Design of O(ϵ) dwell-time graph for stability of singularly perturbed hybrid linear systems

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Abstract— **The paper deals with singularly perturbed hybrid systems. It proposes a methodology for building a graph defining all the rules that ensure the origin is a stable equilibrium in presence of a dwell-time of order of the parameter defining the ratio between the two time-scales of the system. In this framework one can also treat the corresponding problem for interesting particular cases such as: singularly perturbed switched linear systems without impulses, one scale hybrid systems or one scale switched systems. A numerical example illustrates the theoretical results completing the paper.**

Index Terms— **Singular Perturbation, Switched systems, Reset systems, Dwell-time.**

I. Introduction

Stability and stabilizability of switched linear systems have been well studied since the nineties and one may find many contributions in the literature on this topic (see [1], [2] and references therein). Recently, an improvement of these contributions has been proposed by several researchers that included in the analysis a reasoning borrowed from graph theory [3], [4], [5]. The switching sequences are constrained by an automaton and the main objective is to take into account these constraints to obtain less conservative stability conditions. In other words, the switching rules are formulated as a path in some given graph: each mode is seen as a vertex of a graph and a switch from one mode to another is possible only if a link between the corresponding vertices exists in the graph.

In this context we consider that a question of interest is the following: given a set of linear subsystems, is it possible to build an automaton such that stability is guaranteed for of all the switching sequences constrained by this automaton? To the best of our knowledge, there is no contribution in the literature that answers this question.

The design result proposed in [6] allows to exhibit a switching sequence that stabilizes a switched system under dwell time constraints. The proposed algorithm uses graph theory but it does not allow to build the automaton of all constrained stabilizing switching sequences. In the present work, we consider a more general class of switched linear systems that includes impulses and different time-scales dynamics. In other words, we analyze the class of singularly perturbed switched impulsive linear systems. This class of hybrid singularly perturbed systems appear in various domains of science [7], [8] and engineering [9], [10]. Results on stability analysis of hybrid singularly perturbed systems emphasizing the necessity of dwell-time constraints were obtained in [11], [10], [12].

In this paper we develop a methodology for building a graph defining all the rules that ensure that the origin is a stable equilibrium in presence of a dwell-time of order of the parameter defining the ratio between the two time-scales of the singularly perturbed switched impulsive linear system. Our result contains as particular cases several interesting problems. When the singularly perturbed switched linear systems are without impulses, the aforementioned graph contains only dynamical modes whose slow dynamics share a common Lyapunov function. When we deal with switched impulsive systems evolving on one time-scale the methodology can be used to build the graph defining all the rules that do not require a dwell-time to ensure the origin is a stable equilibrium of the overall system. In the case of switched linear systems without jumps, the graph contains all the modes that share a common Lyapunov function.

The proposed methodology is based on the work in [12] where we have presented dwell-time conditions ensuring stability for a class of singularly perturbed switched impulsive linear systems in which the nature of the variable is switch-dependent. Precisely, these conditions are the sum of two-terms: the first one is present also in the case of linear switched systems evolving on one time-scale while the second is of order of the parameter defining the ratio between the two time-scales of the singularly perturbed system at hand.

The paper is organized as follows. Section II introduces the main concepts and formulates the problem under consideration. Some preliminary results concerning the dwell-time condition ensuring stability for singularly perturbed hybrid systems are recalled in Section III. In Section IV we present our main results on the design of the graph defining all the rules that ensure that the origin is a stable equilibrium in presence of a dwell-time of order of the parameter defining the ratio between the two time-scales of the singularly perturbed hybrid linear system. The reformulation of the main results in some particular cases is done in Section V. Finally, we provide a numerical illustration of the proposed results in Section VI before some concluding remarks.

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NOTATION

Throughout this paper, $\mathbb{R}_+$, $\mathbb{R}^n$ and $\mathbb{R}^{n\times m}$ denote respectively, the set of nonnegative real numbers, the n dimensional Euclidean space and the set of all $n \times m$ real matrices. The identity matrix of dimension n is denoted by $\mathbf{I}_n$. We also denote by $\mathbf{0}_{n,m} \in \mathbb{R}^{n\times m}$ the matrix whose components are all 0. For a matrix $A \in \mathbb{R}^{n\times n}$, $\|A\|$ denotes the spectral norm *i.e.* induced 2 norm. $A > \mathbf{0}$ ($A < \mathbf{0}$) means that A is positive definite (negative definite). We write $A^\top$ and A^{-1} to respectively denote the transpose and the inverse of A. The matrix A is said to be Hurwitz if all its eigenvalues have negative real parts. A is said to be Schur if all its eigenvalues have modulus smaller than one. We also use $x(t_k^+) = \lim_{\delta\to 0,\ \delta>0} x(t_k+\delta)$.

II. PROBLEM FORMULATION

A. Graph representation of switched impulsive systems

Let us consider a switched system of the form:

$$\begin{pmatrix} \dot{x}(t) \\ \varepsilon\dot{z}(t) \end{pmatrix} = A^{\sigma_k} \begin{pmatrix} x(t) \\ z(t) \end{pmatrix},\ \forall t \in (t_k, t_{k+1}],\ k \in \mathbb{N} \quad (1)$$

with impulsive dynamics:

$$\begin{pmatrix} x(t_k^+) \\ z(t_k^+) \end{pmatrix} = J^{\nu_k} \begin{pmatrix} x(t_k) \\ z(t_k) \end{pmatrix}, \quad \forall k \geq 1 \quad (2)$$

where $x(t) \in \mathbb{R}^{n_x}$, $z(t) \in \mathbb{R}^{n_z}$, $0 = t_0 < t_1 < \ldots$ are the instants of discrete events (switches, impulses or both), $\sigma_k \in \mathcal{I}$ and $\nu_k \in \mathcal{J}$ with $\mathcal{I}$ and $\mathcal{J}$ finite sets of indices, A^i and J^j are matrices of appropriate dimensions for all $i \in \mathcal{I}$, $j \in \mathcal{J}$, and $\varepsilon > 0$ is a small parameter characterizing the time scale separation between the slow dynamics of x and the fast dynamics of z.

