\pi(S)$ of support $S \in \mathcal{P}(E)$ we have

$$\omega_\partial(\{S\}) \leq \omega_\partial[\pi(S)],. \tag{5}$$

which implies $\omega_\partial(\{S\}) \leq \wedge\omega_\partial[\pi(S)]$ when $\pi(S)$ spans the set of all p.p. of support S.

Super-modularity is defined by replacing $\leq$ by $\geq$ in the inequality (5).

Examples of inf-modular energies For each energy, we define some functional on sets S, plus a law of composition which extends the functional to p.p.. We consider also a numerical function f on the space E

inf-modular ω	perimeter, Euler constant of S	sum composition
super-modular	variance of f in S	sum composition
	$(\sup f - \inf f)$ in S	sup composition

The area of S, and the integral of $f^\alpha(x)$ over S, for any real number α, with sum composition, are both inf- and super-modular.

inf-modularity Vs subadditivity Inf-modularity is to be compared with subadditivity. Remember that a set-wise energy $\omega : \mathcal{P}(E) \to \mathbb{R}$ is subadditive when $\omega(A \cup B) \leq \omega(A) + \omega(B)$, $A, B \in \mathcal{P}(E)$. This property, and more generally the sub-modularity, serves as substitute for the convexity when dealing with the subsets of E [4], .

For comparing inf-modularity versus subadditivity we must match sets and partial partitions in some sense. In particular, if $T_j, 1 \leq j \leq p$ are the classes of the p.p. $\pi(S)$ and if $\omega_\partial[\pi(S)]$ is $\leq$ to the sum of the $\omega_\partial(T_j)$ then the inf-modular ω_∂ is also subadditive.

For example, in a hierarchy of partitions of the Euclidean plane, the perimeters ω_∂ of the classes generate an inf-modular and subadditive energy on the partial partitions. Unlike, the super-modular energy ω_∂ =(sup f − inf f) is neither subadditive nor superadditive.

4 Constrained minimization by Lagrangian

In this section we present the classical Lagrangian based approach [5] applied to constrained minimization in hierarchies [10], [6], and we show by means of a counter example that it does not result in the expected minimal cut [8] [15]. The need for a constrained minimal cut appears when the cuts are subject to conditions. Suppose for example that one wishes to transmit an image f at a given cost (defined by an upper-bound C of the number of transmitted bits), but with the best possible quality [10]. A hierarchy H of segmentations of f has been performed, so that the question comes down to extract the minimal cut w.r.t. a so called *objective energy* ω_φ among a set of cuts π which satisfy the cost constraint $\omega_\partial(\pi) \leq C$ for some *constraint energy* ω_∂.

Consider the hierarchy H of Figure 3, where the objective energy ω_φ and the constraint energy ω_∂ are depicted for the classes. Their extensions to p.p. are obtained by addition of the classes, i.e. $\omega(\pi(S)) = \sum_{T_j \sqsubseteq \pi(S)} \omega(T_j)$. Further when we have equal parent and child energies, we pick the parent. Both ω_φ and ω_∂ are climbing, and ω_∂ is also inf-modular (they have the same features as in [6], [10]). We want to find the cut π^*_φ of minimal energy ω_φ which satisfies $\omega_\partial(\pi^*_\varphi) \leq C$, where C is a cost here set to 7.5.

Introduce the family of Lagrangians $\{\omega(\lambda) = \omega_\varphi + \lambda\omega_\partial, \lambda \geq 0\}$. The minimal cut of the energetic lattice $(\omega(\lambda), H)$ is denoted by $\pi^*(\lambda)$. For the chosen features of ω_φ and ω_∂, the energy $\omega_\varphi[\pi^*(\lambda)]$ increases with λ, and $\omega_\partial[\pi^*(\lambda)]$ decreases [15]. We have:

λ	min cut $\pi^*(\lambda)$	$\omega[\lambda, \pi^*(\lambda)]$	$\omega_\varphi[\pi^*(\lambda)]$	$\omega_\partial[\pi^*(\lambda)]$
$0 \leq \lambda < 2$	(a, b, c, d, e, f)	$6 + 9\lambda$	6	9
$2 \leq \lambda < 3.5$	(a, b, c, d, i)	$8 + 8\lambda$	8	8
$3.5 \leq \lambda$	(g, h, i)	$15 + 6\lambda$	15	6

As a result, ω_∂ is never equal to the cost $C = 7.5$ at any time. Finally, the λ-cut $\pi^*(\lambda^*)$ which minimizes $\omega_\varphi(\pi^*(\lambda)\ )$ while satisfying $\omega_\partial(\pi^*(\lambda)) \leq C$, is (g, h, i).