In the following we call transition the discrete event that takes place at any time t_k. Therefore a switch from the mode $i \in \mathcal{I}$ to mode $i' \in \mathcal{I}$ accompanied by a jump $j \in \mathcal{J}$ is called transition. In order to represent all the possible sequences of transitions we can construct a graph with $|\mathcal{I}|$ vertices representing the dynamical modes of the system. For any $i, j \in \mathcal{I}$ we consider $|\mathcal{J}|$ directed links (i, j) corresponding to the event defined by a switch from mode i to mode j accompanied by an impulse represented by one of the jump matrices $J^\nu,\ \nu \in \mathcal{J}$. We assume that $J^1 = \mathbf{I}$ meaning that one of the previously introduced arcs defines a non-impulsive switch from mode i to mode j. Moreover, $\forall i \in \mathcal{I}$ the graph contains $|\mathcal{J}|-1$ self-loop (i,i) each of them corresponding to an event defined by an impulse represented by one of the jump matrices $J^\nu,\ \nu \in \mathcal{J} \setminus \{1\}$.

Example 1: To illustrate the graph introduced before let us consider $\mathcal{I} = \{1, 2, 3\}$ and $\mathcal{J} = \{1, 2\}$. When a switch without state's jump occurs we use the jump matrix $J^1 = \mathbf{I}$. The arcs associated with this events are represented in blue in Figure 1. When a switch is accompanied by state's jumps we are using the jump matrix $J^2 \neq \mathbf{I}$ and the corresponding arcs are represented in red in Figure 1. The arcs having the same source and sink are associated with state's jumps without change of dynamics (switch).

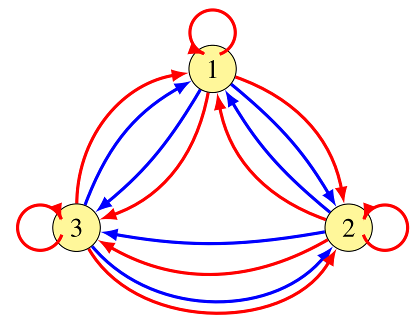

Fig. 1. Illustration of all the transitions associated with a switched impulsive system having 3 modes and 2 jump matrices. Blue arcs are associated with the jump matrix $J^1 = \mathbf{I}$ and red arcs are associated with the jump matrix $J^2 \neq \mathbf{I}$.

In [12] we have characterized the dwell time guaranteeing that zero is globally asymptotically stable equilibrium of (1)-(2) for any event sequence satisfying it.

This paper addresses the reverse problem which consists of finding the subgraph of all transitions guaranteeing that zero is globally asymptotically stable equilibrium of (1)-(2) when the event-rule only satisfies a $O(\varepsilon)$-dwell-time constraint

$$t_{k+1} - t_k \geq \mathcal{O}(\varepsilon), \quad \forall k \in \mathbb{N}.$$

This subgraph is referred in the sequel as the **$\mathcal{O}(\varepsilon)$-subgraph**. For the system in Example 1 a possible representation of this subgraph may be the one in Fig. 2.

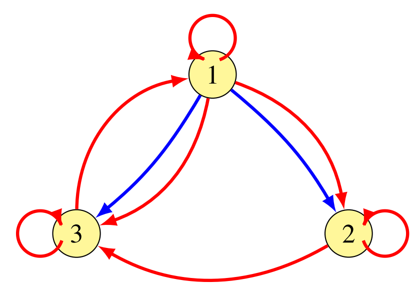

Fig. 2. Illustration of a possible $\mathcal{O}(\varepsilon)$-subgraph associated with a switched impulsive system having 3 modes and 2 jump matrices. Blue arcs are associated with the jump matrix $J^1 = \mathbf{I}$ and red arcs are associated with the jump matrix $J^2 \neq \mathbf{I}$.

Definition 1: A **directed path of length** p in a given digraph (*i.e.* directed graph) $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is a union of directed edges $\bigcup_{k=1}^{p}(i_k, j_k)$ such that $i_{k+1} = j_k$, $\forall k \in \{1, \ldots, p-1\}$. The node j is **connected** with node i in a digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ if there exists at least a directed path in $\mathcal{G}$ from i to j (i.e. $i_1 = i$ and $j_p = j$). A **strongly connected digraph** is such that any two distinct elements are connected. A **strongly connected component** of a digraph is a maximal subset of $\mathcal{V}$ such that any of its two distinct nodes are connected. For a given graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, the subgraph **induced** by a subset of nodes $\mathcal{U} \subseteq \mathcal{V}$ is the graph $\big(\mathcal{U}, \mathcal{E} \cap (\mathcal{U} \times \mathcal{U})\big)$.

Remark 1: In Fig. 2 we have drawn a strongly connected induced spanning subgraph (*i.e.* a subgraph containing all the vertices of the original graph) but in general the $\mathcal{O}(\varepsilon)$-subgraph is a induced spanning subgraph which is not necessarily strongly connected. This means that, when we require stability with an $\mathcal{O}(\varepsilon)$ time between consecutive events and the system starts in a given mode $i \in \mathcal{I}$, some dynamical modes $j \neq i,\ j \in \mathcal{I}$ of the system are forbidden.

III. Preliminaries

A. Change of variable

For $i \in \mathcal{I}$, $j \in \mathcal{J}$, let

$$A^i = \begin{pmatrix} A^i_{11} & A^i_{12} \\ A^i_{21} & A^i_{22} \end{pmatrix},\ J^j = \begin{pmatrix} J^j_{11} & J^j_{12} \\ J^j_{21} & J^j_{22} \end{pmatrix},$$

where $A^i_{11}, J^j_{11} \in \mathbb{R}^{n_x \times n_x}$, and A^i_{22}, A^i_{12}, A^i_{21}, J^j_{22}, J^j_{12}, J^j_{21} are of appropriate dimensions.

Let us impose the following standard assumption [13] in the singular perturbation theory framework:

Assumption 1: A^i_{22} is non-singular for all $i \in \mathcal{I}$.

Then, we perform the following time dependent change of variable:

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = P_{\sigma_k} \begin{pmatrix} x(t) \\ z(t) \end{pmatrix},\ \forall t \in [t_k, t_{k+1}),\ k \in \mathbb{N} \quad (3)$$

where, for all $i \in \mathcal{I}$

$$P_i = \begin{pmatrix} \mathbf{I}_{n_x} & \mathbf{0}_{n_x, n_z} \\ (A^i_{22})^{-1} A^i_{21} & \mathbf{I}_{n_z} \end{pmatrix}.$$

Using (3), the continuous dynamics (1) in the variables x, y becomes:

$$\begin{pmatrix} \dot{x}(t) \\ \varepsilon \dot{y}(t) \end{pmatrix} = \begin{pmatrix} A^{\sigma_k}_0 & B^{\sigma_k}_1 \\ \varepsilon B^{\sigma_k}_2 & A^{\sigma_k}_{22} + \varepsilon B^{\sigma_k}_3 \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}, \quad \forall t \in [t_k, t_{k+1}),\ k \in \mathbb{N} \quad (4)$$

where for all $i \in \mathcal{I}$ one has

$$A^i_0 = A^i_{11} - A^i_{12}(A^i_{22})^{-1} A^i_{21},\ B^i_1 = A^i_{12},$$
$$B^i_2 = (A^i_{22})^{-1} A^i_{21} A^i_0,\ B^i_3 = (A^i_{22})^{-1} A^i_{21} A^i_{12}.$$

Similarly, the jump map (2) is rewritten in the x, y variables as:

$$\begin{pmatrix} x(t_k) \\ y(t_k) \end{pmatrix} = R^{\sigma_{k-1} \xrightarrow{\nu_k} \sigma_k} \begin{pmatrix} x(t^-_k) \\ y(t^-_k) \end{pmatrix}, \quad \forall k \geq 1 \quad (5)$$

where for all $i, i' \in \mathcal{I}$, $j \in \mathcal{J}$,

$$R^{i \xrightarrow{j} i'} = P_{i'} J^j P^{-1}_i = \begin{pmatrix} R^{i \xrightarrow{j} i'}_{11} & R^{i \xrightarrow{j} i'}_{12} \\ R^{i \xrightarrow{j} i'}_{21} & R^{i \xrightarrow{j} i'}_{22} \end{pmatrix}$$

with

$$R^{i \xrightarrow{j} i'}_{11} = J^j_{11} - J^j_{12}(A^i_{22})^{-1} A^i_{21}, R^{i \xrightarrow{j} i'}_{12} = J^j_{12},$$
$$R^{i \xrightarrow{j} i'}_{21} = (A^{i'}_{22})^{-1} A^{i'}_{21} (J^j_{11} - J^j_{12}(A^i_{22})^{-1} A^i_{21}) + J^j_{21} - J^j_{22}(A^i_{22})^{-1} A^i_{21},$$
$$R^{i \xrightarrow{j} i'}_{22} = (A^{i'}_{22})^{-1} A^{i'}_{21} J^j_{12} + J^j_{22}.$$

Assumption 2: A^i_0 and A^i_{22} are Hurwitz for all $i \in \mathcal{I}$.

This assumption implies that there exist symmetric positive definite matrices $Q^i_s \geq \mathbf{I}_{n_x}$, $Q^i_f \geq \mathbf{I}_{n_z}$, $i \in \mathcal{I}$, and positive numbers λ_s and λ_f such that for all $i \in \mathcal{I}$:

$$A^{i^\top}_0 Q^i_s + Q^i_s A^i_0 \leq -2\lambda_s Q^i_s$$
$$A^{i^\top}_{22} Q^i_f + Q^i_f A^i_{22} \leq -2\lambda_f Q^i_f$$

IV. The subgraph of $\mathcal{O}(\varepsilon)$ dwell-time rules

A. Main results

For each $i \in \mathcal{I}$, let $b^i_1 = \left\|(Q^i_s)^{\frac{1}{2}} B_1 (Q^i_f)^{-\frac{1}{2}}\right\|$, $b^i_2 = \left\|(Q^i_f)^{\frac{1}{2}} B_2 (Q^i_s)^{-\frac{1}{2}}\right\|$ and $b^i_3 = \left\|(Q^i_f)^{\frac{1}{2}} Q_f B_3 (Q^i_f)^{-\frac{1}{2}}\right\|$. Let ε^i_1 be defined as

$$\varepsilon^i_1 = \frac{\lambda_f}{\frac{(b^i_1 + b^i_2)^2}{4\lambda_s} + b^i_3} \quad (6)$$

then it follows from [12, Proposition 1] that the i-th linear dynamics of (4) is Lyapunov stable, for $\varepsilon \in (0, \varepsilon^i_1]$. Let $\varepsilon_2 \in (0, \min(\varepsilon^i_1, \frac{\lambda_f}{\lambda_s}))$ and introduce $\beta^i_1 = \frac{\sqrt{(b^i_2)^2 + (b^i_3)^2}}{\lambda_f}$, $\beta^i_2 = \frac{b^i_1}{\lambda_f - \varepsilon_2 \lambda_s}$ and $\beta^i_3 = \frac{b^i_1 \beta^i_1}{\lambda_s}$. The stability analysis of system (4)-(5) is carried out using the following functions

$$\begin{cases} W_s(t) = \sqrt{x(t)^\top Q^{\sigma_k}_s x(t)} \\ W_f(t) = \sqrt{y(t)^\top Q^{\sigma_k}_f y(t)} \end{cases},\ \forall t \in [t_k, t_{k+1}),\ k \in \mathbb{N}.$$