Consider now the two other cuts $\pi = (g, c, d, i)$, and $\pi' = (a, b, h, i)$ which are not minimal cuts $\pi^*(\lambda)$. Cut π obviously provides a better minimum than the minimal λ-cut (g, h, i) since $\omega_\partial(\pi) = 7$ (still below the cost $C = 7.5$), for an energy $\omega_\varphi(\pi) = 11.5$ (also smaller objective $\omega_\varphi(g, h, i) = 15$). And it is the same for π', since $\omega_\partial(\pi') = 7$ and $\omega_\varphi(\pi') = 11.5$. There are thus *several* constrained minimal cuts for the energy ω_φ, and none of them belongs the sequence $\{\pi^*(\lambda), \lambda \geq 0\}$ of λ-minimal cuts! And we cannot take their infimum $\pi \wedge \pi' = (a, b, c, d, i)$ because $\omega_\partial(\pi \wedge \pi') = 8$, which is above the constraint $C = 7.5$. Uniqueless is lost even when the $\omega(\lambda)$ are strictly ioncreasing. We could bring the minimal λ-cut (g, h, i) closer to one

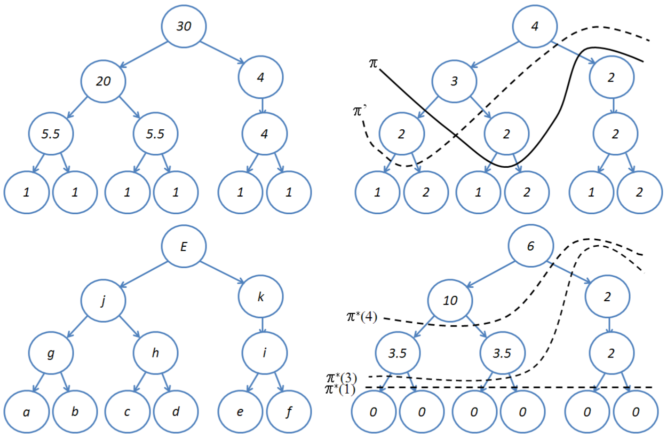


Figure 3: Bottom left, a hierarchy$H$. The two trees of the top row indicate the two energies$\omega_\varphi$and ω_∂, written in the corresponding classes. π and π' are two cuts of H. Bottom right, the values indicated in the nodes are the energies obtained by equating parent and child energies. Their level sets give the minimal cuts w.r.t. $\omega(\lambda)$. They are drawn in dotted lines for $\lambda = 1, 3, 4$ as π_1^*, π_3^* and π_4^*.

of the optimal solutions π or π' by techniques of small perturbations, but we cannot, by no mean, reduce the number of these optimal solutions[1]

A similar trouble also arises in convex optimization [4], where the dual Lagrangian serves only as an upper bound on the optimum corresponding to the primal Lagrangian. In our words, $\pi^*(\lambda^*)$ provides only an upper-bound of the constrained minimal cuts.

5 Optimization by Disseminated Energy (ODE ?)

The above Lagrange type approach fails in finding the minimal cut, because the problem is not defined enough . It is implicitly assumed that a cut is characterized by its energy. Now we saw that the two different cuts may have the same conditional minimal energy (e.g. π and π' in figure 3).

The initial intuition of Lagrange and Euler was to substitute an absolute minimum to the conditional one by changing the space. When dealing with numerical functions in Euclidean spaces, the Lagrangian is the tool which translates this intuition (and it works!), but it is not well adapted to a space of partitions. Can we express the same intuition by different means?