The next result characterizes the variation of W_s and W_f during the continuous dynamics between two events:

Lemma 1: Under Assumption 2, let $\varepsilon \in (0, \varepsilon_2)$, and let $\tau_k = t_{k+1} - t_k$ for a sequence $(t_k)_{k \geq 0}$ of event times. Then for all $k \in \mathbb{N}$,

$$W_s(t^-_{k+1}) \leq W_s(t_k)(\mathrm{e}^{-\lambda_s \tau_k} + \varepsilon \beta^{\sigma_k}_3) + W_f(t_k)\varepsilon(\beta^{\sigma_k}_2 + \beta^{\sigma_k}_3)$$
$$W_f(t^-_{k+1}) \leq W_s(t_k)\varepsilon \beta^{\sigma_k}_1 + W_f(t_k)\big(\mathrm{e}^{-\frac{\lambda_f}{\varepsilon}\tau_k} + \varepsilon \beta^{\sigma_k}_1\big).$$

Proof: This is a refined version of Lemma 4 in [12]. ■

In the following we characterize the behavior of W_s and W_f when the event $i \xrightarrow{j} i'$ (*i.e.* switch from mode i to mode i' joined by a state jump defined by the matrix J^j) occurs.

For all $i, i' \in \mathcal{I}$ and $j \in \mathcal{J}$ we introduce the following supplementary notation:

$$\begin{aligned} \gamma^{i \xrightarrow{j} i'}_{11} &= \left\|(Q^{i'}_s)^{\frac{1}{2}} R^{i \xrightarrow{j} i'}_{11} (Q^i_s)^{-\frac{1}{2}}\right\|, \\ \gamma^{i \xrightarrow{j} i'}_{12} &= \left\|(Q^{i'}_s)^{\frac{1}{2}} R^{i \xrightarrow{j} i'}_{12} (Q^i_f)^{-\frac{1}{2}}\right\|, \\ \gamma^{i \xrightarrow{j} i'}_{21} &= \left\|(Q^{i'}_f)^{\frac{1}{2}} R^{i \xrightarrow{j} i'}_{21} (Q^i_s)^{-\frac{1}{2}}\right\|, \\ \gamma^{i \xrightarrow{j} i'}_{22} &= \left\|(Q^{j}_f)^{\frac{1}{2}} R^{i \xrightarrow{j} i'}_{22} (Q^i_f)^{-\frac{1}{2}}\right\|. \end{aligned} \quad (7)$$

Then, we have the following result:

Lemma 2: Let a sequence $(t_k)_{k\geq 0}$ of event times, then for all $k \geq 1$,

$$W_s(t_k) \leq \gamma_{11}^{\sigma_{k-1}\xrightarrow{\nu_k}\sigma_k} W_s(t_k^-) + \gamma_{12}^{\sigma_{k-1}\xrightarrow{\nu_k}\sigma_k} W_f(t_k^-)$$
$$W_f(t_k) \leq \gamma_{21}^{\sigma_{k-1}\xrightarrow{\nu_k}\sigma_k} W_s(t_k^-) + \gamma_{22}^{\sigma_{k-1}\xrightarrow{\nu_k}\sigma_k} W_f(t_k^-).$$

Proof: A similar result can be found in [12]. ■

In order to keep the notation simple, we introduce the positive matrix parameterized by $\tau > 0$ and $i \in \mathcal{I}$:

$$M_\tau^i = \begin{pmatrix} \mathrm{e}^{-\lambda_s \tau} + \varepsilon\beta_3^i & \varepsilon(\beta_2^i + \beta_3^i) \\ \varepsilon\beta_1^i & \mathrm{e}^{-\frac{\lambda_f}{\varepsilon}\tau} + \varepsilon\beta_1^i \end{pmatrix}.$$

Let us also consider the positive matrix parameterized by $i, i' \in \mathcal{I}$ and $j \in \mathcal{J}$:

$$\Gamma^{i\xrightarrow{j}i'} = \begin{pmatrix} \gamma_{11}^{i\xrightarrow{j}i'} & \gamma_{12}^{i\xrightarrow{j}i'} \\ \gamma_{21}^{i\xrightarrow{j}i'} & \gamma_{22}^{i\xrightarrow{j}i'} \end{pmatrix}.$$

Lemma 3: Under Assumption 2, let $\varepsilon \in (0, \varepsilon_2)$, and let $\tau_k = t_{k+1} - t_k$ for a sequence $(t_k)_{k\geq 0}$ of event times. Then for all $k \in \mathbb{N}$,

$$\begin{pmatrix} W_s(t_k) \\ W_f(t_k) \end{pmatrix} \leq \Gamma^{\sigma_{k-1}\xrightarrow{\nu_k}\sigma_k} M_{\tau_k}^{\sigma_{k-1}} \begin{pmatrix} W_s(t_{k-1}) \\ W_f(t_{k-1}) \end{pmatrix}.$$

Proof: Straightforward from Lemma 1 and 2. ■

Remark 2: It is noteworthy that the matrix $\Gamma^{i\xrightarrow{j}i'} M_\tau^i$ characterize the behavior of the system (1)-(2) when the event $i \xrightarrow{j} i'$ occurs after an evolution of τ seconds in mode i. Consequently we want to find all the events $i \xrightarrow{j} i'$ ensuring that $\Gamma^{i\xrightarrow{j}i'} M_\tau^i$ is Schur for $\tau = \mathcal{O}(\varepsilon)$.

For our main results let us introduce the following set of events:

$$\bar{\mathcal{E}} = \left\{ i \xrightarrow{j} i' \mid (\gamma_{11}^{i\xrightarrow{j}i'} < 1) \vee \left((\gamma_{11}^{i\xrightarrow{j}i'} = 1) \wedge (\gamma_{12}^{i\xrightarrow{j}i'} = 0) \right) \right\}$$

Proposition 1: Let Assumption 2 holds. There exists $\varepsilon_3 > 0$ such that for all $\varepsilon \in (0, \varepsilon_3)$ we can choose $\tau = \eta(\varepsilon) = \mathcal{O}(\varepsilon)$ such that $\Gamma^{i\xrightarrow{j}i'} M_\tau^i$ is Schur $\forall\ i \xrightarrow{j} i' \in \bar{\mathcal{E}}$.

Proof: Let us remark that

$$\Gamma^{i\xrightarrow{j}i'} M_\tau^i = \begin{pmatrix} \gamma_{11}^{i\xrightarrow{j}i'} e^{-\lambda_s \tau} + \varepsilon\delta_1^{i\xrightarrow{j}i'} & \gamma_{12}^{i\xrightarrow{j}i'} e^{-\frac{\lambda_f}{\varepsilon}\tau} + \varepsilon\delta_2^{i\xrightarrow{j}i'} \\ \gamma_{21}^{i\xrightarrow{j}i'} e^{-\lambda_s \tau} + \varepsilon\delta_3^{i\xrightarrow{j}i'} & \gamma_{22}^{i\xrightarrow{j}i'} e^{-\frac{\lambda_f}{\varepsilon}\tau} + \varepsilon\delta_4^{i\xrightarrow{j}i'} \end{pmatrix}$$

where

$$\begin{aligned} \delta_1^{i\xrightarrow{j}i'} &= \gamma_{11}^{i\xrightarrow{j}i'} \beta_3^i + \gamma_{12}^{i\xrightarrow{j}i'} \beta_1^i, \\ \delta_2^{i\xrightarrow{j}i'} &= \gamma_{11}^{i\xrightarrow{j}i'} (\beta_2^i + \beta_3^i) + \gamma_{12}^{i\xrightarrow{j}i'} \beta_1^i, \\ \delta_3^{i\xrightarrow{j}i'} &= \gamma_{21}^{i\xrightarrow{j}i'} \beta_3^i + \gamma_{22}^{i\xrightarrow{j}i'} \beta_1^i, \\ \delta_4^{i\xrightarrow{j}i'} &= \gamma_{21}^{i\xrightarrow{j}i'} (\beta_2^i + \beta_3^i) + \gamma_{22}^{i\xrightarrow{j}i'} \beta_1^i. \end{aligned} \tag{8}$$

The positive matrix $\Gamma^{i\xrightarrow{j}i'} M_\tau^i$ is Schur if and only if there exists $p \in \mathbb{R}_+^2$, such that component-wise $\left(\Gamma^{i\xrightarrow{j}i'} M_\tau^i\right)^\top p < p$ (see e.g. [14]). Let us look for p under the form $(1, a)^\top$ with $a > 0$. Then, $\left(\Gamma^{i\xrightarrow{j}i'} M_\tau^i\right)^\top p < p$ is equivalent to

$$\left\{ \begin{array}{lll} \gamma_{11}^{i\xrightarrow{j}i'} e^{-\lambda_s \tau} + \varepsilon\delta_1^{i\xrightarrow{j}i'} + a\gamma_{21}^{i\xrightarrow{j}i'} e^{-\lambda_s \tau} + a\varepsilon\delta_3^{i\xrightarrow{j}i'} & < & 1 \\ \gamma_{12}^{i\xrightarrow{j}i'} e^{-\frac{\lambda_f}{\varepsilon}\tau} + \varepsilon\delta_2^{i\xrightarrow{j}i'} + a\gamma_{22}^{i\xrightarrow{j}i'} e^{-\frac{\lambda_f}{\varepsilon}\tau} + a\varepsilon\delta_4^{i\xrightarrow{j}i'} & < & a \end{array} \right.$$

which is also equivalent to

$$\left\{ \begin{array}{lll} \tau & > & \frac{1}{\lambda_s} \ln\left(\frac{\gamma_{11}^{i\xrightarrow{j}i'} + a\gamma_{21}^{i\xrightarrow{j}i'}}{1 - \varepsilon\delta_1^{i\xrightarrow{j}i'} - a\varepsilon\delta_3^{i\xrightarrow{j}i'}} \right) \\ \tau & > & \frac{\varepsilon}{\lambda_f} \ln\left(\frac{\gamma_{12}^{i\xrightarrow{j}i'} + a\gamma_{22}^{i\xrightarrow{j}i'}}{a - \varepsilon\delta_2^{i\xrightarrow{j}i'} - a\varepsilon\delta_4^{i\xrightarrow{j}i'}} \right) \end{array} \right. . \tag{9}$$

We emphasize that the right hand side of the inequalities in (9) makes sense only for the values of ε ensuring that

$$1 - \varepsilon\delta_1^{i\xrightarrow{j}i'} - a\varepsilon\delta_3^{i\xrightarrow{j}i'} > 0 \text{ and } a - \varepsilon\delta_2^{i\xrightarrow{j}i'} - a\varepsilon\delta_4^{i\xrightarrow{j}i'} > 0.$$