The energetic lattices offer an alternative approach, where both constraint and minimization can directly be defined on partitions and no longer on partitions via their energies. In

[1]The lack of unicity, observed here on a toy example, becomes catastrophic for larger partitions. One easily proves that a binary hierarchy, with p levels and connected classes has more than $2^{2^{p-1}}$ classes[16]. Even for a range of energies of 10^6 the correspondence "energy-partition" is far from being one-to-one. To give an idea, the number of cuts of a hierarchy with six levels only equals one hundred times the number of the humans on the earth (which was $7,125$ billions in 2013)

this alternative method, we keep the same types of energies as in Lagrange formulation: an objective energy ω_φ and a constraint energy ω_∂, both climbing, with ω_∂ inf-modular. But the other inputs are different:

- the energy ω_φ holds on a braid the cuts of a braid$B$, and the energy ω_∂ holds on the cuts of a hierarchy H, possibly different from B but with the same leaves;
- the previous *scalar* cost C is replaced by a cost *cut* π_C, i.e. by a *partition* of the space $\Pi(\omega_\varphi, B)$.

The families $\Pi(B)$ and $\Pi(H)$ of all cuts of B and H are the matter of three different lattices which interact:

- $\Pi(B) = \Pi(\leq, B)$, lattice w.r.t. the refinement ordering $\leq$, with the leaves as minimal cut and E as maximal one. The set $\Pi(H) = \Pi(\leq, H)$ is usually a sub-lattice of $\Pi(\leq, B)$
- $\Pi(\omega_\varphi, B)$ (resp. $\Pi(\omega_\partial, H)$), lattice w.r.t. the energetic ordering w.r.t. ω_φ, of order $\preceq_\varphi$ and infimum $\curlywedge_{\omega_\varphi}$ (resp. the energetic ordering w.r.t. ω_∂, of order $\preceq_\partial$ and infimum $\curlywedge_{\omega_\partial}$). The minimal cut of $\Pi(\omega_\varphi, B)$ is written $\pi_\varphi^* = \curlywedge_{\omega_\varphi}\{\pi, \pi \in \Pi(\omega_\varphi, B)\}$.

The following two problems will be treated separately:

Problem 5. Characterize the family Π_C of cuts of H which are $\preceq_{\omega_\partial}$ than π_C,

Problem 6. Find the minimal cut in the energetic lattice $\Pi(\omega_\varphi, B)$ subject to be $\preceq_{\omega_\partial} \pi_C$.

Note that the constraint condition is stated in the energetic lattice $\Pi(\omega_\partial, H)$, though the minimization occurs in the lattice $\Pi(\omega_\varphi, B)$. Unlike, in the classical Lagrange approach, the same lattice of numbers is used for both constraint condition and minimization.

6 Problem 5

The problem 5 concerns the relations between the two lattices $\Pi(\leq, H)$ and $\Pi(\omega_\partial, H)$. It does not involve either the energy ω_φ or the energetic lattice $\Pi(\omega_\varphi, B)$, but uniquely ω_∂.

6.1 An equivalence between energetic ordering ω_∂ and refinement ordering

In H, the energetic ordering w.r.t. ω_∂ and that by refinement are linked. We can see it by comparing $\pi = \pi(E) \in \Pi_C$ to π_C. Consider the class S_C of π_C. As H is a hierarchy, S_C is

1. either the support of a p.p. $\pi(S_C)$ of $\pi(E)$, i.e. $\pi(S_C) = \pi(E) \sqcap S_C$
2. or a class of a p.p. $\pi(S)$ of π_C whose support is the class S of π, i.e. $\pi(S) = \pi(E) \sqcap S$, with $S_C \subseteq S$.

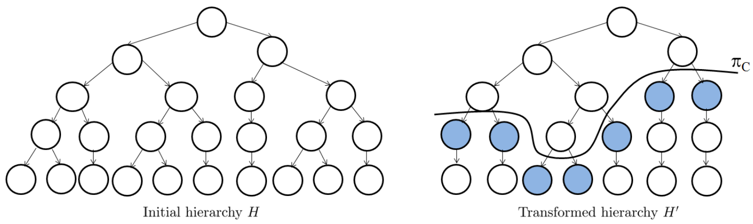


Figure 4: Left: initial hierarchy H; Right:new hierarchy H', formed by all classes above π_C, and completed by the leaves.