This inequalities hold for all $i, i' \in \mathcal{I}, j \in \mathcal{J}$ if

$$1 - \varepsilon\delta_1 - a\varepsilon\delta_3 > 0 \text{ and } a - \varepsilon\delta_2 - a\varepsilon\delta_4 > 0 \tag{10}$$

where

$$\delta_1 = \max_{i,i' \in \mathcal{I}, j \in \mathcal{J}} \delta_1^{i\xrightarrow{j}i'}, \quad \delta_2 = \max_{i,i' \in \mathcal{I}, j \in \mathcal{J}} \delta_2^{i\xrightarrow{j}i'},$$
$$\delta_3 = \max_{i,i' \in \mathcal{I}, j \in \mathcal{J}} \delta_3^{i\xrightarrow{j}i'}, \quad \delta_4 = \max_{i,i' \in \mathcal{I}, j \in \mathcal{J}} \delta_4^{i\xrightarrow{j}i'}.$$

We point out that (10) is used to define the value of ε_3 and for all $\varepsilon \in (0, \varepsilon_3)$ the matrix $\Gamma^{i\xrightarrow{j}i'} M_\tau^i$ is Schur for $\tau = \eta(\varepsilon)$ where

$$\begin{aligned} \eta(\varepsilon) = \max_{i\xrightarrow{j}i' \in \bar{\mathcal{E}}} \Big\{ & \frac{1}{\lambda_s} \ln\Big(\frac{\gamma_{11}^{i\xrightarrow{j}i'} + a\gamma_{21}^{i\xrightarrow{j}i'}}{1 - \varepsilon\delta_1^{i\xrightarrow{j}i'} - a\varepsilon\delta_3^{i\xrightarrow{j}i'}} \Big), \\ & \frac{\varepsilon}{\lambda_f} \ln\Big(\frac{\gamma_{12}^{i\xrightarrow{j}i'} + a\gamma_{22}^{i\xrightarrow{j}i'}}{a - \varepsilon\delta_2^{i\xrightarrow{j}i'} - a\varepsilon\delta_4^{i\xrightarrow{j}i'}} \Big) \Big\} \end{aligned} . \tag{11}$$

It remains to show that $\eta(\varepsilon) = \mathcal{O}(\varepsilon)$.

It is noteworthy that $\gamma_{11}^{i\xrightarrow{j}i'} < 1$ implies that there exists $a > 0$ such that $\gamma_{11}^{i\xrightarrow{j}i'} + a\gamma_{21}^{i\xrightarrow{j}i'} < 1$ as well. In this case $\frac{1}{\lambda_s} \ln\left(\frac{\gamma_{11}^{i\xrightarrow{j}i'} + a\gamma_{21}^{i\xrightarrow{j}i'}}{1 - a\varepsilon\delta_3^{i\xrightarrow{j}i'} - \varepsilon\delta_1^{i\xrightarrow{j}i'}} \right) = \mathcal{O}(\varepsilon)$ and $\frac{\varepsilon}{\lambda_f} \ln\left(\frac{\gamma_{12}^{i\xrightarrow{j}i'} + a\gamma_{22}^{i\xrightarrow{j}i'}}{a - \varepsilon\delta_2^{i\xrightarrow{j}i'} - a\varepsilon\delta_4^{i\xrightarrow{j}i'}} \right) = \mathcal{O}(\varepsilon)$ yielding $\eta(\varepsilon) = \mathcal{O}(\varepsilon)$.

If $\gamma_{11}^{i\xrightarrow{j}i'} = 1$ we have to choose $a = \underline{a}\varepsilon = \mathcal{O}(\varepsilon)$ with $\underline{a} > 0$ in order to have $\frac{1}{\lambda_s} \ln\left(\frac{\gamma_{11}^{i\xrightarrow{j}i'} + a\gamma_{21}^{i\xrightarrow{j}i'}}{1 - \varepsilon\delta_1^{i\xrightarrow{j}i'} - a\varepsilon\delta_3^{i\xrightarrow{j}i'}} \right) = \mathcal{O}(\varepsilon)$.

If $a = \underline{a}\varepsilon = \mathcal{O}(\varepsilon)$ with $\underline{a} > 0$, it can be easily checked that if $\gamma_{12}^{i\xrightarrow{j}i'} \neq 0$ one obtains $\frac{\varepsilon}{\lambda_f} \ln\left(\frac{\gamma_{12}^{i\xrightarrow{j}i'} + a\gamma_{22}^{i\xrightarrow{j}i'}}{a - \varepsilon\delta_2^{i\xrightarrow{j}i'} - a\varepsilon\delta_4^{i\xrightarrow{j}i'}} \right) = \mathcal{O}(\varepsilon \ln(\varepsilon))$.

When $\gamma_{12}^{i\xrightarrow{j}i'} = 0$ and $a = \underline{a}\varepsilon = \mathcal{O}(\varepsilon)$ with $\underline{a} > 0$ yields $\frac{\varepsilon}{\lambda_f} \ln\left(\frac{\gamma_{12}^{i\xrightarrow{j}i'} + a\gamma_{22}^{i\xrightarrow{j}i'}}{a - \varepsilon\delta_2^{i\xrightarrow{j}i'} - a\varepsilon\delta_4^{i\xrightarrow{j}i'}} \right) = \mathcal{O}(\varepsilon)$ since one has that

$$\ln\left(\frac{\gamma_{12}^{i\xrightarrow{j}i'} + a\gamma_{22}^{i\xrightarrow{j}i'}}{a - \varepsilon\delta_2^{i\xrightarrow{j}i'} - a\varepsilon\delta_4^{i\xrightarrow{j}i'}} \right) = \ln\left(\frac{\underline{a}\gamma_{22}^{i\xrightarrow{j}i'}}{\underline{a} - \delta_2^{i\xrightarrow{j}i'} - \underline{a}\varepsilon\delta_4^{i\xrightarrow{j}i'}} \right).$$

■

Theorem 1: Let Assumption 2 holds. It exists $\varepsilon_3 > 0$ such that, for all $\varepsilon \in (0, \varepsilon_3)$ the graph $(\mathcal{I}, \bar{\mathcal{E}})$ is an $\mathcal{O}(\varepsilon)$- subgraph for the system (1)-(2).