Suppose we are in the first case. By inf-modularity of ω_∂ we have $\omega_\partial(S_C) \leq \omega_\partial(\pi(S_C))$. But, as $\pi \preceq_{\omega_\partial} \pi_C$ we have also $\omega_\partial(\pi(S_C)) \leq \omega_\partial(S_C)$. Now by singularity of ω_∂, these two energies cannot be equal. Therefore the first case is impossible and

$$\pi(E) \geq \pi_C \tag{6}$$

Conversely, if the inequality (6) is true, then each class S of $\pi(E)$ contains a p.p. $\pi_C(S)$ of π_C and by inf-modularity of ω_∂, we have $\omega_\partial(S) \leq \omega_\partial(\pi_C(S))$, hence $\pi \preceq_{\omega_\partial} \pi_C$. We can state:

Proposition 7. *When energy ω_δ is inf-modular, then the set Π_C of the cuts of H which are $\preceq_{\omega_\partial}$ than π_C is identical to the family of all cuts of H coarser than π_C for the refinement ordering:*

$$\pi \preceq_{\omega_\partial} \pi_C \quad \Leftrightarrow \quad \pi \geq \pi_C \qquad \pi \in \Pi(H) \tag{7}$$

Proposition 7 answers Problem 5. The key equivalence (7) suggests to interpret the classes of π_C as the set of leaves of a new hierarchy H', identical to H above and on π_C, but where all classes below π_C are removed (see Figure 4). The cuts π of H' are exactly those of H that satisfy the constraint $\pi \preceq_{\omega_\partial} \pi_C$. The set $\Pi_C = \Pi(\leq, H')$ turns out to be a complete lattice for the refinement order, with π_C and E as extremal elements, hence a (pseudo) sub-lattice of $\Pi(\leq, H)$ [2]. In the sub-lattice $\Pi(\leq, H')$ absolute minimizations replace the constrained ones of lattice $\Pi(\leq, H)$ [3].

Dualities The canonical dualities inherent to lattices lead to three variant of Proposition 7. Introduce the hierarchy H'' which is identical to H below and on π_C, but where all classes above π_C are removed. We have the following truth table:

	$\pi \preceq_{\omega_\partial} \pi_C$	$\pi \succeq_{\omega_\partial} \pi_C$
inf-modular ω_∂	H'	H''
super-modular ω_∂	H''	H'

(8)

[2](“pseudo” because Π_C and $\Pi(\leq, H)$ have not the same leaves

[3]Lagrange's intuition is met again, but differently!

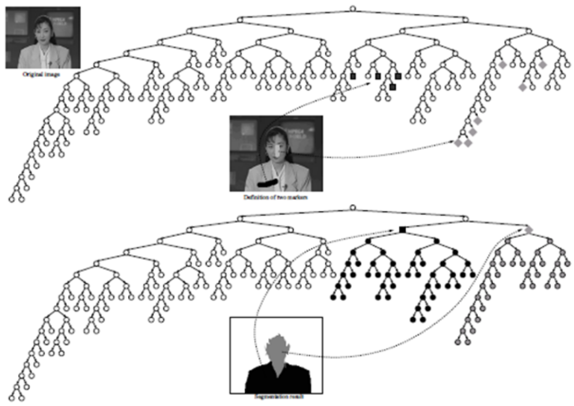


Figure 5: Example of marker & propagation strategy (by courtesy of Ph. salembier). Top: hierarchy of partitions where two markers are indicated. Bottom: result of the propagation process.

An example The "Marker & propagation" strategy introduced by Salembier and Garrido [10] is an interactive tool where some leaves are manually marked as belonging to phase α, or β, etc., and automatically propagated to their maximal extension w.r.t. the following energy ω_∂, firstly over the classes S:

$$\begin{aligned} \omega_\partial(S) &= 0 \quad \text{if } \; S \text{ contains markers of one type only} \\ \omega_\partial(S) &= 1 \quad \text{if } \; S \text{ has no marker} \\ \omega_\partial(S) &= 2 \quad \text{if } \; S \text{ contains markers of more than one type,} \end{aligned} \tag{9}$$

and secondly extended to p.p. by supremum, i.e.

$$\omega_\partial(\pi) = \vee\{\omega_\partial(T_j), T_j \sqsubseteq \pi\} \; + \varepsilon \qquad \pi \in \mathcal{D}(E).$$

The arbitrary small number $\varepsilon > 0$ serves to ensure singularity. The energy ω_∂ is climbing and super-modular. Take the leaves partition for cost cut π_C. According to Proposition 7, and to table (9), the infimum $\curlywedge_{\omega_\partial}$, obtained by dynamic program, is also the largest cut (for the refinement) whose non leave classes contain one type of marker only.

6.2 Scalar cost for classes

Let us go back to Problem 5. Can we replace, the cost partition π_C by a numerical cost C acting on the classes? A model of this type is studied in [8] section 2.40, and stated as follows:

Problem 8. Let H be a hierarchy on E, and C a non negative cost. Given the inf-modular energy ω_∂ , with $\omega_\partial(E) \leq C$, find the family Π'_C of cuts π of H:

$$\pi = \sqcup\{S \mid \omega_\partial(S) \leq C\} \tag{10}$$

whose each class S has an energy $\leq C$.

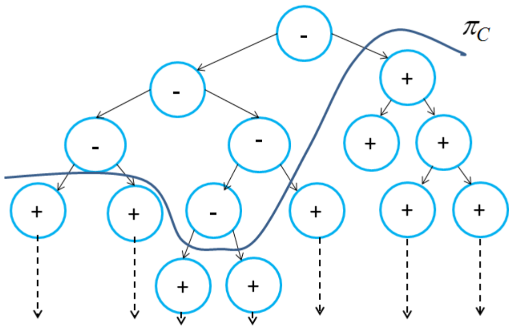


Figure 6: Minimal cut π_C. The + classes S satisfy $\omega_\partial(S) > C$, and the - classes $\omega_\partial(S) \leq C$.

Problem 8 turns out to be, indeed, a particular case of Problem 5. Span top-down the hierarchy H , and stop at each class S whose all ancestors (S included) have energies $\leq C$ and whose all siblings have an energy $> C$. Such limit class S_C does exist, since $\omega_\partial(E) \leq C$, are disjoint and partition the space E. The various S_C generate the partition $\pi_C = \sqcup S_C$, as shown in the toy example of figure 6. Now the cuts of H larger than π_C for the refinement ordering are the only ones which satisfy relation (10). We find Problem 5 again, and the family Π'_C is thus equivalent to:

$$\Pi'_C = \{\pi \geq \pi_C\} = \Pi_C$$

Note that one can make variable the cost parameter C, in accordance with the inf-modularity condition [8].

An example of Problem 8 is proposed by P. Soille in [17] for the super-modular variant. A non negative function f is defined on E, in association with H. The energy ω_∂ at each node S is given by

$$\omega_\partial(S) = \max[f(x), x \in S] - \min[f(x), x \in S] \tag{11}$$

and extended to p.p. by supremum. The mapping $\omega_\partial : \mathcal{D}(E) \to \mathbb{R}^+$ is obviously super-modular, and the largest $\pi \preceq_{\omega_\partial} \pi_C$ is given by $\pi = \vee\{\pi \leq \pi_C, \pi \in \Pi(\leq, H)\}$. For Soille, H is generated by a condition on $\mid f'(x) \mid$which is more and more relaxed along the increasing levels. He thus maximizes a gradient feature subject to the constraint of a bounded variation of the function on each class. But one can apply the super-modular energy ω_∂ of Relation (11) to any hierarchy.