Remark 3: It is noteworthy that under Assumption 2, 0 is a stable equilibrium point of the switched impulsive system (1)-(2) if a dwell-time $t_{k+1} - t_k = \mathcal{O}(1)$ is imposed. Theorem 1 says that when we can only impose $t_{k+1} - t_k = \mathcal{O}(\varepsilon)$ the stability is guaranteed if the events/transitions are defined by the $\mathcal{O}(\varepsilon)$-subgraph $(\mathcal{I}, \bar{\mathcal{E}})$.

It is worth pointing out that for all i the transition $i \xrightarrow{1} i$ belongs the $\mathcal{O}(\varepsilon)$-subgraph. However, we disregard them because neither a switch nor a jump occurs at this transition time.

V. Particular cases

In this section we look how the main results of the previous section change in the particular situations of switched singularly perturbed systems or hybrid/switched systems evolving on one time-scale.

A. Switched singularly perturbed systems

Let us consider here the class of switched singularly perturbed systems described by

$$\begin{pmatrix} \dot{x}(t) \\ \varepsilon \dot{z}(t) \end{pmatrix} = A^{\sigma_k} \begin{pmatrix} x(t) \\ z(t) \end{pmatrix}, \ \forall t \in [t_k, t_{k+1}), \ k \in \mathbb{N} \tag{12}$$

where $x(t) \in \mathbb{R}^{n_x}$, $z(t) \in \mathbb{R}^{n_z}$, $0 = t_0 < t_1 < \dots$ are the switching instants and $\sigma_k \in \mathcal{I}$. In this case $\mathcal{J} = \{1\}$ and $J^1 = \mathbf{I}$. Consequently, the matrix $R^{i \to i'}$ in (5) is defined as $\begin{pmatrix} \mathbf{I}_{n_x} & \mathbf{0}_{n_x, n_z} \\ R_{21}^{i \to i'} & \mathbf{I}_{n_z} \end{pmatrix}$ yielding $\gamma_{11}^{i \to i'} = \left\| (Q_s^{i'})^{\frac{1}{2}} (Q_s^i)^{-\frac{1}{2}} \right\|$ and $\gamma_{12}^{i \to i'} = 0, \forall i, i' \in \mathcal{I}$.

Corollary 1: Under Assumption 2, the strongly connected components of the $\mathcal{O}(\varepsilon)$-subgraph $(\mathcal{I}, \bar{\mathcal{E}})$ associated with system (12) are given by the nodes characterized by reduced (slow) systems sharing common Lyapunov functions.

B. Switched systems

When the processes involved in (12). evolves on a single time scale the dynamics is described by

$$\dot{x}(t) = A^{\sigma_k} x(t), \ \forall t \in [t_k, t_{k+1}), \ k \in \mathbb{N} \tag{13}$$

where $0 = t_0 < t_1 < \dots$ are the switching instants and $\sigma_k \in \mathcal{I}$. In this case, instead of $\mathcal{O}(\varepsilon)$-subgraph we may be interested in $\mathcal{O}(0)$-subgraph (*i.e.* the subgraph defining the switching rules that do not require any dwell-time). In this case the matrix $R^{i \to i'} = \mathbf{I}$ and Corollary 1 rewrites as:

Corollary 2: The strongly connected components of the $\mathcal{O}(0)$-subgraph $(\mathcal{I}, \mathcal{E}_0)$ associated with system (13) are given by the nodes characterized by systems sharing common Lyapunov functions.

C. Hybrid systems

Let us consider now the following hybrid dynamics:

$$\begin{cases} \dot{x}(t) = A^{\sigma_k} x(t) \\ x(t_k) = J^{\nu_k} x(t_k^-) \end{cases}, \ \forall t \in [t_k, t_{k+1}), \ k \in \mathbb{N} \tag{14}$$

where $0 = t_0 < t_1 < \dots$ are the times of discrete events (switches and/or impulses), $\sigma_k \in \mathcal{I}$ and $\nu_k \in \mathcal{J}$ with $\mathcal{I}$ and $\mathcal{J}$ finite sets of indices. In this case one has $R^{i \xrightarrow{j} i'} = J^j$ and Proposition 1 writes as follows:

Corollary 3: The $\mathcal{O}(0)$-subgraph associated with (14) contains all the links $i \xrightarrow{j} i'$ satisfying $\left\| (Q^{i'})^{\frac{1}{2}} J^j (Q^i)^{-\frac{1}{2}} \right\| \leq 1$ with Q^i the matrix defining the Lyapunov function associated with the mode $i \in \mathcal{I}$.

VI. Illustrative example

Let us consider $\mathcal{I} = \{1, 2, 3, 4\}$, $\mathcal{J} = \{1, 2\}$. For all $k \in \mathbb{N}$ we define the following dynamics:

$$\begin{aligned} \begin{pmatrix} \dot{x}(t) \\ \varepsilon \dot{z}(t) \end{pmatrix} &= A^i \begin{pmatrix} x(t) \\ z(t) \end{pmatrix}, \forall t \in [t_k, t_{k+1}), \ i \in \mathcal{I} \\ \begin{pmatrix} x(t_k) \\ z(t_k) \end{pmatrix} &= J^j \begin{pmatrix} x(t_k^-) \\ z(t_k^-) \end{pmatrix}, \quad \forall k \geq 1, \ j \in \mathcal{J} \end{aligned} \tag{15}$$

where

$$A^1 = \begin{pmatrix} -2 & 0.5 \\ -0.5 & -1 \end{pmatrix}, \ A^2 = \begin{pmatrix} -1 & 0.5 \\ -3 & -1 \end{pmatrix},$$
$$A^3 = \begin{pmatrix} 1 & -2 \\ 1 & -0.5 \end{pmatrix}, \ A^4 = \begin{pmatrix} -2.5 & -1 \\ 2 & -2 \end{pmatrix},$$
$$J^1 = \mathbf{I}_2, \quad J^2 = \begin{pmatrix} 0.4 & 0.5 \\ 0.4 & 0.3 \end{pmatrix}$$