7 Problem 6

7.1 Case of a hierarchy

We firstly treat the case when the braid B is identical the the hierarchy H. Then, owing to the equivalence (7), the problem 6 amounts to find the minimal cut of H' w.r.t. ω_φ, a question that we already know how to treat. The minimization is understood in the ω_φ-energetic lattice $\Pi(\omega_\varphi, H')$ relative to hierarchy H'. This minimal cut π'^*_φ is unique, since the energy ω_φ is climbing, and its energy $\omega_\varphi(\pi'^*_\varphi)$ is minimal in the energetic lattice $\Pi(\omega_\varphi, H)$, conditional upon $\pi \preceq_{\omega_\partial} \pi_C$.

Proposition 9. *When ω_φ and ω_∂ are climbing energies, and in addition ω_∂ is inf-modular, then the minimal cut $\pi^*_\varphi(\pi_C)$ of the ω_φ-energetic lattice $\Pi(\omega_\varphi, H)$, subject to be $\preceq_{\omega_\partial} \pi_C$, is also the minimal cut in the energetic lattice $\Pi(\omega_\varphi, H')$.*

(similar statements for the three other cases of table (8). The optimal cut$\pi^*_\varphi(\pi_C)$ is computed by a dynamic programing where the climbing process is restricted to the regions of H which are above π_C.

The particular case when, in addition, ω_φ is super-modular is somehow trivial. It suffices to apply Proposition 7 by replacing "ω_φ sup-modular" by "ω_∂ inf-modular" since then the constrained minimal cut $\pi^*_\varphi(\pi_C)$ coincides with π_C (Proposition 7). As an example, we can revisit the counter-example of figure 3. The two cuts π and π' are minimal, in Lagrange sense, for the scalar cost $C = 7.5$, with $\omega_\partial(\pi) = \omega_\partial(\pi') = 7$ though $\omega_\partial(\pi \wedge \pi') = 8$ (i.e. above the cost). If now the scale cost is replaced by the cut $\pi_C = (a, b, c, d, i)$, then both π and π' belong to $\Pi(\leq, H')$ as well as their refinement infimum $\pi \wedge \pi'$, which turns out be π_C.

7.2 A constraint ω_∂ for describing proximity

The spatialization brought by the partition constraint π_C may serve to express proximity criteria such as an image segmentation close to some ground truth, or a population who whishes to live near cities, or near the sea, etc.. Denote by A this pole of attraction, hence a set, and by f_A a convex numerical function which decrease rapidly near A and more slowly with the distance to A. For example, one can take $f_A = [1 - \frac{d(x,A)}{d_{\max}(x,A)}]^\alpha$ where $d(x, A)$ stands for the distance from point $x \in E$ to set A [13]. The exponent $\alpha \geq 0$ which appears in f_A modulates the importance of the pole A. For $\alpha = 0$ the optimal cut $\pi^*_\varphi(\pi_C)$ is ∂A itself, for $0 < \alpha < 1$, the positive influence of the proximity reduces as α increases, and for $\alpha > 1$ the close classes are disadvantaged.

The continuation is exactly that described in section 6.2 with the energy ω_∂ of relation (11) for $f = f_A$, and with a given upper-bound C. The parameter C represents the largest variation of f_A acceptable in each class S. Therefore, for $0 < \alpha < 1$ the minimal cut π_C of the energetic lattice $\Pi(\omega_\partial, H)$ has small classes near the pole A, and larger ones as moving away from A.

7.3 Two examples

Here are two examples of optimization by disseminated energy which involve constraints of proximity.

Minimization subject to ground truth It happens that some manual drawing A accompanies an image to segment, and indicates some essential features to preserve [13]. Suppose that a hierarchy of segmentations H has been computed on the image (by increasing watersheds, or by connected filters, etc.), and that one seeks the optimal the cut w.r.t. Mumford and Shah energy, denoted here by ω_φ. We want to improve the optimization by tacking the drawing A into account.

The process comprises two steps:

1. Calculate the minimal cut π_C of $\Pi(\omega_\partial, H)$ for the super-modular energy ω_∂ defined by(11) and for a given upper-bound C,

2. calculate the minimal cut $\pi^*_\varphi(\pi_k)$ of the lattice $\Pi(\leq, H'')$ of the cuts $\leq \pi_C$.

Space distribution of a service The administrative structure of France comprises seven levels. This hierarchy H is composed of the quarters (in cities), the communes, the communes communities, the counties, the departments, the regions, and the country itself. Because of their climate, or the proximity of the sea, some regions attract retired persons, a phenomenon which leads to some distortions. For example, the distribution of the physicians is concentrated around the cities, whereas the recent retired populations live in housing estates, usually created in the countryside (see C. Voiron's works). How to find an optimal matching?

Consider an attractive region R. Some statistical surveys to the physicians allow us to allocate a quality index $\omega_\varphi(S)$ to each class S of H. This index is a positive number which depends on the proximity to the city center, to services (schools, entertainment), to the sea, to the highways, etc.. Some of these parameters are defined at the level of the quarter, other for larger classes. In addition, the index takes into account the tax advantages given by some communes or some departments. All in all, the energy ω_φ can increase or decrease when going up in the administrative hierarchy. We admit that the quality index of each p.p. is the sum of the indexes of its classes: this makes sense and permit to interpret ω_φ as an energy on H.

The constraint energy ω_∂ is defined like in the previous example, from the convex function f_A of the distance to the housing estates A, with C as maximal acceptable distance. The minimization process is then led in two steps as in the previous example.

7.4 Logical combinations of the constraints

In the multi-constraints situations, the minimization of ω_φ becomes subject to p constrained energies $\omega_{\partial j}$ and p cost partitions π_{Cj}, $1 \leq j \leq p$. We draw from Relation (7) that, for ω_∂ inf-modular, the logical intersection (resp. union) of p constraints $\pi \preceq_{\omega_{\partial j}} \pi_{C_i}$ is equivalent to the unique condition $\pi \geq \vee\{\pi_{Cj}, 1 \leq j \leq p\}$ (resp. $\pi \geq \wedge\{\pi_{Cj}, 1 \leq j \leq p\}$), which brings us back to the scalar case. Note that the situation when one constraint at least is satisfied cannot be solved by a Lagrangian based approach.

Multi-constrained situations occur with multi-spectral images [2], [18] or also with color images. One seeks the largest partition into classes where each color of f varies at most by C_1 for the reds, C_2 for the greens, and C_3 for the blues. A solution can be the following. A hierarchy H of segmentations of the image color f has been obtained from the luminance. It yields the refinement lattice $\Pi(\leq, H)$. The energy ω_φ to minimize is climbing, e.g. Mumforf and Shah functional, and holds on the luminance. The constraints are three sup-modular functions

$$\omega_{\partial j}(S) = \max[f_j(x), x \in S] - \min[f_j(x), x \in S] \qquad j \in [1, 2, 3] \tag{12}$$

which must be $\leq kj$ respectively. These three conditions yield to three constraint partitions π_{Cj} of refinement supremum $\pi_C = \vee \pi_{Cj}$. The problem amounts now to find the minimal cut in the energetic lattice $\Pi(\preceq_\varphi, H'')$ where H'' is the partial hierarchy identical to H below and on π_C, and where all classes above π_C are removed.

8 Conclusion

An alternative solution to the constrained optimization problem has been proposed. The classical technique, which goes back to Lagrange, consists in

1. giving a numerical value to each element (e.g. in giving an energy to each partition), and minimizing this numerical function,
2. setting the problem in a new space, where the constrained optimization becomes an absolute one.

We have shown, by the toy example of figure 3, that in case of partitions, the first point was not sufficient. A partition should not be equivalent to a significant energy. Indeed, Lagrange method works if we introduce some additional constraint which restricts the domain of the solutions.

In the proposed alternative, the partitions are not compared via their energies, but by an ordering relation which diretcly holds on these partitions, and generates the so called energetic lattice. In the energetic lattice perspective, the second point, above, is expressed by the creation of a new hierarchy, where the energetic minima become absolute.

The two methods do not exactly address the same family of applications. If the numerical value (energy, cost) is the dominant parameter of the problem, then lagrange approach is probably better adapted, though it remains under-determined. If the spatial dissemination of the energy is the most important feature of the problem, then the proposed altenative by energetic lattices will be more adapted.

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