The following table summarizes the system's data:

	Mode 1	Mode 2	Mode 3	Mode 4
A_0	-2.25	-2.5	-3	-3.5
A_{22}	-1	-1	-0.5	-2
B_1	0.5	0.5	-2	-1
B_2	-1.125	-7.5	6	3.5
B_3	0.25	1.5	4	1
Q_s	1	1	1	1.2
Q_f	1	1.1	1	1.05
b_1	0.5	0.47	2	1.06
b_2	1.125	7.86	6	3.27
b_3	0.25	1.65	4	1.05
β_1	1.152	8.03	7.21	3.43
β_2	0.52	0.5031	2.1	1.12
β_3	0.5762	3.83	14.4	3.67
ε_1	1.098	0.052	0.05	0.17

We fix $\lambda_s = \lambda_f = 1$, $\varepsilon_2 = 0.05$, $\varepsilon = 10^{-3}$ and we search the associated $(\mathcal{I}, \bar{\mathcal{E}})$ graph. Applying Theorem 1 one obtains that

$$\begin{aligned} \bar{\mathcal{E}} = \{ & 1 \xrightarrow{2} 1, \ 1 \xrightarrow{1} 2, \ 1 \xrightarrow{2} 2, \ 1 \xrightarrow{1} 3, \ 1 \xrightarrow{2} 3, \ 1 \xrightarrow{2} 4, \\ & 2 \xrightarrow{1} 1, \ 2 \xrightarrow{1} 3, \quad 3 \xrightarrow{1} 1, \ 3 \xrightarrow{1} 2, \\ & 4 \xrightarrow{1} 1, \ 4 \xrightarrow{2} 1, \ 4 \xrightarrow{1} 2, \ 4 \xrightarrow{2} 2, \ 4 \xrightarrow{1} 3, \ 4 \xrightarrow{2} 3, \ 4 \xrightarrow{2} 4 \}. \end{aligned}$$

A graphical representation of this $\mathcal{O}(\varepsilon)$-subgraph is given in Fig. 3 below.

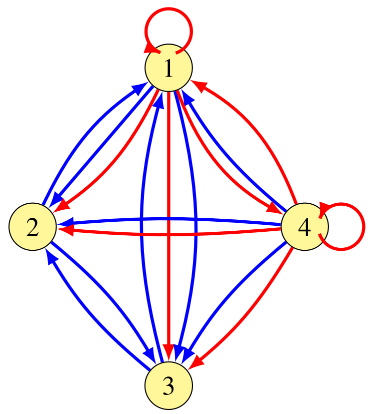

Fig. 3. Illustration of all the transitions associated with the switched impulsive system (15) having 4 modes and 2 jump matrices. Blue arcs are associated with the jump matrix J^1 and red arcs are associated with the jump matrix J^2.

Fig. 4 illustrates the behavior of system (15) with a dwell-time $\tau = 0.0145$ and events belonging to $\bar{\mathcal{E}}$. Basically one observes that 0 is asymptotically stable equilibrium point.

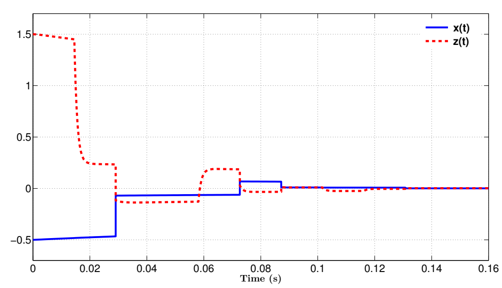

Fig. 4. The event sequence is obtained by repeating the following finite string of transitions: $2 \xrightarrow{1} 1 \xrightarrow{2} 3 \xrightarrow{1} 2 \xrightarrow{1} 1 \xrightarrow{2} 4 \xrightarrow{2} 2$

It is important to note that when the event-sequence contains also elements outside $\bar{\mathcal{E}}$ the stability can be lost (see Fig. 5 for an illustration).

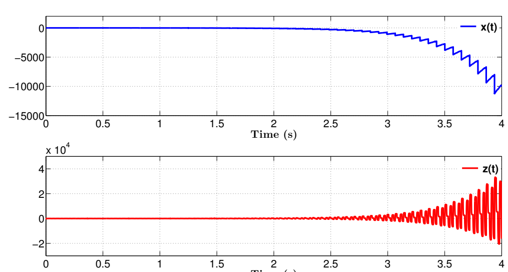

Fig. 5. The event sequence is obtained by repeating the following finite string of transitions: $3 \xrightarrow{2} 2 \xrightarrow{1} 1 \xrightarrow{1} 3 \xrightarrow{1} 2 \xrightarrow{1} 3$. Note that only the first transition ($3 \xrightarrow{2} 2$) does not belong to $\bar{\mathcal{E}}$ in the finite string that is repeated.

VII. Conclusion

In this paper we proposed a methodology to build a graph defining all the rules that ensure that the origin is a stable equilibrium in presence of a dwell-time of order of the parameter defining the ratio between the two time-scales of the singularly perturbed switched impulsive linear system under consideration. We also treated the corresponding problem for interesting particular cases such as: singularly perturbed switched linear systems without impulses, one scale hybrid systems or one scale switched systems. A numerical example illustrates the theoretical results completing the paper.

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